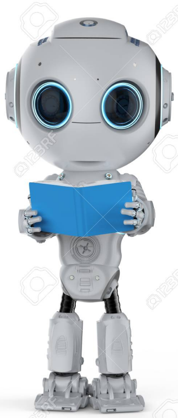
Machine Learning
CS60050

Computational Learning Theory



(Training and Testing)

Learning From Data

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Lecture 5: **Training versus Testing**

Sponsored by Caltech's Provost Office, E&AS Division, and IST • Tuesday, April 17, 2012

Outline

- From training to testing

- Illustrative examples

- Key notion: break point

- Puzzle

The final exam

Testing:

$$\mathbb{P}\left[\,|E_{\text{in}} - E_{\text{out}}| > \epsilon\,\right] \leq \;\; 2 \;\; e^{-2\epsilon^2 N}$$

Training:

$$\mathbb{P}\left[\,|E_{\text{in}} - E_{\text{out}}| > \epsilon\,\right] \leq 2\,M\,e^{-2\epsilon^2 N}$$

Where did the M come from?

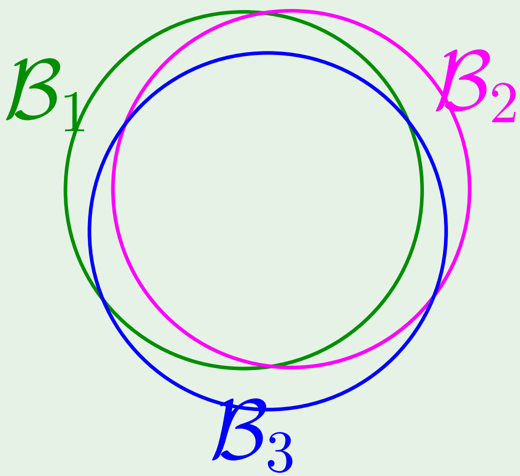
The $\mathcal{B}$ad events $\mathcal{B}_m$ are

$$\text{``}|E_{\text{in}}(h_m) - E_{\text{out}}(h_m)| > \epsilon\text{''}$$

The union bound:

$$\mathbb{P}[\mathcal{B}_1 \textbf{ or } \mathcal{B}_2 \textbf{ or } \cdots \textbf{ or } \mathcal{B}_M]$$

$$\leq \underbrace{\mathbb{P}[\mathcal{B}_1] + \mathbb{P}[\mathcal{B}_2] + \cdots + \mathbb{P}[\mathcal{B}_M]}_{\text{no overlaps: } M \text{ terms}}$$



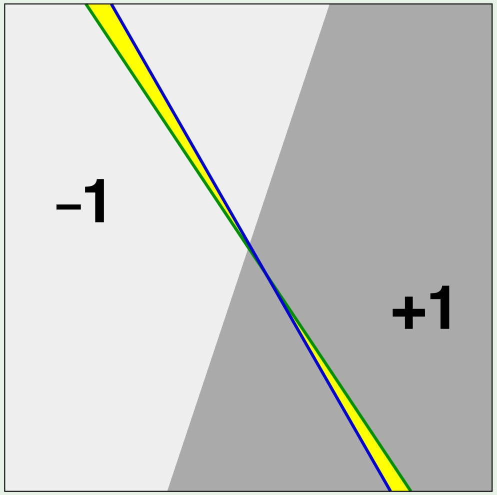
Can we improve on M?

Yes, bad events are *very* overlapping!

ΔE_{out}: change in $+1$ and -1 areas

ΔE_{in}: change in labels of data points

$$|E_{\text{in}}(h_1) - E_{\text{out}}(h_1)| \approx |E_{\text{in}}(h_2) - E_{\text{out}}(h_2)|$$

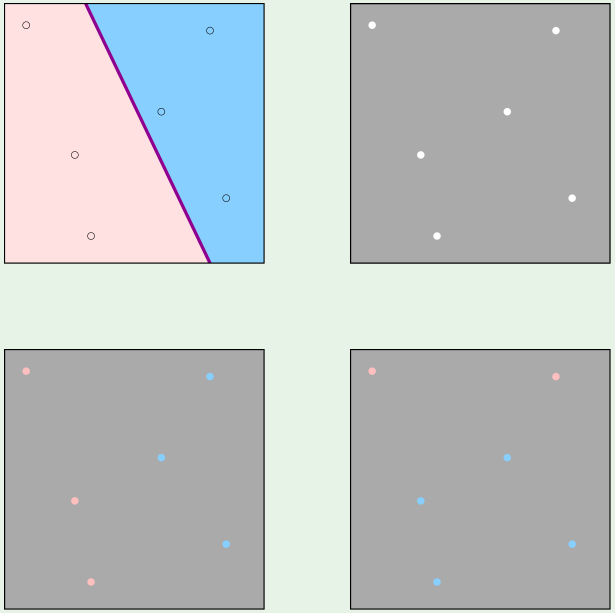


What can we replace M with?

Instead of the whole input space,

we consider a finite set of input points,

and count the number of *dichotomies*



Dichotomies: mini-hypotheses

A hypothesis $h : \mathcal{X} \to \{-1, +1\}$

A dichotomy $h : \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\} \to \{-1, +1\}$

Number of hypotheses $|\mathcal{H}|$ can be infinite

Number of dichotomies $|\mathcal{H}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N)|$ is at most 2^N

Candidate for replacing M

The growth function

The growth function counts the most dichotomies on any N points

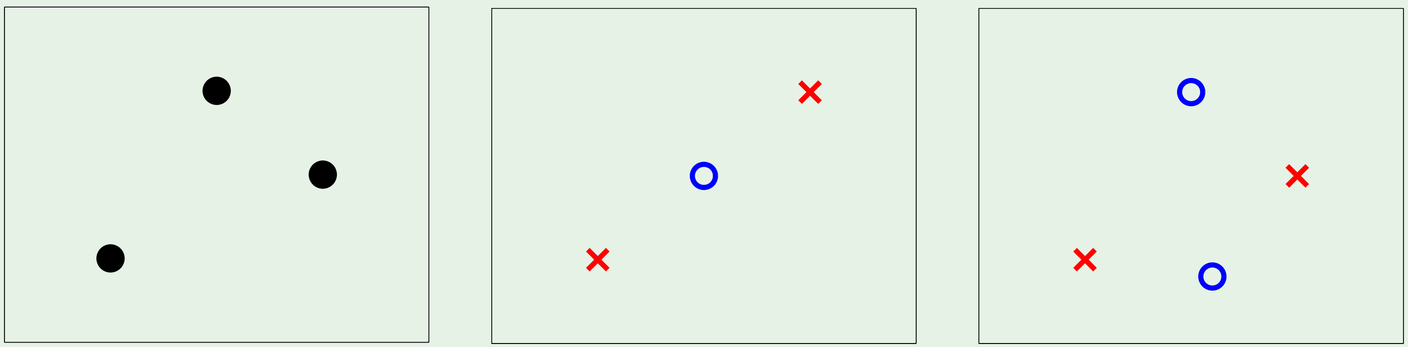
$$m_{\mathcal{H}}(N) = \max_{\mathbf{x}_1,\cdots,\mathbf{x}_N \in \mathcal{X}} |\mathcal{H}(\mathbf{x}_1,\cdots,\mathbf{x}_N)|$$

The growth function satisfies:

$$m_{\mathcal{H}}(N) \leq 2^N$$

Let's apply the definition.

Applying $m_{\mathcal{H}}(N)$ definition - perceptrons



$N = 3$ $\qquad$ $N = 3$ $\qquad$ $N = 4$

$$m_{\mathcal{H}}(3) = 8 \qquad\qquad m_{\mathcal{H}}(4) = 14$$

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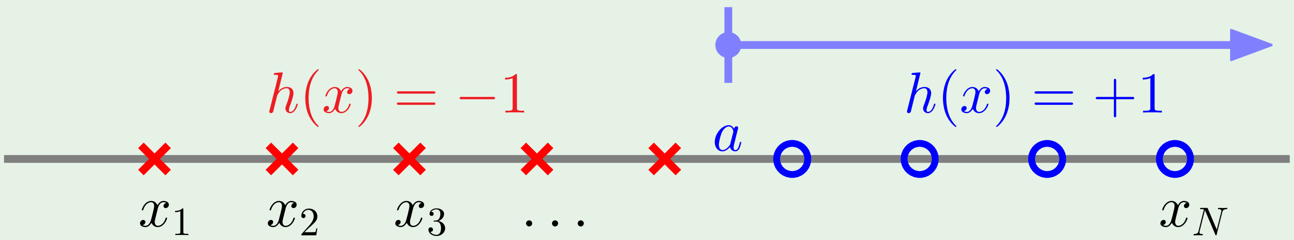
Example 1: positive rays

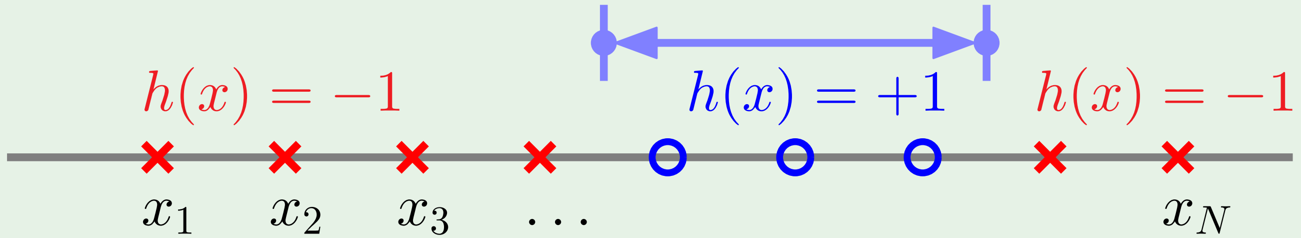


$\mathcal{H}$ is set of $h\colon \mathbb{R} \to \{-1, +1\}$

$h(x) = \text{sign}(x - a)$

$m_{\mathcal{H}}(N) = N + 1$

Example 2: positive intervals



$\mathcal{H}$ is set of $h \colon \mathbb{R} \to \{-1, +1\}$

Place interval ends in two of $N+1$ spots

$$m_{\mathcal{H}}(N) = \binom{N+1}{2} + 1 = \tfrac{1}{2}N^2 + \tfrac{1}{2}N + 1$$

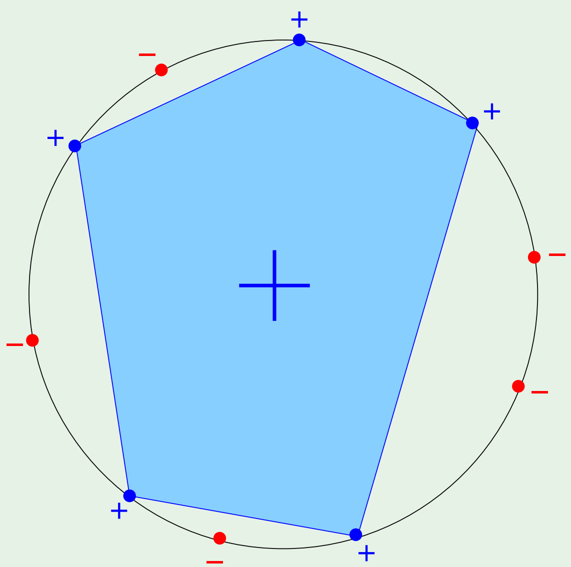
Example 3: convex sets

$\mathcal{H}$ is set of $h\colon \mathbb{R}^2 \to \{-1, +1\}$

$h(\mathbf{x}) = +1$ is convex

$\color{red}{m}_{\mathcal{H}}(N) = 2^N$

The N points are 'shattered' by convex sets



The 3 growth functions

- $\mathcal{H}$ is positive rays:

$$m_{\mathcal{H}}(N) = N + 1$$

- $\mathcal{H}$ is positive intervals:

$$m_{\mathcal{H}}(N) = \tfrac{1}{2}N^2 + \tfrac{1}{2}N + 1$$

- $\mathcal{H}$ is convex sets:

$$m_{\mathcal{H}}(N) = 2^N$$

Back to the big picture

Remember this inequality?

$$\mathbb{P}\left[\,|E_{\text{in}} - E_{\text{out}}| > \epsilon\,\right] \leq 2M\,e^{-2\epsilon^2 N}$$

What happens if $m_{\mathcal{H}}(N)$ replaces M?

$m_{\mathcal{H}}(N)$ polynomial $\implies$ Good!

Just prove that $m_{\mathcal{H}}(N)$ is polynomial?

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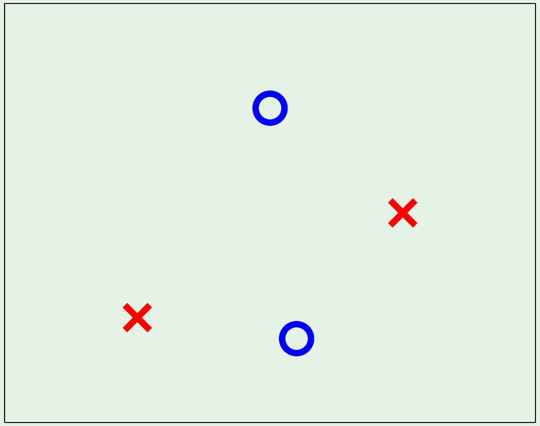
Break point of $\mathcal{H}$

Definition:

If no data set of size k can be shattered by $\mathcal{H}$, then k is a *break point* for $\mathcal{H}$

$$m_{\mathcal{H}}(k) < 2^k$$

For 2D perceptrons, $k = 4$



A bigger data set cannot be shattered either

Break point - the 3 examples

- Positive rays $m_{\mathcal{H}}(N) = N + 1$

break point $k =$ 2

- Positive intervals $m_{\mathcal{H}}(N) = \frac{1}{2}N^2 + \frac{1}{2}N + 1$

break point $k =$ 3

- Convex sets $m_{\mathcal{H}}(N) = 2^N$

break point $k =$'∞'

Main result

No break point $\implies$ $m_{\mathcal{H}}(N) = 2^N$

Any break point $\implies$ $m_{\mathcal{H}}(N)$ is **polynomial** in N

Puzzle

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	●
○	●	○
●	○	○

Thank You!