Notes on the open-loop Nash equilibrium

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August 31, 2022

(Notation follows Ben's memorandum dd. March 27, 2020. I am assuming that everything can be written in per capita form and I am not bothering with accounting identities.)

- We start from the view of the home country. The two main assumption are:

  1. Both countries commit to a strategy prior to $t = 0$. Via the magic of common knowledge, the countries know the strategy of their opponent. This implies that the home country treats all actions of the foreign country as given. Essentially the home country plays the best response against those actions.

  2. Since capital is perfectly mobile, the firms still operate in competitive markets over which countries have limited control. The countries influence the capital stock partially through the capital tax, but for the rest they are bound to market constraints. The world capital stock is allocated to the home country and the foreign country such that the rate of return on capital is equal.

- First, we recursively formulate welfare to keep the welfare function compact. As in our previous paper, we have:

  $$V_t = \mathcal{U}(c_t^y, c_t^o, P_t^w) + \omega V_{t+1}$$

  where

  $$\mathcal{U}(c_t^y, c_t^o, P_t^w) = \log c_t^y + \frac{\beta}{\omega}[\log c_t^o + \chi \log(Q^* - P_t^w)]$$

  Typically, we assume that $\beta = \omega$. The home country maximizes:

  $$V_0 = \sum_{t=0}^{\infty} \omega^t \mathcal{U}(c_t^y, c_t^o, P_t^w)$$

- Instead of focusing on policy instruments, assume that the social planner can set consumption level for the current young and old as he likes (subject to feasibility constraints).[1] The budget constraint for the home country is

$$w_t + a_t(1 + r_t) + \theta_t(y_t - w_t) = c_t^y + c_t^o + a_{t+1} + g_t \tag{1}$$

Note $a_t \equiv s_{t-1}$ to ensure that all time indices start at $t = 0$. Moreover, this is the assets that home country brings over from the previous period (The a stands for assets).[2] The LHS of the budget constraint is income derived from labor, owning assets (plus renting it out to the firms) and tax revenue. The RHS are the four things the government can spend money on (consumption for the young generation, consumption for the old generation, assets for the next period and government spending) . Let λ_t be the associated present-value Lagrange multiplier for this constraint.

- We take an approach with many choice variables of which many will be redundant. This means that we get more but simpler equations. There are two categories of choice variables, dynamic and static. The dynamic choice variables are a_{t+1} and P_{t+1}^w. We impose no bounds on these variables, although e.g. P_{t+1}^w has both a lower and an upper bound (resp. 0 and Q^*): the assumption is that these constraints are never binding. Note that we use '$t+1$' for the dynamic choice variables to get slightly more readable first-order conditions.

- The static choice variables are c_t^y, c_t^o, g_t, k_t, w_t, r_t, y_t, θ_t, k_t^w. We assume that the non-negativity on c_t^y and c_t^o are never binding and that the constraint $\theta_t \leq 1$ is also never binding. There is no non-negativity constraint on θ_t. The Lagrange multiplier for the non-negativity constraints for g_t, k_t, w_t, r_t, y_t and k_t^w are φ_t^g, φ_t^k, φ_t^w, φ_t^r, φ_t^y and φ_t^a, respectively.

- We have the following constraints (see Ben's notes on the market equilibrium), the associated Lagrange-multiplier in parentheses.

$$k_t^w = \pi a_t + (1-\pi)\tilde{a}_t \qquad (\varepsilon_t^a) \tag{2}$$

$$k_t = k_t^w F(\theta_t, \tilde{\theta}_t, \pi) \qquad (\varepsilon_t^k) \tag{3}$$

$$y_t = \Omega_y e^{-\xi P_t^w} k_t^\alpha \qquad (\varepsilon_t^y) \tag{4}$$

$$w_t = (1-\alpha) y_t \qquad (\varepsilon_t^w) \tag{5}$$

$$r_t = \frac{\alpha(1-\theta_t) y_t}{k_t} - \delta \qquad (\varepsilon_t^r) \tag{6}$$

$$P_{t+1}^w = (1-\zeta) P_t^w + \Omega_d L^w \left[\pi y_t e^{-\eta \pi L^w g_t} + (1-\pi)\tilde{y}_t e^{-\eta(1-\pi) L^w \tilde{g}_t}\right] \qquad (\mu_t) \tag{7}$$

where

$$F(\theta, \tilde{\theta}, \pi) = \frac{(1-\theta)^{1/(1-\alpha)}}{\pi(1-\theta)^{1/(1-\alpha)} + (1-\pi)(1-\tilde{\theta})^{1/(1-\alpha)}}$$

[1] Implicitly we assume that the home country has sufficient policy instruments to force the current generations consumption levels in a certain direction.

[2] Instead of talking about savings from the previous period, we talk about financial assets available in the current period.

Observe that partial derivatives are denoted by F_θ etc. Observe that assets of the home country and the foreign country are claims on world capital, see (2) where π is the fraction of the world population living in the home country. The actual capital stock that ends up in the home country is given by (3), where F is a function that equalizes the rate of return in both countries.

- Finally a_0 and P_0^w are given.
- Ergo the Lagrangian is:

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \omega^t \Bigg\{ & \log c_t^y + \frac{\beta}{\omega}[\log c_t^o + \chi \log(Q^* - P_t^w)] \\
& + \lambda_t(w_t + a_t(1+r_t) + \theta_t(y_t - w_t) - c_t^y - c_t^o - a_{t+1} - g_t) \\
& + \mu_t \left(P_{t+1}^w - (1-\zeta)P_t^w - \Omega_d L^w \left[\pi y_t e^{-\eta\pi L^w g_t} + (1-\pi)\tilde{y}_t e^{-\eta(1-\pi)L^w \tilde{g}_t}\right]\right) \\
& + \varepsilon_t^a(k_t^w - \pi a_t - (1-\pi)\tilde{a}_t) \\
& + \varepsilon_t^k(k_t - k_t^w F(\theta_t, \tilde{\theta}_t, \pi)) \\
& + \varepsilon_t^y(y_t - \Omega_y e^{-\xi P_t^w} k_t^\alpha) \\
& + \varepsilon_t^w(w_t - (1-\alpha)y_t) \\
& + \varepsilon_t^r\left(r_t + \delta - \frac{\alpha(1-\theta_t)y_t}{k_t}\right) \\
& + \varphi_t^g g_t + \varphi_t^k k_t + \varphi_t^w w_t + \varphi_t^r r_t + \varphi_t^y y_t + \varphi_t^a k_t^w \Bigg\}
\end{aligned}
$$

Note that the Lagrange multipliers are expressed in present values.

- Below are the first-order conditions. Note that the complementarity conditions, e.g. $\varphi_t^g g_t = 0$, $\varphi_t^g \geq 0$ and $g_t \geq 0$, can be summarized by the Fisher-Burmeister function: $\Phi(\varphi_t^g, g_t) = 0$.[3]

$$\frac{\partial \mathcal{L}}{\partial c_t^y} = 0 \implies \frac{1}{c_t^y} = \lambda_t \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial c_t^o} = 0 \implies \frac{\beta}{\omega c_t^o} = \lambda_t \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \implies \lambda_t = (1+r_{t+1})\omega\lambda_{t+1} - \omega\pi\varepsilon_{t+1}^a \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial P_{t+1}^w} = 0 \implies \mu_t = \omega(1-\zeta)\mu_{t+1} + \frac{\beta\chi}{Q^* - P_{t+1}^w} - \omega\varepsilon_{t+1}^y \xi \Omega_y e^{-\xi P_{t+1}^w} k_{t+1}^\alpha \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial g_t} = 0 \implies \lambda_t = \eta\Omega_d(\pi L^w)^2 \mu_t y_t e^{-\eta\pi L^w g_t} + \varphi_t^g \tag{12}$$

[3] The Fisher-Burmeister function is $\Phi(a,b) = a + b - \sqrt{a^2+b^2}$. It is well-know results that $\Phi(a,b) = 0$ if and only if $a \geq 0$, $b \geq 0$ and $ab = 0$.

$$\frac{\partial \mathcal{L}}{\partial k_t} = 0 \implies \varepsilon_t^k - \alpha \varepsilon_t^y \Omega_y e^{-\xi P_t^w} k_t^{\alpha-1} + \varepsilon_t^r \frac{\alpha(1-\theta_t) y_t}{k_t^2} + \varphi_t^k = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial w_t} = 0 \implies (1-\theta_t)\lambda_t + \varepsilon_t^w + \varphi_t^w = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial r_t} = 0 \implies a_t \lambda_t + \varepsilon_t^r + \varphi_t^r = 0 \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial y_t} = 0 \implies \begin{array}{c} \lambda_t \theta_t - \mu_t \Omega_d L^w \pi e^{-\eta \pi L^w g_t} + \varepsilon_t^y - (1-\alpha)\varepsilon_t^w \\ -\varepsilon_t^r \frac{\alpha(1-\theta_t)}{k_t} + \varphi_t^y = 0 \end{array} \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_t} = 0 \implies \lambda_t (y_t - w_t) + \varepsilon_t^r \frac{\alpha y_t}{k_t} - \varepsilon_t^k k_t^w F_\theta(\theta_t, \tilde{\theta}_t, \pi) = 0 \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial k_t^w} = 0 \implies \varepsilon_t^a - \varepsilon_t^k F(\theta_t, \tilde{\theta}_t, \pi) + \varphi_t^a = 0 \tag{18}$$

plus

$$\Phi(\varphi_t^g, g_t) = 0 \tag{19}$$

$$\Phi(\varphi_t^k, k_t) = 0 \tag{20}$$

$$\Phi(\varphi_t^w, w_t) = 0 \tag{21}$$

$$\Phi(\varphi_t^r, r_t) = 0 \tag{22}$$

$$\Phi(\varphi_t^y, y_t) = 0 \tag{23}$$

$$\Phi(\varphi_t^a, k_t^w) = 0 \tag{24}$$

- Note that for the foreign country the equations are almost identical. However in equilibrium they need to make the same choices for k_t^w, P_{t+1}^w. Therefore (2) and (7) are not repeated and these variables do not receives tildes (to indicate that these variables pertain to the foreign country). The same is true for the non-negativity constraint multiplier φ_t^a. The associated Lagrange-multipliers, ε_t^a and μ_t, are potentially different.

- Note that we do not need to add the constraint that $r_t = \tilde{r}_t$ as capital is allocated such that return on capital is equal in both countries (this is what the F-function does).

- The additional equations:

$$\tilde{w}_t + \tilde{a}_t(1+\tilde{r}_t) + \tilde{\theta}_t(\tilde{y}_t - \tilde{w}_t) = \tilde{c}_t^y + \tilde{c}_t^o + \tilde{a}_{t+1} + \tilde{g}_t \tag{25}$$

$$\tilde{k}_t = k_t^w F(\tilde{\theta}_t, \theta_t, 1-\pi) \tag{26}$$

$$\tilde{y}_t = \Omega_y e^{-\xi P_t^w} \tilde{k}_t^\alpha \tag{27}$$

$$\tilde{w}_t = (1-\alpha)\tilde{y}_t \tag{28}$$

$$\tilde{r}_t = \frac{\alpha(1-\tilde{\theta}_t)\tilde{y}_t}{\tilde{k}_t} - \delta \tag{29}$$

$$\frac{1}{\tilde{c}^y_t} = \tilde{\lambda}_t \tag{30}$$

$$\frac{\beta}{\omega\tilde{c}^o_t} = \tilde{\lambda}_t \tag{31}$$

$$\tilde{\lambda}_t = (1+\tilde{r}_{t+1})\omega\tilde{\lambda}_{t+1} - \omega(1-\pi)\tilde{\varepsilon}^a_{t+1} \tag{32}$$

$$\tilde{\mu}_t = \omega(1-\zeta)\tilde{\mu}_{t+1} + \frac{\beta\chi}{Q^* - P^w_{t+1}} - \omega\tilde{\varepsilon}^y_{t+1}\xi\Omega_y e^{-\xi P^w_{t+1}}\tilde{k}^{\alpha}_{t+1} \tag{33}$$

$$\tilde{\lambda}_t = \eta\Omega_d((1-\pi)L^w)^2\tilde{\mu}_t\tilde{y}_t e^{-\eta(1-\pi)L^w\tilde{g}_t} + \tilde{\varphi}^g_t \tag{34}$$

$$\tilde{\varepsilon}^k_t - \alpha\tilde{\varepsilon}^y_t\Omega_y e^{-\xi P^w_t}\tilde{k}^{\alpha-1}_t + \tilde{\varepsilon}^r_t\frac{\alpha(1-\tilde{\theta}_t)\tilde{y}_t}{\tilde{k}^2_t} + \tilde{\varphi}^k_t = 0 \tag{35}$$

$$(1-\tilde{\theta}_t)\tilde{\lambda}_t + \tilde{\varepsilon}^w_t + \tilde{\varphi}^w_t = 0 \tag{36}$$

$$\tilde{a}_t\tilde{\lambda}_t + \tilde{\varepsilon}^r_t + \tilde{\varphi}^r_t = 0 \tag{37}$$

$$\tilde{\lambda}_t\tilde{\theta}_t - \tilde{\mu}_t\Omega_d L^w(1-\pi)e^{-\eta(1-\pi)L^w\tilde{g}_t} + \tilde{\varepsilon}^y_t - (1-\alpha)\tilde{\varepsilon}^w_t - \tilde{\varepsilon}^r_t\frac{\alpha(1-\tilde{\theta}_t)}{\tilde{k}_t} + \tilde{\varphi}^y_t = 0 \tag{38}$$

$$\tilde{\lambda}_t(\tilde{y}_t - \tilde{w}_t) + \tilde{\varepsilon}^r_t\frac{\alpha\tilde{y}_t}{\tilde{k}_t} - \tilde{\varepsilon}^k_t k^w_t F_\theta(\tilde{\theta}_t, \theta_t, 1-\pi) = 0 \tag{39}$$

$$\tilde{\varepsilon}^a_t - \tilde{\varepsilon}^k_t F(\tilde{\theta}_t, \theta_t, 1-\pi) + \varphi^a_t = 0 \tag{40}$$

$$\Phi(\tilde{\varphi}^g_t, \tilde{g}_t) = 0 \tag{41}$$

$$\Phi(\tilde{\varphi}^k_t, \tilde{k}_t) = 0 \tag{42}$$

$$\Phi(\tilde{\varphi}^w_t, \tilde{w}_t) = 0 \tag{43}$$

$$\Phi(\tilde{\varphi}^r_t, \tilde{r}_t) = 0 \tag{44}$$

$$\Phi(\tilde{\varphi}^y_t, \tilde{y}_t) = 0 \tag{45}$$

- All in all, we have 45 variables and 45 equations (per time period). Observe that P^w_0, a_0 and $\tilde{a}_0$ need starting values. This leads to $45(T+1)+3$ equations and variables, where we truncate at $t = T$.
- A programming issue: Define a dynamic equation as an equation which contains different time indices and a static equation as an equation where all time indices are the same. The dynamic equations for the state variables determine values for a_{T+1}, $\tilde{a}_{T+1}$ and P^w_{T+1} when we truncate at $t = T$. However the dynamic equations for the

multipliers λ_t, μ_t, $\tilde{\lambda}_t$, and $\tilde{\mu}_t$, i.e. (10), (11), (32) and (33), do not determine values for λ_{T+1}, μ_{T+1}, $\tilde{\lambda}_{T+1}$, and $\tilde{\mu}_{T+1}$ unless we specify values for ε^a_{T+1}, ε^y_{T+1}, $\tilde{\varepsilon}^a_{T+1}$ and $\lambda\varepsilon^y_{T+1}$. We impose that λ_{T+1}, μ_{T+1}, $\tilde{\lambda}_{T+1}$, and $\tilde{\mu}_{T+1}$ are at the steady-state values. Then we use the static equations for $t = T + 1$ to determine values for ε^a_{T+1}, ε^y_{T+1}, $\tilde{\varepsilon}^a_{T+1}$ and $\tilde{\varepsilon}^y_{T+1}$. Note that given (1) a_{T+1}, $\tilde{a}_{T+1}$ and P^w_{T+1} and (2) λ_{T+1}, μ_{T+1}, $\tilde{\lambda}_{T+1}$, and $\tilde{\mu}_{T+1}$ at the steady-state values, we are left with 38 static equations and variables. That hopefully is enough for a unique solution.

- Calibration: Ben's memorandum gives values for the parameters that work well for the market equilibrium, so we just copy those.