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Achieving Cramer–Rao Lower Bounds in Sensor Network Estimation

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Abstract—Achievable bounds for parametric estimation using sensor networks are known for some specific problems. In this article, achievable bounds for the general parametric estimation problem with fixed and varying parameters in the network and for various noise conditions on sensor nodes are studied. The general Cramer–Rao lower bounds (CRLBs) of parameter estimates using sensor array network are derived. Procedures for reaching these bounds can be designed based on the obtained results. The accuracy of derived relationships is confirmed by an example. It is shown that it is possible to develop algorithms of moderate complexity and communication cost achieving the CRLB.

Index Terms—Sensor signals processing, sensor network, parameter estimation, Cramer-Rao lower bound (CRLB).

I. INTRODUCTION

Wireless sensor networks (WSNs) attract novel applications in diverse fields of life and science every day. There are many facilitators of this development among them cheap hardware, sensors, communication devices, growing quality of batteries, energy harvesting ability [1]–[6], etc. Numerous open challenges related to energy consumption, accurate synchronization and communication often overshadows the main reason for the WSN systems development, i.e., accurate estimation of parameters used in decision making process [7]–[12]. Parametric estimation is one of crucial problems in radar systems. Distributed sensor (antenna) networks are used frequently in radar imaging and targets estimation and detection [13]–[15]. Fusing of local observations for parametric estimation is studied in [16], while statistical strategies for efficient signal detection and parameter estimation are considered in [17]. Parametric estimation in ad hoc wireless sensor networks is presented in [18]. In this paper, we propose general framework for comparison of estimation strategies in sensor networks with achievable bounds. Estimation strategy that can be used in numerous practical systems with centralized processing or with nodes having capability of processing data gathered from several neighbor nodes is given.

The letter is organized as follows. Models of (wireless) sensor network and observations are presented in Section II. Derivation of the Cramer-Rao lower bounds (CRLBs) is done in Section III. Simple observation model for testing derived expressions is given in Section IV, with algorithm estimating parameters achieving the CRLB. Conclusions are summarized in Section V.

II. MODEL OF NETWORK AND RECORDINGS

We assume network with M sensor nodes and recordings that can be written as

$$x_m(n) = A_m f(n; \alpha, \beta_m) + \nu_m(n), \quad m \in [1, M]. \tag{1}$$

Our goal is to estimate the following parameters: $\alpha-$ constant for entire network; β_m- different and mutually independent for each node; and A_m- independent and random on each node. Signals are corrupted by a white Gaussian noise that is independent between nodes with variance σ^2, $E\{\nu_{m_1}(n_1)\nu_{m_2}(n_2)\} = \sigma^2\delta(m_1 - m_2)\delta(n_1 - n_2)$. In practice, signal-to–noise ratio (SNR) on sensor nodes, $\mathrm{SNR}_m = A_m^2/\sigma_m^2$, varies due to attenuation of signal amplitude and/or changes in noise strength. For simpler derivations we assume that noise variance is constant with varying signal strength in networks and N samples available on each node. This model is quite general and can be extended to more parameters (both constant and varying), or parameter variation according to a deterministic law. This model appears in frequency estimation that is constant in sensor network (in communications or power networks), with varying phases and amplitudes on nodes [7], [8], or measurements in agriculture where air temperature is constant while soil humidity varies. Sometimes, sensors can make independent parameters estimation. Then, constant (or slowly varying) parameters are usually (spatially) filtered. However, these strategies give suboptimal results and they can lead to error propagation from nodes with outliers to other with accurate estimates.

III. CRLB DERIVATION

The log-likelihood ratio (LLR) [19] for model (1) is

$$\Lambda(\gamma) = -\frac{1}{2} NM \ln(\pi) - NM \ln(\sigma)$$
$$\frac{1}{2\sigma^2} \sum_n \sum_{m=1}^{M} [x_m(n) - \gamma_{m+1} f(n; \gamma_1, \gamma_{M+1+m})]^2 \tag{2}$$

where γ is vector of parameters of interest $\{\gamma_j | j \in [1, 2M+1]\}$, $\gamma_1 = \alpha, \gamma_{m+1} = A_m, m \in [1, M]$, and $\gamma_{M+1+m} = \beta_m, m \in [1, M]$. The next step is determining the Fisher information matrix (FIM) $\mathbf{F}$ with expectations of the second-order partial derivatives of the LLR with respect to the elements of the signal vector parameters [19]:

$$F_{ij} = -E\left\{\frac{\partial^2 \Lambda(\gamma)}{\partial \gamma_i \partial \gamma_j}\right\}, \{i, j\} \in [1, 2M+1]. \tag{3}$$

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Here, the FIM can be written as

$$\mathbf{F} = \begin{bmatrix} d & \mathbf{b} & \mathbf{e} \\ \mathbf{b}^T & \mathbf{a} & \mathbf{c} \\ \mathbf{e}^T & \mathbf{c} & \mathbf{f} \end{bmatrix} \tag{4}$$

where scalar d, vectors $\mathbf{b}$ and $\mathbf{e}$ of M elements, matrices $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{f}$, are $M \times M$ diagonal (with off-diagonal elements equal to zero) are introduced due to brevity reasons and represent partial derivatives of the FIM

$$d = F_{11} = -E\left\{\frac{\partial^2 \Lambda(\gamma)}{\partial \gamma_1^2}\right\} = \sum\nolimits_{m=1}^{M} d_m$$

$$d_m = \frac{1}{\sigma^2}\sum_n \gamma_{m+1}^2 \left[\frac{\partial f(n;\gamma_1,\gamma_{M+1+m})}{\partial \gamma_1}\right]^2. \tag{5}$$

$$b_j = F_{1,j+1} = \frac{1}{\sigma^2}\sum_n \gamma_{m+1} f(n;\gamma_1,\gamma_{M+1+j}) \times \tag{6}$$

$$\frac{\partial f(n;\gamma_1,\gamma_{M+1+j})}{\partial \gamma_1},\ j \in [1, M], \tag{7}$$

$$e_j = F_{1,j+M+1} = \frac{1}{\sigma^2}\sum_n \gamma_{j+1}$$

$$\times \frac{\partial f(n;\gamma_1,\gamma_{M+j+1})}{\partial \gamma_1}\frac{\partial f(n;\gamma_1,\gamma_{M+j+1})}{\partial \gamma_{M+j+1}},\ j \in [1, M]. \tag{8}$$

$$a_j = \frac{1}{\sigma^2}\sum_n f^2(n;\gamma_1,\gamma_{M+j+1}),\ j \in [1, M] \tag{9}$$

$$c_j = \frac{1}{\sigma^2}\sum_n \gamma_{j+1} f(n;\gamma_1,\gamma_{M+j+1})$$

$$\times \frac{\partial f(n;\gamma_1,\gamma_{M+j+1})}{\partial \gamma_{M+j+1}},\ j \in [1, M] \tag{10}$$

$$f_j = \frac{1}{\sigma^2}\sum_n \gamma_{j+1}^2 \left[\frac{\partial f(n;\gamma_1,\gamma_{M+j+1})}{\partial \gamma_{M+j+1}}\right]^2,\ j \in [1, M]. \tag{11}$$

The CRLBs are diagonal elements of the FIM inverse [19]

$$CRLB = \left[\mathbf{F}^{-1}\right]_{ii}, \quad i \in [1, 2M+1] \tag{12}$$

where $[\mathbf{F}^{-1}]_{ii}$ are diagonal elements of matrix $\mathbf{F}^{-1}$. Matrix $\mathbf{F}$ has simple structure but determination of its inverse is not straightforward and it involves the Schur complement [20]. After some derivations it follows:

$$CRLB\{\alpha\} = [\mathbf{F}^{-1}]_{11}$$

$$= \left(d - \sum_i \frac{b_i^2 f_i - 2b_i e_i c_i + e_i^2 a_i}{a_i f_i - c_i^2}\right)^{-1} \tag{13}$$

$$CRLB\{A_m\} = [\mathbf{F}^{-1}]_{(m+1,m+1)}$$

$$= \frac{a_m}{a_m f_m - c_m^2} + CRLB\{\alpha\}\sum_{i=1}^{M}\left[\frac{b_i f_i - e_i c_i}{a_i f_i - c_i^2}\right]^2 \tag{14}$$

$$CRLB\{\beta_m\} = [\mathbf{F}^{-1}]_{(M+m+1,M+m+1)}$$

$$= \frac{a_m}{a_m f_m - c_m^2} + CRLB\{\alpha\}\sum_{i=1}^{M}\left[\frac{-b_i c_i + e_i a_i}{a_i f_i - c_i^2}\right]^2. \tag{15}$$

For each particular considered problem the CRLBs can be easily determined by replacing (5)–(11) into (13)–(15). In addition, we can notice some other important issues. Without need for detailed mathematical derivations we know that the variance and CRLB are nonnegative. It means that $CRLB\{\alpha\} \geq 0$, i.e., it holds that

$$d_m \geq \frac{b_m^2 f_m - 2b_m e_m c_m + e_m^2 a_m}{a_m f_m - c_m^2}. \tag{16}$$

Thus, each additional sensor decreases the CRLB in parameters estimation regardless of fact that the SNR at some sensors can be so low that its estimates are unreliable. To summarize each added sensor node decreases the CRLB, i.e., achievable mean squared errors (MSE) in signal parameters estimation.

IV. ILLUSTRATIVE EXAMPLE

Consider simple sensor system with recordings

$$x_m(n) = A_m(\alpha n^2 + \beta_m n + 1) + \nu_m(n). \tag{17}$$

Recordings are quadratic functions with constant parameter α and varying amplitudes A_m and β_m, $m \in [1, M]$. Such polynomial model is considered in [21] for temperature sensors and it is common in sensor calibration process. Perfect synchronization in sensor network is assumed [22]. Assume that signal parameters are available in the interval $n \in [-N/2, N/2]$ with relatively large N. Elements of the Fisher matrix are, in this case

$$d = \frac{1}{\sigma^2}\sum_n\sum_m A_m^2 n^4 \approx \frac{1}{\sigma^2}\frac{N^5}{80}\sum_m A_m^2 \tag{18}$$

$$b_m \approx \frac{A_m N^3}{4\sigma^2}\left(\frac{\alpha N^2}{20} + \frac{1}{3}\right), e_m = 0, \tag{19}$$

$$f_m \approx \frac{A_m^2 N^3}{12\sigma^2}, a_m \approx \frac{\alpha^2 N^5}{80\sigma^2} + \frac{b_m^2 N^3}{12\sigma^2} + \frac{N}{\sigma^2} + \frac{aN^3}{6\sigma^2} \tag{20}$$

$$c_m = \frac{1}{\sigma^2}\sum_n A_m[\alpha n^2 + \beta_m n + 1]n \approx \frac{A_m b_m N^3}{12\sigma^2}. \tag{21}$$

By evaluation of the FIM inverse for this case we obtain the CRLBs as

$$CRLB\{\alpha\} \approx 9\alpha^2\sigma^2 \Big/ \left(4N\sum_{m=1}^{M} A_m^2\right) \tag{22}$$

$$CRLB\{A_m\} \approx 9A_m^2\sigma^2 \Big/ \left(4N\sum_{m=1}^{M} A_m^2\right) \tag{23}$$

$$CRLB\{b_m\} \approx \frac{9b_m^2\sigma^2}{4N\sum_{m=1}^{M} A_m^2} + \frac{12\sigma^2}{A_m^2 N^3}. \tag{24}$$

The CRLBs expressions are asymptotic for $N \to \infty$, i.e., all terms $N^{-k}, k > 1$ are neglected except $12\sigma^2/A_m^2 N^3$ since its contribution can be significant for nodes with small amplitude $A_m \ll 1$ (more precisely $A_m|\beta_m| < 0.1$). Also, it is clear that any additional sensor decreases achievable CRLBs for all parameters. This is similar to sinusoidal signals estimation in sensor networks [7], [8]. The main question following these derivations is how to design estimation algorithm that is able to achieve the CRLB. In our opinion, the procedure following obtained derivations and results from [7], [8], should be the following: 1) Estimate the parameter that is constant in the network (in our case

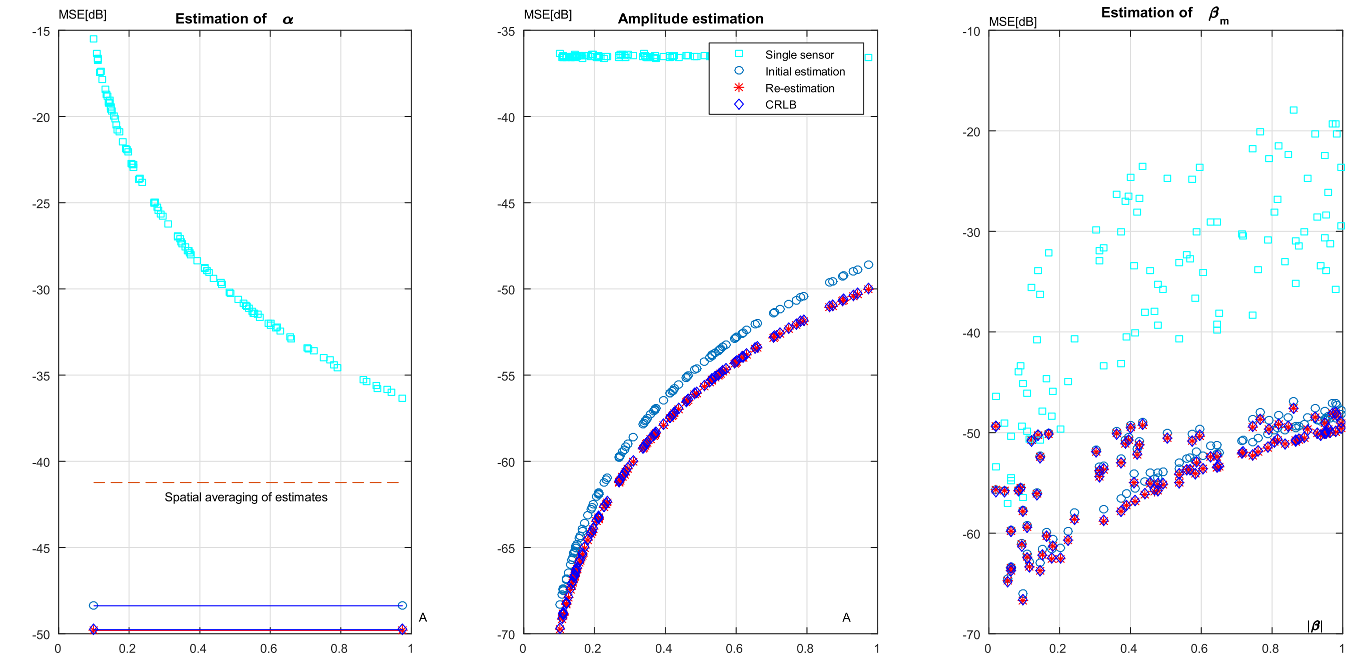

Fig. 1. Parameter estimation in test sensor system. Estimation of $\alpha-$ left. Estimation of A_m- middle. Estimation of β_m- right. Independent estimation for each node–(cyan) squares. Estimation in the first algorithm run using data from all sensors–(blue) circles. Estimation after the second run–(red) stars. CRLBs–(blue) diamonds. Dashed line in α is the estimation denotes averaging of estimates obtained on all nodes.

α) by using all available recordings. 2) Estimate other parameters (in our case A_m and β_m) by plugging estimate of α. 3) Refine estimate of α based on $\hat{A}_m$ and $\hat{\beta}_m$. If required steps 2 and 3 can be repeated several times. These steps can be performed for test system as follows. Sum signals from all available sensors is

$$x(n) = \sum_m x_m(n). \tag{25}$$

Polynomial interpolation of $x(n)$ can be used to estimate α but here it can be expressed simply as

$$\hat{\alpha} = \frac{c_1 \sum_n x(n) - 240 \sum_n n^2 x(n)}{c_1 \sum_n n^2 x(n) - c_2 \sum_n x(n)}, \tag{26}$$

where $c_1 = 20(N+1)(N+2)$ and $c_2 = (3N^2 + 6N - 4)(N+1)(N+2)$. The next step is estimating varying parameters in the network. It can be done on each node but it is more convenient to perform estimation on the processing node in order to reduce communication cost. This estimation can be done by minimizing the cost function:

$$J(A_m, \beta_m) = \sum_n [x_m(n) - A_m(\hat{\alpha} n^2 + \beta_m n + 1)]^2$$
$$\frac{\partial J(A_m, \beta_m)}{\partial A_m} = 0, \quad \frac{\partial J(A_m, \beta_m)}{\partial \beta_m} = 0. \tag{27}$$

The estimates of β_m and A_m are obtained as

$$\hat{\beta}_m = \frac{\sum_n x_m(n) n}{\sum_n n^2} \frac{\hat{\alpha}^2 \sum_n n^4 + 2\hat{\alpha} \sum_n n^2 + N}{\hat{\alpha} \sum_n x_m(n) n^2 + \sum_n x_m(n)} \tag{28}$$

$$\hat{A}_m = \frac{\hat{\alpha} \sum_n x_m(n) n^2 + \sum_n x_m(n)}{\hat{\alpha}^2 \sum_n n^4 + 2\hat{\alpha} \sum_n n^2 + N}. \tag{29}$$

Now, we can re-estimate parameter $\hat{\alpha}$ by weighted sum on the processing node [7]

$$x(n) = \sum_m \hat{A}_m x_m(n) \tag{30}$$

and using (26). The proposed algorithm is obviously simple. It includes three sums $\sum_n x(n)$, $\sum_n x(n)n$, $\sum_n x(n)n^2$, in (26), three sums in (28), (29), for each m, and additional three for weighted aggregated signal (30). Sums $\sum_{n=-N/2}^{N/2} n^2 = \frac{1}{12} N(N+1)(N+2)$, $\sum_{n=-N/2}^{N/2} n^4 = \frac{1}{240} N(N+1)(N+2)(3N^2+6N-4)$ are calculated in advance and stored. Therefore, in our realization it is required to evaluate $2(M+3)$ sums of N elements what is the most demanding operation in the algorithm, i.e., complexity of this algorithm is of order $O(NM)$. Obviously, there is a need for just single data transfer from recording unit toward processing node minimizing communication costs. For example, for $N = M = 100$, we have less than 300.000 additions what is at least an order of magnitude less, in energy consumption, than data transfer from recording nodes toward processing unit for distance of 100 m. The data transfer is performed only once per data frame. The procedure can be repeated for next data frames. Final results or control sequence can be transferrer to distant nodes if required.

Example: Network with $M = 100$ sensors, and signals recorded in the interval $n \in [-50, 50]$ (101 samples for each sensor) according to (17) is considered. Parameters are set as: $\alpha = 1$, $A_m = 0.1 \cdot 10^{r_m}$, where r_m and β_m are selected randomly in range $[-1, 1]$ according to the uniform law ($A_m \in [0.01, 1]$). We have considered 10.000 trials with various realizations of the additive Gaussian noise with variance $\sigma^2 = 0.01$. Obtained results are given in Fig. 1. Left subplot depicts the MSE in estimation of constant parameter α, the middle is for estimation of amplitudes both given as function of amplitudes on sensor nodes, and the right is for parameters β_m as function of $|\beta_m|$, while the MSE in estimation performed separately on each node is depicted with (cyan) squares. The MSE in estimation of parameter α using aggregated signal is given with (blue) circles. MSEs in estimation of A_m and β_m obtained with estimate $\hat{\alpha}$ are given in the same manner. Re-estimation of all parameters in the next round is depicted with (red) stars, while the corresponding CRLBs are given with (blue) diamonds. It is impossible to distinguish between the MSEs after algorithm re-run and the CRLBs confirming reaching the achievable limits. Estimation of α parameter averaging estimates from all nodes obtained separately gives $\mathrm{MSE} = -41.2$ dB depicted with dashed line (achievable bound for $M = 15$ nodes) that is 8.5 dB above the CRLB. This clearly demonstrates that (spatial) filtering of estimates in sensor network cannot produce achievable bounds that are possible with the used hardware. Situation can be even more difficult when outliers appear.

V. CONCLUSION

The CRLBs in estimation of constant and varying parameters in sensor network with varying SNRs in recordings are derived. Each additional sensor node decreases the achievable MSE in parameters estimation, even in the case when added node produces outliers in estimate considered separately. The procedure that can be used more or less generally in similar systems is proposed. In the first stage, estimation of constant parameters for entire network is performed followed by estimation of other parameters. In the next iteration, signal amplitude estimate is used for weighting in the process of re-estimation of parameter(s). In practice, situation could be more complicated but general rules should be as proposed here. In our opinion, all other strategies are unable to reach the achievable bounds.

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