20 Marks

Date 17/2/18

Fourier Series

A single valued function which used in applied mathematic can be expressed in the form
$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots b_1 \sin x + b_2 \sin 2x \dots$

within a desired range of values of variables. Such a series is known as fourier Series.

The fourier series for the funct^n $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$

Period will be $\alpha + 2\pi - \alpha = 2\pi$ (1)

is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx \qquad a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx$$

(2)

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cdot \sin nx\, dx$$

Case 1: taking $\alpha = 0$ the interval becomes $0 < x < 2\pi$ and eqn^n (2) reduce

$$a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cdot \cos nx\, dx$$

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Note

Case 2: taking $\alpha = -\pi$ than interval become $-\pi < x < \pi$ eqn ② reduce.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

e.g.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx$$

Q Find a fourier Series to represent a funⁿ $f(x) = x - x^2$, $-\pi < x < \pi$ and hence reduced at $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

Solⁿ we know that

The fourier Series formula is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad — ①$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx \qquad \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{3\pi^2 - 2\pi^3}{6}\right] - \left(\frac{3\pi^2 + 2\pi^3}{6}\right)$$

$$= \frac{1}{\pi}\left[\frac{3\pi^2 - 2\pi^3}{6}\right]$$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$$= \frac{3\pi - \pi^2}{6} = \frac{2\pi^2}{6}$$

$\sin n\pi = 0$

$\sin\cos n\pi = (-1)^n$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\cancel{(x - x^2)\frac{\sin nx}{n}}^{\,0}\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1 - 2x)\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[0 - \left[(1 - 2x)\frac{-\cos nx}{n^2} - \int_{-\pi}^{\pi} 2\frac{\cos nx}{n^2}\,dx\right]\right]$$

$$= \frac{1}{\pi}\left[(1 - 2\pi)\frac{\cos n\pi}{n^2} - (1 + 2\pi)\frac{\cos n\pi}{n^2} - \left[\frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(1 - 2\pi)\frac{\cos n\pi}{n^2} - (1 + 2\pi)\left(\frac{\cos n\pi}{n^2}\right) - 0\right]$$

$$= \frac{1}{\pi}\left[\cancel{1 - 2\pi - 1 - 2\pi}\,(1 - 2\pi) - (1 + 2\pi)\right]\frac{\cos n\pi}{n^2}$$

$$= \frac{1}{\pi}\left[-4\pi\right)\frac{\cos n\pi}{n^2}$$

$$\Rightarrow a_n = \frac{-4(-1)^n}{n^2} \qquad b_n = \frac{-2(-1)^n}{n}$$

Sin cos

cos sin

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Function having point of discontinuity

Note

At a point of finite discontinuity at $x = c$ there is a finite jump both the limits on the left i.e. $f(c-0)$ and the limit on the right i.e. $f(c+0)$ exists and are different. At such point the fourier series give the value of $f(x)$ as the arithmetic mean of these two limits i.e. at $x = c$

e.g.

mean $f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$

Q. Find the fourier series $F(x)$ if $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$ hence reduce $\frac{1}{(1)^2} + \frac{1}{(3)^2} + \frac{1}{(5)^2} \ldots = \frac{\pi^2}{8}$

We know that the fourier series formula

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—(1)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 - (-\pi \cdot \pi) + \frac{\pi^2}{2}\right]$$

$$= \left[-\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$b_n = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

$\cos n\pi = (-1)^n$

$\sin n\pi = 0$

$$\boxed{a_0 = -\frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x\left[\frac{\sin nx}{n}\right]_{0}^{\pi} - \int\frac{\sin nx}{n}\right]\right]$$

$$\frac{1}{\pi}\left[-\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x\left[\frac{\sin nx}{n}\right]_{0}^{\pi} + \left[\frac{\cos nx}{n^2}\right]_{0}^{\pi}\right]\right]$$

$$\frac{1}{\pi}\left[\left[\frac{-\pi\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{x\sin nx}{n}\right]_{0}^{\pi} + \left[\frac{\cos nx}{n^2}\right]_{0}^{\pi}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2}\cos nx\right]_{0}^{\pi}$$

$$a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \ldots$$

$b_n = \frac{1}{n}$

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Note:

eg.

Putting the value of a_0, a_n and b_n then the eq^n ① will be

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi}\{(-1)^n - 1\}\cos nx + \sum_{n=1}^{\infty} \frac{1}{n}\{1-2(-1)^n\}\sin nx \quad -\alpha$$

~~Let~~ now $f(x)$ is discontinuous at $x=0$

Put $x=0$ in eq^n α we find.

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\left\{-\frac{2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} - \cdots\right\} \quad -②$$

At discontinuous

$$f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$$

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right]$$

~~Putting value~~

$$f(0) = \frac{1}{2}\left(-\pi + 0\right)$$

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

$\therefore$

$f(0-0) = -\pi$

$f(0+0) = 0$

$$\boxed{f(0) = -\frac{\pi}{2}}$$

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi}{4} - \frac{\pi}{2} = -\frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$-\frac{\pi}{4} = -\frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

Change Of Intervals

In many engineering Prob the period of funct^n is not 2π but some other interval say $(2c)$.

Let the $f(x)$ be defined in interval $(-c, c)$ now we ~~t~~ want to change the funct^n to the period 2π so that we can use the formula of a_0, a_n and b_n.

$\because 2c$ is the interval of the variable $= x$

$\therefore 1$ is the interval of " $= \frac{x}{2c}$

$\therefore 2\pi$ " " " " " $= \frac{x}{2c} \times 2\pi$

$= \frac{\pi x}{c}$

So we put $z = \frac{\pi x}{c}$

$\Rightarrow x = \frac{cz}{\pi}$

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Note

Thus the function $f(x)$ of the period $2c$ is transferred to the funct^n $f\left(\frac{cz}{\pi}\right)$ or $f(z)$ of period 2π

$\therefore$ the fourier series is defined as

e.g.

$$f(z) = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + a_1 \cos z + a_2 \cos 2z + \cdots + b_1 \sin z + b_2 \sin 2z$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right) dz$$

$z = \frac{\pi x}{c}$

$dz = \frac{\pi}{c} dx$

Given z to x

If $z \Rightarrow 0$ $\quad x \Rightarrow 0$

If $z = 2\pi$ $\quad x \Rightarrow 2c$

$$a_0 = \frac{1}{\pi}\int_0^{2c} f(x) \frac{\pi}{c} dx$$

$$a_0 = \frac{1}{c}\int_0^{2\pi} f(x) dx$$

$a_n =$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right) \cos nz \, dz$$

$$a_n = \frac{1}{\pi}\int_0^{2c} f(x) \cos n\left(\frac{\pi x}{c}\right) \frac{\pi}{c} dx$$

$$= \frac{1}{c}\int_0^{2c} f(x) \cos\left(\frac{n\pi x}{c}\right) dx$$

Similarly

$$b_n = \frac{1}{c}\int_0^{2c} f(x) \sin\left(\frac{n\pi x}{c}\right) dx$$

Thus the fourier series defined in the interval $0 - 2c$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where $a_0 = \frac{1}{c}\int_0^{2c} f(x) dx$

$$a_n = \frac{1}{c}\int_0^{2c} f(x) \cos n\left(\frac{\pi x}{c}\right) dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x) \sin\left(\frac{n\pi}{c}\right) dx$$

Even And Odd function.

A funct^n $f(x)$ is said to be even if $f(-x) = f(x)$

e.g. $\cos x$, $\sec x$, x^2, x^4 etc.

A funct^n $f(x)$ is said to be odd if $f(-x) = -f(x)$

e.g. $\sin x$, $\operatorname{cosec} x$, x, x^3, x^5 etc.

We know the the property of definite integral.

$$\int_{-c}^{c} f(x) dx = 2\int_0^c f(x) dx, \text{ if } f(x) \text{ is even}$$

$$= 0, \text{ if } f(x) \text{ is odd}$$

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Tut

Noti

Periods:

$f(x+T) = f(x)$ has period of T

$f(x) = \sin x$

$f(2\pi + x) = \sin(2\pi + x)$

$= \sin x = f(x)$ with period 2π

e.g. Funt^n have finite no. of max and minima

" " " of discontinuity

and is defn in the defn pr p interval in which it

1. The convergence of fourier Series.

① If $f(x)$ and $f'(x)$ are continious on the interval $0-2L$ // $0-2\pi$ than the fourier Series converges to $f(x)$

② If ~~x = a~~ funt^n ~~is~~ $f(x)$ is discontinious at $x = c$ than

$$f(c) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$$

③ The value of $f(x)$ at end points is given be

$$= \frac{1}{2}\left[f(c) - f(-c)\right]$$

Q. $f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ 0 & 0 < 0 < \pi \end{cases}$

The fourier Series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)\,dx + \int_{0}^{\pi} dx\right]$$

$$= \frac{1}{\pi}\left[\left[\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{-\pi}^{0}\right]$$

$$= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]$$

$$= +\frac{\pi}{2}$$

Si cos.

a_n - $-\sin$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)\cos nx + \int_{0}^{\pi} (0)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi\int_{-\pi}^{0} \cos nx + \int_{-\pi}^{0} x\cos nx\right]$$

$$= \frac{1}{\pi}\left[\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{x\sin nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0} \frac{\sin nx}{n}\right]$$

$$= \frac{1}{\pi}\left[\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{x\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{\cos nx}{n^2}\right]_{-\pi}^{0}\right]$$

$$= \frac{1}{\pi}\left[0 + 0 + \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$a_n = \frac{1}{n^2}\left(1 - (-1)^n\right)$$

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Not

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\sin nx\,dx + \int_{0}^{\pi}(-x)\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\pi\sin nx + \int_{-\pi}^{0} x\sin nx\right]$$

eg

$$= \frac{1}{\pi}\left[\pi\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{-x\cos nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{-\cos nx}{n}\right]$$

$$= \frac{1}{\pi}\left[\pi\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{-x\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n^2}\right]_{-\pi}^{0}\right]$$

$$= \frac{1}{\pi}\left[\pi\left[\frac{-1}{n} + \frac{(-1)^n}{n}\right] + \left[-\frac{(-\pi)(-1)^n}{n}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left((-1)^n - 1\right) - \frac{\pi}{n}(-1)^n\right]$$

$$= \frac{1}{\pi}\frac{\pi}{n}\left((-1)^n - 1 - (-1)^n\right)$$

$$= \frac{1}{n}(-1)$$

$$\boxed{b_n = \frac{-1}{n}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi}(1-(-1)^n)\cos nx - \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Point of discontinuous

$$f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$$

For $f = 0$

$\pi + x$ $\quad -\pi \quad 0$

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] \quad 0 \quad -\pi$$

$$= \frac{1}{2}\left[\pi + x + 0\right]$$

$$f(0) = \left(\frac{\pi}{2}\right) \quad \frac{\pi}{2}$$

$$f(0) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi}(1-(-1)^n)\cos nx - \quad \because \sin nx = 0.$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left[\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} \cdots\right]$$

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\frac{\pi}{2} - \frac{\pi}{4} = \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\frac{\pi}{4} = \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right] \quad \text{Ans}$$

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We can also find single

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_n, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,(\cos nx + \sin bx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,e^{inx}\,dx$$

It should be single value funt^n

Expansion of even and odd periodic funt^n

We know that a periodic funt^n $f(x)$ defined in a interval (defined in interval) $(-c, c)$ can be represented by fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

odd funt^n

Case 1: When $f(x)$ is an even function, than

$$a_0 = \frac{2}{c}\int_{0}^{c} f(x)\,dx$$

$$a_n = \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = 0$$

Thus, if periodic funt^n $f(x)$ is even its fourier series contains only cosine term i.e fourier series represented by

$$\cancel{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} f(x)\cos\frac{n\pi x}{c}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}$$

where $a_0 = \frac{2}{c}\int_{0}^{c} f(x)\,dx$

$$a_n = \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

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This series is also called the cosine series or half Range Co-Sine Series.

Cosine formula is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c}\,dx$$

Case 2: When $f(x)$ is an odd function than

$a_0 = 0$ by property of definite intl.

$a_n = 0$ by property of definite integral $\because \cos\frac{n\pi x}{c}$ is even and $f(x)$ is odd

$$b_n = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\,dx.$$

$\because f(x)$ is odd and $\sin\frac{n\pi x}{c}$ $\therefore$ as a whole it is even fun^n

Thus if a periodic funt^n is an odd fun^n its fourier series contain on sine term. Therefore the series is

$$\boxed{\cancel{f(x) = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\,dx}}$$

$$b_n = 2 \quad \boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}}$$

we only check funt^n even or odd if range is $\boxed{-c < x < c}$

$f(x) \begin{cases} -\pi & 0 < x < \pi \\ x & -\pi < x < 0 \end{cases} \rightarrow$ we can't check even or odd since interval breaks.

$$b_n = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\,dx$$

This series is known as Sine Series or Half Range Sine Series.

Q. Find the Fourier series of $f(x) = x$ $\quad -\pi < x < \pi$

$-c < x < c$

$\because f(x)$ is odd fun. $\therefore a_0 = 0$ & $a_n = 0$

2π

$2c$ $\quad \boxed{c = \pi}$

$$b_n = \frac{2}{c}\int_0^c$$

$$b_n = \frac{-2}{n}(-1)^n$$

Q. ~~Ans~~

Q: A periodic funⁿ of period 4 is defined as

$$f(x) = |x| \quad , -2 < x < 2 \quad = 4 \quad \Big\} = 2c = 4$$
$$-c < x < c \quad = 2c \qquad \boxed{c = 2}$$

$$\boxed{c = 2}$$

For $f(x) = |x| \quad 0 < x < 2 \Rightarrow$

For this we can say whether its even or odd.

we know that the fourier Series formula is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

here funⁿ $f(x)$ is even funⁿ $\therefore$ $b_n = 0$

$$\cancel{a_0 = \frac{2}{c}\int_0^c f(x)\,dx}$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx \qquad |x| = \begin{cases} -x & -2<x<0 \\ x & 0<x<2 \end{cases}$$

$$= \frac{1}{2}\int_{-2}^{2} |x|\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} -x\,dx + \int_{0}^{2} x\,dx\right]$$

$$= \frac{1}{2}\left[\left[\frac{-x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2}\right]$$

$$= \frac{1}{2}\left[+2 + 2\right]$$

$$\frac{4}{2} = 2$$

$$\frac{2}{c}\int_0^c |x| \qquad \left[\frac{x^2}{2}\right]_0^2 \qquad \frac{4}{2} \qquad \boxed{= 2}$$

$$\cancel{= 4}$$

$$\frac{4}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x) \cos \frac{n\pi x}{c}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} |x| \cos \frac{n\pi x}{c}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} (-x) \cos \frac{n\pi x}{2}\,dx + \int_{0}^{2} x \cos \frac{n\pi x}{2}\,dx\right]$$

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$\sin \to$
$\cos \to -\sin$

Q. Obtain the fourier series for the funct^n f

$f(x) = x \sin x \qquad 0 < x < 2\pi \quad \Big\} \; 2c = 2\pi$
$0 < x < 2c$ $\qquad \boxed{c = \pi}$

We know the the fourier series formula is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

where

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx \qquad = \boxed{a_0 = -2}$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx$$

$$= \frac{1}{\pi}\left[-x \frac{\cos x}{n}\right]_0^{2\pi} - \int_0^{2\pi} -\cos x$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x) \cos \frac{n\pi x}{c}\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x \sin x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left[\frac{1}{2}(2 \sin x \cos nx)\right] dx.$$

$\cos(n+1)2\pi = 1$
$\cos(n-1)2\pi = +1$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x \sin(1+n)x \, dx + \int_0^{2\pi} x \sin(1-n)x \, dx\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{-x\cos(1+n)x}{1+n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\cos(1+n)x}{1+n}\, dx + \left[\frac{-x\cos(1-n)x}{1-n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\cos(1+n)x\, dx}{(1+n)}\right]$$

$$\frac{1}{2\pi}\left[\frac{-2\pi}{(1+n)} + \left[\frac{\sin(1+n)x}{(1+n)^2}\right]_0^{2\pi} + \left\{\frac{-2\pi}{1-n}\right\} + \left[\frac{\sin(1-n)x}{(1-n)^2}\right]_0^{2\pi}\right]$$

$$\frac{1}{2\pi}\left[\frac{-2\pi}{1+n} - \frac{-2\pi}{1-n}\right]$$

$$\frac{1}{2\pi}\left[\frac{-2\pi(1-n) - 2\pi(1+n)}{1-n^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi + \cancel{2\pi n} - 2\pi + \cancel{2\pi n}}{1+n^2}\right]$$

$$\frac{1}{2\pi}\left(\frac{-4\pi}{1+n^2}\right)$$

$$a_n = \frac{-2}{1+n^2} = \boxed{\frac{2}{n^2-1}} \qquad \boxed{n \neq 1}$$

$$\boxed{a_n = \frac{2}{n^2-1}}$$

so we can say $n \neq -1$
$\therefore \sum_{n=1}^{\infty}$ is defined from $n = 1$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx$$

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$\boxed{b_1 = \pi}$

$\boxed{b_1 = 0}$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = -\frac{1}{2}$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}(2 \sin x \cos x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2} \sin 2x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[\frac{-x \cos 2x}{2}\right]_0^{2\pi} - \int \cos$$

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Laplace TRANSFORM

Integral Transform:

Let $K(s,t)$ be a funct^n of two variable s & t. where s is a parameter (may be real or complex) independent of t. Then the integral transform is defined as

$$I\{f(t)\} = F(s) = \int_{-\infty}^{\infty} K(s,t) f(t)\, dt \quad -A$$

The function $K(s,t)$ is called "Kernel" of the transformation. If "Kernel" $K(s,t)$ is defined as

$$K(s,t) = \begin{cases} 0 & , t < 0 \\ e^{-st} & , t \geq 0 \end{cases}$$

Then the eq^n A Reduced

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

This is called the Laplace Transform

GAMMA FUNCTION

We know the ~~the~~ def^n of Gamma funct^n is

$$\boxed{\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx} \quad , n > 0$$

$$\Gamma 1 = \int_0^{\infty} e^{-x}\, dx \rightarrow \left[-e^{-x}\right]_0^{\infty} = 1$$

$$\boxed{\begin{aligned} e^{-\infty} &= 0 \\ e^{\infty} &= \text{not defined} \end{aligned}}$$

Replacing n by $n+1$ — Reduction formula

$$\Gamma(n+1) = \int_0^{\infty} \underset{II}{e^{-x}}\, \underset{I}{x^n}\, dx$$

$$= \left\{x^n - e^{-x}\right\}_0^{\infty} + \int_0^{\infty} n x^{n-1} e^{-x}\, dx$$

$$= \left\{-x^n e^{-x}\right\}_0^{\infty} + n \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

$$\Gamma(n+1) = n \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

$$\boxed{\Gamma(n+1) = n\Gamma n} \Rightarrow \text{reduction funct}^n$$

$$\boxed{\Gamma n = (n-1)\Gamma(n-1)}$$

eg:- $\Gamma 2 = 1\Gamma 1 = 1!$

$\Gamma 3 = 2\Gamma 2 = 2!\,\Gamma 1 = 2!$

$\Gamma 4 = 3\Gamma 3 = 3 \cdot 2\Gamma 2 = 3 \cdot 2 \cdot 1 = 3!$

$$\boxed{\Gamma(n+1) = n!}$$

$$\boxed{\Gamma \tfrac{1}{2} = \sqrt{\pi}}$$

$$\boxed{\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx}$$

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Laplace Transform

Let $f(t)$ be a funt^n defined for all the value of t than the Laplace transform is defined as

$$\boxed{L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

where s is a parameter this is real or complex no. In general the parameter s is taken to be a real the no.

IMPORTANT RESULTS

1) $L[1] = \frac{1}{s}$

we known that the defn^n of Laplace transform^n is

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[1] = \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty \Rightarrow \left[\frac{1}{s}\right]$$

2) $L[t^n] = \frac{n!}{s^{n+1}}$, $n = 0, 1, 2, 3 \cdots$

$= \frac{\Gamma(n+1)}{s^{n+1}}$ if n is in fraction

we know that the defin^n of Laplace transform is

$$L[f(t)] = F(x) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\, dt$$

Put $st = x$

$\Rightarrow t = \frac{x}{s}$

$dt = \frac{dx}{s}$

if $t \to 0$ $x \to 0$

$t \to \infty$ $x \to \infty$

$$L[t^n] = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx \quad \text{— (1)}$$

we know that

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx \quad \text{— (2)}$$

$$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

3) $L(e^{at}) = \frac{1}{s-a}$, $s > a$

we know that Laplace

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

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$$L(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt$$

$$\int_0^\infty e^{(a-s)t}\, dt \quad \text{or} \quad \int_0^\infty e^{-(s-a)t}\, dt$$

✗ it is not defined.

✓ because it is defined. $s - a > 0$

$$L(e^{at}) = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{-e^{-(s-a)t}}{s-a} \right]_0^\infty$$

$$\boxed{L(e^{at}) = \frac{1}{s-a}}$$

4. $L[\sin at] = \dfrac{a}{s^2+a^2}, \quad s > 0$

5. $L[\cos at] = \dfrac{s}{s^2+a^2}, \quad s > 0$

6. $L[\cosh at] = \dfrac{s}{s^2-a^2}, \quad s^2 > a^2$

7. $L[\sinh at] = \dfrac{a}{s^2-a^2} \quad s^2 > a^2$

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PROPERTIES OF LAPLACE TRANSFORM

1) Linearity Property

$$L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2 L[f_2(t)]$$

2 First Shifting Property :-

If Laplace transform is

$$L[e^{at} f(t)] =$$

$$\text{If } L[f(t)] = F(s)$$

Then

$$L[e^{at} f(t)] = F(s-a)$$

We know that the defn of La

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt \quad \text{let } s-a = u$$

$$= \int_0^\infty e^{-ut} f(t)\, dt$$

$$= f(u) = F(s-a) \text{ proved}$$

Formulas

$$\Rightarrow L[e^{at} t^n] = \frac{n!}{(s-a)^{n+1}}$$

Here $f(t) = t^n$

$$L[t^n] = \frac{n!}{s^{n+1}} = f(s)$$

$$L[e^{at} t^n] = \frac{n!}{(s-a)^{n+1}}$$

$$L[e^{at} f(t)] = f(s-a)$$

$$\Rightarrow L[e^{at} \cos bt] =$$

let $f(t) = \cos bt$

$$L[\cos bt] = \frac{s}{s^2+b^2} = f(s)$$

$$L[e^{at} f(t)] = f(s-a)$$

replace s by (s-a)

$$L[e^{at} \cos bt] = \frac{s-a}{(s-a)^2+b^2}$$

$$\Rightarrow L[e^{at} \sin bt] = \frac{b}{(s-a)^2+b^2}$$

let $f(t) = \sin bt$

$$L[\sin bt] = \frac{b}{s^2+a^2} = f(s)$$

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$\frac{1-\cos 2t}{2}$ $\cos 2t = 1 - 2\sin^2 t$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\cos A =$ $3\sin\theta - 4\sin^3\theta$

Q. Find Laplace transform of
(i) $\sin 2t \sin 3t$
(ii) $\cos^2 2t$
(iii) $\sin^3 2t$

(i) $\sin 2t \sin 3t$
we know than

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin 2t \sin 3t = \frac{1}{2}\left[\cos t - \cos 5t\right]$$

taking laplace transform.

$$L\left[\sin 2t \sin 3t\right] = \frac{1}{2} L\left[\cos t - \cos 5t\right]$$

$$= \frac{1}{2}\left\{L\left[\cos t\right] - L\left[\cos 5t\right]\right\}$$

$\because L[\cos at] = \frac{s}{s^2+a^2}$

$$= \frac{1}{2}\left\{\frac{s}{s^2+1^2} - \frac{s}{s^2+25}\right\}$$

Q2 $\cos^2 2t$
we know that

$\cos^2 2t = 2\cos^2 t - 1$
replace t by 2t
$\cos 4t = 2\cos^2 2t - 1$
$\frac{\cos 4t + 1}{2}$

taking Laplace transform both side

$$L\left[\cos^2 2t\right] = \frac{1}{2} L\left[\cos 4t + 1\right]$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos at = \frac{s}{s^2+a^2}$

By linearity Property

$$= \frac{1}{2}\left\{L\left[\cos 4t\right] + L\left[1\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{s}{s^2+a^2} \quad \frac{s}{s^2+4^2} + \frac{1}{s}\right\}$$

$$= \frac{1}{2}\left\{\frac{s}{s^2+16} + \frac{1}{s}\right\}$$

4) $\sin^3 2t$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

taking Laplace transform

$$L\left[\sin^3 2t\right] = \frac{3}{4} L\left[\sin 2t\right] - \frac{1}{4} L\left[\sin 6t\right]$$

$$= \frac{3}{4}\left[\left\{\frac{2}{s^2+4}\right\} - \left\{\frac{1}{4}\ \frac{6}{s^2+36}\right\}\right] \quad \because L[\sin at] = \frac{a}{s^2+a^2}$$

Find Laplace transform of.
(i) $e^{-3t}(2\cos 2t - 3\sin 5t)$
(ii) $e^{3t}\sin^2 t$
(iii) $e^{4t}\sin 2t \cos t$

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(i) $e^{-3t}(2\cos 5t - 3\sin 5t)$

Here, $f(t) = 2\cos 5t - 3\sin 5t$

$L[2\cos 5t - 3\sin 5t] = 2L[\cos 5t] - 3L[\sin 5t]$

we know

$L[\cos at] = \frac{s}{s^2+a^2}$

$L[\sin at] = \frac{a}{s^2+a^2}$

$\therefore$

$= 2\left\{\frac{s}{s^2+25}\right\} - 3\left\{\frac{5}{s^2+25}\right\}$

$= \frac{2s}{s^2+25} - \frac{15}{s^2+25} = \frac{2s-15}{s^2+25}$

$= \frac{2s-15}{s^2+25} = f(s)$

$L[e^{at}f(t)] = f(s-a)$

$\therefore L[e^{-3t}(2\cos 5t - 3\sin 5t)] = \frac{2(s+3)-15}{(s+3)^2+25}$

$= \frac{2s-9}{(s+3)^2+25}$

(ii) $e^{3t}\sin^2 t$

$\because \cos 2t = 1 - 2\sin^2 t$

Here $f(t) = \sin^2 t$

$\therefore \sin^2 t = \frac{1-\cos 2t}{2}$

$f(t) = \frac{1-\cos 2t}{2}$

$\because L[1] = \frac{1}{s}$

$L[f(t)] = \frac{1}{2}\left[L[1] - L[\cos 2t]\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \left[\frac{1}{2s} - \frac{s}{2(s^2+4)}\right]$ $\because L[\cos at] = \frac{s}{s^2+a^2}$

$L[e^{3t}\sin^2 t] =$

$L[e^{at}\sin f(t)] = f(s-a)$

replace s by $(s-a)$

$L\{e^{3t}\sin^2 t\} = \left[\frac{1}{2(s-3)} - \frac{(s-3)}{2(s-3)^2+8}\right]$

$\frac{1}{2}(\sin(A+B) + \sin(A-B))$

$= \frac{1}{2}\left(\frac{1}{s-3} - \frac{s-3}{(s-3)^2+4}\right)$

(iii) $e^{4t}\sin 2t\cos t$

Here $f(t) = \sin 2t\cos t$

$\because$ we know

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$\sin A\cos B =$

$[\sin A\cos B] = \frac{1}{2}\left[\sin 3t + \sin t\right]$

$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$

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replace s by $(s-4)$ i.e $s-a$

$$\therefore y_2 = \frac{1}{2}\left[\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)+1}\right]$$

Find Laplace Xrm of $f(t)$

$$f(t) = \begin{cases} \frac{t}{\tau}, & 0 < t < \tau \\ 1, & t > \tau \end{cases}$$

We know that a defn of Laplace transform is

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\tau e^{-st}\frac{t}{\tau}\,dt + \int_\tau^\infty e^{-st}\,1.\,dt$$

$$= \left[\frac{t}{\tau}\frac{e^{-st}}{-s}\right]_0^\tau - \int \frac{1}{\tau}\frac{e^{-st}}{-s}\,dt$$

$$= \left[\frac{-t}{\tau}\frac{e^{-st}}{s}\right]_0^\tau + \frac{1}{\tau s}\left[\frac{e^{-st}}{-s}\right]_0^\tau$$

$$= \left[\frac{-\tau}{\tau}\frac{e^{-\tau t}}{s}\right] + \frac{1}{\tau s}\left[\frac{e^{-s\tau}}{-s} - \frac{e^{1}}{-s}\right]$$

$$= \left[\frac{-e^{-s\tau}}{s}\right] - \frac{e^{-s\tau}}{\tau s^2} + \frac{1}{s}$$

$$= \left\{\frac{1-e^{-s\tau}}{\tau s^2}\right\}$$

Change Of Scale Property :

If Laplace transform i.e

$$L[f(t)] = f(s)$$

Then

$$L[F(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

From basic defn of Laplace transform.

$$L[F(at)] = \int_0^\infty e^{-st}F(at)\,dt$$

Put $at = x$

$t = \frac{x}{a}$

$dt = \frac{dx}{a}$

$$L[F(at)] = \int_0^\infty e^{-s\left(\frac{x}{a}\right)}F(x)\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}(x)}F(x)\,dx$$

If $t \to 0$ $x \to 0$

$t \to \infty$ $x \to \infty$

$\frac{s}{a} = u$

$$= \frac{1}{a}\int_0^\infty e^{-ux}f(x)\,dx$$ will be $= \frac{1}{a}f(u)$

$$= \frac{1}{a}f\left(\frac{s}{a}\right)$$

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14/03/2018

→ Laplace transform of derivative

If $L[f(t)] = f(s)$ then

$L[f^{n}(t)] = s^{n} f(s)$ — n is derivative

$L[f^{n}(t)] = s^{n} f(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots f^{n-1}(0)$

where $f^{n}(t) = \frac{d^{n}}{dt^{n}} f(t)$ — n^{th} derivative

Not in syllabus

Proof:

$L[f(t)] = f(s)$ — (1)

From basic defin^{n}

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \int_0^\infty \underset{\substack{\downarrow \\ 2}}{e^{-st}} \underset{\substack{\downarrow \\ II}}{\left(\frac{df}{dt}\right)} dt$$

$$= \left\{ e^{-st} f(t) \right\}_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\, dt$$

$$= \{0 - f(0)\} + s \int_0^\infty e^{-st} f(t)\, dt$$

$$= \{0 - f(0)\} + s f(s) \quad \text{From 1 eq}^{n}$$

$$L[f'(t)] = s f(s) - f(0) \quad \text{— (2)}$$

for, put $n=1$ in main eq^{n}

For second^{n} derivative let $G(t) = f'(t)$ — (3)

$L[G'(t)] = s g(s) - G(0)$ From eq (2).

$\because L[f(t)] = f(s)$
$L[G(t)] = g(s)$

$\therefore L[G'(t)] = s g(s) - G(0)$

$L[G'(t)] = s L[G(t)] - G(0)$

$L[G'(t)] = s L[f'(t)] - G(0)$

$\because G'(t) = f''(t)$ from (3)

$L(f''(t)) = s L[f'(t)] - f'(0)$

$= s[s f(s) - f(0)] - f'(0)$

$L[f''(t)] = s^2 f(s) - s f(0) - f'(0)$ — (4)

Similarly.

$L[f'''(t)] = s^3 f(s) - s^2 f(0) - s f'(0) - f''(0)$

In general

$L[f^{n}(t)] = s^{n} f(s) - s^{n-1} f(0) - s^{n-2} f(0) \cdots f^{n-1}(0)$

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LAPLACE TRANSFORM OF INTEGRAL

If Laplace transform of

if $L[F(t)] = f(s)$

$$\boxed{L\left[\int_0^t F(t)\,dt\right] = \frac{1}{s} f(s)}$$

Multiplication by powers of t^n :-

If $L[F(t)] = f(s)$ then

$$L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s), \quad n = 1, 2, 3, \ldots$$

Proof:-

Let $L[F(t)] = f(s)$ & $\frac{d^n}{ds^n} f(s) = f^n(s)$

we know that the basic defn of is

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

differentiating w.r.t s we find,

$$\frac{d}{ds} f(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} F(t)\,dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left\{e^{-st} F(t)\right\} dt$$

By taking rule of differentiating under sign of integral

$$\frac{d}{ds} f(s) = \int_0^\infty (-t) e^{-st} F(t)\,dt$$

$$(-1)\frac{d}{ds} f(s) = \int_0^\infty e^{-st}\{t F(t)\}\,dt$$

$$-1 \frac{d f(s)}{ds} = L[t \cdot F(t)]$$

This theorem is true for $n=1$

By Mathematical induction Method let us suppose that the theorem is true for $n=r$.

$$(-1)^r \frac{d^r}{ds^r} f(s) = L[t^r F(t)]$$

$$(-1)^r \frac{d^r}{ds^r} f(s) = \int_0^\infty e^{-st} t^r F(t)\,dt$$

differentiating w.r.t s

$$(-1)^r \frac{d^{r+1}}{ds^{r+1}} f(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} t^r F(t)\,dt\right]$$

$$= \int_0^\infty \left[\frac{\partial}{\partial s}\left(e^{-st} t^r F(t)\right)\right] dt$$

$$= \int_0^\infty (-t) e^{-st} t^r F(t)\,dt$$

$$(-1)^{r+1} \frac{d^{r+1}}{ds^{r+1}} f(s) = \int_0^\infty e^{-st}\{t^{r+1} F(t)\}\,dt$$

$$= L[t^{r+1} F(t)]$$

$(-1)^{r+1}$

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$$(-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}} f(s) = L[t^{n+1} F(t)] dt$$

This proves that this theorem is true for $n+1$ if it is true for $n=n$ hence it is true for any +ve integer n

Theorem

If $L[F(t)] = f(x)$ Than

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s)\, ds$$

Provided the integral exists

Proof:

We know that the def^n of Laplace transform

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt.$$

integrating both side w.r.t ~~dt~~ S b/w the limit S to ∞

$$\int_s^\infty f(s)\, ds = \int_s^\infty \left\{ \int_0^\infty e^{-st} F(t)\, dt \right\} ds$$

By changing order of integration because S & t are independent.

$$= \int_0^\infty \int_s^\infty e^{-st} F(t)\, ds\, dt$$

$$\int_s^\infty f(s)\, ds = \int_0^\infty F(t) \left\{ \int_s^\infty e^{-st} ds \right\} dt$$

$$= \int_0^\infty F(t) \left\{ \frac{e^{-st}}{-t} \right\}_s^\infty dt$$

$$= \int_0^\infty F(t) \left\{ \frac{e^{-st}}{t} \right\} dt$$

$$\int_s^\infty f(s)\, ds = \int_0^\infty e^{-st} \left\{ \frac{F(t)}{t} \right\} dt$$

$$\int_s^\infty f(s)\, ds = L\left[\frac{F(t)}{t}\right]$$ Proved

Q. Find Laplace transform of

(i) $\left(\frac{1-e^t}{t}\right)$ (ii) $\left\{\frac{\cos at - \cos bt}{t}\right\}$

We know $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\, ds$

Here $F(t) = 1 - e^t$

$$L(F(t)) = L[1 - e^t] = L[1] - L[e^t]$$

$$= \left\{\frac{1}{s} - \frac{1}{s-1}\right\}$$

$$= \frac{s-1-s}{s(s-1)} = \frac{-1}{s(s-1)} = f(s)$$

We know that

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s)\, ds$$

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$$\therefore L\left[\frac{1-e^{t}}{t}\right] = \int_{s}^{\infty} \frac{-1}{s(s-1)}\, ds$$

$$= \int_{s}^{\infty} \left\{\frac{1}{s} - \frac{1}{s-1}\right\} ds$$

$$= \left\{\log s - \log(s-1)\right\}_{s}^{\infty}$$

$$= \left[\log \frac{s}{s-1}\right]_{s}^{\infty}$$

$$= \left[\log \frac{1}{1-\frac{1}{s}}\right]_{s}^{\infty}$$

$$= \left[\log 1 - \log\left(1-\frac{1}{s}\right)\right]_{s}^{\infty}$$

$$= 0 - 0 + \log\left(1-\frac{1}{s}\right)$$

$$L\left[\frac{1-e^{t}}{t}\right) = \log\left(1-\frac{1}{s}\right)$$

$$= \left(\log \frac{s-1}{s}\right)$$

Eg. $\left(\frac{\cos at - \cos bt}{t}\right) = L(t)$

if $\left[\frac{1-e^{t}}{t^2}\right] = \left[\frac{\frac{1-e^{t}}{t}}{t}\right] = \int_{s}^{\infty} \frac{\log(s-1)}{s}\, ds$

$$\left[\frac{\cos at - \cos bt}{t}\right) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{\frac{1}{2}}$$

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Obtain the Laplace Transform of

Q. $t^2 e^t \sin 4t$

Here $F(t) = \sin 4t$

$$L[F(t)] = [\sin 4t] = \frac{4}{s^2+16} = f(s)$$

we know that the first Shifting Property of Laplace transfor is

$$\Rightarrow L[e^{at} F(t)] = f(s-a)$$

$$\therefore L[e^{at} \sin 4t] = \frac{4}{(s-a)^2+16} = f(s)$$

we also know that

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(x)$$

$$L[t^2 (e^{at} \sin 4t)] = (-1)^2 \frac{d^2}{ds^2}\left[\frac{4}{s^2-2s+17}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2-2s+17)\cdot 0 - 4(2s-2)}{(s^2-2s+17)^2}\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{(s^2-2s+17)^2 \times -8 + 4(2s-2)\left(2(s^2-2s+17)(2s-2)\right)}{(s^2-2s+17)^4}\right]$$

$$= \left[\frac{(s^2-2s+17)\cdot(-8) + 4(2s-2)2(2s-2)}{(s^2-2s+17)^3}\right]$$

$$= \frac{(s^2-2s+17)\cdot(-8) + 8(4s^2-4s-4s+4)}{(s^2-2s+17)^3}$$

$$= \frac{-8s^2+16s-136+32s^2-32s-32s+32}{(s^2-2s+17)^3}$$

$$= \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

Q. $\frac{\sin 2t}{t}$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds$$

$F(t) = \sin 2t$

$$L[F(t)] = \frac{2}{s^2+4} = f(s)$$

$$= \int_s^\infty \left(\frac{2}{s^2+4}\right) ds$$

$$\frac{1}{x^2+a^2} \quad \frac{1}{(x^2+a^2)}$$

$$= \frac{2}{2}\tan^{-1}\left(\frac{s}{2}\right)$$

$$= \left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{2}$$

$$= \left[\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right]$$

$$= \boxed{\cot^{-1}\frac{s}{2}}$$

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Q. Find the Laplace transform of F

$$t\,F(t) = \int_0^t \frac{\sin t}{t}\,dt \qquad f(s) = L[F(t)]$$

$$L\left[\frac{\sin t}{t}\right] = L\left[\frac{F(t)}{t}\right]$$

$$F(t) = \sin t$$

$$L[F(t)] = \frac{1}{s^2+1} = f(s)$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \cot^{-1} s = f(s)$$

$\therefore$ we also know that

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} f(s)$$

$$L\left[\int_0^t \frac{\sin t}{t} f\,dt\right] = \frac{1}{s}\cot^{-1} s$$

$$\frac{\Gamma(n+1)}{s^{n+1}}$$

Q. Find the Laplace transform of

$$L[\sin\sqrt{t}]$$

we know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Replacing x by $\sqrt{t}$

$$\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \frac{t^{9/2}}{9} - \cdots$$

taking Laplace transform both the side.

$$L[\sin\sqrt{t}] = L[t^{1/2}] + \frac{1}{3!}L[t^{3/2}] + \frac{1}{5!}L[t^{5/2}] - \frac{1}{7!}L[t^{7/2}] - \cdots$$

$$= \frac{\Gamma 3/2}{s^2} - \frac{1}{3!}\frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{5}\frac{\Gamma 7/2}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma 9/2}{s^{9/2}} + \cdots$$

$$\text{Use } \Gamma n = (n-1)\Gamma n-1 \qquad s^{-3/2}$$

$$= \frac{1/2\,\Gamma 1/2}{s^{3/2}} - \frac{1}{3!}\frac{3/2\cdot\frac{1}{2}\Gamma 1/2}{s^{5/2}} + \frac{1}{5!}\frac{5/2\cdot 3/2\cdot 1/2\,\Gamma 1/2}{s^{7/2}} - \frac{1}{7!}\frac{7/2\cdot 5/2\cdot 3/2\cdot\frac{1}{2}\Gamma 1/2}{s^{9/2}} + \cdots$$

$$= \frac{1/2\,\Gamma 1/2}{s^{3/2}}\left[1 - \frac{1}{3!}\frac{3/2}{s} + \frac{1}{5!}\frac{5/2\cdot 3/2}{s^2} - \frac{1}{7!}\frac{7/2\cdot 5/2\cdot 3/2}{s^3} + \cdots\right]$$

$$= \frac{1/2\,\Gamma 1/2}{s^{3/2}}\left[1 - \frac{1}{2^2 s} + \frac{1}{(2^2 s)^2\,2!} - \frac{1}{(2^2 s)^3\,3!} + \cdots\right]$$

$$\text{let } = x = \frac{1}{2^2 s}$$

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$\sin\sqrt{t} \Rightarrow \cos\sqrt{t}$

$$L\{\sin\sqrt{t}\} = \frac{1}{2}\frac{\sqrt{\pi/2}}{s^{3/2}}\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} - \dots\right]$$

$$= \frac{1}{2}\frac{\sqrt{\pi/2}}{s^{3/2}}\left\{e^{-x}\right\}$$

$$= \frac{1}{2}\frac{\sqrt{\pi/2}}{s^{3/2}}\left\{e^{-\frac{1}{2^2 s}}\right\}$$

$$\boxed{L[\sin\sqrt{t}] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}}$$

Find Laplace transform of

Q. $\frac{\cos\sqrt{t}}{\sqrt{t}}$

take $F(t) = \sin\sqrt{t}$

$\therefore F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$

we know that

$$L[F'(t)] = sf(s) - F(0)$$

$$\frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\left(\frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}\right) - F(0)$$

$$= \left(\frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}\right) - 0$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

Find the Laplace transform of $f(t)$

$$f(t) = \begin{cases} 0 & , t < 2\pi/3 \\ \cos\left(t - \frac{2\pi}{3}\right) & , t > \frac{2\pi}{3} \end{cases}$$

we know that the definat^n of Laplace transform is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{2\pi/3} e^{-st} \cdot 0\, dt + \int_{2\pi/3}^\infty e^{-st}\cos(t - 2\pi/3)\, dt$$

$$= \int_{2\pi/3}^\infty e^{-st}\cos\left(t - \frac{2\pi}{3}\right) dt$$

Put $t - \frac{2\pi}{3} = u$

$dt = du$

If $t \to \frac{2\pi}{3}$ $u \to 0$

$t \to \infty$ $u \to \infty$

$$= \int_0^\infty e^{-s(u + 2\pi/3)}\cos u\, du$$

$$= e^{-\frac{2\pi s}{3}}\int_0^\infty e^{-su}\cos u\, du$$

for basic def^n

$$= e^{-\frac{2\pi s}{3}} L[\cos t]$$

$$L[f(t)] = e^{-\frac{2\pi s}{3}}\left\{\frac{s}{s^2+1}\right\}$$

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21/03/2018

UNIT - STEP FUNCTION

The unit step function is defined as

$$u(t-a) = \begin{cases} 0 & , t < a \\ 1 & , t > 0 \end{cases} \Big/ a \geq 0$$

where a is always +ve

Transform of unit Step - funct^n

The basic def^n of Laplace transform is

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt$$

$$\therefore L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt$$

But $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0 & , t < a \\ 1 & , t \geq a \end{cases}$$

$$L[u(t-a)] = \int_0^a e^{-st} 0 \cdot dt + \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty \qquad e^{-\infty} = 0$$

$$= \frac{e^{-s\infty}}{-s} - \frac{e^{-sa}}{-s}$$

$$= \frac{e^{-sa}}{-s}$$

$$\boxed{L[u(t-a)] = \frac{1}{s} e^{-as}}$$

$e^{a \times f}$

Second Shifting Property.

If Laplace transf of $L[F(t)] = f(s)$

Then,

$$L[F(t-a) \cdot u(t-a)] = e^{-as} f(s)$$

From basic def^n of Laplace transf

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt$$

$$L[F(t-a) \cdot u(t-a)] = \int_0^\infty e^{-st} F(t-a) \cdot u(t-a)\, dt$$

but $u(t-a) = \begin{cases} 0 & , t < a \\ 1 & , t > a \end{cases} \Big/ a \geq 0$

$$= \int_0^a e^{-st} F(t-a) \cdot 0 \cdot dt + \int_a^\infty e^{-st} F(t-a) \cdot 1\, dt$$

$$= \int_0^\infty e^{-st} F(t-a)\, dt \qquad \text{Put } t-a = x, \; dt = dx$$

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$$\mathcal{L}[f(t-a)u(t-a)] = \int_a^\infty e^{-st} f(t-a)\,dt$$

$t-a=x$

$t=x+a$

$$= \int_0^\infty e^{-s(x+a)} f(x)\,dx$$

$$= e^{-as}\int_0^\infty e^{-sx} f(x)\,dx \qquad \mathcal{L}[f(x)] = f$$

$$= e^{-as} f(s)$$

Q. Find Laplace transform of

(i) $f(t) = \begin{cases} (t-1), & 1<t<2 \\ (3-t), & 2<t<3 \end{cases}$

(ii) $e^{-t}\{1-u(t-2)\}$

Sol. $f(t) = (t-1)\{u(t-1)-u(t-2)\}$

$+(3-t)\{u(t-2)-u(t-3)\}$

$f(t) = (t-1)u(t-1) - (t-1)u(t-2) - (t-3)u(t-2) + (t-3)u(t-3)$

$f(t) = (t-1)u(t-1) - (t-2)\{(t-1)u - (t-3)u\}$

$+ (t-3)u(t-3)$

$\mathcal{L}[(t-2)^2 u(t-2)] = e^{-2s}\mathcal{L}[t^2]$

$[(t-2)^2 u(t-2)] = e^{-2s}\mathcal{L}[t^2]$

$(t-1)u(t-1) - (t-2)(ut - u - ut - 3u) + (t-3)u(t-3)$

$= (t-2)u(t-1+t-3)$

we know that the sol.

$\mathcal{L}[f(t)]$

$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}f(s)$

$f(t) = (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$

taking laplace to both the side

$\mathcal{L}[f(t)] = \mathcal{L}[(t-1)u(t-1)] - 2\mathcal{L}[(t-2)u(t-2)]$
$+ \mathcal{L}[(t-3)u(t-3)]$

$= e^{-s}\mathcal{L}[t] - 2e^{-2s}\mathcal{L}[t] + e^{-3s}\mathcal{L}[t]$

$$= \boxed{e^{-s}\frac{1}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}}$$

$$\boxed{\because \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}}$$

(ii) $\mathcal{L}[e^{-t}\{1-u(t-2)\}]$

$\mathcal{L}[e^{-t}\{1-u(t-2)\}] = \mathcal{L}[e^{-t}] - \mathcal{L}[e^{-t}u(t-2)]$

$= \left(\frac{1}{s+1}\right) - \mathcal{L}[e^{-t+2-2}u(t-2)]$

$= \frac{1}{s+1} - \mathcal{L}[e^{-t-2+2}u(t-2)]$

$= \left(\frac{1}{s+1}\right) - \mathcal{L}[e^{-2}e^{-(t-2)}u(t-2)]$

$= \left(\frac{1}{s+1}\right) - e^{-2}\mathcal{L}[e^{-(t-2)}u(t-2)]$

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$\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$

we know the

$\mathcal{L}[f(t-a)\,u(t-a)] = e^{-as} f(s)$

$= \frac{1}{s+1} - e^{-2}\left\{ e^{-2s}\,\mathcal{L}[e^{-t}] \right\}$

$= \frac{1}{s+1} - e^{-2}\left\{ e^{-2s} \frac{1}{s+1} \right\}$

$= \frac{1}{s+1}\left(1 - e^{-2} e^{-2s}\right)$

$= \frac{1}{s+1}\left(1 - e^{-2s}\right)$

2nd method $\mathcal{L}[1 - u(t-2)] = \mathcal{L}[1] - \mathcal{L}[u(t-2)]$

$= \left(\frac{1}{s} - \frac{e^{-2s}}{s}\right) = f(s)$

we know that first shifting property is

$\mathcal{L}(e^{at} f(t)) = f(s-a)$

$\mathcal{L}[e^{-t}\{1 - u(t-2)\}] = \boxed{\frac{1}{s+1} - \frac{e^{-2(s+1)}}{s+1}}$

Evaluation of integrals

Prob: Evaluate $\int_0^\infty t\, e^{-3t} \sin t \, dt$

$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$

$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$\mathcal{L}[t \sin t] = (-1)\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\}$

$= (-1)\left\{\frac{(s^2+1)0 - 1(2s)}{(s^2+1)^2}\right\}$

$= \frac{2s}{(s^2+1)^2}$

$\mathcal{L}[e^{at} f(t)] = f(s-a)$

~~$\int_0^\infty e^{-st} \sin t \sin t \, dt$~~

By def^n of laplace.

$\int_0^\infty e^{-st}(t \sin t)\,dt = \frac{2s}{(s^2+1)^2}$

Put $s = 3$

$= \frac{3 \times 2}{(9+1)^2} = \frac{3}{50}$

H.W.

① Prove that $\int_0^\infty t \cdot e^{-3t} \cos 4t \, dt = \frac{-7}{625}$

② $\int_0^\infty \left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = \log 3$ Hint: $\frac{f(t)}{t} =$

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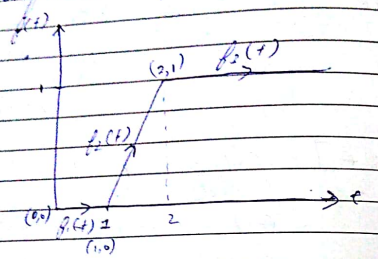
23/3/2018

Q. Express the following function in terms of unit step funct" and find its Laplace transform



$$f(t) = \begin{cases} f_1(t) & , 0 < t < 1 \\ f_2(t) & 1 < t < 2 \\ f_3(t) & t > 2 \end{cases}$$

$f_1(t)$: $y - 0 = \frac{0}{1-0}(\cdot$

$f_1(t) = y = 0$

$y - 1 = \frac{1-0}{2-1}(x - 0)$

$y - 1 = 1(x)$

$y = x + 1$

$f_2(t) = t - 1$

$f_3(t) = 1$

$$f(t) = \begin{cases} 0 & , 0 < t < 1 \\ t-1 & , 1 < t < 2 \\ 1 & , t > 2 \end{cases}$$

$\therefore f(t) = 0\{u(t-0) - u(t-1)\} +$

$+ (t-1)\{u(t-1) - u(t-2)\}$

$+ 1\{u(t-2)\}$

$f(t) = (t-1)\{u(t-1) - u(t-2)\} + \{u(t-2)\}$

$= (t-1)u(t-1) - (t-1)u(t-2) + u(t-2)$

$(t-1)u(t-1) - \{(t-2)\}\{(t-1) + 1\}$

Taking $(t-1)u(t-1) - (t-2)u(t-2)$

INVERSE LAPLACE TRANSFORM

1) $L[1] = \frac{1}{s} \Rightarrow L^{-1}\left[\frac{1}{s}\right] = 1$

2) $L[F(t)] = f(s) \Rightarrow L^{-1}[f(s)] = F(t)$

3) $L[\sin at] = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

$\Rightarrow L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at.$

$\Rightarrow L^{-1}\left[\frac{1}{s-a}\right] \Rightarrow e^{at} L\left[\frac{1}{s}\right] = e^{at}$

$\Rightarrow L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!} \Rightarrow L[t^n] = \frac{n!}{s^{n+1}}$

by $n-1$

$\Rightarrow L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)}$

$\sqrt{n+1} = n\sqrt{n}, \quad n > 0$

$\sqrt{n} = \frac{\sqrt{n+1}}{n}, \quad n < 0$

Given a function

$-\pi/2$

$\frac{-1+1}{-1/2} \quad \frac{\pi/2}{-1/2}$

Find Laplace of $L[t^{-3/2}]$

$L[t^{-3/2}] = \frac{\sqrt{n+1}}{s^{n+1}} = \frac{\sqrt{-1/2}}{s^{-1/2}}$

$\sqrt{n} = \frac{\sqrt{n+1}}{n} = 2\sqrt{\pi}$

$= \frac{-2\sqrt{\pi}}{s^{-1/2}} = -2\sqrt{\pi s}$

Q. (i) $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$ (ii) $L^{-1}\left[\frac{s+2}{s^2+4s+13}\right]$

(i) $f(s) = \left(\frac{s^2-3s+4}{s^3}\right)$

$= \left(\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right)$

taking Laplace inverse

$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] - 4L^{-1}\left[\frac{1}{s^2}\right]$

we know $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

$= 1 - 3(t) + 2t^2$

$\boxed{= 1 - 3t + 2t^2}$

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(ii) $\frac{s+2}{s^2-4s+13}$

$= \frac{s+2}{(s-2)^2+9}$

$= \frac{s-2+4}{(s-2)^2+9} \Rightarrow \frac{s-2}{(s-2)^2+9} + \frac{4}{(s-2)^2+9}$

taking $L^{-1}[f(s)]$

$L^{-1}\left(\frac{s+2}{s^2-4s+13}\right) = L^{-1}\left(\frac{s-2}{(s-2)^2+9}\right) + L^{-1}\left[\frac{4}{(s-2)^2+9}\right]$

$L[e^{at}f(t)] = f(s-a)$

$L^{-1}[f(s-a)] = e^{at}f(t)$

$= e^{at}L^{-1}[f(s)]$

$= e^{2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] + \frac{4}{3}e^{2t}\left[\frac{1}{s^2+3^2}\right]$

$= e^{2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] + \frac{4}{3}e^{2t}\left[\frac{3}{s^2+3^2}\right]$

$= \left[e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t\right]$

METHOD OF Partial Fraction

(i) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (ii) $\frac{4s+5}{(s-1)^2\cdot(s+2)}$

(i) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6} \rightarrow$ factor $\rightarrow s^3-s^2-5s^2+5s+6s-6$

$s^2(s-1)-5s(s-1)+6(s-1)$

$(s-1)(s^2-5s+6)$

$(s-1)(s-2)(s-3)$

$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$

$A = \frac{1}{2}$ $B = -1$ $C = 5/2$

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$$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{1}{2(s-1)} - \frac{1}{(s-2)} + \frac{5}{2(s-3)}$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s-2}\right] + \frac{5}{2}L^{-1}\left[\frac{1}{s-3}\right]$$

$$= \frac{e^{t}}{2}L^{-1}\left[\frac{1}{s}\right] - e^{2t}L^{-1}\left[\frac{1}{s}\right] + \frac{5}{2}e^{3t}L^{-1}\left[\frac{1}{s}\right]$$

$$= \frac{e^{t}}{2} - e^{2t} + \frac{5}{2}e^{3t}$$

(ii) $$\frac{4s-5}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+2)}$$

$A = \frac{1}{3}$ $B = 3$ $C = -\frac{1}{3}$

$$\frac{4s-5}{(s-1)^2(s+2)} = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}$$

taking inverse transform

$$= \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{(s-1)^2}\right) - \frac{1}{3}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{3}e^{at}L^{-1}\left[\frac{1}{s}\right] + 3e^{t}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3}e^{-2t}\left[\frac{1}{s}\right]$$

$$= \frac{1}{3}e^{at} + 3e^{t}\left[\frac{t^{n+1}}{(n+1)!}\right] - e^{-2t}$$

Q. Find Laplace transform of

(i) $$\frac{5s+3}{(s-1)(s^2+2s+5)}$$ (ii) $$\frac{s}{s^4+4a^4}$$

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+2s+5}$$

$A = 1$ $B = -1$ $C = 2$ — ?

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{(s-1)} + \frac{-s+2}{s^2+2s+5}$$

taking inverse

$$= L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left[\frac{-s+2}{s^2+2s+5}\right]$$

$$= e^{t} + L^{-1}\left[\frac{-s+2}{(s+1)^2+4}\right]$$

$$= e^{t} + L^{-1}\left[\frac{-(s+1)+3}{(s+1)^2+4}\right]$$

$$= e^{t} - L^{-1}\left(\frac{(s+1)}{(s+1)^2+2^2}\right) + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^{t} - e^{-t}L^{-1}\left[\frac{s}{(s)^2+2^2}\right] + 3\frac{e^{-t}}{2}L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

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(ii) $\frac{s}{s^4+4a^4}$

$$s^4+4a^4 = (s^2+2a^2)^2-(2as)^2$$

$$= (s^2+2as+2a^2)\cdot(s^2-2as+2a^2)$$

$$= \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$$

$$\frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

$$s = (As+B)(s^2-2as+2a^2)+(Cs+D)(s^2+2as+2a^2)$$

$A=0 \quad B=\frac{-1}{4a} \quad C=0 \quad D=\frac{1}{4a}$

$$= \frac{-1}{4a(s^2+2as+2a^2)} + \frac{1}{4a(s^2-2as+2a^2)}$$

taking inverse Laplace inverse both side we

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{4a}L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right] + \frac{1}{4a}L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]$$

$$= \frac{1}{4a}e^{-at}L^{-1}\left[\frac{1}{s^2+a^2}\right] + \frac{1}{4a}e^{at}L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{4a^2}e^{-at}L^{-1}\left[\frac{a}{s^2+a^2}\right] + \frac{1}{4a^2}e^{at}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{4a^2}e^{-at}\sin at + \frac{1}{4a^2}e^{at}\sin at$$

$$= \frac{1}{4a^2}\sin at\left(e^{at}+e^{-at}\right)$$

$$\boxed{= \frac{1}{4a^2}\sin at \sinh at}$$

Find inverse Laplace transform of

(i) $\log\left(\frac{s+1}{s-1}\right)$ (ii) $\cot^{-1}\left(\frac{s}{a}\right)$ (iii) $\frac{1}{s(s^2+a^2)}$

(i) ~~$\log\frac{(s+1)}{s-1}$~~ $\log\left(\frac{s+1}{s-1}\right)$

$f(s) = \log\left(\frac{s+1}{s-1}\right) \Rightarrow \log(s+1) - \log(s-1)$

differentiating

$f'(s) = \frac{1}{(s+1)} - \frac{1}{(s-1)}$

taking L^{-1}

$L^{-1}[f'(s)] = e^{-t} - e^{t}$ — (1)

$$L^{-1}[f'(s)] = e^{-t} - e^{t}$$

$\because$ we know $L[t^n f(t)] = (-1)^n f^n(s)$

$$L[t\,F(t)] = -f'(s)$$

$$L^{-1}[f'(s)] = -t\,F(t)$$

$$L^{-1}[f'(s)] = -t\,L^{-1}[f(s)]$$

$\downarrow$ from eqn (1).

$$e^{-t} - e^{t} = -t\,L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$$

$$L^{-1}\left[\log\frac{s+1}{s-1}\right] = \frac{e^{-t} - e^{t}}{-t}$$

$$L^{-1}\left[\log\frac{(s+1)}{(s-1)}\right] = \frac{e^{t} - e^{-t}}{t}$$

(ii) $f(s) = \cot^{-1}\left(\frac{s}{a}\right)$

$$f'(s) = \frac{-a}{s^2+a^2}$$

$$L^{-1}[f'(s)] = L^{-1}\left[\frac{-a}{s^2+a^2}\right]$$

$$L^{-1}[f'(s)] = -\sin at \quad -(1)$$

$$-\sin at = -t\,L^{-1}[f(s)]$$

$$-\sin at = -t\,L^{-1}\left[\cot^{-1}\left(\frac{s}{a}\right)\right]$$

$$L^{-1}\left\{\cot^{-1}\left(\frac{s}{a}\right)\right\} = \frac{\sin at}{t}$$

(iii) $\frac{1}{s(s^2+a^2)}$ $\quad f(s) = \left[\frac{1}{s}\left(\frac{1}{s^2+a^2}\right)\right]$ $\quad \because \frac{1}{s}f(s)$

$$\because L\left[\int_0^t F(t)\,dt\right] = \frac{1}{s}f(s)$$

$$L^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2+a^2}\right)\right] = \quad \Downarrow \quad L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t F(t)\,dt$$

$$L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t L^{-1}[f(s)]\,dt$$

$$L^{-1}\left(\frac{1}{s}\left(\frac{1}{s^2+a^2}\right)\right) = \int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right)dt$$

$$= \frac{1}{a}\int_0^t \sin at\,dt$$

$$= \frac{-1}{a^2}\left[\cos at\right]_0^t$$

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Second Shifting Property of Laplace transform

$\Rightarrow$ if $L^{-1}[f(s)] = F(t)$ than

$$L^{-1}[e^{-as} f(s)] = g(t)$$
$$= F(t-a)\,U(t-a)$$
$$= F(t-a)\,H(t-a)$$

where $g(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

when $U(t-a)$ & $H(t-a)$ is a Heaviside unit step funct^n

$$U(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

Q. $L^{-1}\left[\dfrac{e^{-3s}}{s+2}\right]$ (ii) $L^{-1}\left[\dfrac{4e^{-(s\pi/2)}}{s^2+16}\right]$

(iii) $L^{-1}\left[\dfrac{e^{-2s}}{s^2}\right]$

$L[F(t)] = f(x)$
$L^{-1}[f(x)] = f(t)$

(i) $L^{-1}\left[\dfrac{e^{-3s}}{s+2}\right]$

$$L^{-1}\left[e^{-3s}\frac{1}{s+2}\right] = F(t-3)\,U(t-3)$$

$$f(s) = \frac{1}{s+2}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+2}\right] = L^{-1}\left[\frac{1}{s-(-2)}\right] = e^{-2t} = f(t)$$

$$L^{-1}\left[e^{-3s}\frac{1}{s+2}\right] = e^{-2(t-3)}\,U(t-3)$$

(ii) $L^{-1}\left[\dfrac{4e^{-(s\pi/2)}}{s^2+16}\right] = L^{-1}\left[e^{-\pi/2\, s}\dfrac{4}{s^2+16}\right]$

$$f(s) = \frac{4}{s^2+16}$$

$$L^{-1}\left[e^{-\pi/2\, s}\frac{4}{s^2+16}\right] = F(t-\tfrac{\pi}{2})\,U(t-\tfrac{\pi}{2})$$

$$L^{-1}[f(s)] = f(t)$$

$$L^{-1}\left[\frac{4}{s^2+16}\right] = \sin 4t = f(t)$$

$$L^{-1}\left[e^{-\pi/2\, s}\frac{4}{s^2+16}\right] = \sin 4(t-\tfrac{\pi}{2})\,U(t-\tfrac{\pi}{2})$$

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(iii) $L^{-1}\left(\frac{e^{-2s}}{s^2}\right) \Rightarrow f(t-2)\,u(t-2)$

$f(s) = \frac{1}{s^2}$

$L^{-1}[f(s)] = f(t)$

$L^{-1}\left[\frac{1}{s^2}\right] = t$

$f(t-2) = (t-2)$

$L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = (t-2)\,u(t-2)$ Ans

Convolution Theorem

State: Let $F(t)$ & $G(t)$ be two function of a class A defined on real line

~~Then~~

Let $L^{-1}[f(s)] = F(t)$

$L^{-1}[g(s)] = G(t)$ then

$$\boxed{L^{-1}[f(s)\cdot g(s)] = \int_0^t F(x)\,G(t-x)\,dx = F * G}$$

Proof:-

we shall prove that ~~replace the~~

$$L\left[\int_0^t F(x)\,G(t-x)\,dx\right] = f(s)\cdot g(s)$$

→ by conv def of Laplace.

$$L\left[\int_0^t F(x)\,G(t-x)\,dx\right] = \int_0^\infty e^{-st}\left[\int_0^t F(x)\,G(t-x)\,dx\right]dt$$

$$L\left[\int_0^t F(x)\,G(t-x)\,dx\right] = \int_{t=0}^\infty e^{-st}\left[\int_{x=0}^t F(x)\,G(t-x)\,dx\right]dt$$

By changing the order of integ^n

$$L\left[\int_0^t F(x)\,G(t-x)\,dx\right] = \int_{x=0}^\infty\left[\int_{t=x}^\infty e^{-st}F(x)\,G(t-x)\,dt\right]dx$$

$$= \int_0^\infty\left[\int_x^\infty e^{-st+sx-sx}F(x)\,G(t-x)\,dt\right]dx$$

$$= \int_0^\infty e^{-sx}\left[\int_x^\infty e^{-s(t-x)}G(t-x)\,dt\right]F(x)\,dx$$

$$= \int_0^\infty e^{-sx}F(x)\,dx\int_0^\infty e^{-sz}G(z)\,dz$$

By Property of definite integral

i.e. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b [G(z)\,dz]$

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$$= \int_0^\infty e^{-sx} f(x)\,dx \int_0^\infty e^{-s^2} G(z)\,dz$$

$$L\left[\int_0^t f(x)\,G(t-x)\,dx\right] = \int_0^\infty e^{-st} f(t)\,dt \int_0^\infty e^{-st} G(t)\,dt$$

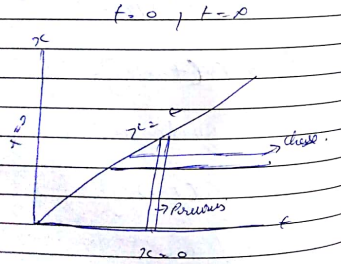
By Laplace transf. of f^n

$$\therefore L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = f(s)$$

$$= f(s)\cdot g(s)$$

$$\boxed{L\left[\int_0^t f(x)\,G(t-x)\,dx\right] = f(s)\cdot g(s)}$$ Proved

* change of limits. $x=0,\ x=t$ previously
$t=0,\ t=\infty$



Convolution Theorem

$$L\left[\int_0^t f_1(x)\,f_2(t-x)\,dx\right] = f_1(s)\,f_2(s)$$

we know

$$L[f_1(t)] = f_1(s)$$
$$L[f_2(t)] = f_2(s)$$

$$L^{-1}[f_1(s)\,f_2(s)] = \int_0^t f_1(x)\,f_2(t-x)\,dx$$

Using convolution theorem

① $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right] \Rightarrow L^{-1}\left[\dfrac{s}{(s^2+a^2)}\,\dfrac{s}{(s^2+b^2)}\right]$

$$f_1(s) = \frac{s}{s^2+a^2} \qquad f_2(s) = \frac{s}{s^2+b^2}$$

$$L^{-1}[f_1(s)] = \cos at = F_1(t) \qquad L^{-1}[f_2(s)] = \cos bt = F_2(t)$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t F_1(x)\,F_2(t-x)\,dx$$

sub. t by x, t by $t-x$

$$= \int_0^t \cos ax \cos b(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t 2\cos \underbrace{\boxed{ax}}_{A} \cos \underbrace{b(t-x)}_{B}\,dx$$

$$= \frac{1}{2}\int_0^t \cos(ax+bt-bx) + \cos(ax-bt+bx)\,dx$$

$$= \frac{1}{2}\int_0^t \cos\{(a-b)x+bt\} + \cos\{(a+b)x-bt\}\,dx$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)x+bt\}}{a-b} + \frac{\sin\{(a+b)x-bt\}}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)t+bt\}}{a-b} + \frac{\sin\{(a+b)t-bt\}}{a+b} - \frac{\sin\{\cancel{(a-b)}\,b(t)\}}{a-b} - \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)(\sin at - \sin bt) + (a-b)(\sin at + \sin bt)}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[2a\sin at - \right.$$

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$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = L^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{(s^2+1)}\right]$$

$$f_1(s) = \frac{1}{s^3} \qquad f_2(s) = \frac{1}{s^2+1}$$

$$L^{-1}[f(s)] = \sin at = \sin t = f_2(t)$$

$$L^{-1}[f(s)] = \frac{t^2}{2} = f_1(t)$$

$$L^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{(s^2+1)}\right] = \int_0^t f_1(x)\, f_2(t-x)\, dx$$

$$= \int_0^t \frac{x^2}{2}\cdot \sin(t-x)\, dx$$

$$= \frac{1}{2}\int_0^t x^2 \sin(t-x)\, dx$$

$$f_2(s) = \frac{1}{s^2} \qquad f_1(s) = \frac{1}{s^2+1}$$

$$L^{-1}[f_1(s)] = \frac{t^2}{2} \qquad L^{-1}[f_1(s)] = \sin t = f_2(t)$$

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t f_1(x)\, f_2(t-x)\, dx$$

$$= \frac{1}{2}\int_0^t \sin t\,(t-x)^2\, dx$$

$$= \int_0^t (t-x)^2 \cdot \cos t$$

$\sin A\cos B + \cos A\sin B$

$\sin A\cos B - \sin B\cos A$

$2\sin A\cos B$

$$= \frac{1}{2}\int_0^t \sin t\,(t^2+x^2-2tx)\, dx$$

$$= \frac{1}{2}\int_0^t \left[t^2\sin t + x^2\sin t - \sin t\, 2tx\right] dx$$

$$= \frac{1}{2}\int_0^t (t^2+x^2-2tx)(-\cos x)\Big|_0^t - \int (2x-2t)(-\sin x)\, dx$$

$$+ 2\cos x\Big]_0^t$$

$$\frac{1}{2}\left[0 - 0 + 2\cos t\right] - \left[-t^2 + 2\cos 0\right]$$

$$= \frac{1}{2}\left[\left[2\cos t\right] + t^2 - 2\right]$$

$$\cos t + \frac{t^2}{2} - 1$$

Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] \Rightarrow L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+a^2)}\right]$

$$f_1(s) = \frac{1}{(s^2+a^2)} \qquad f_2(s) = \frac{s}{(s^2+a^2)}$$

$$L^{-1}[f_1(s)] = \frac{\sin at}{a} = F_1(t) \qquad L^{-1}[f_2(s)] = \cos at = F_2(t)$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_0^t F_1(x)\, F_2(t-x)\, dx$$

$$= \frac{1}{a}\int_0^t 2\,\frac{\sin ax}{a}\cos a(t-x)\, dx$$

$$= \frac{1}{2a}\left[\sin(ax+at-ax) + \sin(ax-at+ax)\right] dx$$

$$= \frac{1}{2a}\left[\sin at + \sin(2ax-at)\right] dx$$

$$= \frac{1}{2a}\left[x\sin at + \frac{\cos(2ax-at)}{2a}\right]_0^t$$

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$$= \frac{1}{2a}\left[x \sin at + \frac{\cos(2ax - at)}{2a} \right]_0^t$$

$$= \frac{1}{2a}\left[\left[t \sin at - \frac{\cos at}{2a} \right] - \left[0 - \frac{\cos at}{2a} \right] \right]$$

$$= \frac{1}{2a}\left[t \sin at - \frac{\cos at}{2a} + \frac{\cos at}{2a} \right]$$

$$\boxed{= \frac{1}{2a} t \sin at}$$

Q. $L^{-1}\left[\frac{1}{(s^2+a^2)^2} \right] = \frac{1}{2a^3}\left(\sin ax - at \cos ax \right)$

Q. $L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)} \right]$

Q. $L^{-1}\left[\frac{1}{(s^2+a^2)^2} \right] = L^{-1}\left[\frac{1}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)} \right]$

$f_1(s) = \frac{1}{(s^2+a^2)}$ $\qquad$ $f_2(s) = \frac{1}{(s^2+a^2)}$

$L^{-1}[f_1(s)] = \frac{\sin at}{a} = f_1(t)$ $\qquad$ $L^{-1}[f_2(s)] = \frac{\sin at}{a} = f_2(t)$

$L^{-1}\left(\frac{1}{(s^2+a^2)^2} \right) = \int_0^t f_1(x)\, f_2(t-x)\, dx$

$\sin = \cos$

$\cos = \sin$

$2 \sin A \sin B$

$\cos(A+B) = \sin A \sin B - \cos A \cos B$

$\cos(A-B) = \sin A \sin B + \cos A \cos B$

$$= \int_0^t \frac{\sin ax}{a} \frac{\sin a(t-x)}{a}\, dx$$

$$= \frac{1}{a^2}\int_0^t \sin ax \, \sin(t-x)\, dx$$

$$= \frac{1}{2a^2}\int_0^t \cos(ax + at - ax) + \cos(ax - at + ax)\, dx$$

$$= \frac{1}{2a^2}\int_0^t \cos at + \cos(2ax - at)\, dx$$

$$= \frac{1}{2a^2}\left[\cancel{\sin at} + x \cos at + \frac{\sin(2ax - at)}{2a} \right]_0^t$$

$$= \frac{1}{2a^2}\left[\left[t \cos at + \frac{\sin at}{2a} \right] - \left[0 + \frac{\sin at}{2a} \right] \right]$$

$$= \frac{1}{2a^2}\left[t \cos at + \frac{\sin at}{a} \right]$$

$$= \frac{1}{2a^2}\left[\frac{at \cos at + \sin at}{a} \right]$$

$$= \frac{1}{2a^3}\left[at \cos at + \sin at \right]$$

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$\sin = \cos\theta$

(2) $L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)}\right] = L^{-1}\left(\frac{1}{(s^2+1)}\frac{2}{(s^2+2^2)}\right)$

$f_1(s) = \frac{1}{s^2+1}$ $\qquad f_2(s) = \frac{2}{s^2+2^2}$

$L^{-1}[f(s)] = \underset{F_1(t)}{\sin t}$ $\qquad L^{-1}[f(s)] = \underset{F_2(t)}{\sin 2t}$

$L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)}\right] = \int_0^t F_1(x)\, F_2(t-x)\, dx$

$= \int_0^t \sin t \, \sin 2(t-x)\, dx$

$= \frac{1}{2}\int_0^t 2\sin t \, \sin 2(t-x)\, dx$

$= \frac{1}{2}\int_0^t \cos(t+2t-2x) + \cos(t-2t+2x)\, dx$

$= \frac{1}{2}\int_0^t \cos(3t-2x) + \cos(2x-t)\, dx$

$= \frac{1}{2}\left[\frac{\sin(3t-2x)}{-2} + \frac{\sin(2x-t)}{2}\right]_0^t$

$= \frac{1}{2}\left\{\left[\frac{-\sin(t)}{-2} + \frac{\sin t}{2}\right] - \left[\frac{-\sin 3t}{2} - \frac{\sin(t)}{2}\right]\right\}$

$= \frac{1}{2}\left[\frac{\sin t}{2} + \frac{\sin t}{2} + \frac{\sin 3t}{2} + \frac{\sin t}{2}\right]$

$= \frac{1}{4}\left[\sin 3t + \sin t\right]$

Application of Laplace transform.

1) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$

2) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

It is used for finding sol^n of ordinary linear diff eq^n with const co-efficient. This method gives the sol^n without finding the sol^n

Steps for solving ordinary differential eq^n using L[ ] transform.

① Take L[ ] for both side using initial cond^n this gives an algebric eq^n

② Solve the algebraic equ^n to get $\bar{Y} = L[Y]$ in terms of s

③ Taking L^{-1}[ ] from both side which gives Y which is desire sol^n

Using L

1) $L[f(t)] = f(s)$

$L[y] = Y(s) = \bar{y}$

$L[f'(t)] = s f(s) - f(0)$

$\therefore [y''] = s^2\bar{y} - s^2 y(0) - s y'(0) - y''(0)$

Q. $y'' + 9y = 6\cos 3t$

$y(0) = 2 \qquad y'(0) = 0$

Sol. Taking Laplace both side.

$L[y''] + 9L[y] = 6\,L[\cos 3t]$

$s^2\bar{y} - s y(0) - y'(0) + 9\bar{y} = \frac{6s}{s^2+9}$

Put value $y(0) = 2$ & $y'(0) = 0$

$s^2\bar{y} - 2s + 9\bar{y} = \frac{6s}{s^2+9}$

$(s^2+9)\bar{y} = \frac{6s}{s^2+9} + 2s$

$f(s) = \bar{y} = \frac{6s}{(s^2+9)^2} + \frac{2s}{s^2+9}$

Taking inverse

$L^{-1}[\bar{y}] = 6L^{-1}\left(\frac{s}{(s^2+9)^2}\right) + 2L^{-1}\left(\frac{s}{s^2+9}\right)$

$= \frac{6}{6} + \sin 3t + 2\cos 3t$

Q. $y'' + 4y = 0 \qquad y(0) = 1 \quad y(0) = 6$

Sol. Taking $L[\,]$

$L[y''] + 4L[y] = L(0)$

$s^2 f(y) - s y(0) - y'(0) + 4\bar{y} = 0$

$s^2\bar{y} - s y(0) - y'(0) + 4\bar{y} = 0$

$s^2\bar{y} - s - 6 + 4\bar{y} = 0$

$(s^2+4)\bar{y} = 6 + s$

$\bar{y} = \frac{6+s}{s^2+4}$

taking

$L^{-1}(\bar{y}) = L^{-1}\left(\frac{6+s}{s^2+4}\right)$

$L^{-1}(\bar{y}) = L^{-1}\left(\frac{6}{s^2+4} + \frac{s}{s^2+4}\right)$

$= 6\,\frac{1}{2}\sin 2t + \cos 2t$

Q. Solve: $y'' + 2y' - 3y = 3 \qquad y(0) = 4$

$y'(0) = -7$

$L^{-1}(y'') + 2L[y'] - 3L(y) = 3$

$s^2\bar{y} - s y(0) - y'(0) + 2(s\bar{y} - y(0)) - 3\bar{y} = 3L[1]$

$s^2\bar{y} - 4s + 7 + 2s\bar{y} - 8 - 3\bar{y} = \frac{3}{s}$

$(s^2 + 2s - 3)\bar{y} = \frac{3}{s} + 4s + 1$

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Solution of Simultaneous diff. eqn using $L[\ ]$ transform

Solve

$\frac{dx}{dt} + y = 0$ $\quad x(0) = 1$

$\frac{dy}{dt} - x = 0$ $\quad y(0) = 0$

when 0

$x' + y = 0$

taking Laplace both side

$L[x'] + L[y] = 0$

$s\bar{x} - x(0) + \bar{y} = 0$

$s\bar{x} - 1 + \bar{y} = 0$

$s\bar{x} + \bar{y} = 1$ — ③

$y' - x = 0$

$L[y'] - L[x] = 0$

$s\bar{y} - y(0) - \bar{x} = 0$

$s\bar{y} - \bar{x} = 0$ — ④

$s\bar{x} + \bar{y} = 1$

$s\bar{y} - \bar{x} = 0$

$s\bar{x} + \bar{y} = 0$

$s^2\bar{y} - s\bar{x} = 0$

$s^2\bar{y} + \bar{y} = 0$

$L^{-1}[\bar{y}] = L^{-1}\left[\frac{1}{s^2+1}\right]$ $\quad \bar{y} =$

$\bar{y} = \cos t$

2) $(D^2 - 3)x - 4y = 0$

$x + (D^2 + 1)y = 0$

$x'' - 3x - 4y = 0$ — ① $\quad x = y = \frac{dx}{dt} = 0$

$x + y'' + y = 0$ — ②

$L[x''] - 3L[x] - 4L[y] = 0$

$s^2\bar{x} - sx(0) - x'(0) - 3\bar{x} - 4\bar{y} = 0$

$s^2\bar{x} - 2 - 3\bar{x} - 4\bar{y} = 0$

$(s^2 - 3)\bar{x} = 4\bar{y} + 2$ — ③

$L[x] + L[y''] + L[y] = 0$

$\bar{x} + s^2\bar{y} - sy(0) - y'(0) + \bar{y} = 0$

$\bar{x} + s^2\bar{y} + \bar{y} = 0$

$(s^2 + 1)\bar{y} + \bar{x} = 0$ — ④

$(s^2+1)(s^2-3)\bar{y} + (s^2-3)\bar{x} = 0$

$-(s^2-3)\bar{x} = 4\bar{y} + 2$

$(s^2+1)(s^2-3)\bar{y} = 4\bar{y} + 2$

$(s^2+1)(s^2-3)\bar{y} + 4\bar{y} - 2 = 0$

$\bar{y} = \frac{-2}{(s^2+1)(s^2-3)+4} = \frac{-2}{s^4 + s^2 - 3s^2 - 3 + 4}$

$\bar{y} = \frac{-2}{s^4 - 2s^2 + 1} = \frac{-2}{}$

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3/03/2018

Matrices

Types of Matrices

1. Square Matrix :- A matrix has n rows and n columns is called Square matrix of order n.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$

The diagnol of this matrix containing the elements 1, 5, 9, is called the principle diagnol.

A Square matrix is said to be singular if its determinant is zero otherwise its is non-singular.

2) Scalar Matrix :-

A diagnol matrix whose all the principle diagnol elements are equal is called a Scalar Matrix.

e.g $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ or $\begin{bmatrix} 3 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 9 & 3 \end{bmatrix}$

3) Upper triangular Matrix :-

A square Matrix in which all elements below the principle diagnol are zero is called upper triangular Matrix. e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 8 \end{bmatrix}$

4. Lower triangular Matrix :-

A square Matrix in which all elements above principle diagonal are ~~zero~~ zero is called the lower triangular Matrix. e.g $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 0 & 9 \end{bmatrix}$

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(7) Rank of a Matrix:

⇒ Echelon Form

A Matrix is said to be echelon form if the following cond^n are satisfied

(i) Any given matrix to convert in an Echelon form we applied either row transformation or column transformation. Both transformation are not applied.

(ii) If A matrix is square form then the row then the row echelon form will form the upper triangular.

And the column Echelon form is a lower triangular Matrix

(iii) The rank of Matrix in echelon form is equal to the no. non zero rows of the matrix

or

The no of non-zero column of the matrix

Q. Find rank of Matrix $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

(1) $R_2 \to R_2 - R_1$ $\quad R_3 \to R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$$

(2) $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is the form of upper Δ'lar matrix this matrix is echelon form having 3 non zero rows &

$\therefore$ Rank $(A) = 3$

$\rho(A) = 3$

⇒ 1

Reduce the Matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$ $\quad R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

$$\begin{array}{r} -24 \\ +17 \\ \hline -10 \end{array} \qquad \begin{array}{r} -32 \\ 17 \\ \hline 15 \end{array}$$

$R_4 \to R_4 - 5R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & -10 & -15 \end{bmatrix}$$

$R_4 \to R_4 - 5R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\rho(A) = 2}$ $\quad$ Rank $(\rho) = 2$ Ans

⇒ Normal Form of a Matrix / ~~common~~ Canonical form

Every non-zero matrix of rank (r) can be reduced by a sequence of elementary (either row or column operation or both row & column opert^n) operations to the form

The normal form or $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ is called canonical form

we convert the given matrix into unit matrix (identity matrix) any how either row ~~map~~ or colum or both operation.

Q) Reduce ~~to~~ the matrix into its normal form and find its rank. $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\begin{array}{r} 24 \\ -7 \\ \hline 17 \end{array}$$

$R_2 \to R_2 - 2R_1$ $\quad R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$C_4 \to C_2 + C_3$; $C_3 \to C_3 + 2C_2$ $C_4 \to C_4 + 4C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_4 \to R_4 - R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to 5R_3 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -2 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

27

27

42

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 6C_2$

$C_4 \to C_4 + 2C_2$

$C_3 \to \frac{C_3}{33}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 22C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $(A) = 3$ $\left[\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 0 \end{array}\right]$

$\therefore$ the rank of Matrix is 3

3) Rank of Matrix $= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 4 \end{bmatrix}$

Reducing it into normal form $C_2 \to C_2 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & -1 & 2 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

$C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

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$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 3 & 0 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 3 & 0 & 1 \\ -1 & 0 & 0 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 3 & 0 & 1 \\ 0 & 2 & 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 2 & 0 & 6 \end{bmatrix}$$

$C_4 \to C_4 - C_2$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$R_3 = R_3 + 2R_2$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 = \frac{C_2}{-1}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{I_2}$$

rank of $A = 2$

4/03/2018

INVERSE

Compute the inverse of a matrix from elementary transformation

we know that $A = IA$ — (1)

applied elementary row transformation on A of the left side and I on the right side so that A is reduce to indentity Matrix I and I of R.H.S is reduced to another Matrix B we get

$$I = BA$$

This implies that $\Rightarrow IA^{-1} = BAA^{-1}$

$$\Rightarrow IA^{-1} = BI$$

$$A^{-1} = B$$

Find inverse of matrix using elementary transform"

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Let $A = IA$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

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