Dynamical Systems

An introduction

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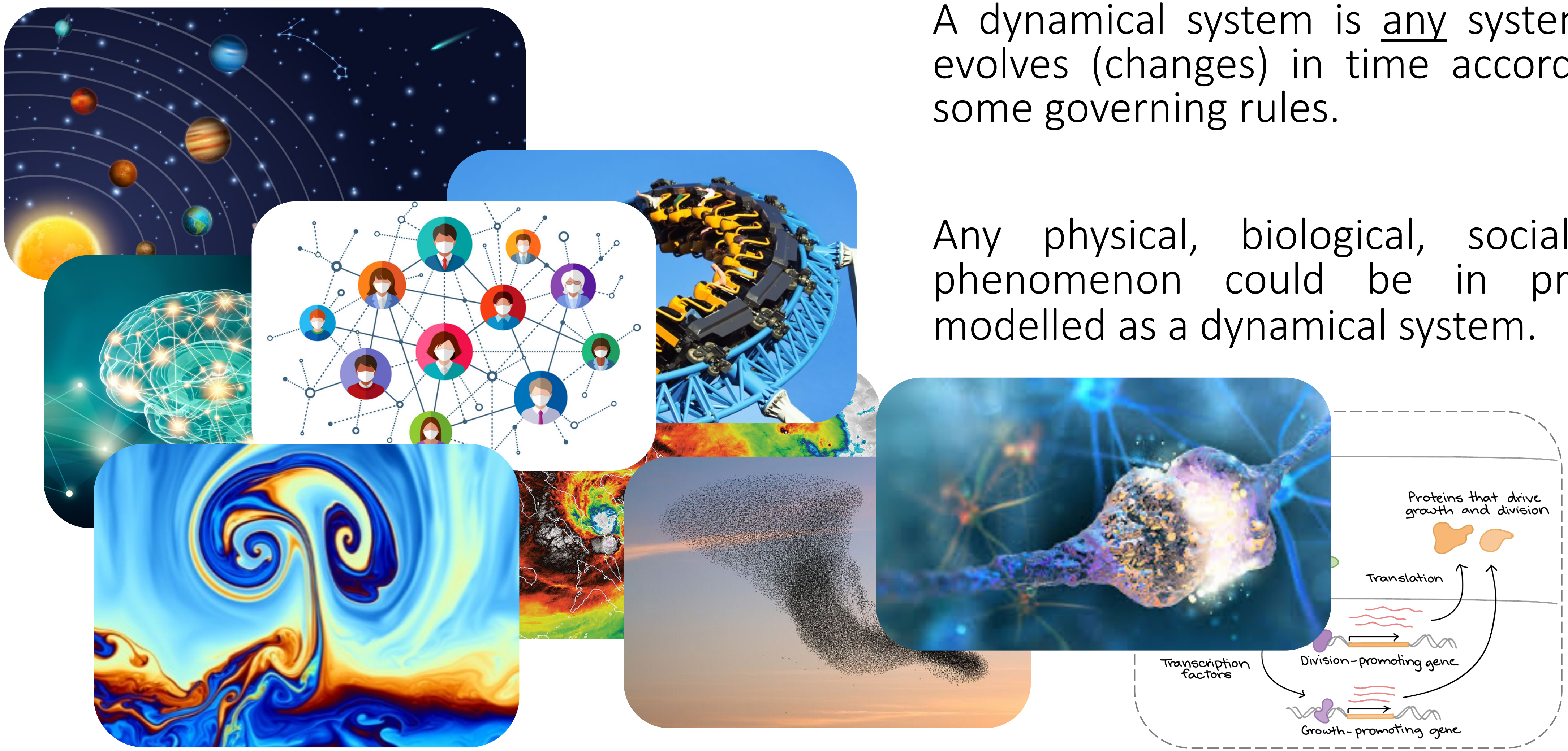
Dynamical systems

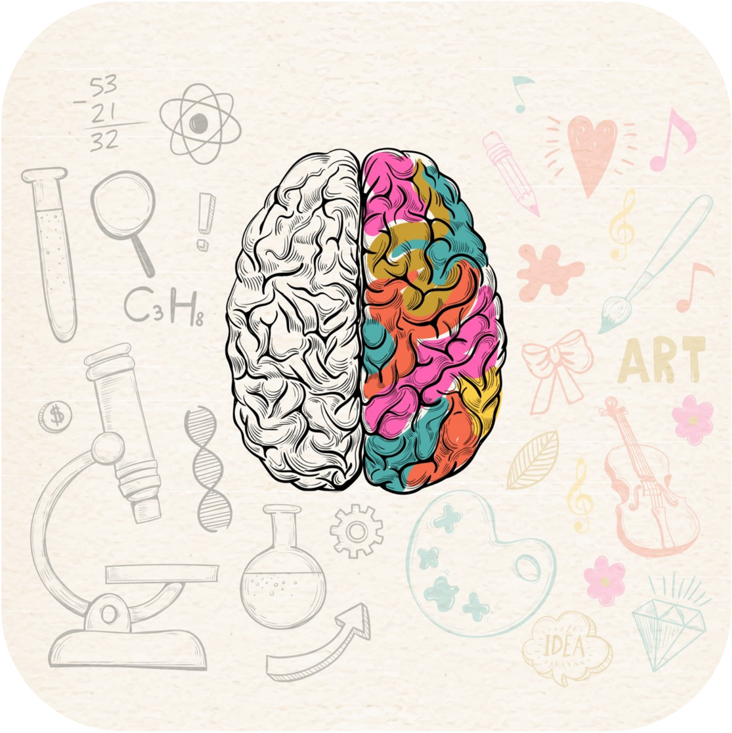
A dynamical system is <u>any</u> system that evolves (changes) in time according to some governing rules.

Any physical, biological, social, etc. phenomenon could be in principle modelled as a dynamical system.



Dynamical systems classification

- Low or high dimensional
- Linear or nonlinear
- Discrete or continuous
- Single or multiscale
- Stochastic or deterministic
- Autonomous or nonautonomous (unforced/forced)



A dynamical view of the world

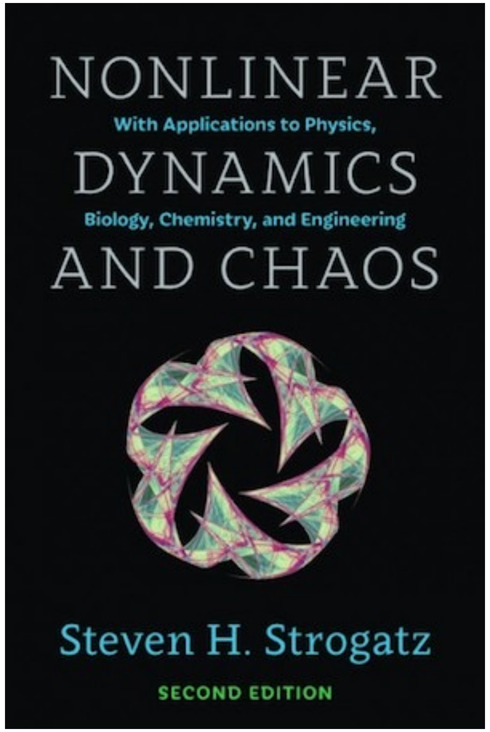


Figure 1.3.1

Number of variables →

Nonlinearity ↓

	$n = 1$	$n = 2$	$n \geq 3$	$n >> 1$	Continuum
Linear	*Growth, decay, or equilibrium* Exponential growth RC circuit Radioactive decay	*Oscillations* Linear oscillator Mass and spring RLC circuit 2-body problem (Kepler, Newton)	Civil engineering, structures Electrical engineering	*Collective phenomena* Coupled harmonic oscillators Solid-state physics Molecular dynamics Equilibrium statistical mechanics	*Waves and patterns* Elasticity Wave equations Electromagnetism (Maxwell) Quantum mechanics (Schrödinger, Heisenberg, Dirac) Heat and diffusion Acoustics Viscous fluids
			The frontier		
Nonlinear	Fixed points Bifurcations Overdamped systems, relaxational dynamics Logistic equation for single species	Pendulum Anharmonic oscillators Limit cycles Biological oscillators (neurons, heart cells) Predator-prey cycles Nonlinear electronics (van der Pol, Josephson)	*Chaos* Strange attractors (Lorenz) 3-body problem (Poincaré) Chemical kinetics Iterated maps (Feigenbaum) Fractals (Mandelbrot) Forced nonlinear oscillators (Levinson, Smale) Practical uses of chaos Quantum chaos ?	*Spatio-temporal complexity* Coupled nonlinear oscillators Lasers, nonlinear optics Nonequilibrium statistical mechanics Nonlinear solid-state physics (semiconductors) Josephson arrays Heart cell synchronization $\boxed{\text{Neural networks}}$ Immune system Ecosystems Economics	Nonlinear waves (shocks, solitons) Plasmas Earthquakes General relativity (Einstein) Quantum field theory Reaction-diffusion, biological and chemical waves Fibrillation $\boxed{\text{Epilepsy}}$ Turbulent fluids (Navier-Stokes) $\boxed{\text{Life}}$

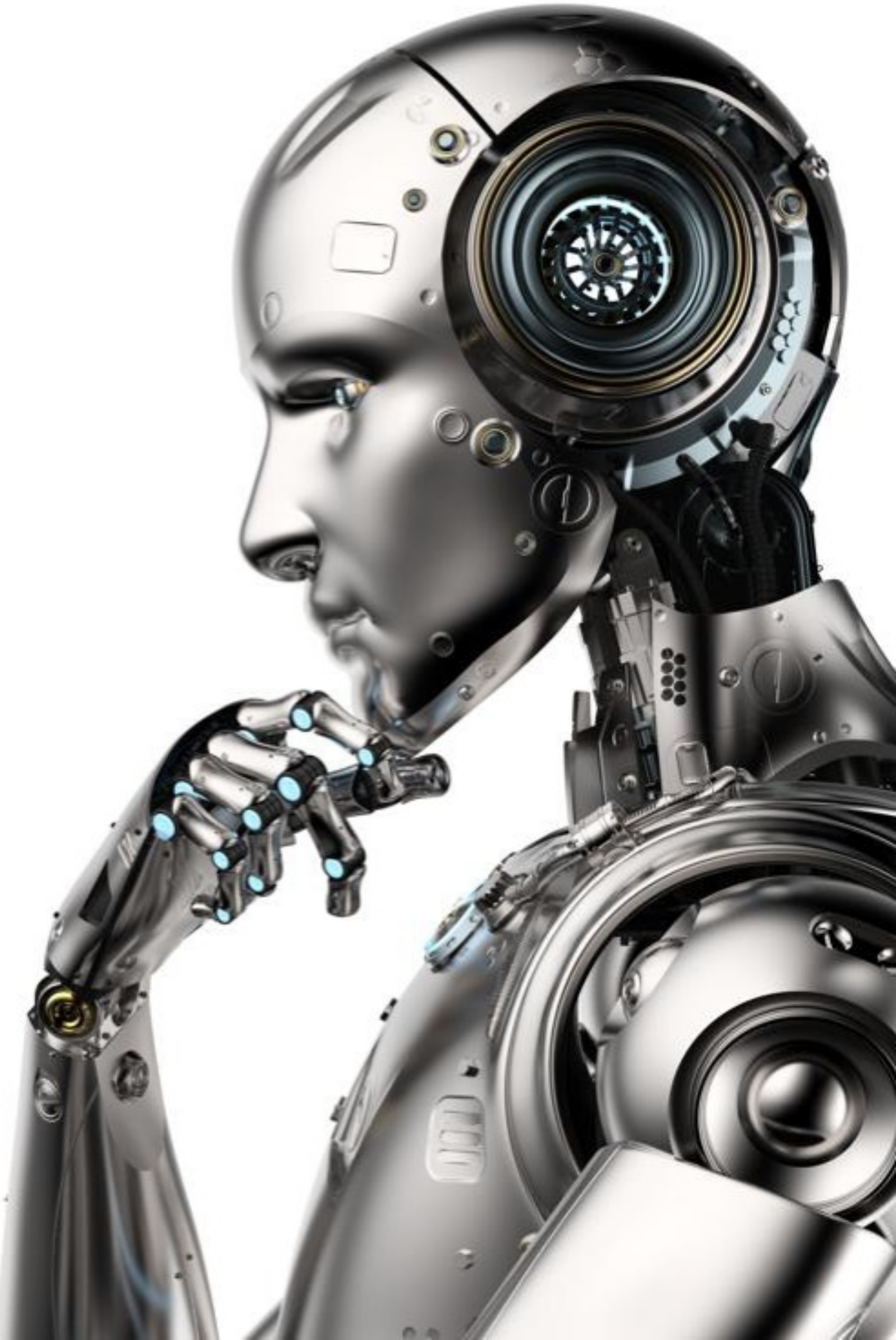
Statistical *vs* dynamical modeling

- Goal: to determine which variables contribute to an observable: do variables X, Y and Z contribute to variable A?
- Explain observables in terms of each other
- They can have either predictive (machine learning) and <u>statistical</u> explanatory power (sometimes both, depending the case)

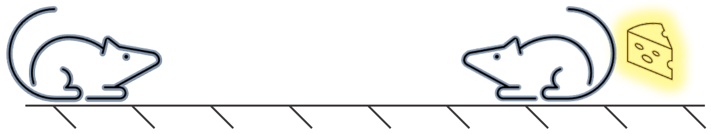
- Goal: to determine how variables contribute to an observable: how do variables X, Y and Z interact to produce A?
- Based on the current understanding of the underlying “physics” - Requires positing “invisible” causes (hidden/latent variables)
- They have both predictive and <u>*mechanistic*</u> explanatory power

Practical uses of DS theory and models

- ❖ Understanding – to gain insight
- ❖ Prediction of future behavior
- ❖ Design and optimization
- ❖ Control of the system



What is a differential equation?

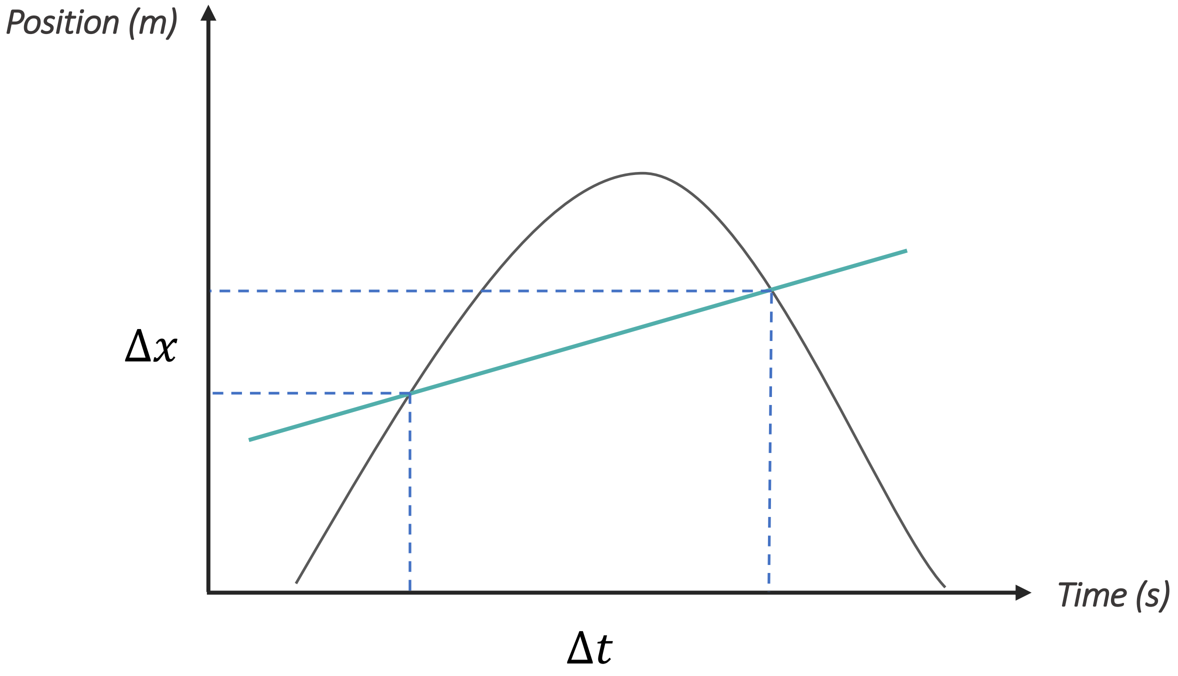


x: position t: time

v: velocity

Average velocity $$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

Instantaneous velocity



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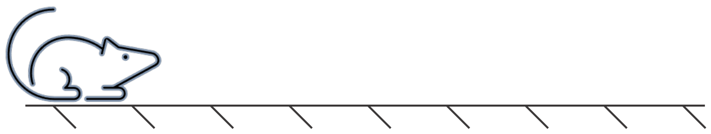

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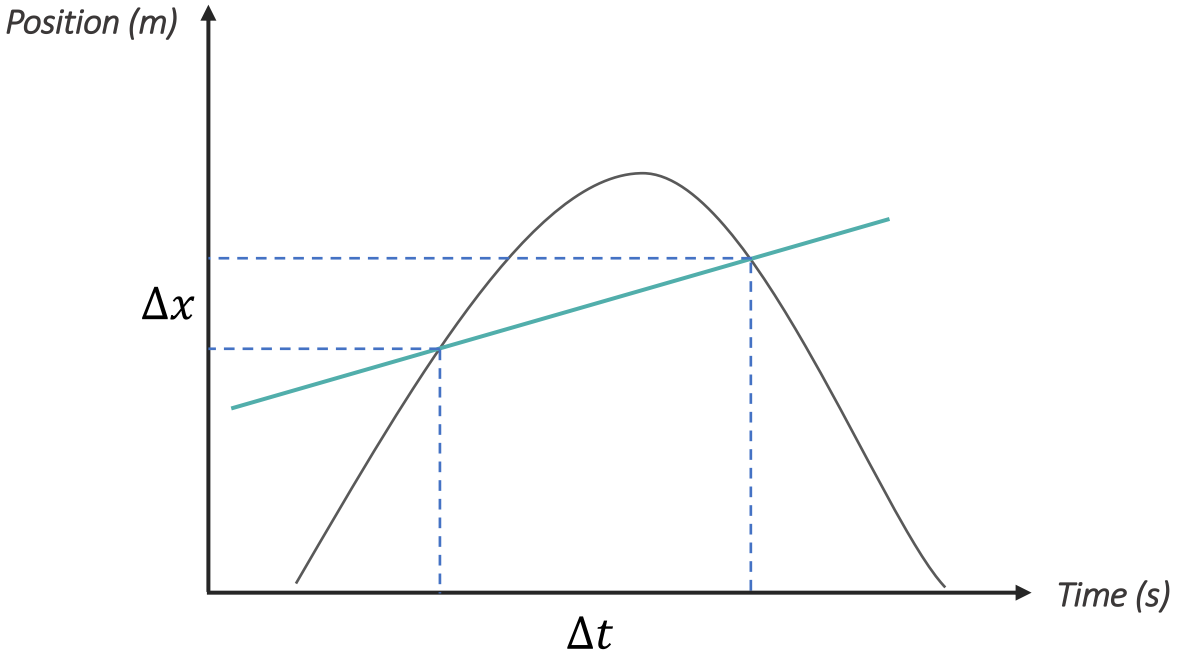
Instantaneous velocity



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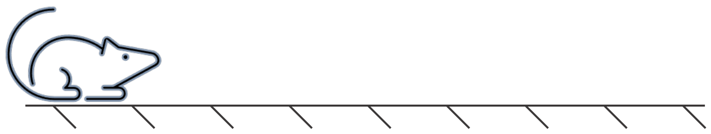

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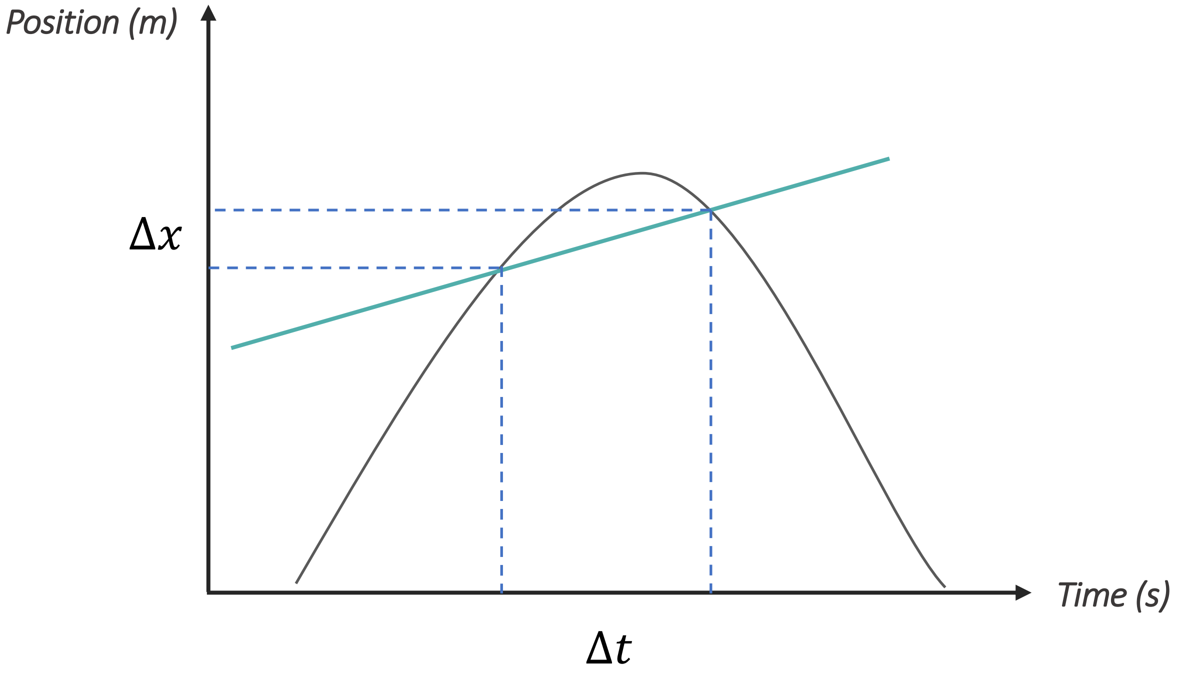

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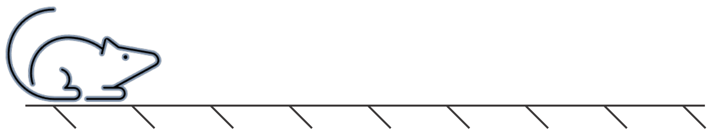
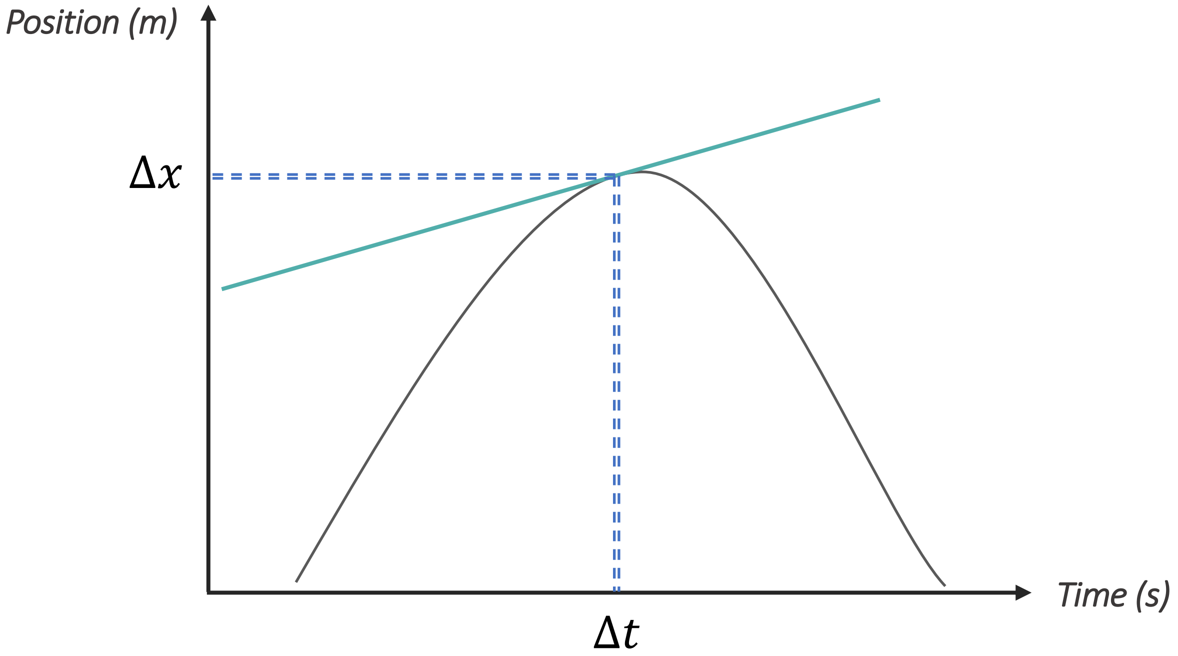


x: position $\quad$ t: time

v: velocity

Average velocity $\quad \bar{v} = \dfrac{\Delta x}{\Delta t} = \dfrac{x_f - x_i}{t_f - t_i}$

Instantaneous velocity $\quad v = \dfrac{dx}{dt} = \lim_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}$



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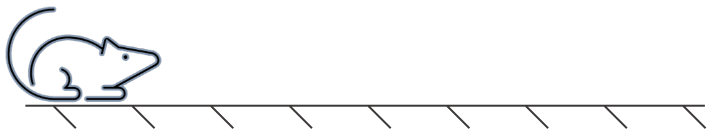

x: position

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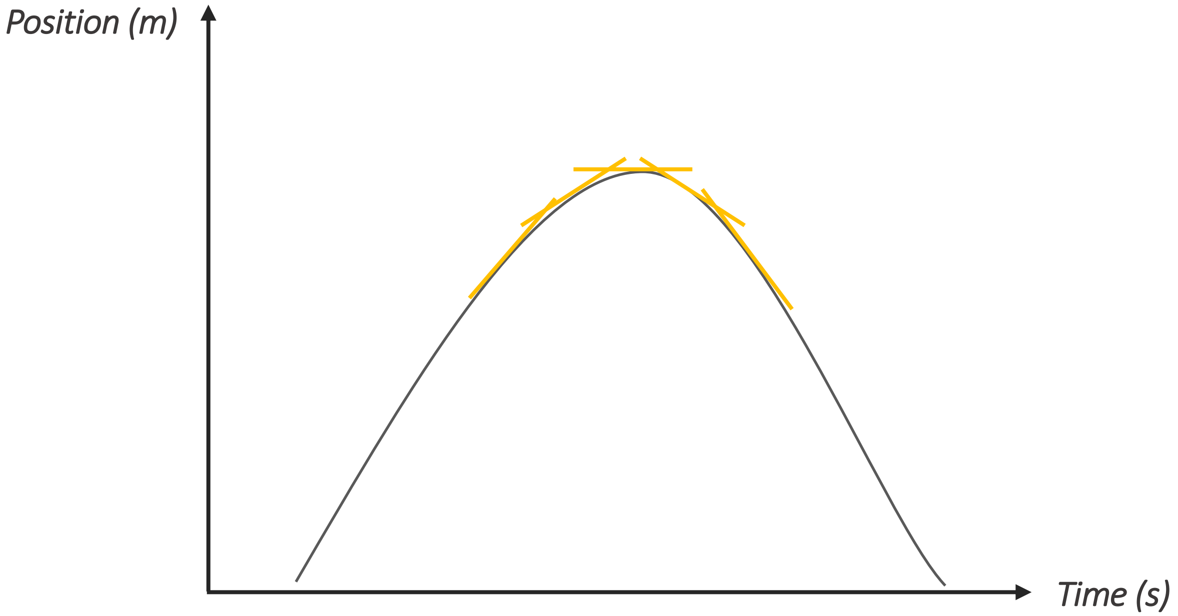

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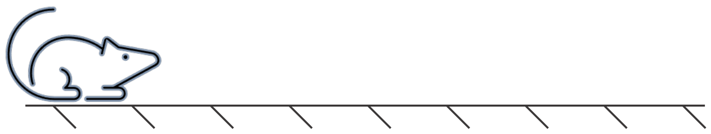


x: position

t: time

v: velocity

T: total time

L: maze length

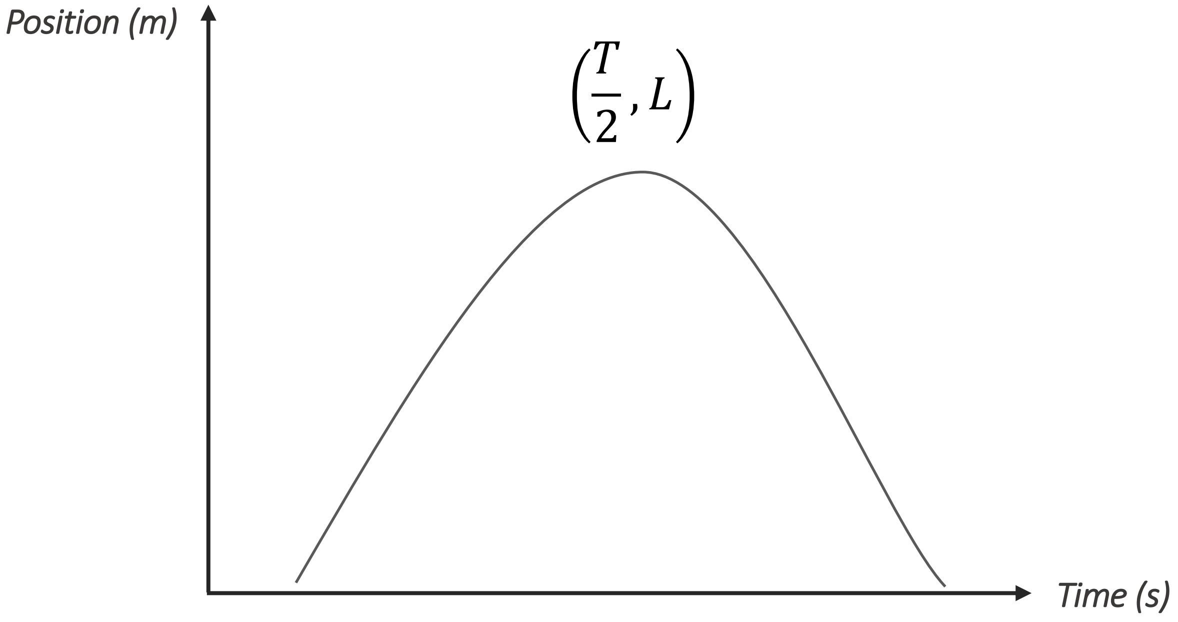
$$x(t) = -\left(t - \frac{T}{2}\right)^2 + L$$

$$= -t^2 + Tt - \frac{T^2}{4} + L$$

$$= -t^2 + Tt + C$$

$$v = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$v(t) = \frac{dx}{dt} = -2t + T$$



Dynamical systems as [coupled] differential equations

$$\mathbf{x} = [x_1, x_2, \dots, x_n] \qquad \mathbf{x} \in \mathbb{R}^n$$

$$\left.\begin{aligned}
\dot{x}_1 &= \frac{dx_1}{dt} = f_1(x_1, \dots, x_n) \\
\dot{x}_2 &= \frac{dx_2}{dt} = f_2(x_1, \dots, x_n) \\
&\vdots \\
\dot{x}_n &= \frac{dx_n}{dt} = f_n(x_1, \dots, x_n)
\end{aligned}\right\}$$

dynamics

state

actuation control
(external input)

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, t, u; \beta)$$

parameters

time

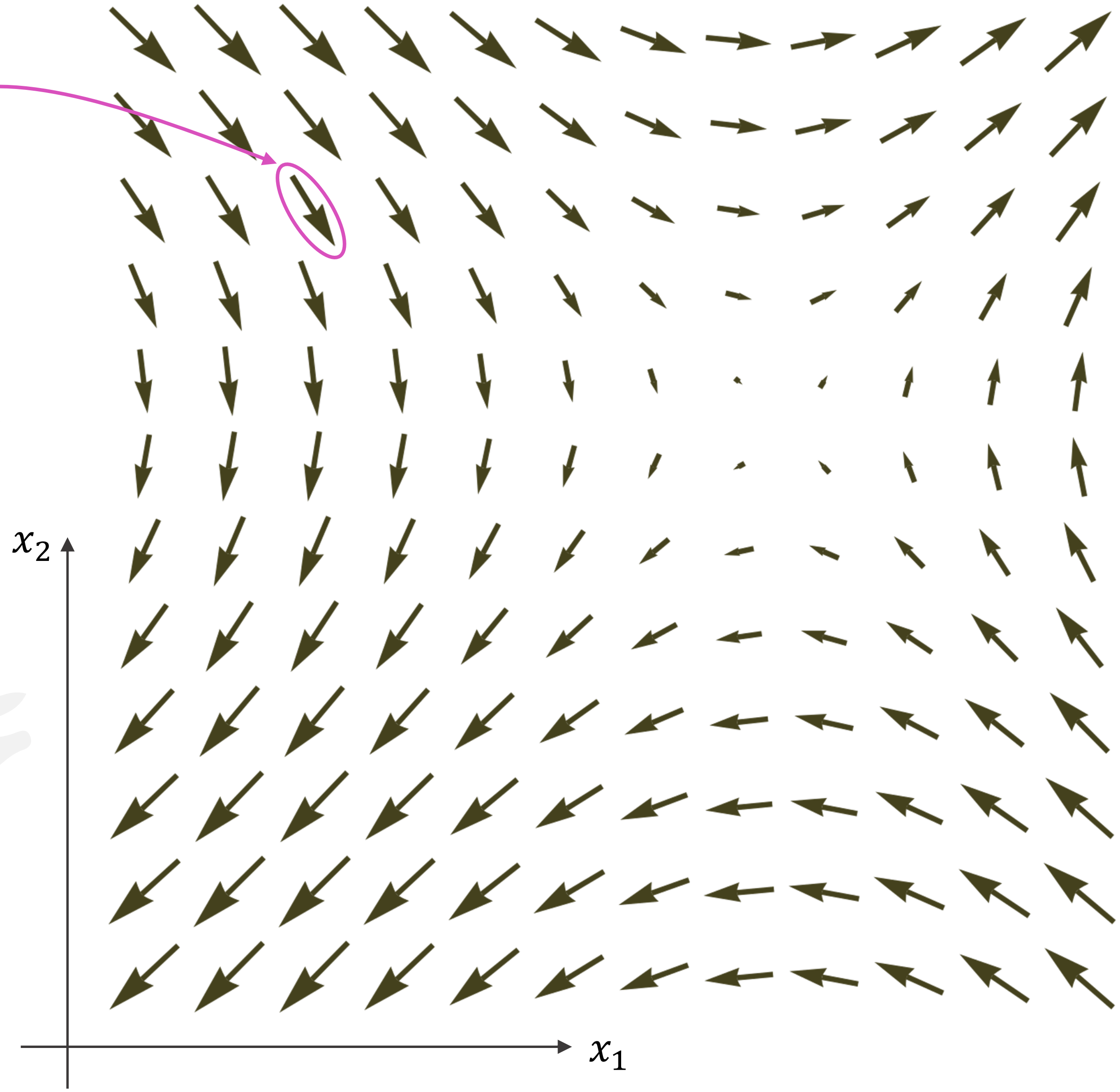
The overarching goal of DS theory

- We are not interested in single trajectories (i.e., $\mathbf{x}(t)$). Rather, we want to describe the behavior of the entire system in the limit $\lim_{t\to\infty} \mathbf{x}(t)$
- How sensitive is the system to small perturbations?
- How do the dynamics of the system depend on its parameters?
- How does the long-term behavior of the system depend on the initial conditions?

$$\mathbf{x}^{\mathrm{T}} = [x_1, x_2] \qquad \mathbf{x} \in \mathbb{R}^2$$

$$\left.\begin{aligned} \dot{x}_1 &= \frac{dx_1}{dt} = f_1(x_1, x_2) \\ \dot{x}_2 &= \frac{dx_2}{dt} = f_2(x_1, x_2) \end{aligned}\right\} \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

Dynamical systems describe vector fields



Linear vs nonlinear dynamical systems

LINEAR

Additive relationships among variables.

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{A}\mathbf{x}$$

$$\mathbf{x}(t) = \mathbf{x}_0 e^{\boldsymbol{A}t}$$

NONLINEAR

Nonadditive relationships among variables, particularly exponential and multiplicative interactions.

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

Sometimes, there is no explicit solution. Therefore, numerical integration methods are often used to visualize the system's trajectories given some initial conditions!!!

CHAOS

Linear vs nonlinear dynamical systems

$$\frac{dx_1}{dt} = -x_1 - 2x_2$$

$$\frac{dx_2}{dt} = 2x_1 - x_2$$

$$\frac{dx_3}{dt} = -x_3$$

$$\frac{dx_1}{dt} = -x_1 x_2 - 2x_2$$

$$\frac{dx_2}{dt} = 2x_1 - x_2 x_3^2$$

$$\frac{dx_3}{dt} = -x_3$$

$$\frac{dx_1}{dt} = x_1 - x_2$$

$$\frac{dx_2}{dt} = \frac{e_1^x - x_2^2}{x_3}$$

$$\frac{dx_3}{dt} = \cos(x_3)$$

$$\frac{dx_1^2}{dt} = x_{1^2} - x_2$$

$$\frac{dx_2}{dt} = 5x_1^2 + x_2$$

$$\frac{dx_3}{dt} = x_3 \tan\theta$$

CONTENT

1. Example of a linear dynamical system (3D)
2. Fixed points and stability of linear systems (2D)
3. Stability of nonlinear systems
4. Bifurcation analysis (Hodgkin-Huxley model)
5. Applications in neuroscience

A simple linear system (3D)

$$\mathbf{x}^{\mathbf{T}} = [x_1, x_2, x_3] \qquad \mathbf{x} \in \mathbb{R}^3$$

$$\boldsymbol{A} = \begin{bmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -x_1 - 2x_2 \\ \frac{dx_2}{dt} &= 2x_1 - x_2 \\ \frac{dx_3}{dt} &= -x_3 \end{aligned}\right\} \frac{d\mathbf{x}}{dt} = \boldsymbol{A}\mathbf{x}$$

$$\boxed{\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

$$\mathbf{x}(t) = \mathbf{x}_0 e^{\boldsymbol{A}t}$$

The Euler method

$V_0 = 5$ L

$\frac{dQ}{dt} = 2$ L/s

$V_1 = ?$

$V_1 = V_0 + \frac{dQ}{dt}\Delta t$

$= 5 + 2(1) = 7$L

$$x_{t+1} = x_t + \frac{dx}{dt}\,\Delta t$$

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \frac{d\mathbf{x}}{dt}\,\Delta t$$

A simple linear system (3D)

$$\mathbf{x}^{\mathbf{T}} = [x_1, x_2, x_3] \qquad \mathbf{x} \in \mathbb{R}^3$$

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{A}\mathbf{x} = \begin{bmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

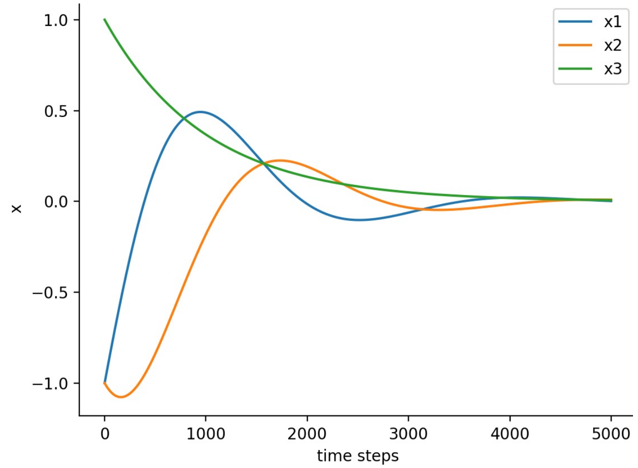
analytical solution

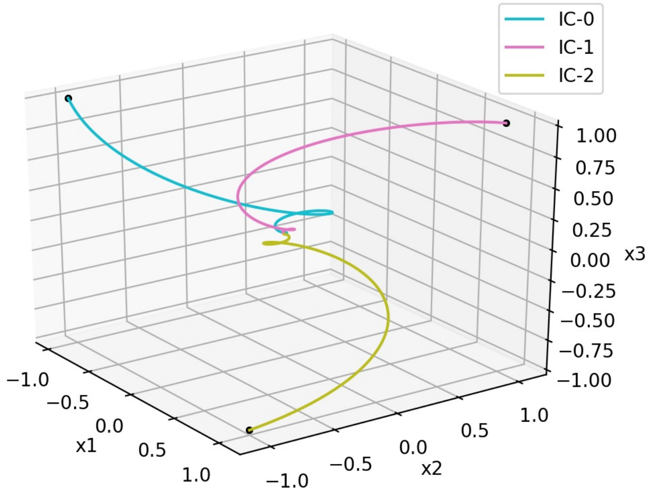
$$\mathbf{x}(t) = \mathbf{x}_0 e^{At}$$

numerical integration

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \frac{d\mathbf{x}}{dt}\Delta t$$

$$\mathbf{x_0} = [-1, -1, 1]$$



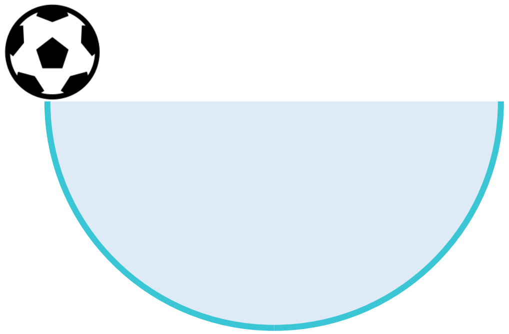


Phase diagram/ plot/ portrait/ plane

Fixed points!

The fixed points of a system are points (in state space) where the dynamics of the system do not change!

$$\frac{d\mathbf{x}}{dt} = 0$$



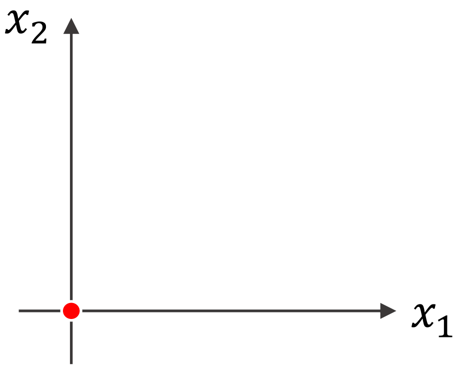
Stability of linear systems (2D)

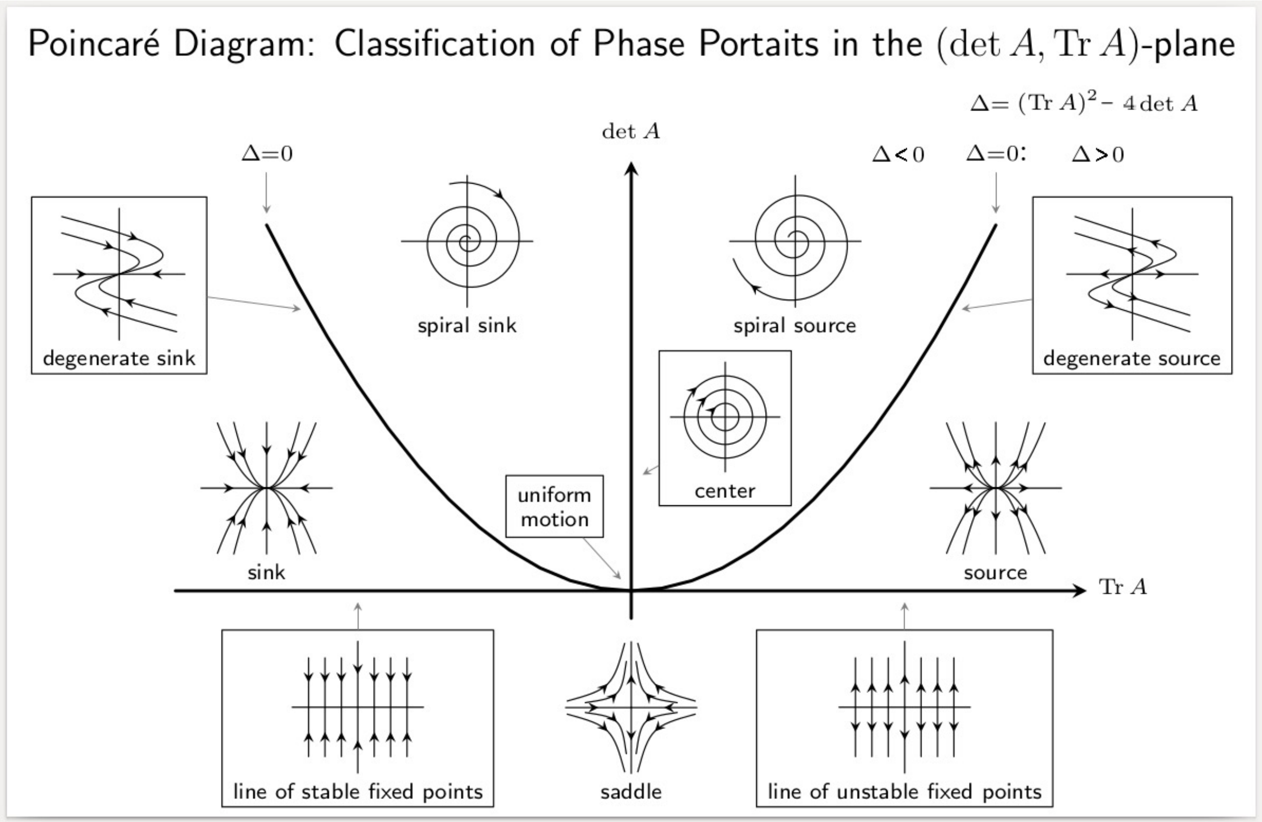
$$\frac{d\mathbf{x}}{dt} = \boldsymbol{A}\mathbf{x}$$

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d\mathbf{x}}{dt} = 0 \qquad \mathbf{x}^* = [0,0]$$

Phase diagram





$$\det(\mathbf{A}) = |\mathbf{A}| = ad - bc \qquad \operatorname{Tr}(\mathbf{A}) = a + d$$

$$\Delta = \left(\operatorname{Tr}(\mathbf{A})\right)^2 - 4\det(\mathbf{A})$$

Stability of linear systems (2D)

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{A}\mathbf{x} \qquad \boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{x}(t) = \mathbf{x_0} e^{At}$$

$$= \mathbf{x_0} V \begin{bmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\lambda} \end{bmatrix} V^{-1}$$

Characteristic polynomial

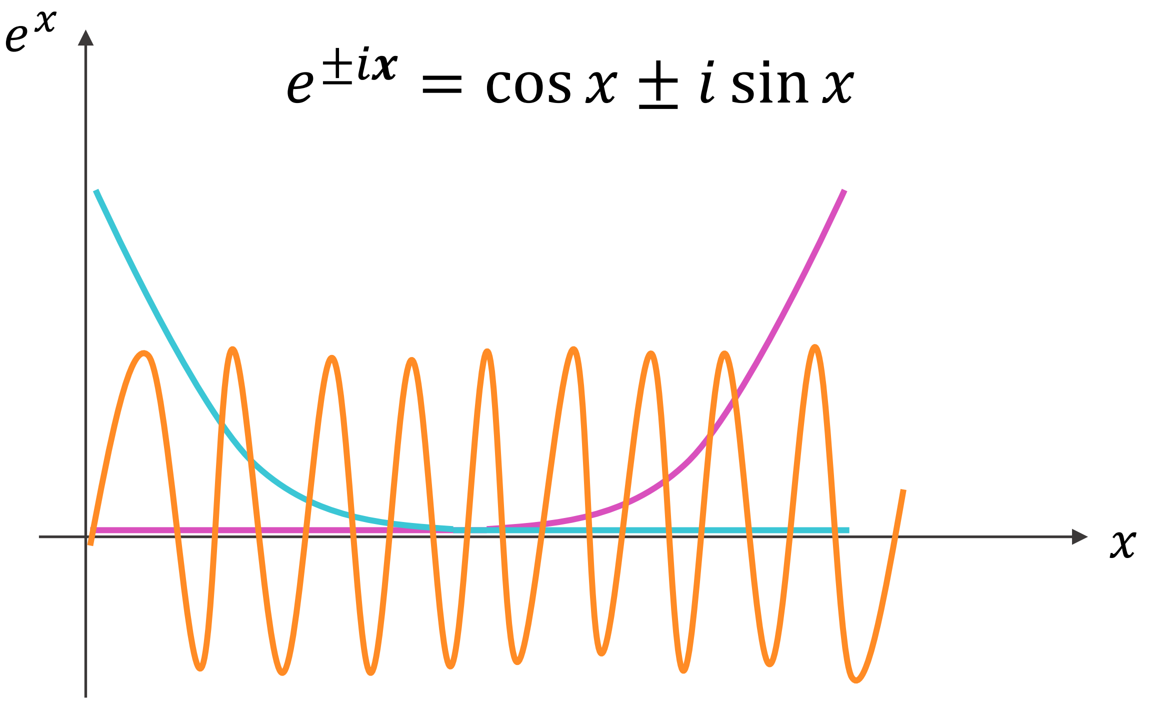
$$(a - \lambda)(d - \lambda) - cb = 0$$

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = 0$$

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \qquad \boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}$$

$$y(x) = e^{\boldsymbol{x}}$$



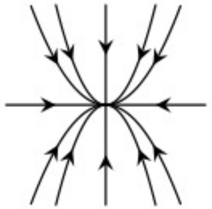
Take home message: the behavior and stability of a linear system depends on the eigenvalues (λ) of matrix $\boldsymbol{A}$!!!

Stability of linear systems (2D)

$$\mathbf{x}(t) = \mathbf{x_0} V \begin{bmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\lambda} \end{bmatrix} V^{-1}$$

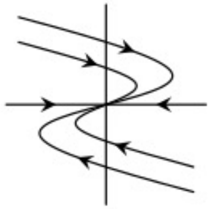
$\boldsymbol{\lambda_1, \lambda_2} \in \mathbb{R}$

$\lambda_1 \neq \lambda_2 < 0$



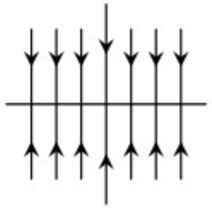
sink

$\lambda_1 = \lambda_2 < 0$



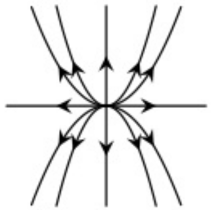
degenerate sink

$\lambda_1 = 0, \lambda_2 < 0$



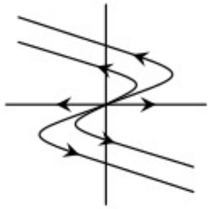
line of stable fixed points

$\lambda_1 \neq \lambda_2 > 0$



source

$\lambda_1 = \lambda_2 > 0$



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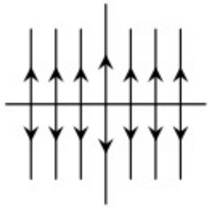
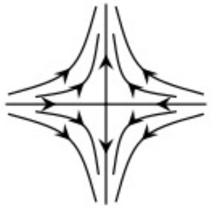
$\lambda_1 = 0, \lambda_2 > 0$



line of unstable fixed points

$\lambda_1 > 0, \lambda_2 < 0$

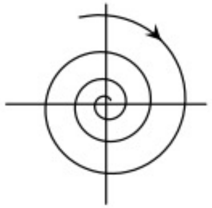


saddle

$\boldsymbol{\lambda_1, \lambda_2} \in \mathbb{C}$

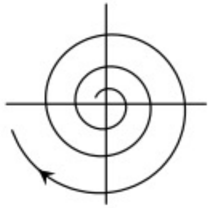
$\lambda_1 = \overline{\lambda_2} = a \pm ib \quad \mathrm{Re}(\lambda) = a \quad \mathrm{Im}(\lambda) = b$

$\mathrm{Re}(\lambda) < 0$



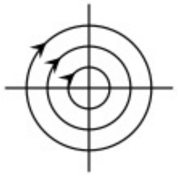
spiral sink

$\mathrm{Re}(\lambda) > 0$



spiral source

$\mathrm{Re}(\lambda) = 0$

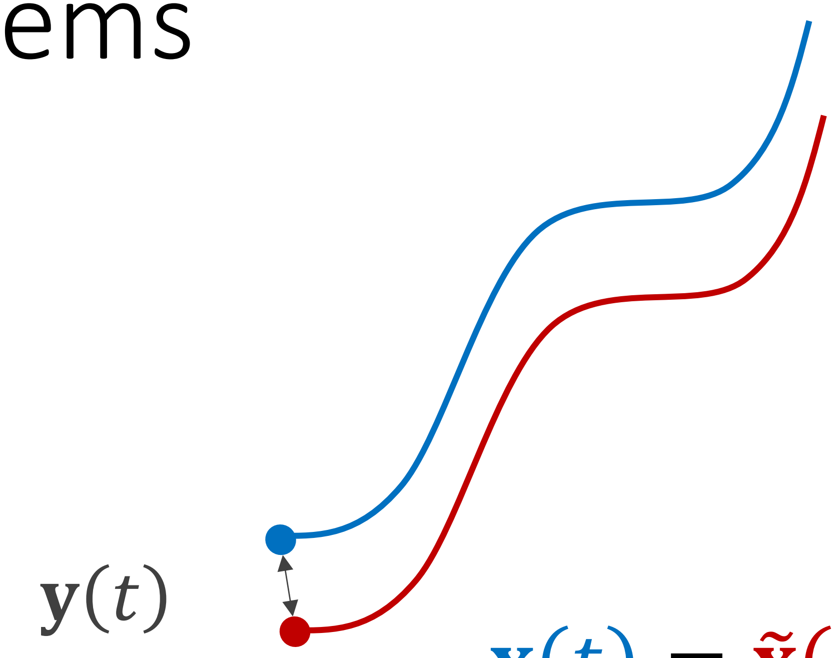


center

Stability of nonlinear systems

$$\frac{d\mathbf{x}}{dt} = \cancel{A\mathbf{x}} \qquad \frac{d\mathbf{x}}{dt} = f(\mathbf{x}(t); \beta)$$

$$\frac{d\mathbf{x}}{dt} = 0 \qquad \mathbf{x}^* = [\cancel{0,0}]$$



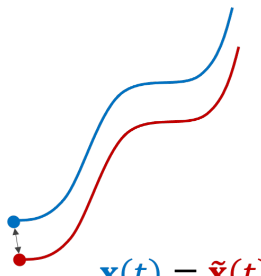
$$\mathbf{x}(t) = \tilde{\mathbf{x}}(t) + \mathbf{y}(t)$$

Both $\mathbf{x}(t)$ *and* $\tilde{\mathbf{x}}(t)$ *are solutions of* $\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$

Original system: $$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

A solution of the system ($\tilde{\mathbf{x}}(t)$) perturbed by a small amount ($\mathbf{y}(t)$):

$$\mathbf{x}(t) = \tilde{\mathbf{x}}(t) + \mathbf{y}(t)$$



$\mathbf{x}(t) = \tilde{\mathbf{x}}(t) + \mathbf{y}(t)$

We derive w.t.r. to time on both sides of the equation and replace $\frac{d}{dt}$ by f :

$$\frac{d\mathbf{x}}{dt} = \frac{d(\tilde{\mathbf{x}} + \mathbf{y})}{dt}$$

$$f(\mathbf{x}) = f(\tilde{\mathbf{x}} + \mathbf{y})$$

Taylor expanding around $\tilde{\mathbf{x}}$:

$$f(\tilde{\mathbf{x}} + \mathbf{y}) = f(\tilde{\mathbf{x}}) + \frac{df(\tilde{\mathbf{x}})}{dt}\boldsymbol{y} + \varepsilon\cancel{(\mathbf{y}^2})$$

$$f(x + \delta x) = f(x) + f'(x)\,\delta x + \varepsilon(\delta x^2)$$
$$\approx f(x) + f'(x)\,\delta x$$

$$f(\tilde{\mathbf{x}} + \mathbf{y}) = f(\tilde{\mathbf{x}}) + \frac{df(\tilde{\mathbf{x}})}{dt}\boldsymbol{y}$$

$$\cancel{f(\tilde{\mathbf{x}})} + f(\mathbf{y}) = \cancel{f(\tilde{\mathbf{x}})} + \frac{df(\tilde{\mathbf{x}})}{dt}\boldsymbol{y}$$

$$\frac{d\mathbf{y}}{dt} = Df(\tilde{\mathbf{x}})\boldsymbol{y}$$

If $\tilde{\mathbf{x}}$ is a fixed point, then $\frac{d\mathbf{x}}{dt} = 0$ and $Df(\tilde{\mathbf{x}})$ becomes a constant matrix

$$Df(\tilde{\mathbf{x}}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Jacobian matrix of the vector field

- Evaluating the stability (of the fixed points) of a nonlinear system becomes a matter of analyzing the linear system defined by $\frac{d\mathbf{y}}{dt} = \widetilde{\boldsymbol{A}}\boldsymbol{y}$, where $\widetilde{\boldsymbol{A}}$ is a constant matrix that corresponds to the Jacobian evaluated at a fixed point !!!

The unforced Duffing oscillator (2D)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_1 - x_1^3 - \delta x_2 \qquad \delta > 0$$

1. Find the fixed points $\quad \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = 0$

$$\dot{x}_1 = 0 \leftrightarrow x_2 = 0$$

$$x_2 = 0 \rightarrow \dot{x}_2 = x_1 - x_1^3 = 0$$

$$x_1(1 - x_1^2) = 0$$

FP: $x_1 = 0, \; x_2 = 0$

FP: $x_1 = \pm\sqrt{1}, \; x_2 = 0$

We need to analyze the stability of the system in these <u>**three**</u> fixed points

2. Calculate the Jacobian

$$Df(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$Df(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 & -\delta \end{bmatrix}$$

$$Df([0,0]) = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix}$$

$$Df([1,0]) = \begin{bmatrix} 0 & 1 \\ -2 & -\delta \end{bmatrix}$$

$$Df([-1,0]) = \begin{bmatrix} 0 & 1 \\ -2 & -\delta \end{bmatrix}$$

Wiggins 1990, p. 6; Ott 1993, p.3

The unforced Duffing oscillator (2D)

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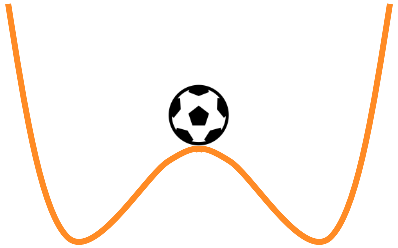
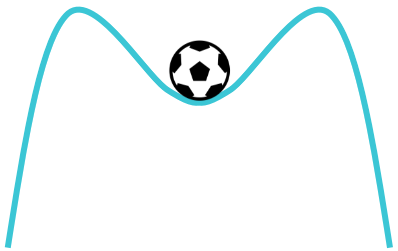


$\delta < 0$



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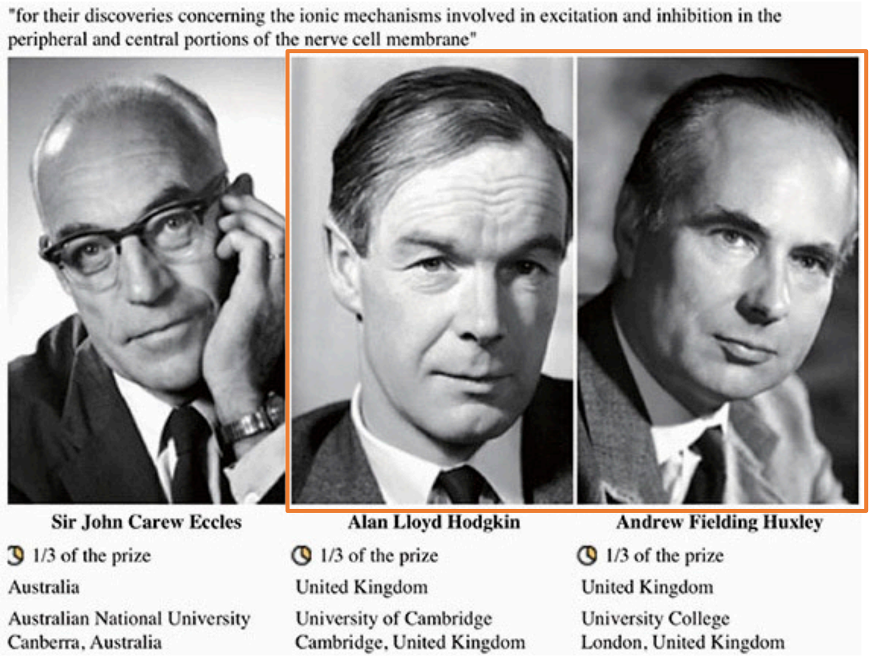
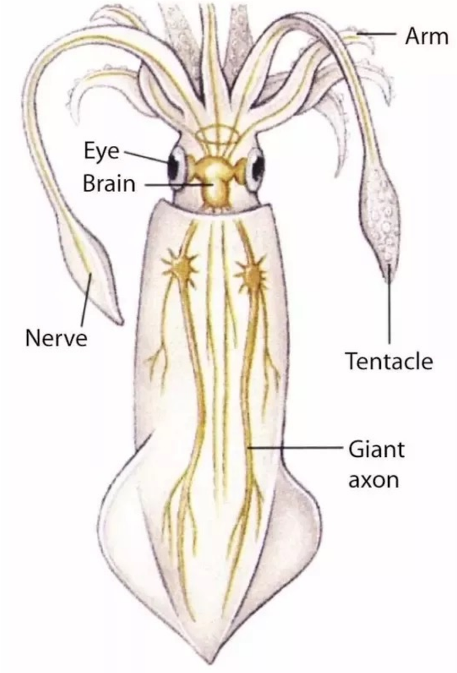
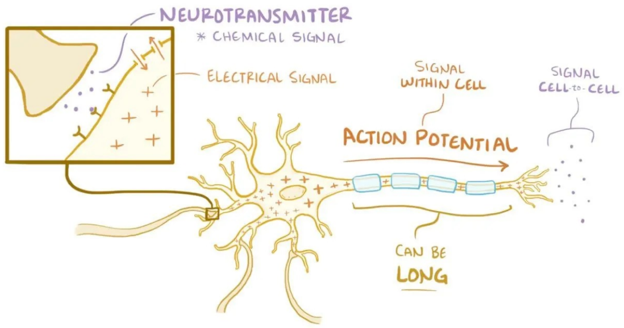
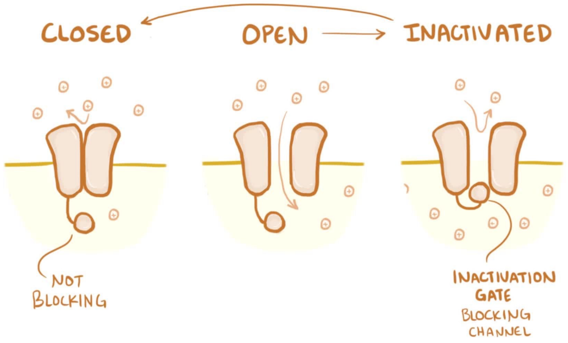
The Hodgkin-Huxley neuron model









The Hodgkin-Huxley neuron model

Total current = capacitive current + Ionic current (Na, K, leak channels)

$$I = C_m \frac{dV_m}{dt} + g_K n^4 (V_m - E_K) + g_{Na} m^3 h (V_m - E_{Na}) + g_l (V_m - E_l)$$

$$C_m \frac{dV_m}{dt} = I - g_K n^4 (V_m - E_K) - g_{Na} m^3 h (V_m - E_{Na}) - g_l (V_m - E)$$

$$\frac{dn}{dt} = \alpha_n(V_m)(1-n) - \beta_n(V_m) n$$

$$\frac{dm}{dt} = \alpha_m(V_m)(1-m) - \beta_m(V_m) m$$

$$\frac{dh}{dt} = \alpha_h(V_m)(1-h) - \beta_h(V_m) h$$

How many state variables do we need to describe this system?

This is just to model the membrane potential of a single neuron!!!

The simplified Morris-Lecar model

$$C_m \frac{dV_m}{dt} = I_{ext} - g_K n(V_m - E_K) - g_{Ca} m_\infty (V_m - E_{Ca}) - g_l(V_m - E_l)$$

bifurcation parameter

$$\frac{dn}{dt} = \frac{\phi(n_\infty - n)}{\tau_n}$$

$$m_\infty(V_m) = \frac{1}{2}\left[1 + \tanh\left(\frac{V_m - V_1}{V_2}\right)\right]$$

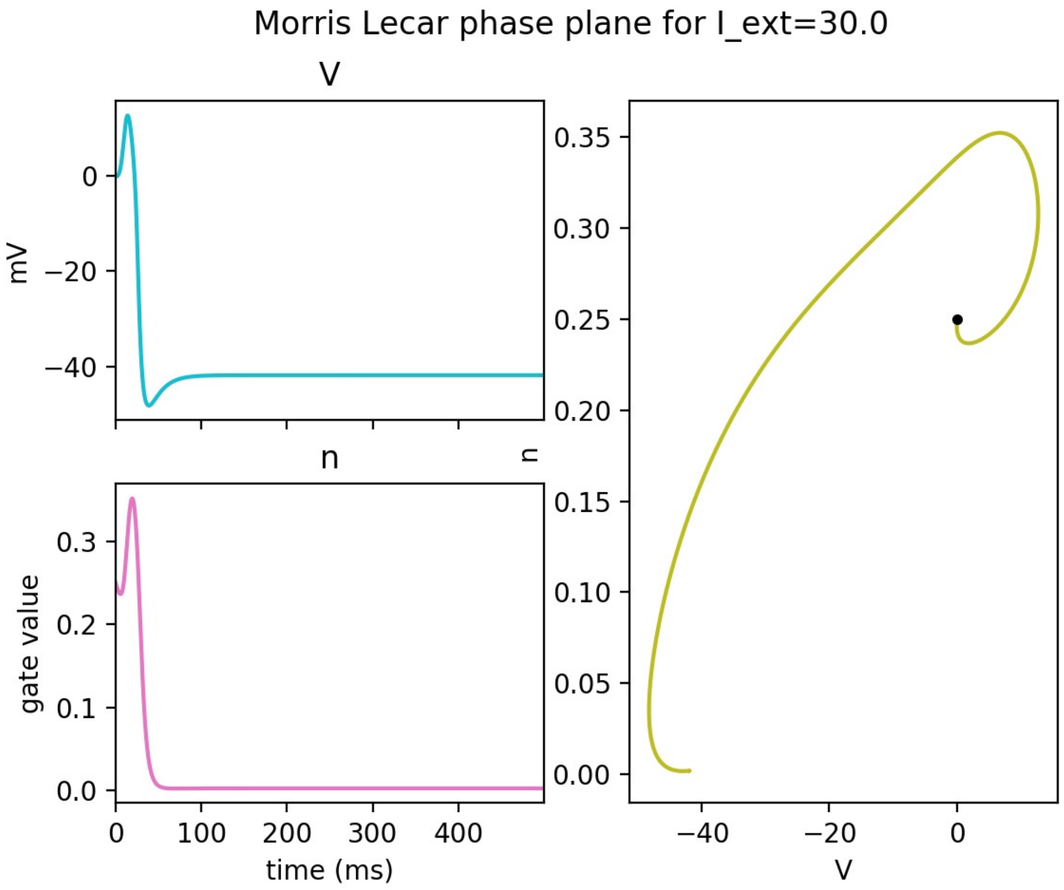
$$\tau_n(V_m) = \frac{1}{\cosh\left(\frac{V_m - V_3}{2V_4}\right)}$$

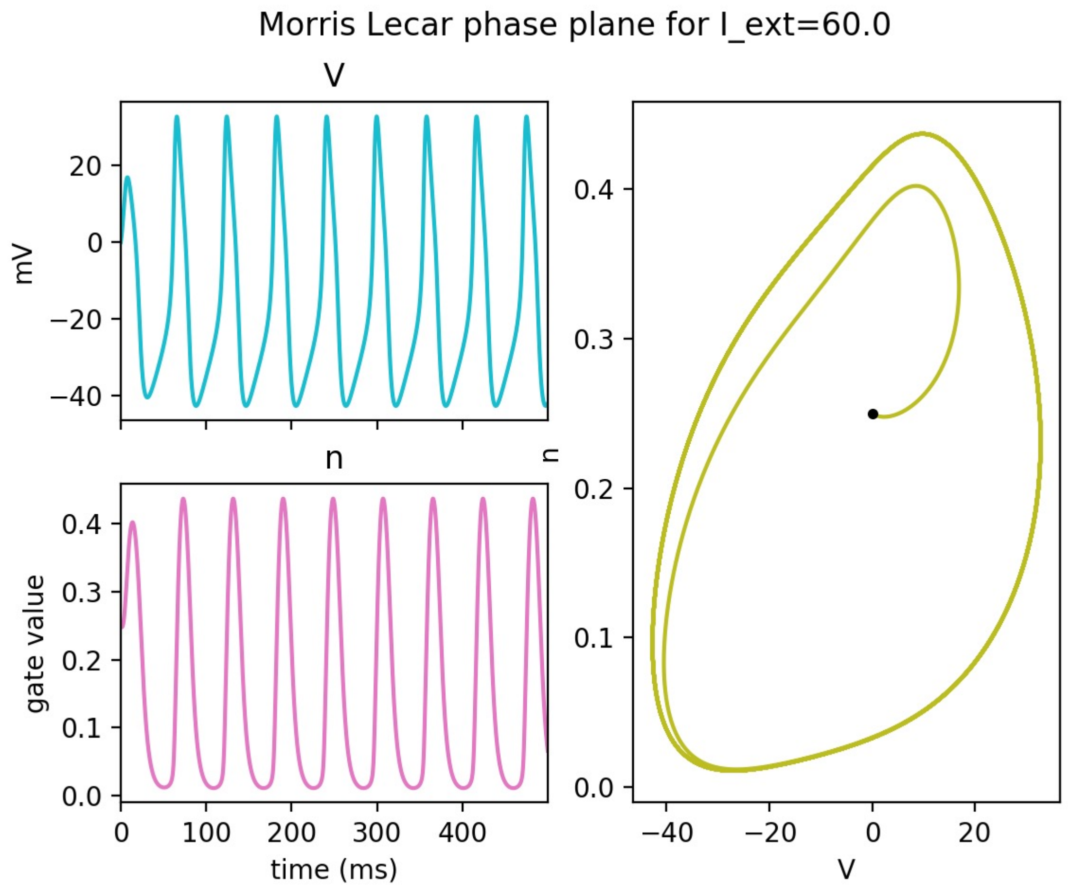
$$n_\infty(V_m) = \frac{1}{2}\left[1 + \tanh\left(\frac{V_m - V_3}{V_4}\right)\right]$$

How many state variables do we need to describe this system?

http://www.scholarpedia.org/article/Morris-Lecar_model

Bifurcation theory





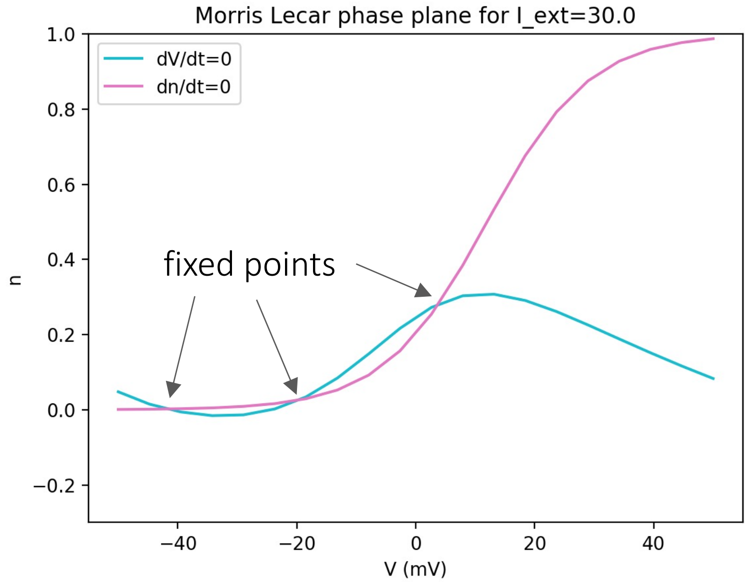
https://www.youtube.com/watch?v=ovJcsL7vyrk

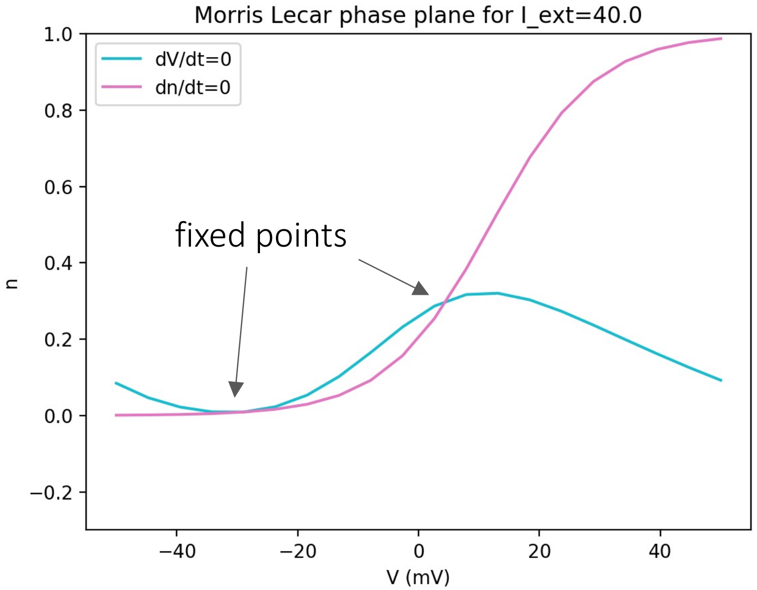
Nullclines
+ phase plane analysis

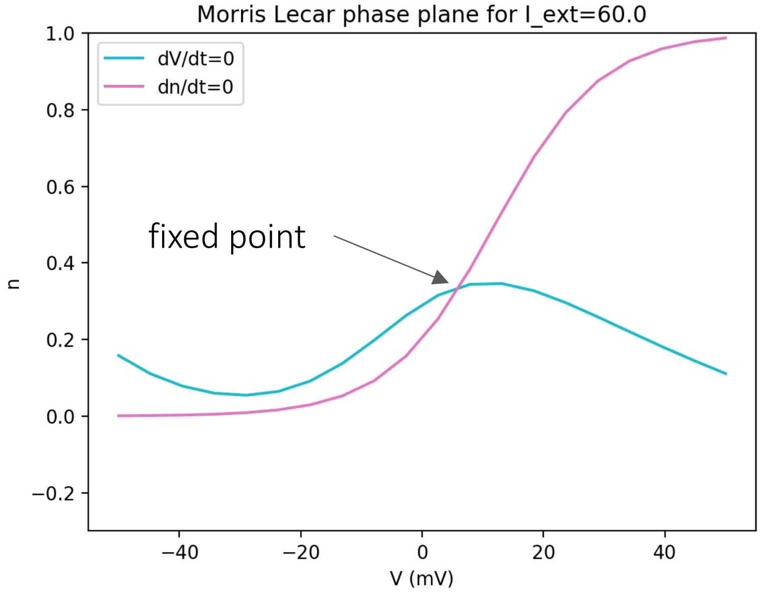
$$\frac{dV_m}{dt} = \frac{I_{ext} - g_K n(V_m - E_K) - g_{Ca} m_\infty (V_m - E_{Ca}) - g_l(V_m - E_l)}{C_m} = 0$$

$$\frac{dn}{dt} = \frac{\phi(n_\infty - n)}{\tau_n} = 0$$

- There is a nullcline for every state variable.
- For instance, the nullcline of V_m corresponds to the set of paired (V_m, n) values for which $\frac{dV_m}{dt} = 0$, and the nullcline of n corresponds to the set of paired (V_m, n) values for which $\frac{dn}{dt} = 0$.
- The intersection of the nullclines correspond to the fixed points of the system.

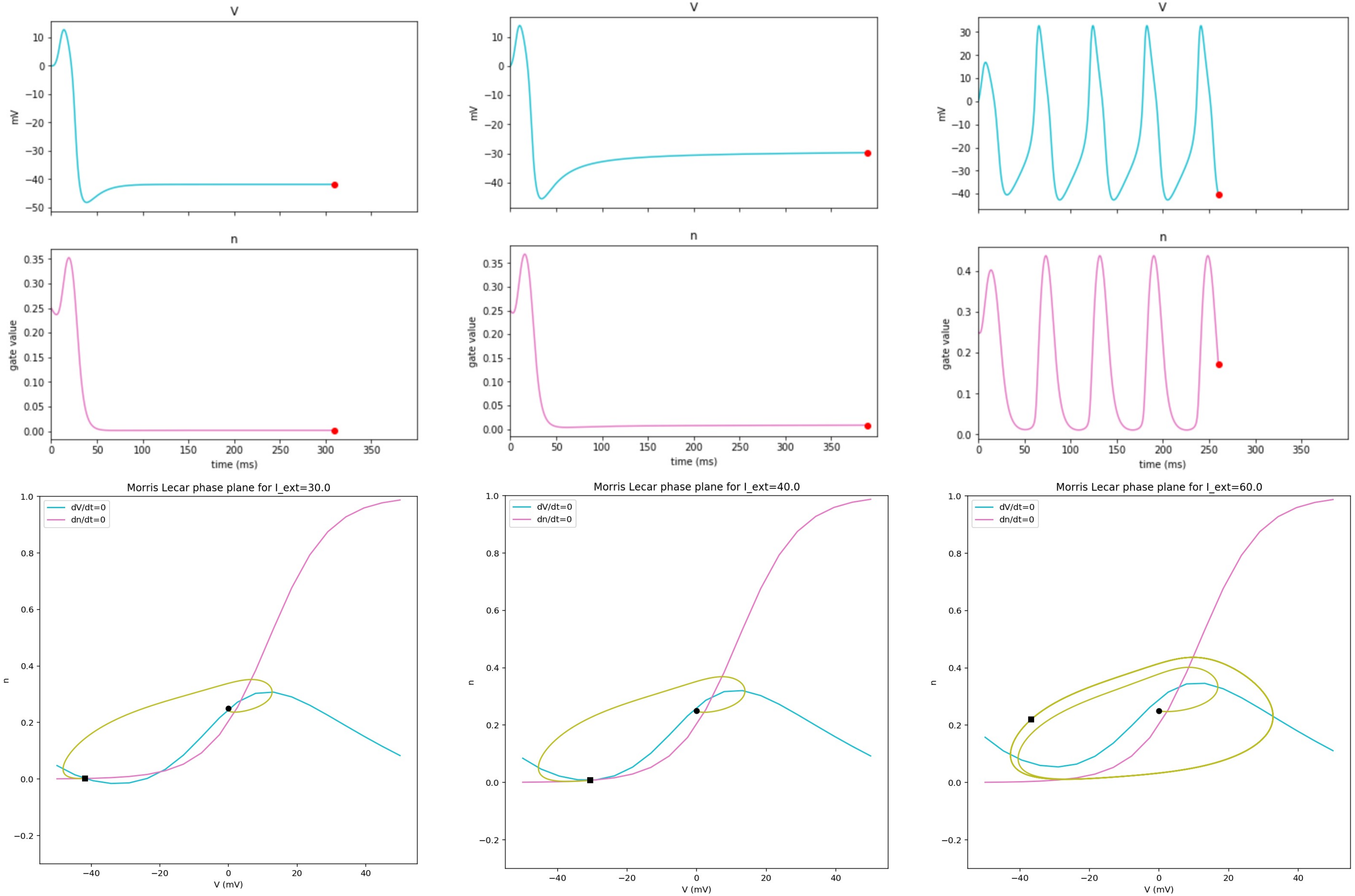






The Saddle-Node to Limit Cycle (SNLC) bifurcation

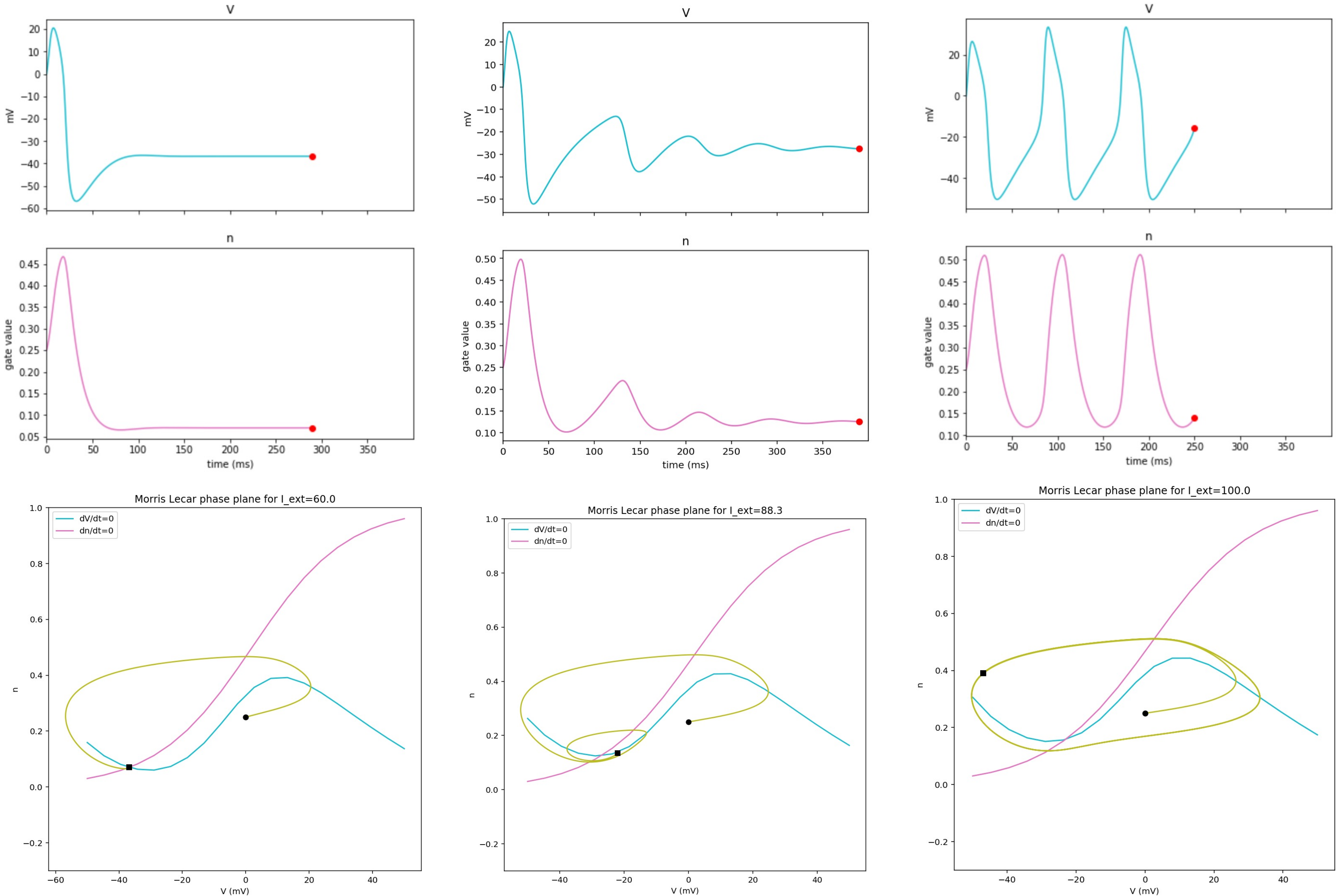
$I_{ext} = 30$ $\qquad$ $I_{ext}^* \approx 40$ $\qquad$ $I_{ext} = 60$



In a SNLC bifurcation a saddle and a node (stable fixed point) collide and disappear at I_{ext}^* to give rise to a stable limit cycle.

The Hopf bifurcation

$I_{ext} = 60$ $\quad$ $I_{ext}^{*} \approx 88$ $\quad$ $I_{ext} = 100$



In a Hopf bifurcation a stable fixed point looses stability at I_{ext}^{*} and gives rise to a stable limit cycle.

Uses of DS theory in Neuroscience

- Modeling of different dynamical phenomena at different spatiotemporal scales:
  - At the single neuron level, integrators and resonators (type I and type II neurons)
  - At the circuit level, phenomena like multi-stability, synchronization, competition, intermittency and resonance
  - At the large-scale level (whole-brain), resting-state dynamics and functional connectivity and patterns (resting–state networks)
  - Self-organized criticality
- Modelling aspects of cognition
  - Learning and memory with attractor dynamics
  - Creativity and chaos
  - Criticality and optimal information processing/transmission
  - Computing with dynamics (perception, working memory)
- Modelling of brain disease states
  - Epilepsy
  - Psychiatric disorders (hallucinations)
- Modelling effects of stimulation treatments

$$\mathbf{x}(t) = \mathbf{x}_0 e^{\boldsymbol{A}t}$$

$$\left. \boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1} \qquad \boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \right\} \text{Diagonalization of matrix A}$$

$$e^{\boldsymbol{A}} = e^{\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}} = \boldsymbol{V}e^{\boldsymbol{D}}\boldsymbol{V}^{-1}$$

Chaos and the butterfly effect