Assignment 4: Relations, Functions and Introducction to SML

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1 Problem 2

Lets assume the following:

1. Let's assume that the functions that satisfy the given conditions are all functions such that $f(x) \leq x$.
2. Since the function is injective and the natural numbers are countable. Under the first condition we must also assume that all numbers $n \in \mathbb{N}$ in the codomain are mapped by some f(x) $x \in \mathbb{N}$ — $x \leq n$. In other words all numbers smaller than a certain number are mapped by functions of a number smaller or equal than the number.

Let's now suppose there is some $f(x)$ — $f(x) \leq x \; \forall \; x \leq N$. But for some $n > N$, f(n)¿n:

$$f^2(n) \leq \frac{n + f(n)}{2}$$

For the above to be true, $f^2(n) < f(n)$ since $n < f(n)$. This means $f^2(n)$ must be in the interval [N,f(n)]. Now let's consider the case $f(n)$:

$$f^3(n) \leq \frac{f(n) + f^2(n)}{2}$$

We know that for this equivalence to hold:

$$f^3(n) \leq f(n) \text{ or } f^3(n) \leq f^2(n)$$

But since we know that $f^2(n) \leq f(n)$ then is evident that $f^3 \leq f(n)$ must be true for the above inequality to hold. Since the function must be injective, the last statement implies that $f^3(n)$ must be in the interval [N,f(n)] as well. We can do this for the next k steps, namely $f^k(x)$ and conclude that $f^k(n)$ must belong to the interval [N,f(n)]. This means we can always choose k to be bigger than f(n)-N and conclude that the function $f(x)$ can't be injective since all the natural numbers in the interval [N,f(n)] will already be mapped by some $f^i(n)$.

Conclusion The proof above states that the set {f(n)} of functions that satisfy the inequality above must be defined as follows:

$$\{f(n)|f : \mathbb{N} \to \mathbb{N} \wedge f(n) \leq n \wedge \text{ f(n) is injective}\}$$

The only function in such set is $f(n) = n$.