1   1D examples

2   Second order differential equation

2.1   Producing our differential equation

Let's produce our own differential equation so we know the exact solution. Let's try with the following polynomial

$$f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6 \tag{1}$$

Let's get the firs two derivates

$$\begin{aligned}f'(x) &= 3x^2 - 12x + 11 \\ f''(x) &= 6x - 12x\end{aligned}$$

And our second order differential equation is

$$\frac{d^2 f(x)}{dx^2} = 6x - 12x \tag{2}$$

This form is known as the **strong** form of the partial differential equation. Of course, later we'll have to proportionate the appropriate boundary conditions to get the function shown in 1. Then let's say our problem is

$$\frac{d^2 f(x)}{dx^2} = 6x - 12x \quad \text{in } \mathbb{R}$$

$$f(x) = 0 \quad \text{on } x = 1 \tag{3}$$

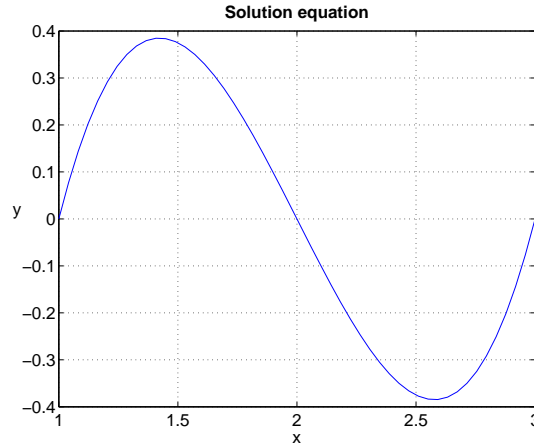$$f'(x)n_x = \frac{df(x)}{dx}n_x = 2 \quad \text{on } x = 3$$

Figure 1: **Example's objective solution**

2.2   The weak form

In order to use the finite element method to solve this equation, the first step is to write the differential equation in the **weak** or **variational** form. To do so, we require the Green's Theorem in 1D

$$\int_\Omega \frac{\partial}{\partial x}\frac{\partial u}{\partial x} w \, d\Omega + \int_\Omega \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} \, d\Omega = \int_\Gamma \frac{\partial u}{\partial x} n_x \, w \, d\Gamma$$

Because we'll be working in 1D, then we can use d instead of ∂

$$\int_\Omega \frac{d}{dx}\frac{du}{dx}w\,d\Omega + \int_\Omega \frac{du}{dx}\frac{dw}{dx}\,d\Omega = \int_{\Gamma_D} \frac{du}{dx}n_x\,w\,d\Gamma + \int_{\Gamma_N} \frac{du}{dx}n_x\,w\,d\Gamma \tag{4}$$

As you can see the boundary Γ was split into Γ_D and Γ_N, which are the places where the **essential** (in this case Dirichlet) and **natural** (in this case Neumann) boundary conditions, respectively, are imposed. w is known as **test function** and works as a weight to do an average version of the original (the strong form) differential equation.

Using the equations 4 and 2 and in our case $u = f(x)$

$$\int_\Omega (6x-12)w\,d\Omega + \int_\Omega \frac{df(x)}{dx}\frac{dw}{dx}\,d\Omega = \int_{\Gamma_D}\frac{df(x)}{dx}n_x\,w\,d\Gamma + \int_{\Gamma_N}\frac{df(x)}{dx}n_x\,w\,d\Gamma$$

Reordering

$$\int_\Omega \frac{df(x)}{dx}\frac{dw}{dx}\,d\Omega + \int_\Omega (6x-12)w\,d\Omega - \int_{\Gamma_D}\frac{df(x)}{dx}n_x\,w\,d\Gamma - \int_{\Gamma_N}\frac{df(x)}{dx}n_x\,w\,d\Gamma = 0$$

In order to impose easily the dirichlet boundary conditions we'll use a w such that $w = 0$ on Γ_D. So the final weak form version is

$$\int_\Omega \frac{df(x)}{dx}\frac{dw}{dx}\,d\Omega + \int_\Omega (6x-12)w\,d\Omega - \int_{\Gamma_N}\frac{df(x)}{dx}n_x\,w\,d\Gamma = 0 \tag{5}$$

Then the problem using the weak form would be

$$\int_\Omega \frac{df(x)}{dx}\frac{dw}{dx}\,d\Omega + \int_\Omega (6x-12)w\,d\Omega - \int_{\Gamma_N}\frac{df(x)}{dx}n_x\,w\,d\Gamma = 0 \quad \text{in } \mathbb{R}$$

$$f(x) = 0 \quad \text{on } x = 1 \tag{6}$$

$$f'(x)n_x = \frac{df(x)}{dx}n_x = 2 \quad \text{on } x = 3$$

2.3 Discretization and the Galerkin method

Now its time to do the discretization part and use the famous Galerkin method. First over a finite element method the solution will be interpolated like this

$$f(x) \approx \sum_{i=1}^{n} N_i(x_i)f(x_i) \tag{7}$$

The x_i are the location of the nodes that define the element and $N_i(x_i)$ are the shape functions (known as **trial functions**). Because we are in 1D our elements are bars and at least we need two nodes to define it.

MISSING FIGURE

So our finite element approximation will be, if we use two nodes

$$f(x) \approx \sum_{i=1}^{2} N_i(x_i)f(x_i) = N_1(x_1)f(x_1) + N_2(x_2)f(x_2)$$

The Galerkin method implies to use $w = N_i(x)$. Considering this and the finite element approximation over an element we get

$$\int_{\Omega^e} \frac{df(x)}{dx}\frac{dw}{dx}\,d\Omega + \int_{\Omega^e} (6x-12)w\,d\Omega - \int_{\Gamma_N^e}\frac{df(x)}{dx}n_x\,w\,d\Gamma = 0$$

$$\int_{\Omega^e} \frac{d\left(\sum_{i=1}^{2} N_i(x_i)f(x_i)\right)}{dx}\frac{dN_i(x)}{dx}\,d\Omega + \int_{\Omega^e} (6x-12)N_i(x)\,d\Omega$$

$$-\int_{\Gamma_N^e}\frac{df(x)}{dx}n_x\,N_i(x)\,d\Gamma = 0 \tag{8}$$

You may be wondering why we ketp the $f(x)$ in the last term on the left side of the equation. Well this is because this integral is for the Neumann boundary conditions as you can see in equation 6. This term will be not zero only when a Neumann condition is applied on the edges of a element. From the last formulae we get a system of equations, because $i = 1, 2$.

$$
\begin{bmatrix}
\int_{\Omega^e} \dfrac{dN_1(x)}{dx}\dfrac{dN_1(x)}{dx}\, d\Omega & \int_{\Omega^e} \dfrac{dN_1(x)}{dx}\dfrac{dN_2(x)}{dx}\, d\Omega \\
\int_{\Omega^e} \dfrac{dN_2(x)}{dx}\dfrac{dN_1(x)}{dx}\, d\Omega & \int_{\Omega^e} \dfrac{dN_2(x)}{dx}\dfrac{dN_2(x)}{dx}\, d\Omega
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
=
$$
$$
\begin{bmatrix} -\bar{q}_1 \int_{\Gamma_N^e} N_1(x)\, d\Gamma \\ \bar{q}_2 \int_{\Gamma_N^e} N_2(x)\, d\Gamma \end{bmatrix}
-
\begin{bmatrix} \int_{\Gamma_N^e} (6x - 12)N_1(x)\, d\Gamma \\ \int_{\Gamma_N^e} (6x - 12)N_2(x)\, d\Gamma \end{bmatrix}
\tag{9}
$$

This is the system of equations for each element. Note we substitute $f(x_i) = y_i$ and $\frac{df(x_i)}{dx} = \bar{q}_i$. We can take out from the integral the $f(x_i)$ because his value does not change.

Another thing is n_x. In this case $n_x(x_1) = -1$ and $n_x(x_2) = 1$, considering $x_1 \le x_2$. The normal's direction is taken always pointing outwards.

2.4   The shape functions

If you can remember we set $w = N_i(x)$ but one condition that w has to respect is $w = 0$ on Γ_D. Also remember the finite element approximation $f(x) \approx \sum_{i=1}^{2} N_i(x_i) f(x_i)$. One way to follow this rule is make each N_i take value 1 in his node and 0 on the other ones.

In this case is easy to "guess" the expressions for N_1 and N_2 taking in account the rule

$$
N_1(x) = \frac{x_2 - x}{x_2 - x_1} \quad N_2(x) = \frac{x - x_1}{x_2 - x_1}
\tag{10}
$$

The next step is substitute the expressions for the shape functions in eq. 9

$$
\begin{bmatrix}
\int_{x_1}^{x_2} \dfrac{1}{(x_2 - x_1)^2}\, dx & -\int_{x_1}^{x_2} \dfrac{1}{(x_2 - x_1)^2}\, dx \\
-\int_{x_1}^{x_2} \dfrac{1}{(x_2 - x_1)^2}\, dx & \int_{x_1}^{x_2} \dfrac{1}{(x_2 - x_1)^2}\, dx
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
=
$$
$$
\begin{bmatrix} -\bar{q}_1 \dfrac{x_2 - x}{x_2 - x_1} \Big]_{x_1}^{x_2} \\ \bar{q}_2 \dfrac{x - x_1}{x_2 - x_1} \Big]_{x_1}^{x_2} \end{bmatrix}
-
\begin{bmatrix} (6x - 12)\dfrac{x_2 - x}{x_2 - x_1} \Big]_{x_1}^{x_2} \\ (6x - 12)\dfrac{x - x_1}{x_2 - x_1} \Big]_{x_1}^{x_2} \end{bmatrix}
$$

Doing the integrals we get

$$
\begin{bmatrix}
\dfrac{1}{x_2 - x_1} & -\dfrac{1}{x_2 - x_1} \\
-\dfrac{1}{x_2 - x_1} & \dfrac{1}{x_2 - x_1}
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
=
\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix}
-
\begin{bmatrix} -(6x_1 - 12) \\ 6(x_2 - 12) \end{bmatrix}
\tag{11}
$$

3   Bar with axial force