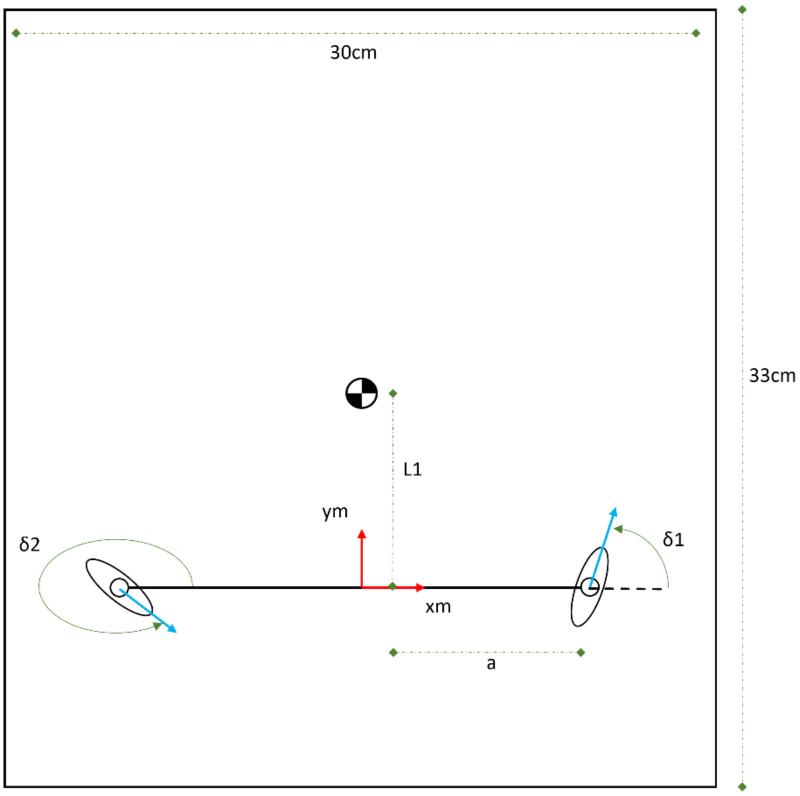
Dynamic model computation

Dynamic model diagram



Dynamic Model
Known equations:

$$m = 2kg$$

$$v = [\dot{x} \quad \dot{y} \quad o]^T$$

$$\omega = \dot{\theta}\overrightarrow{z_m}$$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

$$v_1 = R\dot{\varphi}_1 \qquad v_2 = R\dot{\varphi}_2$$

$$\dot{x} = v_1 c\delta_1 c\theta - v_2 s\delta_2 s\theta$$

$$\dot{y} = v_1 c\delta_1 s\theta - v_2 s\delta_2 c\theta$$

$$\dot{\theta} = \frac{1}{2a}(v_1 s\delta_1 - v_2 s\delta_2)$$

We use the Lagrange Theorem; we first calculate the kinetic energy:

$$E = \frac{1}{2}\left(mv^T v + \omega^T I \omega + 2ms^T(v \times \omega)\right)$$

$$E = \frac{1}{2}\left(m(\dot{x}^2 + \dot{y}^2) + \dot{\theta}^2 I_{zz} + 2\begin{bmatrix}0 & l_1 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}\dot{y} \\ \dot{\theta}\dot{x} \\ 0\end{bmatrix}\right)$$

$$E = \frac{1}{2}\left(m(\dot{x}^2 + \dot{y}^2) + \dot{\theta}^2 I_{zz} + 2l_1\dot{\theta}\dot{x}\right)$$

The potential energy is defined as follows:

$$U = \begin{bmatrix}0 & 0 & g & 0\end{bmatrix}\begin{bmatrix}c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}ms_x \\ ms_y \\ ms_z \\ m\end{bmatrix} = \begin{bmatrix}0 & 0 & g & gt_z\end{bmatrix}\begin{bmatrix}ms_x \\ ms_y \\ ms_z \\ m\end{bmatrix}$$

$$U = g(ms_z + mt_z)$$

Therefore, the Lagrangian is defined as:

$$L = E - U$$

$$L = \frac{1}{2}\left(m(\dot{x}^2 + \dot{y}^2) + \dot{\theta}^2 I_{zz} + 2l_1\dot{\theta}\dot{x}\right) - g(ms_z + mt_z)$$

(1)

$$\tau = \frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right) - \left(\frac{dL}{dq}\right)$$

Dividing the torque equation in several smaller sections:

$$\left(\frac{dL}{dq}\right) = 0$$

$$\frac{d\dot{x}^2}{d\dot{\varphi}_1} = 2(R^2\dot{\varphi}_1 c^2\delta_1 c^2\theta - R^2\dot{\varphi}_2 s\delta_2 c\delta_1 s\theta c\theta)$$

$$\frac{d\dot{y}^2}{d\dot{\varphi}_1} = 2(R^2\dot{\varphi}_1 c^2\delta_1 s^2\theta - R^2\dot{\varphi}_2 c\delta_1 s\delta_2 c\theta s\theta)$$

$$\frac{d\dot{\theta}^2}{d\dot{\varphi}_1} = \frac{1}{2a^2}(R^2\dot{\varphi}_1 s^2\delta_1 - R^2\dot{\varphi}_2 s\delta_1 s\delta_2)$$

$$\dot{\theta}\dot{x} = \frac{1}{2a}(R^2\dot{\varphi}_1^2 c\delta_1 s\delta_1 c\theta - R^2\dot{\varphi}_1\dot{\varphi}_2 s\delta_1 s\delta_2 s\theta - R^2\dot{\varphi}_1\dot{\varphi}_2 c\delta_1 s\delta_2 c\theta + R^2\dot{\varphi}_2^2 s^2\delta_2 s\theta)$$

$$\frac{d(\dot{\theta}\dot{x})}{d\dot{\varphi}_1} = \frac{R^2}{2a}(2\dot{\varphi}_1 c\delta_1 s\delta_1 c\theta - \dot{\varphi}_2 s\delta_1 s\delta_2 s\theta - \dot{\varphi}_2 c\delta_1 s\delta_2 c\theta)$$

$$\left(\frac{d\dot{x}^2}{d\dot{\varphi}_2}\right) = 2(R^2\dot{\varphi}_2\,\mathrm{s}^2\,\delta_2\,\mathrm{s}^2\,\theta - R^2\dot{\varphi}_1\mathrm{c}\delta_2 s\delta_2 c\theta s\theta)$$

$$\left(\frac{d\dot{y}^2}{d\dot{\varphi}_2}\right) = 2(R^2\dot{\varphi}_2\,\mathrm{s}^2\,\delta_2\,\mathrm{c}^2\,\theta - R^2\dot{\varphi}_1\mathrm{c}\delta_1 s\delta_2 c\theta s\theta)$$

$$\left(\frac{d\dot{\theta}^2}{d\dot{\varphi}_2}\right) = \frac{1}{2a^2}(R^2\dot{\varphi}_2\,\mathrm{s}^2\,\delta_2 - R^2\dot{\varphi}_1 s\delta_1 s\delta_2)$$

$$\left(\frac{d\dot{\theta}\dot{x}}{d\dot{\varphi}_2}\right) = \frac{R^2}{2a}(-\dot{\varphi}_1 s\delta_1 s\delta_2 s\theta - \dot{\varphi}_1 c\delta_1 s\delta_2 c\theta + 2\dot{\varphi}_2 s^2\delta_2 s\theta)$$

Finally, we substitute these equations into (1) and derivate with respect to time to obtain the torques:

$$\begin{aligned}
\tau_1 = \frac{1}{2}m\big(&4R^2 s\delta_1 s\delta_2 s\theta c\theta\dot{\delta}_1\dot{\varphi}_2 - 4R^2 s\delta_1 s^2\theta c\delta_1\dot{\delta}_1\dot{\varphi}_1 - 4R^2 s\delta_1 c\delta_1 c^2\theta\dot{\delta}_1\dot{\varphi}_1 \\
&+ 4R^2 s\delta_2 s^2\theta c\delta_1\dot{\varphi}_2\dot{\theta} - 4R^2 s\delta_2 s\theta c\delta_1 c\theta\ddot{\varphi}_2 - 4R^2 s\delta_2 c\delta_1 c^2\theta\dot{\varphi}_2\dot{\theta} + 2R^2 s^2\theta c^2\delta_1\ddot{\varphi}_1 \\
&- 4R^2 s\theta c\delta_1 c\delta_2 c\theta\dot{\delta}_2\dot{\varphi}_2 + 2R^2 c^2\delta_1 c^2\theta\ddot{\varphi}_1\big) \\
&+ \frac{R^2 l_1}{2a}\big(-2s^2\delta_1 c\theta\dot{\delta}_1\dot{\varphi}_1 + s\delta_1 s\delta_2 c\theta\dot{\delta}_1\dot{\varphi}_2 - 2s\delta_1 s\theta c\delta_1\dot{\varphi}_1\dot{\theta} + 2s\delta_1 c\delta_1 c\theta\ddot{\varphi}_1 \\
&- s^2\delta_2 s\theta\ddot{\varphi}_2 - s^2\delta_2 c\theta\dot{\varphi}_2\dot{\theta} + s\delta_2 s\theta c\delta_1\dot{\varphi}_2\dot{\theta} - 2s\delta_2 s\theta c\delta_2\dot{\delta}_2\dot{\varphi}_2 - s\delta_2 c\delta_1 c\theta\ddot{\varphi}_2 \\
&+ 2c^2\delta_1 c\theta\dot{\delta}_1\dot{\varphi}_1 - c\delta_1 c\delta_2 c\theta\dot{\delta}_2\dot{\varphi}_2\big) \\
&+ \frac{I_{zz}}{4a^2}\big(R^2 s^2\delta_1\ddot{\varphi}_1 - R^2 s\delta_1 s\delta_2\ddot{\varphi}_2 + 2R^2 s\delta_1 c\delta_1\dot{\delta}_1\dot{\varphi}_1 - R^2 s\delta_1 c\delta_2\dot{\delta}_2\dot{\varphi}_2 \\
&- R^2 s\delta_2 c\delta_1\dot{\delta}_1\dot{\varphi}_2\big)
\end{aligned}$$

$$\begin{aligned}
\tau_2 = \frac{m}{2}\big(&2R^2 s\delta_1 s\delta_2 s\theta c\theta\dot{\delta}_1\dot{\varphi}_1 + 2R^2 s^2\delta_2 s^2\theta\ddot{\varphi}_2 + 2R^2 s^2\delta_2 s\theta c\theta\dot{\delta}_2\dot{\varphi}_1 + 2R^2 s^2\delta_2 c^2\theta\ddot{\varphi}_2 \\
&+ 2R^2 s\delta_2 s^2\theta c\delta_1\dot{\varphi}_1\dot{\theta} + 4R^2 s\delta_2 s^2\theta c\delta_2\dot{\delta}_2\dot{\varphi}_2 + 2R^2 s\delta_2 s^2\theta c\delta_2\dot{\varphi}_1\dot{\theta} \\
&- 2R^2 s\delta_2 s\theta c\delta_1 c\theta\ddot{\varphi}_1 - 2R^2 s\delta_2 s\theta c\delta_2 c\theta\ddot{\varphi}_1 - 2R^2 s\delta_2 c\delta_1 c^2\theta\dot{\varphi}_1\dot{\theta} \\
&+ 4R^2 s\delta_2 c\delta_2 c^2\theta\dot{\delta}_2\dot{\varphi}_2 - 2R^2\delta_2 c\delta_2 c^2\theta\dot{\varphi}_1\dot{\theta} - 2R^2 s\theta c\delta_1 c\delta_2 c\theta\dot{\delta}_2\dot{\varphi}_1 \\
&- 2R^2 s\theta c^2\delta_2 c\theta\dot{\delta}_2\dot{\varphi}_1\big) \\
&+ \frac{R^2 L_1}{2a}\big(-s\delta_1 s\delta_2 s\theta\ddot{\varphi}_1 + s\delta_1 s\delta_2 c\theta\dot{\delta}_1\dot{\varphi}_1 - s\delta_1 s\delta_2 c\theta\dot{\varphi}_1\dot{\theta} - s\delta_1 s\theta c\delta_2\dot{\delta}_2\dot{\varphi}_1 \\
&+ 2s^2\delta_2 s\theta\ddot{\varphi}_2 + 2s^2\delta_2 c\theta\dot{\varphi}_2\dot{\theta} - s\delta_2 c\theta c\delta_1\dot{\delta}_1\dot{\varphi}_1 + s\delta_2 s\theta c\delta_1\dot{\varphi}_1\dot{\theta} + 4s\delta_2 s\theta c\delta_2\dot{\delta}_2\dot{\varphi}_2 \\
&- s\delta_2 c\delta_1 c\theta\ddot{\varphi}_1 - c\delta_1 c\delta_2 c\theta\dot{\delta}_2\dot{\varphi}_1\big) + \frac{I_{zz}}{4a^2}(-R^2 s\delta_1 s\delta_2\ddot{\varphi}_1 - R^2 s\delta_1 c\delta_2\dot{\delta}_2\dot{\varphi}_1 \\
&+ R^2 s^2\delta_2\ddot{\varphi}_2 - R^2 s\delta_2 c\delta_1\dot{\delta}_1\dot{\varphi}_1 + 2R^2 s\delta_2 c\delta_2\dot{\delta}_2\dot{\varphi}_2)
\end{aligned}$$