Inequality-Constrained Multivariate Normal Splines with Some Applications in Finance

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A problem of estimating a smooth function $\varphi(x)$ given discrete scattered observations of its values $\{u_i\}_{i=1}^p$ and given it satisfies a set of linear inequality constraint is presented. Often we may treat $\varphi(x)$ as an element of Hilbert space $H(R^n)$. Consider a following data model

$$
\begin{aligned}
(f_i, \varphi) &= u_i\,, \qquad 1 \le i \le \; m, \\
(f_i, \varphi) &\le u_i\,, \qquad m+1 \le i \le p
\end{aligned}
\tag{1}
$$

where f_i — linear continuous functionals.

Problem of reconstruction of the function φ satisfying with (1) is underdefined. To overcome it we introduce a penalty functional J

$$
(J, \varphi) = \|\varphi - \varphi_0\|_H^2 \;, \tag{2}
$$

and find a function φ in the Hilbert space H to minimize J subject to (1) (where φ_0 is a trial function, and $\|\cdot\|_H$ denotes the H norm).

Solution of this problem is a generalized spline of Atteia-Laurent [2]. Following to the work [3] we will call it as a Normal Spline.

Let $H \equiv H^s_\varepsilon$, H^s_ε — a fractional order Sobolev space (it is a kind of the Bessel potentials space [1])

$$
H^s_\varepsilon(R^n) = \left\{ \varphi |\, \varphi \in S' \,,\, (\varepsilon^2 + |\xi|^2)^{s/2} \mathcal{F}[\varphi] \in L_2 \right\} \,,\, \varepsilon > 0 \,,\, s > \frac{n}{2} \,,
$$

where S' — space of L. Schwartz distributions, $\mathcal{F}[\varphi]$ — Fourier transform of φ.

As H^s_ε is a Reproducing Kernel Hilbert Space, we can transform functionals f_i to the canonic form (form of inner product). It allows to reduce the previous problem to the simplest quadratic programming problem. In order to do such reduction we have to know a reproducing kernel of the Hilbert space H^s_ε. In the case when parameter s is under the following condition

$$
s = r + \frac{n+1}{2} \;,\; (r = 0, 1, \ldots) \;.
$$

related reproducing kernel $V_{s_r}(\eta, x)$ can be expressed via elementary functions, namely

$$V_{s_r}(\eta, x) = C \exp(-\varepsilon|\eta - x|) \sum_{k=0}^{r} \frac{(r+k)!}{2^k k!(r-k)!} (\varepsilon|\eta - x|)^{r-k}. \tag{3}$$

Parallelized version of algorithm of solving simplest quadratic programming problem (finding the normal solution of a system of linear equations and inequalities) was developed. Method of the normal splines was applied for investigating some finance data analysis problems.

References

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