**Problem 1.** For the following functions, state either $f(n) \in o(g(n))$, $f(n) \in \omega(g(n))$, or $f(n) \in \Theta(g(n))$.

**a.** $f(n) = (n + \log n)(\sqrt{n} + 5)$, $g(n) = n \log n$

*Solution:* $f(n) = n^{1.5} + 5n + n^{0.5} \log n + 5 \log n$
Because $5n$, $n^{0.5} \log n$, and $5 \log n$ are all in $o(n^{1.5})$, we can show that $f(n) \in \Theta(n^{1.5})$.
Therefore $f(n) \in \omega(n \log n)$.

**b.** $f(n) = 2^{n \lg n}$, $g(n) = 3^n$

*Solution:* $f(n) = (2^{\lg n})^n = n^n \in \omega(3^n)$

**c.** $f(n) = \frac{n}{\lg n}$, $g(n) = \sqrt{n}$

*Solution:* In this case, $f(n)$ is less than $n$ by a factor of $\lg n$; $g(n)$ is less than $n$ by a factor of $\sqrt{n}$. A logarithm is asymptotically smaller than any positive power (or root) of $n$, so we expect that $f(n) \in \omega(g(n))$. We confirm this using the formal definition of $\omega$.

For all $c > 0$ and all $n_0 > 0$, we construct a formula for $n$ in terms of $c$ and $n_0$ such that $n \geq n_0$ and $f(n) > cg(n)$. We proceed by starting from that inequality and solving for $n$.

$$\begin{array}{rcll} \frac{n}{\lg n} & > & c\sqrt{n} & \\ n & > & c\sqrt{n}\lg n & \text{by multiplying each side by } \lg n \\ \sqrt{n} & > & c \lg n & \text{by dividing each side by } \sqrt{n} \\ \sqrt{n} & > & c\sqrt[4]{n} & \text{is sufficient if we choose } n > 2^{16} \text{ so that } \sqrt[4]{n} > \lg n \\ \sqrt[4]{n} & > & c & \text{by dividing each side by } \sqrt[4]{n} \\ n & > & c^4 & \text{by squaring both sides twice} \end{array}$$

Therefore we choose $n = n_0 + c^4 + 2^{16}$.

**d.** $f(n) = \Sigma_{i=1}^{n}(3i^2 \log i + 2i(\log i)^2)$, $g(n) = n^3(\log n)^3$

*Solution:*

$$\begin{array}{rcl} f(n) & = & \Sigma_{i=1}^{n}(3i^2 \log i + 2i(\log i)^2) \\ & = & 3\Sigma_{i=1}^{n}(i^2 \log i) + 2\Sigma_{i=1}^{n}(i(\log i)^2) \\ & \leq & 3\Sigma_{i=1}^{n}(i^2 \log n) + 2\Sigma_{i=1}^{n}(i(\log n)^2) \\ & = & 3(\Sigma_{i=1}^{n} i^2)\log n + 2(\Sigma_{i=1}^{n} i)(\log n)^2 \\ & = & 3(\frac{2n^3+3n^2+n}{6})\log n + 2(\frac{n^2+n}{2})(\log n)^2 \\ & = & n^3 \log n + \frac{3}{2}n^2 \log n + \frac{1}{2}n \log n + n^2(\log n)^2 + n(\log n)^2 \\ & \in & \Theta(n^3 \log n) \\ & \subseteq & o(n^3(\log n)^3) \end{array}$$

**e.** $f(n) = \frac{n}{n^{1/2}}$, $g(n) = \sqrt[4]{n^2}$

*Solution:* $f(n) = \frac{n}{n^{1/2}} = \sqrt{n} = \sqrt[4]{n^2} = g(n)$, therefore $f(n) \in \Theta(g(n))$.

**Problem 2.** Solve the following recurrences using summations, recursion trees, and/or the master method.

**a.**$T(n) = 3T(\frac{1}{4}n) + 1.5^n$

*Solution:* Solution via the master method where $a = 3$, $b = 4$, and $f(n) = 1.5^n$. In this case $f(n) \in \Omega(n^{\log_b a})$ because $f(n)$ is exponential, so we use case 3. We must choose $c < 1$ and $n_0 > 0$ such that for $n \geq n_0$, $af(\frac{n}{b}) \leq cf(n)$.

$$\begin{array}{rcll} af(\frac{n}{b}) & \leq & cf(n) & \\ 3(1.5^{\frac{n}{4}}) & \leq & c1.5^n & \text{by definition of } f(n) \\ 3 & \leq & c1.5^{\frac{3n}{4}} & \text{by dividing both sides by } 1.5^{\frac{n}{4}} \\ 81 & \leq & c^4 1.5^{3n} & \text{by squaring both sides twice} \\ 81 & \leq & 1/3(1.5^{3n}) & \text{by choosing } c = \sqrt[4]{\frac{1}{3}} \\ 243 & \leq & 1.5^{3n} & \text{by multiplying both sides by 3} \\ \log_{1.5} 243 & \leq & 3n & \text{by taking the log base 1.5 of both sides} \\ \frac{\log_{1.5} 243}{3} & \leq & n & \text{by dividing by 3} \end{array}$$

Therefore we choose $c = \sqrt[4]{\frac{1}{3}} \approxeq 0.760$ and $n_0 = \frac{\log_{1.5} 243}{3} \approxeq 4.52$, demonstrating that $T(n) = \Theta(1.5^n)$.

**b.**$T(n) = 2T(\frac{1}{2}n) + \log n$

*Solution:* Solution via the master method where $a = 2$, $b = 2$, and $f(n) = \log n$. In this case, $n^{\log_b a} = n^1$. We can choose any $0 < \epsilon < 1$ to make $f(n) = \log n \in O(n^{1-\epsilon})$, so case 1 applies, and $T(n) \in \Theta(n)$.

**c.**$T(n) = T(\frac{3}{5}n) + \sqrt[3]{n}$

*Solution:* Solution via the master method where $a = 1$, $b = \frac{5}{3}$, and $f(n) = \sqrt[3]{n}$; $n^{\log_b a} = n^0 = 1$. Therefore $f(n) = n^{\frac{1}{3}} \in \Omega(n^{0+\epsilon})$ for any $0 < \epsilon < \frac{1}{3}$, and we try case 3. We must choose $c < 1$ and $n_0 > 0$ such that for any $n \geq n_0$, $af(\frac{n}{b}) \leq cf(n)$.

$$\begin{array}{rcll} af(\frac{n}{b}) & \leq & cf(n) & \\ \sqrt[3]{\frac{3}{5}n} & \leq & c\sqrt[3]{n} & \text{by definition of } f(n) \\ \sqrt[3]{\frac{3}{5}}\sqrt[3]{n} & \leq & c\sqrt[3]{n} & \\ \sqrt[3]{\frac{3}{5}} & \leq & c & \text{by dividing both sides by } \sqrt[3]{n} \end{array}$$

Therefore we choose $n_0 = 1$ and $c = \sqrt[3]{\frac{3}{5}} \approxeq 0.843$ to demonstrate that $T(n) \in \Theta(\sqrt[3]{n})$.

**d.**$T(n) = T(n-1) + n^2 + 3n + 2$

*Solution:* Solution via summation.

$$\begin{array}{rcl} T(n) & = & \Sigma_{i=1}^{n} i^2 + 3i + 2 \\ & = & (\Sigma_{i=1}^{n} i^2) + 3(\Sigma_{i=1}^{n} i) + (\Sigma_{i=1}^{n} 2) \\ & = & \frac{2n^3+3n^2+n}{6} + 3(\frac{n^2+n}{2}) + 2n \\ & = & \frac{1}{3}n^3 + 2n^2 + \frac{11}{3}n \\ & \in & \Theta(n^3) \end{array}$$

**e.**$T(n) = 2T(\frac{1}{4}n) + \sqrt{n}$

*Solution:* Solution via the master method where $a = 2$, $b = 4$, and $f(n) = \sqrt{n}$. Therefore $n^{\log_b a} = \sqrt{n} = f(n)$; case 2 applies and $T(n) \in \Theta(\sqrt{n}\log n)$.

**f.** $T(n) = T(\frac{1}{3}n) + T(\frac{2}{3}n) + n$

*Solution:* Solution by recursion trees. The first layer does $n$ work. The second layer has two nodes of size $\frac{1}{3}n$ and $\frac{2}{3}n$, which adds up to $n$ work again. The first node's children have size $\frac{1}{9}n$ and $\frac{2}{9}n$; the second node's children have size $\frac{2}{9}n$ and $\frac{4}{9}n$. The total is again $n$. Each layer splits up the work, but always totals $n$. The maximum height of the tree is $\log_{\frac{3}{2}} n$ at the leaf reached by taking $\frac{2}{3}$ of the input each time. The minimum height is $\log_3 n$ at the leaf reached by taking $\frac{1}{3}$ each time. The total work is therefore between $n\log_3 n$ and $n\log_{\frac{3}{2}} n$, and is in $\Theta(n\log n)$.

**Problem 3.** Here we examine a sorting algorithm for vectors based on swapping elements that are out of order. The first function, naturally enough, swaps elements that are out of order.

```
(define (sort!-swap-if-needed v i j)
  (define v.i (vector-ref v i))
  (define v.j (vector-ref v j))
  (cond
    [(<= v.i v.j) (void)]
    [else
     (vector-set! v i v.j)
     (vector-set! v j v.i)]))
```

**a.** State the running time of `sort!-swap-if-needed` as a recurrence. Solve the recurrence using summations, recursion trees, or the master method.

*Solution:* $T(n) = 1 \in \Theta(1)$

The `sort!-from/to` function sorts elements that are swapped from position `i` to position `j`, incrementing `j` until the end of the vector.

```
(define (sort!-from/to v i j)
  (cond
    [(>= j (vector-length v)) (void)]
    [else
     (sort!-swap-if-needed v i j)
     (sort!-from/to v i (add1 j))]))
```

**b.** State the running time of `sort!-from/to` as a recurrence. Solve the recurrence using summations, recursion trees, or the master method.

*Solution:* Solution by summation. $T(n) = T(n-1) + 1$, where $n = j - i$. Therefore $T(n) = \Sigma_{i=1}^{n} 1 = n \in \Theta(n)$.

The `sort!-from` function sorts elements that are swapped from position `i` to each position at a greater index; `i` increments until the end of the vector. Finally, `sort!` calls `sort!-from` starting at index 0.

```
(define (sort! v)
  (sort!-from v 0))

(define (sort!-from v i)
  (cond
    [(>= i (vector-length v)) (void)]
    [else
     (sort!-from/to v i (add1 i))
     (sort!-from v (add1 i))]))
```

**c.** State the running time of `sort!` as a recurrence. Solve the recurrence using summations, recursion trees, or the master method.

*Solution:* Solution by summation. $T(n) = T(n-1) + n$, where $n = |v| - i$. $T(n) = \Sigma_{j=1}^{n} j = \frac{1}{2}(n^2 + n) \in \Theta(n^2)$.

**Problem 4.** The *median of medians* algorithm selects the `i`th smallest element of a list `xs`, much like quickselect. The implementation starts with partitioning functions.

```
(define (all-less-than pivot xs)
  (cond
    [(empty? xs) empty]
    [(< (first xs) pivot) (cons (first xs) (all-less-than pivot (rest xs)))]
    [else (all-less-than pivot (rest xs))]))

(define (all-greater-than pivot xs)
  (cond
    [(empty? xs) empty]
    [(> (first xs) pivot) (cons (first xs) (all-greater-than pivot (rest xs)))]
    [else (all-greater-than pivot (rest xs))]))
```

**a.** State the running time of `all-less-than` and `all-greater-than` as recurrences. Solve the recurrences using summations, recursion trees, or the master method.

| |
|---|
| *Solution:* In both cases, $T(n) = T(n-1) + 1 \in \Theta(n)$. |

The `groups-of-five` function splits a list into groups of five (or fewer) elements.

```
(define (groups-of-five xs)
  (cond
    [(empty? xs) empty]
    [(empty? (rest xs)) (list (list (first xs)))]
    [(empty? (rest (rest xs))) (list (list (first xs) (second xs)))]
    [(empty? (rest (rest (rest xs))))
     (list (list (first xs) (second xs) (third xs)))]
    [(empty? (rest (rest (rest (rest xs)))))
     (list (list (first xs) (second xs) (third xs) (fourth xs)))]
    [else (cons (list (first xs) (second xs) (third xs) (fourth xs) (fifth xs))
            (groups-of-five (rest (rest (rest (rest (rest xs)))))))]))
```

**b.** State the running time of `groups-of-five` as a recurrences. Solve the recurrence using summations, recursion trees, or the master method.

| |
|---|
| *Solution:* In this case, $T(n) = T(n-1) + 1$; we consider $n$ to be $\lceil \frac{\lvert xs \rvert}{5} \rceil$. Again, $T(n) = \Theta(n)$. |

The `median` function uses `slow-select` to find the median element of a list. Assume that `slow-select` runs in $n \log n$ time. The `medians-of-five` function takes a list of lists, where each inner list has at most five elements. It produces a list of numbers, where each number is the median of the corresponding list in the input.

```
(define (median five-or-less)
  (slow-select (quotient (length five-or-less) 2) five-or-less))

(define (medians-of-five fives)
  (cond
    [(empty? fives) empty]
    [else (cons (median (first fives))
            (medians-of-five (rest fives)))]))
```

**c.** State the running time of `medians-of-five` as a recurrence. Solve the recurrence using summations, recursion trees, or the master method.

*Solution:* $T(n) = T(n-1) + 1$ once again, where $n$ is the length of `fives`. Here, the call to `median` takes $\Theta(1)$ time because the input is of constant size. The running time for `medians-of-five` is $\Theta(n)$.

Finally, the `select` function itself operates like quickselect, subdividing the input by partitioning. In order to guarantee a good pivot, the algorithm splits the input into groups of five elements, finds the median of each group of five, then uses a recursive call to `select` to find the median of all of those medians. That *median of medians* serves as the pivot for partitioning the input.

```
(define (select i xs)
  (define n (length xs))
  (cond
    [(<= n 5) (slow-select i xs)]
    [else
     (define fives (groups-of-five xs))
     (define medians (medians-of-five fives))
     (define pivot (select (quotient n 10) medians))
     (define left (all-less-than pivot xs))
     (define right (all-greater-than pivot xs))
     (define n1 (length left))
     (define n2 (length right))
     (cond
       [(< i n1) (select i left)]
       [(>= i (- n n2)) (select (- i (- n n2)) right)]
       [else pivot])]))
```

**d.** Argue that the length of `(medians-of-five (groups-of-five xs))` is at most $\lceil \frac{1}{5} n \rceil$.

*Solution:* The `groups-of-five` function returns a list of exactly $\lceil \frac{1}{5} n \rceil$ groups, and `medians-of-five` produces the median of each one. Therefore the length of `(medians-of-five (groups-of-five xs))` is precisely $\lceil \frac{1}{5} n \rceil$.

**e.** Argue that there are no more than $\frac{7}{10} n$ elements in `left`.

*Solution:* If `pivot` is the median of `medians`, then at least half the elements of `medians` are greater than or equal to `pivot`. In the corresponding groups of five elements in `fives`, there must therefore be at least three elements greater than or equal to `pivot`. Three out of five, out of half of `xs`, is three tenths of `xs` that is greater than or equal to `pivot`, and thus not in `left`. Therefore `left` can have at most $\frac{7}{10} n$ elements.

**f.** Argue that there are no more than $\frac{7}{10} n$ elements in `right`.

*Solution:* This follows symmetrically from the argument for `left` above; if there are three tenths of the list that must be greater than or equal to `pivot`, there are likewise three tenths of the list that must be less than or equal to `pivot`, and therefore not in `right`.

**g.** State the running time of `select` as a recurrence. **Note:** this recurrence may have an unusual form, as not every recursion in `select` has the same worst-case input size.

*Solution:* $T(n) = T(\frac{n}{5}) + T(\frac{7}{10} n) + n$

**h.** Solve the recurrence for the running time of `select` using summations, recursion trees, or the master method.

*Solution:* Solution by recursion trees. The first layer does $n$ work. The second layer contributes at most $\frac{n}{5} + \frac{7n}{10} = \frac{9n}{10}$ work. The third layer, $\frac{1}{5}\frac{1}{5}n + \frac{1}{5}\frac{7}{10}n + \frac{7}{10}\frac{1}{5}n + \frac{7}{10}\frac{7}{10}n = \frac{81}{100}n$. In each layer, the work is split into $\frac{1}{5}$ and $\frac{7}{10}$ of the total, giving a consistent reduction by a factor of $\frac{9}{10}$. The size of the tree is therefore bounded by $\Sigma_{i=0}^{\infty}(\frac{9}{10})^i n = \frac{1}{1-\frac{9}{10}} n = 10n \in \Theta(n)$.