Homework 2—Sample Solution

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1. (b) Like quicksort, quickselect has worst-case performance when the pivot is consistently chosen poorly: one partition has $n-1$ elements, and the desired index is in that partition.

   i. State the worst-case running time of quickselect as a recurrence.

   Solution:

   $$T(n) = T(n-1) + n$$

   ii. Solve the recurrence using the master method, recursion trees, or summations.

   Solution: Solution using summations.

   $$T(n) = \sum_{i=1}^{n} i = \frac{n^2+n}{2} \in \Theta(n^2)$$

   (c) Again like quicksort, quickselect has best-case performance when the pivot is consistently chosen well: both partitions have at most $\frac{n}{2}$ elements.

   i. State the best-case running time of quickselect as a recurrence.

   Solution:

   $$T(n) = T(\frac{n}{2}) + n$$

   ii. Solve the recurrence using the master method, recursion trees, or summations.

   Solution: Solution using the master method where $a = 1$, $b = 2$, and $f(n) = n$. Trying case 3 because $n^{\log_b a} = n^0 = 1 \in o(f(n))$. Choosing $\frac{1}{2}$ for ϵ gives us $f(n) \in \Omega(n^{\log_b a+\epsilon}) = \Omega(\sqrt{n})$. Choosing $c = \frac{1}{2}$ gives us $af(\frac{n}{b}) = f(\frac{n}{2}) = \frac{n}{2} = \frac{1}{2}n = cf(n)$. Therefore case 3 applies, and we have a solution for the recurrence.

   $$T(n) \in \Theta(n)$$

2. (b) State the running time of stooge sort as a recurrence.

   Solution:

   $$T(n) = 3T(\frac{2}{3}n) + 1$$

(c) Solve the recurrence using the master method, recursion trees, or summations.

Solution: Solution using the master method where $a = 3$, $b = 1.5$, and $f(n) = 1$. Trying case 1 where $\epsilon = \log_{1.5} 2$; specifically, $\log_b a - \epsilon = \log_{1.5} 3 - \log_{1.5} 2 = 1$. Therefore $f(n) = 1 \in O(n^1) = O(n^{\log_b a - \epsilon})$, so case 1 applies. (Of course, ϵ does not have to be chosen so cleverly to make $\log_b a - \epsilon = 1$; any number between 0 and $\log_{1.5} 3 \approxeq 2.71$ works.)

$$T(n) \in \Theta(n^{\log_{1.5} 3E}) \approxeq \Theta(n^{2.71})$$

3. **Note:** For each case, we note the simplified Θ form of each $f(n)$ first.

   (a) $f(n) = 5n^{1.25} + 3n\log n + 2n\sqrt{n}$. **Note:** $f(n) \in \Theta(n\sqrt{n}) = \Theta(n^{1.5})$

      i. $g(n) = n^2$. **Solution:** $f(n) \in o(g(n))$
      ii. $g(n) = n^{3/2}$. **Solution:** $f(n) \in \Theta(g(n))$
      iii. $g(n) = n\log n$. **Solution:** $f(n) \in \omega(g(n))$
      iv. $g(n) = n$. **Solution:** $f(n) \in \omega(g(n))$

   (b) $f(n) = n(\log \frac{n}{2})^2$. **Note:** $f(n) = n(\log n - \log 2)^2 = n(\log n)^2 - 2\log 2(n\log n) + (\log 2)^2 n \in \Theta(n(\log n)^2)$

      i. $g(n) = n$. **Solution:** $f(n) \in \omega(g(n))$
      ii. $g(n) = n\sqrt{n}$. **Solution:** $f(n) \in o(g(n))$
      iii. $g(n) = n(\log n)^2$. **Solution:** $f(n) \in \Theta(g(n))$
      iv. $g(n) = n\log n$. **Solution:** $f(n) \in \omega(g(n))$

   (c) $f(n) = 2^{2n}$. **Note:** $f(n) = 4^n \in \Theta(4^n)$

      i. $g(n) = n^{65536}$. **Solution:** $f(n) \in \omega(g(n))$
      ii. $g(n) = 2^n$. **Solution:** $f(n) \in \omega(g(n))$
      iii. $g(n) = 3^n$. **Solution:** $f(n) \in \omega(g(n))$
      iv. $g(n) = 4^n$. **Solution:** $f(n) \in \Theta(g(n))$
      v. $g(n) = 5^n$. **Solution:** $f(n) \in o(g(n))$
      vi. $g(n) = n!$. **Solution:** $f(n) \in o(g(n))$

4. (a) $T(n) = 5T(\frac{1}{4}n) + (\frac{5}{4})^n$

Solution: $a = 5$, $b = 4$, and $f(n) = (\frac{5}{4})^n$. An exponential $f(n)$ suggests that case 3 applies. Clearly $f(n) \in \Omega(n^{\log_b a + \epsilon})$ regardless of what ϵ we choose. We must show that $5f(\frac{n}{b}) =\leq c(\frac{5}{4})^n$ for sufficiently large n.

$$\begin{array}{rcll} 5(\frac{5}{4})^{\frac{n}{4}} & \leq & c(\frac{5}{4})^n & \\ 5c^{-1} & \leq & (\frac{5}{4})^{\frac{3}{4}n} & \\ 10 & \leq & (\frac{5}{4})^{\frac{3}{4}n} & \text{if we choose } c = \frac{1}{2}. \\ 10{,}000 & \leq & (\frac{5}{4})^n & \\ 41.3 \approxeq \log_{\frac{5}{4}} 10{,}000 & \leq & n & \end{array}$$

The equation therefore holds for $c = \frac{1}{2}$ and n over 42.

$$T(n) \in \Theta((\frac{5}{4})^n)$$

(b) $T(n) = 9T(\frac{1}{3}n) + n^2\sqrt{\log n}$

Solution: $a = 9$, $b = 3$, and $f(n) = n^2\sqrt{\log n}$. In this case, $n^{\log_b a} = n^2$. This is asymptotically smaller than $f(n)$, which suggests case 3. However, there is no ϵ we can choose such that $n^2\sqrt{\log n} \in \Omega(n^{2+\epsilon})$. Therefore, the master method does not apply.

Note: Exercise 4.6-2 in the text (CLRS, 3rd Edition) extends case 2 of the master method to allow $f(n) \in \Theta(n^{\log_b a}(\log n)^k)$ for any $k \geq 0$; the solution is then $T(n) \in \Theta(n^{\log_b a}(\log n)^{k+1})$. If we use that extension, we can show in this case that $T(n) \in \Theta(n^2(\log n)^{1.5})$.

(c) $T(n) = 2T(\frac{1}{4}n) + \sqrt[3]{n}$

Solution: $a = 2$, $b = 4$, and $f(n) = n^{\frac{1}{3}}$. In this case, $n^{\log_b a} = n^{\frac{1}{2}}$. Since $f(n)$ is slower by exactly a factor of $n^{\frac{1}{6}}$, we use case 1 where $\epsilon = \frac{1}{6}$.

$$T(n) = \Theta(\sqrt{n})$$

(d) $T(n) = 2T(\frac{2}{3}n) + (\log n)^2$

Solution: $a = 2$, $b = 1.5$, and $f(n) = (\log n)^2$. In this case, $n^{\log_b a} = n^{\log_{1.5} 2} \approxeq n^{1.71}$. Case 1 applies where ϵ is any number between 0 and $\log_{1.5} 2$, exclusive.

$$T(n) = \Theta(n^{\log_{1.5} 2}) \approxeq \Theta(n^{1.71})$$