Homework 1

Sample Solution

January 24, 2013

1. See repository on `github.com`.
2. See code in `sample-solution.rkt`.
3. (a) Demonstrate that $f(n) = 2n^2 - 3n + 4 \in O(n^2)$. That is, choose a specific $c > 0$ and a specific $n_0 > 0$. Then, show that for any possible $n > n_0$, it must be true that $f(n) \le cn^2$.

   Solution: Solving for c and n_0 as we go.

$$\begin{array}{rcll} 2n^2 - 3n + 4 & \le & cn^2 & \\ 2n^2 - 3n + 4n & \le & cn^2 & \text{because } 4 \le 4n \text{ if we choose } n_0 \ge 1 \\ 2n^2 + n & \le & cn^2 & \text{by simplification} \\ 2n^2 + n^2 & \le & cn^2 & \text{because } n \le n^2 \text{ given } n_0 \ge 1 \\ 3n^2 & \le & cn^2 & \text{which is true if we choose } c \ge 3 \end{array}$$

   Therefore, the equation holds for $c = 3$ and $n_0 = 1$.

   (b) Demonstrate that $f(n) = 3\sqrt{n} \in O(n)$. That is, choose a specific $c > 0$ and a specific $n_0 > 0$. Then, show that for any possible $n > n_0$, it must be true that $f(n) \le cn$.

   Solution: Solving for c and n_0 as we go.

$$\begin{array}{rcll} 3\sqrt{n} & \le & cn & \\ 3n & \le & cn & \text{because } \sqrt{n} \le n \text{ if we choose } n_0 \ge 1 \\ & & & \text{and this is true if we choose } c \ge 3 \end{array}$$

   Therefore, the equation holds for $c = 3$ and $n_0 = 1$.

   (c) Demonstrate that $f(n) = \Sigma_{i=1}^{n}(2i^2 + 3i + 5) \in O(n^3)$. That is, choose a specific $c > 0$ and a specific $n_0 > 0$. Then, show that for any possible $n > n_0$, it must be true that $f(n) \le cn^3$.

   Solution: Solving for c and n_0 as we go.

$$\begin{array}{rcll} \Sigma_{i=1}^{n}(2i^2 + 3i + 5) & \le & cn^3 & \\ 2(\Sigma_{i=1}^{n} i^2) + 3(\Sigma_{i=1}^{n} i) + 5 & \le & cn^3 & \text{by linearity of } \Sigma \text{ (see Appendix A in the text)} \\ 2(\Sigma_{i=1}^{n} i^2) + 3(\frac{n^2+n}{2}) + 5 & \le & cn^3 & \text{by arithmetic series (see Appendix A again)} \\ 2(\frac{2n^3+3n^2+n}{6}) + 3(\frac{n^2+n}{2}) + 5 & \le & cn^3 & \text{by sum of squares (see Appendix A yet again)} \\ \frac{2}{3}n^3 + \frac{5}{2}n^2 + \frac{11}{6}n & \le & cn^3 & \text{by simplification} \\ \frac{2}{3}n^3 + \frac{5}{2}n^3 + \frac{11}{6}n^3 & \le & cn^3 & \text{because } n \le n^2 \le n^3 \text{ if we choose } n_0 \ge 1 \\ 5n^3 & \le & cn^3 & \text{by simplification; true if we choose } c \ge 5 \end{array}$$

   Therefore, the equation holds for $c = 5$ and $n_0 = 1$.

(d) Demonstrate that $f(n) = \sqrt{n} \not\in O(\log_2 n)$. That is, for any possible $c > 0$ and for any possible $n_0 > 0$, give a formula for some $n > n_0$ and show that n always satisfies $f(n) > c \log_2 n$.

Solution: Solving for n as we go.

$$\begin{array}{rcll}
\sqrt{n} & > & c \log_2 n & \\
\sqrt{n} & > & 4c \log_2 \sqrt[4]{n} & \text{by properties of log} \\
\sqrt{n} & > & 4c \sqrt[4]{n} & \text{because } \sqrt[4]{n} > \log_2 \sqrt[4]{n} \text{ if we choose } \sqrt[4]{n} > 2\text{, meaning } n > 16 \\
\sqrt[4]{n} & > & 4c & \text{divide by } \sqrt[4]{n} \\
n & > & 256c & \text{raise both sides to the power of 4}
\end{array}$$

Therefore, the equation holds for $n = 256c + n_0 + 16$.