Case 1. Contingent pricing without social learning (C N)

(*The equilibrium results in case CN*)

$$p_{CN1} = \frac{q_o\,(2-\delta)^2}{6-4\,\delta};$$

$$D_{CN1} = \frac{q_o\,(1-\delta)}{t\,(3-2\,\delta)};$$

$$p_{CN2} = \frac{q_o\,(2-\delta)}{6-4\,\delta};$$

$$D_{CN2} = \frac{q_o\,(2-\delta)}{2\,t\,(3-2\,\delta)};$$

$$\Pi_{CN} = \frac{q_o^2\,(2-\delta)^2}{4\,t\,(3-2\,\delta)};$$

$$CS_{CN} = \frac{q_o^2\,(4+\delta\,(2-\delta)\,(2-5\,\delta))}{8\,t\,(3-2\,\delta)^2};$$

Case 2. Contingent pricing with social learning (C L)

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(***Proof of Proposition 2(i)***)

(*In the second period, consumer reviews will realize in three types,
including completely positive (P), mixed (M), and completely negative (N)*)
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(*If $q_R^{CL}=q_P^{CL}=\frac{2q_o+p_1+t\ D_1}{2}$ *)

$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4}$; (*The second-period price if $q_R^{CL}=q_P^{CL}$*)

$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t}$; (*The second-period demand if $q_R^{CL}=q_P^{CL}$*)

(*If $q_R^{CL}=q_M^{CL}=\frac{2p_1+t\ D_1}{2}$ *)

$p_{2M} = \frac{2\,p_1 - t\,D_1}{4}$;

$D_{2M} = \frac{2\,p_1 - t\,D_1}{4\,t}$;

(*If $q_R^{CL}=q_N^{CL}=\frac{p_1}{2}$ *)

$p_{2N} = \frac{p_1 - 2\,t\,D_1}{4}$;

$D_{2N} = \frac{p_1 - 2\,t\,D_1}{4\,t}$;

(*Scenario 1: $p_1\le\frac{t\ D_1}{2}$, consumers will not purchase upon observing mixed or completely negative reviews, i.e., $D_{2M}=D_{2N}=0$*)

In[∘]:= $p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4}$;

$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t}$;

In[∘]:= $U_1 = q_o - p_1 - t\,D_1$; (*The expected utility buying in the first period*)

$U_2 = \delta\,\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2P} - t\,D_1\right)$;

(*The expected utility buying in the second period*)

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[∘]=

$$\left\{\left\{D_1 \to \frac{2\,(-2+\delta)\,q_o - \sqrt{\delta^2\,p_1^2 - 8\,\delta\,p_1\,q_o - 8\,(-2+\delta)\,q_o^2}}{t\,\delta}\right\},\right.$$
$$\left.\left\{D_1 \to \frac{2\,(-2+\delta)\,q_o + \sqrt{\delta^2\,p_1^2 - 8\,\delta\,p_1\,q_o - 8\,(-2+\delta)\,q_o^2}}{t\,\delta}\right\}\right\}$$

(*Check which solution is the feasible solution*)

In[∘]:= $D_1 = \frac{2\,(-2+\delta)\,q_o - \sqrt{\delta^2\,p_1^2 - 8\,\delta\,p_1\,q_o - 8\,(-2+\delta)\,q_o^2}}{t\,\delta}$;

Reduce$\left[0 < p_1 \le \frac{t\,D_1}{2}\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[ ]=

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False
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In[ ]:=

$$D_1 = \frac{2\ (-2+\delta)\ q_o + \sqrt{\delta^2\ p_1^2 - 8\ \delta\ p_1\ q_o - 8\ (-2+\delta)\ q_o^2}}{t\ \delta};$$

$$\text{Reduce}\left[0 < p_1 \le \frac{t\ D_1}{2}\ \&\&\ t > 2\ q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

Out[ ]=

$$p_1 > 0\ \&\&\ \left(\left(3\ p_1 < q_o < 2\ p_1 + \frac{1}{2}\ \sqrt{19}\ \sqrt{p_1^2}\ \&\&\ t > 2\ q_o\ \&\&\ 0 < \delta \le \frac{-24\ p_1\ q_o + 8\ q_o^2}{3\ p_1^2 - 8\ p_1\ q_o + 4\ q_o^2}\right)\ ||\right.$$
$$\left(q_o == 2\ p_1 + \frac{1}{2}\ \sqrt{19}\ \sqrt{p_1^2}\ \&\&\ t > 2\ q_o\ \&\&\ 0 < \delta < \frac{-24\ p_1\ q_o + 8\ q_o^2}{3\ p_1^2 - 8\ p_1\ q_o + 4\ q_o^2}\right)\ ||$$
$$\left.\left(0 < \delta < 1\ \&\&\ q_o > 2\ p_1 + \frac{1}{2}\ \sqrt{19}\ \sqrt{p_1^2}\ \&\&\ t > 2\ q_o\right)\right)$$

(*The second solution is feasible,
hence the optimal response function of the first-period demand is as follows*)

In[ ]:=

$$D_1 = \frac{\sqrt{\delta^2\ p_1^2 - 8\ \delta\ p_1\ q_o - 8\ (-2+\delta)\ q_o^2} - 2\ (2-\delta)\ q_o}{t\ \delta};$$

$$\Pi = \text{Simplify}\left[p_1\ D_1 + \frac{2\ q_o - p_1 - t\ D_1}{2\ q_o}\ p_{2\,P}\ D_{2\,P}\right];\ \text{(*The firm's total profit function*)}$$

$$\text{Reduce}\left[D[D[\Pi,\ p_1],\ p_1] \ge 0\ \&\&\ 0 < p_1 \le \frac{t\ D_1}{2}\ \&\&\ 0 < \delta < 1\ \&\&\ t > 2\ q_o > 0\right]$$

(*Determine the sign of $\frac{\partial^2 \Pi}{\partial p_1^2}$*)

Out[ ]=

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False
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(*$\frac{\partial^2 \Pi}{\partial p_1^2}$<0, meaning Π is concave and it has a maximum value at point where $\frac{\partial \Pi}{\partial p_1}$=0*)

(*Construct Karush-Kuhn-Tucker (KKT) conditions*)

In[ ]:=

$$g = \frac{t\ D_1}{2} - p_1;\ \text{(*The constrain of } p_1 \le \tfrac{t\ D_1}{2}\text{*)}$$

$$L = -\left(p_1\ D_1 + \frac{2\ q_o - p_1 - t\ D_1}{2\ q_o}\ p_{2\,P}\ D_{2\,P}\right) - \lambda\ g;\ \text{(*The KKT Lagrange functio*)}$$

In[ ]:=

$$\text{Simplify}[\text{Solve}[\{D[L,\ p_1] == 0,\ \lambda\ g == 0\},\ \{p_1,\ \lambda\},\ \text{Reals}],\ q_o > 0\ \&\&\ 0 < \delta < 1]$$

Out[ ]=

$$\{\{p_1 \to \text{ConditionalExpression}[$$
$$\text{Root}[8\ \delta^4\ \#1^4 + 144\ q_o^4 + 225\ \delta\ q_o^4 - 272\ \delta^2\ q_o^4 + 64\ \delta^3\ q_o^4 + \#1^3\ (-16\ \delta^2\ q_o - 94\ \delta^3\ q_o) + \#1^2$$
$$(124\ \delta\ q_o^2 + 376\ \delta^2\ q_o^2 - 30\ \delta^3\ q_o^2 - 8\ \delta^4\ q_o^2) + \#1\ (-224\ q_o^3 - 480\ \delta\ q_o^3 + 104\ \delta^2\ q_o^3 + 64\ \delta^3\ q_o^3)\ \&,$$
$$1],\ \delta < 0.339\ldots\ ||\ 0.339\ldots < \delta < 0.543\ldots\ ||\ \delta > 0.543\ldots],$$
$$\lambda \to \text{ConditionalExpression}[(-8\ \delta^3\ \text{Root}[8\ \delta^4\ \#1^4 + 144\ q_o^4 + 225\ \delta\ q_o^4 - 272\ \delta^2\ q_o^4 + 64\ \delta^3\ q_o^4 +$$
$$\#1^3\ (-16\ \delta^2\ q_o - 94\ \delta^3\ q_o) + \#1^2\ (124\ \delta\ q_o^2 + 376\ \delta^2\ q_o^2 - 30\ \delta^3\ q_o^2 - 8\ \delta^4\ q_o^2) +$$
$$\#1\ (-224\ q_o^3 - 480\ \delta\ q_o^3 + 104\ \delta^2\ q_o^3 + 64\ \delta^3\ q_o^3)\ \&,\ 1]^2 + 4\ (-8 - 15\ \delta + 8\ \delta^2)\ q_o^2 +$$
$$q_o\ (\delta\ (8 + 47\ \delta)\ \text{Root}[8\ \delta^4\ \#1^4 + 144\ q_o^4 + 225\ \delta\ q_o^4 - 272\ \delta^2\ q_o^4 + 64\ \delta^3\ q_o^4 +$$

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        #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
        #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 1] +
      (8 + 17 δ - 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
              #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
              #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 1]^2 -
         8 δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
             #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
             #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 1] q_o - 8 (-2 + δ) q_o^2))) /
  (2 t δ^2 (δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
         #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
         #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 1] -
     4 q_o - 2 √(δ^2 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
             #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
             #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 1]^2 -
         8 δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
             #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
             #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 1] q_o - 8 (-2 + δ) q_o^2))),
  δ < 0.339… || 0.339… < δ < 0.543… ||
   δ >
    0.543…]},
{p_1 → ConditionalExpression[
  Root[
   8 δ^4 #1^4 +
     144 q_o^4 + 225 δ q_o^4 -
     272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
     #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) +
     #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
     #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2],
  δ < 0.339… || 0.339… < δ < 0.543… ||
   δ >
    0.543…],
 λ → ConditionalExpression[
  (-8 δ^3 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
         #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
         #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2]^2 + 4 (-8 - 15 δ + 8 δ^2) q_o^2 +
    q_o (δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
          #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
          #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2] +
      (8 + 17 δ - 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
              #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
              #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2]^2 -
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        8 δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
            #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
            #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2] q_o - 8 (-2 + δ) q_o^2))) /
  (2 t δ^2 (δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
          #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
          #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2] -
      4 q_o - 2 √(δ^2 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
              #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
              #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2]^2 -
          8 δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
              #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
              #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2] q_o - 8 (-2 + δ) q_o^2))),
  δ < 0.339… || 0.339… < δ < 0.543… || δ > 0.543…]},
{p_1 → ConditionalExpression[
  Root[
   8 δ^4 #1^4 + 144 q_o^4 +
     225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
     #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) +
     #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
     #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &,
   3], 0.339… < δ < 0.543…],
 λ → ConditionalExpression[
  (-8 δ^3 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
          #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
          #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 3]^2 + 4 (-8 - 15 δ + 8 δ^2) q_o^2 +
    q_o (δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
            #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
            #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 3] +
       (8 + 17 δ - 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
                #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
                #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 3]^2 -
            8 δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
                #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
                #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 3] q_o - 8 (-2 + δ) q_o^2))) /
  (2 t δ^2 (δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
          #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
          #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 3] -
      4 q_o - 2 √(δ^2 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
              #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
              #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 3]^2 -
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            8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] qo - 8 (-2 + δ) qo^2))),
    0.339… < δ < 0.543…]}, {p1 → ConditionalExpression[
    Root[
     8 δ^4 #1^4 +
       144 qo^4 + 225 δ qo^4 -
       272 δ^2 qo^4 + 64 δ^3 qo^4 +
       #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
       #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
       #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &,
     4], δ < 0.339… || 0.339… <
      δ <
      0.543…],
  λ → ConditionalExpression[
    (-8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
              #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 + 4 (-8 - 15 δ + 8 δ^2) qo^2 +
       qo (δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] +
          (8 + 17 δ - 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                     #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                     #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 -
              8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] qo - 8 (-2 + δ) qo^2))) /
     (2 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
              #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] -
         4 qo - 2 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 -
            8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] qo - 8 (-2 + δ) qo^2))),
    δ < 0.339… || 0.339… < δ < 0.543…]}, {p1 →
   Undefined,
  λ →
   Undefined},
 {p1 →
```

```
ConditionalExpression[
  2 (-6 + 2 δ + Sqrt[36 - 18 δ + δ^2]) q0
  ───────────────────────────────────── ,
                  3 δ
  δ <
   0.339… ||
  0.339… <
   δ <
   0.543… || δ >
   0.543…],
 λ → ConditionalExpression[
  ((-80 δ^3 + 24 (-12 + Sqrt[36 - 18 δ + δ^2] +
          Sqrt[180 + 5 δ^2 - 24 Sqrt[36 - 18 δ + δ^2] + 4 δ (-21 + Sqrt[36 - 18 δ + δ^2])]) -
      2 δ^2 (-549 + 32 Sqrt[36 - 18 δ + δ^2] +
          12 Sqrt[180 + 5 δ^2 - 24 Sqrt[36 - 18 δ + δ^2] + 4 δ (-21 + Sqrt[36 - 18 δ + δ^2])]) +
      3 δ (-740 + 111 Sqrt[36 - 18 δ + δ^2] +
          17 Sqrt[180 + 5 δ^2 - 24 Sqrt[36 - 18 δ + δ^2] + 4 δ (-21 + Sqrt[36 - 18 δ + δ^2])]))
     q0) / (6 t δ^2 (-12 + 2 δ + Sqrt[36 - 18 δ + δ^2] -
      2 Sqrt[180 + 5 δ^2 - 24 Sqrt[36 - 18 δ + δ^2] + 4 δ (-21 + Sqrt[36 - 18 δ + δ^2])])),
  δ < 0.339… || 0.339… < δ < 0.543… ||
   δ >
   0.543…]}}
```

(*There are 5 solutions, we check each solution if it satisfies conditions*)

(*Solution 1, interior solution*)

In[•]:=

```
p1 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
     #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
     #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1];
λ = (-8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
          #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
          #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 + 4 (-8 - 15 δ + 8 δ^2) qo^2 +
     qo (δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
            #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
            #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] +
        (8 + 17 δ - 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 -
             8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                 #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] qo - 8 (-2 + δ) qo^2))) /
   (2 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
          #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
          #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] -
       4 qo - 2 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 -
           8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
               (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
               (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] qo - 8 (-2 + δ) qo^2)));
Reduce[0 < p1 < (t D1)/2 && t > 2 qo > 0 &&
  (δ < 0.339… || 0.339… < δ < 0.543… || δ > 0.543…) && λ == 0, Reals]
```

Out[•]=

```
False
```

```
(*Solution 2, interior solution*)
```

```
In[ ]:= p1 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
          #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
          #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2];
     λ = (-8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
               #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
               #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2]^2 + 4 (-8 - 15 δ + 8 δ^2) qo^2 +
          qo (δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2] +
             (8 + 17 δ - 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                         #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                         #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2]^2 -
                  8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                        (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                       #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2] qo - 8 (-2 + δ) qo^2))) /
        (2 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
                #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2] -
           4 qo - 2 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                       #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                       #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2]^2 -
                8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                      (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                      (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2] qo - 8 (-2 + δ) qo^2)));
     Reduce[0 < p1 < t D1/2 && t > 2 qo > 0 &&
       (δ < 0.339... || 0.339... < δ < 0.543... || δ > 0.543...) && λ == 0, Reals]
Out[ ]=
     False

     (*Solution 3, interior solution*)
```

```
In[•]:= p1 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
           #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
           #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3];
        λ = (-8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                 #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                 #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 + 4 (-8 - 15 δ + 8 δ^2) qo^2 +
              qo (δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                    #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                    #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] +
                 (8 + 17 δ - 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                          #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                          #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 -
                    8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                          (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                        #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] qo - 8 (-2 + δ) qo^2))) /
             (2 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
                    #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                    #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] -
                 4 qo - 2 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                        #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                        #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 -
                    8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                        (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                        (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] qo - 8 (-2 + δ) qo^2)));
        Reduce[0 < p1 < t D1 / 2 && t > 2 qo > 0 && 0.339... < δ < 0.543... && λ == 0, Reals]
Out[•]=
        False

        (*Solution 4, interior solution*)
```

In[ ]:=
```
p1 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
     #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
     #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4];
λ = (-8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
          #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
          #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 + 4 (-8 - 15 δ + 8 δ^2) qo^2 +
      qo (δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
              #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] +
         (8 + 17 δ - 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                    #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                    #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 -
              8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                   (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] qo - 8 (-2 + δ) qo^2))) /
   (2 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
            #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
            #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] -
        4 qo - 2 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 -
            8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] qo - 8 (-2 + δ) qo^2)));
Reduce[0 < p1 < t D1/2 && t > 2 qo > 0 &&
   (δ < 0.339... || 0.339... < δ < 0.543...) && λ == 0, Reals]
```

Out[ ]=
```
False
```

```
(*Solution 5, boundary solution*)
```

In[◦]:=

$$p_1 = \frac{2\left(-6+2\,\delta+\sqrt{36-18\,\delta+\delta^2}\right)q_o}{3\,\delta};$$

$$\lambda = \Bigg(\bigg(-80\,\delta^3+24\left(-12+\sqrt{36-18\,\delta+\delta^2}+\sqrt{180+5\,\delta^2-24\,\sqrt{36-18\,\delta+\delta^2}+4\,\delta\left(-21+\sqrt{36-18\,\delta+\delta^2}\right)}\right)-2\,\delta^2\left(-549+32\,\sqrt{36-18\,\delta+\delta^2}+12\,\sqrt{180+5\,\delta^2-24\,\sqrt{36-18\,\delta+\delta^2}+4\,\delta\left(-21+\sqrt{36-18\,\delta+\delta^2}\right)}\right)+3\,\delta\left(-740+111\,\sqrt{36-18\,\delta+\delta^2}+17\,\sqrt{180+5\,\delta^2-24\,\sqrt{36-18\,\delta+\delta^2}+4\,\delta\left(-21+\sqrt{36-18\,\delta+\delta^2}\right)}\right)\bigg)q_o\Bigg)\Bigg/\left(6\,t\,\delta^2\left(-12+2\,\delta+\sqrt{36-18\,\delta+\delta^2}-2\,\sqrt{180+5\,\delta^2-24\,\sqrt{36-18\,\delta+\delta^2}+4\,\delta\left(-21+\sqrt{36-18\,\delta+\delta^2}\right)}\right)\right);$$

$$\text{Reduce}\left[p_1 == \frac{t\,D_1}{2}\;\&\&\; t > 2\,q_o > 0\;\&\&\;\left(\delta < 0.339\ldots \;||\; 0.339\ldots < \delta < 0.543\ldots \;||\; 0.543\ldots < \delta < 1\right)\;\&\&\;\lambda > 0,\ \text{Reals}\right]$$

Out[◦]=

$$(\delta < 0\;\&\&\; q_o > 0\;\&\&\; t > 2\,q_o)\;||\;(0 < \delta < 0.339\ldots\;\&\&\; q_o > 0\;\&\&\; t > 2\,q_o)\;||\;(0.339\ldots < \delta < 0.543\ldots\;\&\&\; q_o > 0\;\&\&\; t > 2\,q_o)\;||\;(0.543\ldots < \delta < 1\;\&\&\; q_o > 0\;\&\&\; t > 2\,q_o)$$

(*Hence, the optimal solution in scenario 1 is the boundary solution, i.e., $p_1=\frac{2\,(-6+2\,\delta+\sqrt{36-18\,\delta+\delta^2})\,q_o}{3\,\delta}$, which is the solution of $p_1=\frac{t\,D_1}{2}$*)

```
(*Scenario 2: t D1/2 < p1 ≤ 2t D1,
consumers will not purchase upon observing completely negative reviews, i.e., D2N=0*)
```

In[•]:=
$$p_{2P} = \frac{2 q_o + p_1 - t D_1}{4};$$
$$D_{2P} = \frac{2 q_o + p_1 - t D_1}{4 t};$$
$$p_{2M} = \frac{2 p_1 - t D_1}{4};$$
$$D_{2M} = \frac{2 p_1 - t D_1}{4 t};$$

In[•]:=
$$U_1 = q_o - p_1 - t D_1;$$
$$U_2 = \delta \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right)\right);$$

In[•]:= `Simplify[Solve[U1 == U2, D1]]`

Out[•]=
$$\left\{\left\{D_1 \to \frac{\delta p_1^2 - 8 p_1 q_o - 4 (-2 + \delta) q_o^2}{2 t (\delta p_1 - 2 (-2 + \delta) q_o)}\right\}\right\}$$

In[•]:=
$$D_1 = \frac{\delta p_1^2 - 8 p_1 q_o - 4 (-2 + \delta) q_o^2}{2 t (\delta p_1 - 2 (-2 + \delta) q_o)};$$

```
(*The optimal response function of the first-period demand*)
```

$$\Pi = \text{Simplify}\left[p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P} + \frac{t D_1}{2 q_o} p_{2M} D_{2M}\right];$$

```
(*The firm's total profit function*)
```

$$\text{Reduce}\left[D[D[\Pi, p_1], p_1] \geq 0 \,\&\&\, \frac{t D_1}{2} < p_1 \leq 2 t D_1 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

(*Determine the sign of $\frac{\partial^2 \Pi}{\partial p_1^2}$*)

Out[•]=

```
False
```

(* $\frac{\partial^2 \Pi}{\partial p_1^2}$<0, meaning Π is concave and it has a maximum value at point where $\frac{\partial \Pi}{\partial p_1}$=0*)

```
(*Construct Karush-Kuhn-Tucker (KKT) conditions*)
```

In[•]:=
$$g_1 = p_1 - \frac{t D_1}{2};$$
$$g_2 = 2 t D_1 - p_1;$$

In[•]:=
$$L = -\left(p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P} + \frac{t D_1}{2 q_o} p_{2M} D_{2M}\right) - \lambda_1 g_1 - \lambda_2 g_2;$$

KKT conditions

In[•]:=
$$\text{Simplify}[\text{Solve}[\{D[L, p_1] == 0, \lambda_1 g_1 == 0, \lambda_2 g_2 == 0\}, \{p_1, \lambda_1, \lambda_2\}], q_o > 0 \,\&\&\, 0 < \delta < 1]$$

Out[$\circ$]=

$$\{\{p_1 \to \text{Root}[9\delta^3 \#1^5 + 7424 q_o^5 - 11136 \delta q_o^5 + 5568 \delta^2 q_o^5 - 928 \delta^3 q_o^5 + \#1^4 (12\delta^2 q_o + 70 \delta^3 q_o) + \#1^3 (-960 \delta q_o^2 + 864 \delta^2 q_o^2 - 520 \delta^3 q_o^2) + \#1^2 (-6912 q_o^3 + 1920 \delta q_o^3 - 672 \delta^2 q_o^3 + 720 \delta^3 q_o^3) + \#1 (-6656 q_o^4 + 11712 \delta q_o^4 - 4992 \delta^2 q_o^4 + 400 \delta^3 q_o^4) \&, 1], \lambda_1 \to 0, \lambda_2 \to 0\},$$

$$\{p_1 \to \text{Root}[9\delta^3 \#1^5 + 7424 q_o^5 - 11136 \delta q_o^5 + 5568 \delta^2 q_o^5 - 928 \delta^3 q_o^5 + \#1^4 (12\delta^2 q_o + 70 \delta^3 q_o) + \#1^3 (-960 \delta q_o^2 + 864 \delta^2 q_o^2 - 520 \delta^3 q_o^2) + \#1^2 (-6912 q_o^3 + 1920 \delta q_o^3 - 672 \delta^2 q_o^3 + 720 \delta^3 q_o^3) + \#1 (-6656 q_o^4 + 11712 \delta q_o^4 - 4992 \delta^2 q_o^4 + 400 \delta^3 q_o^4) \&, 2], \lambda_1 \to 0, \lambda_2 \to 0\},$$

$$\{p_1 \to \text{Root}[9\delta^3 \#1^5 + 7424 q_o^5 - 11136 \delta q_o^5 + 5568 \delta^2 q_o^5 - 928 \delta^3 q_o^5 + \#1^4 (12\delta^2 q_o + 70 \delta^3 q_o) + \#1^3 (-960 \delta q_o^2 + 864 \delta^2 q_o^2 - 520 \delta^3 q_o^2) + \#1^2 (-6912 q_o^3 + 1920 \delta q_o^3 - 672 \delta^2 q_o^3 + 720 \delta^3 q_o^3) + \#1 (-6656 q_o^4 + 11712 \delta q_o^4 - 4992 \delta^2 q_o^4 + 400 \delta^3 q_o^4) \&, 3], \lambda_1 \to 0, \lambda_2 \to 0\},$$

$$\{p_1 \to \text{Root}[9\delta^3 \#1^5 + 7424 q_o^5 - 11136 \delta q_o^5 + 5568 \delta^2 q_o^5 - 928 \delta^3 q_o^5 + \#1^4 (12\delta^2 q_o + 70 \delta^3 q_o) + \#1^3 (-960 \delta q_o^2 + 864 \delta^2 q_o^2 - 520 \delta^3 q_o^2) + \#1^2 (-6912 q_o^3 + 1920 \delta q_o^3 - 672 \delta^2 q_o^3 + 720 \delta^3 q_o^3) + \#1 (-6656 q_o^4 + 11712 \delta q_o^4 - 4992 \delta^2 q_o^4 + 400 \delta^3 q_o^4) \&, 4], \lambda_1 \to 0, \lambda_2 \to 0\},$$

$$\{p_1 \to \text{Root}[9\delta^3 \#1^5 + 7424 q_o^5 - 11136 \delta q_o^5 + 5568 \delta^2 q_o^5 - 928 \delta^3 q_o^5 + \#1^4 (12\delta^2 q_o + 70 \delta^3 q_o) + \#1^3 (-960 \delta q_o^2 + 864 \delta^2 q_o^2 - 520 \delta^3 q_o^2) + \#1^2 (-6912 q_o^3 + 1920 \delta q_o^3 - 672 \delta^2 q_o^3 + 720 \delta^3 q_o^3) + \#1 (-6656 q_o^4 + 11712 \delta q_o^4 - 4992 \delta^2 q_o^4 + 400 \delta^3 q_o^4) \&, 5], \lambda_1 \to 0, \lambda_2 \to 0\},$$

$$\left\{p_1 \to \frac{2(-2+\delta) q_o}{-6+\delta}, \lambda_1 \to 0, \lambda_2 \to -\frac{(72 - 636\delta + 278\delta^2 - 117\delta^3 + 16\delta^4) q_o}{16 t (-6+\delta)^4}\right\},$$

$$\left\{p_1 \to -\frac{2\left(6 - 2\delta + \sqrt{36 - 18\delta + \delta^2}\right) q_o}{3\delta},\right.$$

$$\lambda_1 \to \frac{1}{18 t \delta^2 (36 - 18\delta + \delta^2)} \left(17\delta^4 + 3\delta^2 \left(-456 + \sqrt{36 - 18\delta + \delta^2}\right) - 864\left(6 + \sqrt{36 - 18\delta + \delta^2}\right) + 36\delta\left(174 + 23\sqrt{36 - 18\delta + \delta^2}\right) - \delta^3\left(204 + 55\sqrt{36 - 18\delta + \delta^2}\right)\right) q_o, \left.\lambda_2 \to 0\right\},$$

$$\left\{p_1 \to \frac{2\left(-6 + 2\delta + \sqrt{36 - 18\delta + \delta^2}\right) q_o}{3\delta}, \lambda_1 \to \frac{1}{18 t \delta^2 (36 - 18\delta + \delta^2)} \left(17\delta^4 + \delta\left(6264 - 828\sqrt{36 - 18\delta + \delta^2}\right) + 864\left(-6 + \sqrt{36 - 18\delta + \delta^2}\right) - 3\delta^2\left(456 + \sqrt{36 - 18\delta + \delta^2}\right) + \delta^3\left(-204 + 55\sqrt{36 - 18\delta + \delta^2}\right)\right) q_o, \lambda_2 \to 0\right\}\}$$

```
(*There are 8 solutions, we check each solution if it satisfies conditions*)

(*Solution 1, interior solution*)
```

In[$\circ$]:=

$$p_1 = \text{Root}[9\delta^3 \#1^5 + 7424 q_o^5 - 11136 \delta q_o^5 + 5568 \delta^2 q_o^5 - 928 \delta^3 q_o^5 + \#1^4 (12\delta^2 q_o + 70 \delta^3 q_o) + \#1^3 (-960 \delta q_o^2 + 864 \delta^2 q_o^2 - 520 \delta^3 q_o^2) + \#1^2 (-6912 q_o^3 + 1920 \delta q_o^3 - 672 \delta^2 q_o^3 + 720 \delta^3 q_o^3) + \#1 (-6656 q_o^4 + 11712 \delta q_o^4 - 4992 \delta^2 q_o^4 + 400 \delta^3 q_o^4) \&, 1];$$

$$\text{Reduce}\left[\frac{t D_1}{2} < p_1 < 2 t D_1 \&\& D_1 > 0 \&\& t > 2 q_o > 0 \&\& 0 < \delta < 1, \text{Reals}\right]$$

Out[$\circ$]=

False

```
(*Solution 2, interior solution*)
```

In[∘]:= $p_1 = \text{Root}\left[9\,\delta^3\,\#1^5 + 7424\,q_o^5 - 11\,136\,\delta\,q_o^5 + 5568\,\delta^2\,q_o^5 - 928\,\delta^3\,q_o^5 + \#1^4\,\left(12\,\delta^2\,q_o + 70\,\delta^3\,q_o\right) + \#1^3\,\left(-960\,\delta\,q_o^2 + 864\,\delta^2\,q_o^2 - 520\,\delta^3\,q_o^2\right) + \#1^2\,\left(-6912\,q_o^3 + 1920\,\delta\,q_o^3 - 672\,\delta^2\,q_o^3 + 720\,\delta^3\,q_o^3\right) + \#1\,\left(-6656\,q_o^4 + 11\,712\,\delta\,q_o^4 - 4992\,\delta^2\,q_o^4 + 400\,\delta^3\,q_o^4\right)\,\&,\,2\right];$

$\text{Reduce}\left[\frac{t\,D_1}{2} < p_1 < 2\,t\,D_1\,\&\&\,D_1 > 0\,\&\&\,t > 2\,q_o > 0\,\&\&\,0 < \delta < 1,\,\text{Reals}\right]$

Out[∘]=

$q_o > 0\,\&\&\,t > 2\,q_o\,\&\&\,0 < \delta <$ 0.119…

(*Solution 3, interior solution*)

In[∘]:= $p_1 = \text{Root}\left[9\,\delta^3\,\#1^5 + 7424\,q_o^5 - 11\,136\,\delta\,q_o^5 + 5568\,\delta^2\,q_o^5 - 928\,\delta^3\,q_o^5 + \#1^4\,\left(12\,\delta^2\,q_o + 70\,\delta^3\,q_o\right) + \#1^3\,\left(-960\,\delta\,q_o^2 + 864\,\delta^2\,q_o^2 - 520\,\delta^3\,q_o^2\right) + \#1^2\,\left(-6912\,q_o^3 + 1920\,\delta\,q_o^3 - 672\,\delta^2\,q_o^3 + 720\,\delta^3\,q_o^3\right) + \#1\,\left(-6656\,q_o^4 + 11\,712\,\delta\,q_o^4 - 4992\,\delta^2\,q_o^4 + 400\,\delta^3\,q_o^4\right)\,\&,\,3\right];$

$\text{Reduce}\left[\frac{t\,D_1}{2} < p_1 < 2\,t\,D_1\,\&\&\,D_1 > 0\,\&\&\,t > 2\,q_o > 0\,\&\&\,0 < \delta < 1,\,\text{Reals}\right]$

Out[∘]=

False

(*Solution 4, interior solution*)

In[∘]:= $p_1 = \text{Root}\left[9\,\delta^3\,\#1^5 + 7424\,q_o^5 - 11\,136\,\delta\,q_o^5 + 5568\,\delta^2\,q_o^5 - 928\,\delta^3\,q_o^5 + \#1^4\,\left(12\,\delta^2\,q_o + 70\,\delta^3\,q_o\right) + \#1^3\,\left(-960\,\delta\,q_o^2 + 864\,\delta^2\,q_o^2 - 520\,\delta^3\,q_o^2\right) + \#1^2\,\left(-6912\,q_o^3 + 1920\,\delta\,q_o^3 - 672\,\delta^2\,q_o^3 + 720\,\delta^3\,q_o^3\right) + \#1\,\left(-6656\,q_o^4 + 11\,712\,\delta\,q_o^4 - 4992\,\delta^2\,q_o^4 + 400\,\delta^3\,q_o^4\right)\,\&,\,4\right];$

$\text{Reduce}\left[\frac{t\,D_1}{2} < p_1 < 2\,t\,D_1\,\&\&\,D_1 > 0\,\&\&\,t > 2\,q_o > 0\,\&\&\,0 < \delta < 1,\,\text{Reals}\right]$

Out[∘]=

False

(*Solution 5, interior solution*)

In[∘]:= $p_1 = \text{Root}\left[9\,\delta^3\,\#1^5 + 7424\,q_o^5 - 11\,136\,\delta\,q_o^5 + 5568\,\delta^2\,q_o^5 - 928\,\delta^3\,q_o^5 + \#1^4\,\left(12\,\delta^2\,q_o + 70\,\delta^3\,q_o\right) + \#1^3\,\left(-960\,\delta\,q_o^2 + 864\,\delta^2\,q_o^2 - 520\,\delta^3\,q_o^2\right) + \#1^2\,\left(-6912\,q_o^3 + 1920\,\delta\,q_o^3 - 672\,\delta^2\,q_o^3 + 720\,\delta^3\,q_o^3\right) + \#1\,\left(-6656\,q_o^4 + 11\,712\,\delta\,q_o^4 - 4992\,\delta^2\,q_o^4 + 400\,\delta^3\,q_o^4\right)\,\&,\,5\right];$

$\text{Reduce}\left[\frac{t\,D_1}{2} < p_1 < 2\,t\,D_1\,\&\&\,D_1 > 0\,\&\&\,t > 2\,q_o > 0\,\&\&\,0 < \delta < 1,\,\text{Reals}\right]$

Out[∘]=

False

(*Solution 6, boundary solution, which is the solution of p_1=2tD_1*)

In[∘]:= $p_1 = \frac{2\,q_o\,(-2 + \delta)}{-6 + \delta};$

$\lambda_1 = 0;$

$\lambda_2 = -\frac{q_o\,\left(72 - 636\,\delta + 278\,\delta^2 - 117\,\delta^3 + 16\,\delta^4\right)}{16\,t\,(-6 + \delta)^4};$

$\text{Reduce}[\lambda_2 > 0\,\&\&\,p_1 == 2\,t\,D_1\,\&\&\,D_1 > 0\,\&\&\,t > 2\,q_o > 0\,\&\&\,0 < \delta < 1,\,\text{Reals}]$

Out[∘]=

$q_o > 0\,\&\&\,t > 2\,q_o\,\&\&$ 0.119… $< \delta < 1$

(*Solution 7, boundary solution, which is the solution of $p_1=\frac{t\,D_1}{2}$*)

In[∘]:=
$$p_1 = -\frac{2\, q_o \left(6 - 2\,\delta + \sqrt{36 - 18\,\delta + \delta^2}\right)}{3\,\delta};$$

$$\lambda_1 = \frac{1}{18\,t\,\delta^2\left(36 - 18\,\delta + \delta^2\right)}\left(17\,\delta^4 + 3\,\delta^2\left(-456 + \sqrt{36 - 18\,\delta + \delta^2}\right) - 864\left(6 + \sqrt{36 - 18\,\delta + \delta^2}\right) + 36\,\delta\left(174 + 23\,\sqrt{36 - 18\,\delta + \delta^2}\right) - \delta^3\left(204 + 55\,\sqrt{36 - 18\,\delta + \delta^2}\right)\right) q_o;$$

$\lambda_2 = 0;$

$$\text{Reduce}\left[\lambda_1 > 0\ \&\&\ p_1 == \frac{t\,D_1}{2}\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1,\ \text{Reals}\right]$$

Out[∘]=

False

(*Solution 8, boundary solution, which is the solution of $p_1=\frac{t\ D_1}{2}$ *)

In[∘]:=
$$p_1 = \frac{2\, q_o \left(-6 + 2\,\delta + \sqrt{36 - 18\,\delta + \delta^2}\right)}{3\,\delta};$$

$$\lambda_1 = \frac{1}{18\,t\,\delta^2\left(36 - 18\,\delta + \delta^2\right)}\left(17\,\delta^4 + \delta\left(6264 - 828\,\sqrt{36 - 18\,\delta + \delta^2}\right) + 864\left(-6 + \sqrt{36 - 18\,\delta + \delta^2}\right) - 3\,\delta^2\left(456 + \sqrt{36 - 18\,\delta + \delta^2}\right) + \delta^3\left(-204 + 55\,\sqrt{36 - 18\,\delta + \delta^2}\right)\right) q_o;$$

$\lambda_2 = 0;$

$$\text{Reduce}\left[\lambda_1 > 0\ \&\&\ p_1 == \frac{t\,D_1}{2}\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1,\ \text{Reals}\right]$$

Out[∘]=

False

(*Therefore, when 0<δ< 0.119…,

p_1=Root[9 δ^3 #1^5+7424 q_o^5-11136 δ q_o^5+5568 δ^2 q_o^5-928 δ^3 q_o^5+#1^4 (12 δ^2 q_o+70 δ^3 q_o)+
#1^3 (-960 δ q_o^2+864 δ^2 q_o^2-520 δ^3 q_o^2)+#1^2 (-6912 q_o^3+1920 δ q_o^3-672 δ^2 q_o^3+720 δ^3 q_o^3)+
#1 (-6656 q_o^4+11712 δ q_o^4-4992 δ^2 q_o^4+400 δ^3 q_o^4)&,2];

when 0.119…<δ<1, $p_1=\frac{2\ q_o\ (-2+\delta)}{-6+\delta}$;*) (*We define $P_1^{CL}(q_o,\delta)$ =

Root[9 δ^3 #1^5+7424 q_o^5-11136 δ q_o^5+5568 δ^2 q_o^5-928 δ^3 q_o^5+#1^4 (12 δ^2 q_o+70 δ^3 q_o)+
#1^3 (-960 δ q_o^2+864 δ^2 q_o^2-520 δ^3 q_o^2)+#1^2 (-6912 q_o^3+1920 δ q_o^3-672 δ^2 q_o^3+720 δ^3 q_o^3)+
#1 (-6656 q_o^4+11712 δ q_o^4-4992 δ^2 q_o^4+400 δ^3 q_o^4)&,2]*)

```
(***Proof that P_1^CL(q_o,δ) is the product of q_o and a polynomial function of δ***)

(*To determine if P_1^CL(q_o,δ) is a proportional function of q_o,
we need to see if P_1^CL(q_o,δ) can be expressed
as P_1^CL(q_o,δ)=k q_o, where 0<k<1 is a polynomial function of δ*)

(*As defined in the Proof of Proposition 2, f_1(p_1;
   q_o,δ)=7424 q_o^5-11136 q_o^5 δ+5568 q_o^5 δ^2+9 (p_1)^5 δ^3-928 q_o^5 δ^3+
  (p_1)^4 (12 q_o δ^2+70 q_o δ^3)+(p_1)^3 (-960 q_o^2 δ+864 q_o^2 δ^2-520 q_o^2 δ^3)+
  (p_1)^2 (-6912 q_o^3+1920 q_o^3 δ-672 q_o^3 δ^2+720 q_o^3 δ^3)+
  p_1 (-6656 q_o^4+11712 q_o^4 δ-4992 q_o^4 δ^2+400 q_o^4 δ^3), substitute p_1=k q_o into f_1(p_1;
   q_o,δ)*)
```

```
In[•]:= p_1 = k q_o;
f_1 =
 Simplify[7424 q_o^5 - 11136 q_o^5 δ + 5568 q_o^5 δ^2 + 9 (p_1)^5 δ^3 - 928 q_o^5 δ^3 + (p_1)^4 (12 q_o δ^2 + 70 q_o δ^3) +
   (p_1)^3 (-960 q_o^2 δ + 864 q_o^2 δ^2 - 520 q_o^2 δ^3) + (p_1)^2 (-6912 q_o^3 + 1920 q_o^3 δ - 672 q_o^3 δ^2 + 720 q_o^3 δ^3) +
   p_1 (-6656 q_o^4 + 11712 q_o^4 δ - 4992 q_o^4 δ^2 + 400 q_o^4 δ^3)]
```

Out[•]=

$$\left(-928\,(-2+\delta)^3+9\,k^5\,\delta^3+2\,k^4\,\delta^2\,(6+35\,\delta)-8\,k^3\,\delta\left(120-108\,\delta+65\,\delta^2\right)+48\,k^2\left(-144+40\,\delta-14\,\delta^2+15\,\delta^3\right)+16\,k\left(-416+732\,\delta-312\,\delta^2+25\,\delta^3\right)\right)q_o^5$$

```
(*Since q_o^5>0, we can set the polynomial inside the parentheses equal to zero,
and solve the equation for k*)
```

```
In[•]:= Simplify[Solve[-928 (-2 + δ)^3 + 9 k^5 δ^3 + 2 k^4 δ^2 (6 + 35 δ) - 8 k^3 δ (120 - 108 δ + 65 δ^2) +
     48 k^2 (-144 + 40 δ - 14 δ^2 + 15 δ^3) + 16 k (-416 + 732 δ - 312 δ^2 + 25 δ^3) == 0, k]]
```

Out[•]=

```
{{k → Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 +
      (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 + (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
      (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 1]},
 {k → Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 + (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 +
      (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
      (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 2]},
 {k → Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 + (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 +
      (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
      (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 3]},
 {k → Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 + (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 +
      (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
      (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 4]},
 {k → Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 + (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 +
      (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
      (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 5]}}
```

```
(*We then check each solution if it satisfies conditions*)
```

```
In[•]:= k1 = Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 +
            (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 + (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
            (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 1];
        k2 = Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 +
            (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 + (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
            (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 2];
        k3 = Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 +
            (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 + (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
            (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 3];
        k4 = Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 +
            (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 + (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
            (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 4];
        k5 = Root[7424 - 11136 δ + 5568 δ^2 - 928 δ^3 +
            (-6656 + 11712 δ - 4992 δ^2 + 400 δ^3) #1 + (-6912 + 1920 δ - 672 δ^2 + 720 δ^3) #1^2 +
            (-960 δ + 864 δ^2 - 520 δ^3) #1^3 + (12 δ^2 + 70 δ^3) #1^4 + 9 δ^3 #1^5 &, 5];
        Reduce[0 < δ < 0.119... && 0 < k1 < 1]
        Reduce[0 < δ < 0.119... && 0 < k2 < 1]
        Reduce[0 < δ < 0.119... && 0 < k3 < 1]
        Reduce[0 < δ < 0.119... && 0 < k4 < 1]
        Reduce[0 < δ < 0.119... && 0 < k5 < 1]
Out[•]=
        False
Out[•]=
        0 < δ < 0.119...
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False

        (*Therefore, we obtain the unique solution as the second root k2. We thus
         prove that P1^CL(qo,δ) is the product of qo and a polynomial function of δ*)
```

```
(*Scenario 3: p1>2t D1*)
```

In[∘]:=

$$p_{2P} = \frac{2 q_o + p_1 - t D_1}{4};$$

$$D_{2P} = \frac{2 q_o + p_1 - t D_1}{4 t};$$

$$p_{2M} = \frac{2 p_1 - t D_1}{4};$$

$$D_{2M} = \frac{2 p_1 - t D_1}{4 t};$$

$$p_{2N} = \frac{p_1 - 2 t D_1}{4};$$

$$D_{2N} = \frac{p_1 - 2 t D_1}{4 t};$$

In[∘]:=

$$U_1 = q_o - p_1 - t D_1;$$

$$U_2 = \delta \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right) + \frac{p_1}{2 q_o} \left(\frac{p_1}{2} - p_{2N} - t D_1\right)\right);$$

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[∘]=

$$\left\{\left\{D_1 \to \frac{\frac{2 p_1}{-2+\delta} + q_o}{t}\right\}\right\}$$

In[∘]:=

$$D_1 = \frac{q_o (2 - \delta) - 2 p_1}{t (2 - \delta)};$$

$$\Pi = \text{Simplify}\left[p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P} + \frac{t D_1}{2 q_o} p_{2M} D_{2M} + \frac{p_1}{2 q_o} p_{2N} D_{2N}\right];$$

```
(*The firm's total profit*)
```

Reduce[D[D[Π, p_1], p_1] ≥ 0 && $D_1 > 0$ && $p_1 > 2 t D_1$ && $t > 2 q_o > 0$ && $0 < \delta < 1$]

(*Determine the sign of $\frac{\partial^2 \Pi}{\partial p_1^2}$ *)

Out[∘]=

```
False
```

(* $\frac{\partial^2 \Pi}{\partial p_1^2}$<0, meaning Π is concave and it has a maximum value at point where $\frac{\partial \Pi}{\partial p_1}$=0*)

```
(*Construct Karush-Kuhn-Tucker (KKT) conditions*)
```

In[∘]:=

$$g = p_1 - 2 t D_1;$$

$$L = -\left(p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P} + \frac{t D_1}{2 q_o} p_{2M} D_{2M} + \frac{p_1}{2 q_o} p_{2N} D_{2N}\right) - \lambda g;$$

In[∘]:= Simplify[Solve[{D[L, p_1] == 0, $\lambda g == 0$}, {p_1, λ}], $q_o > 0$ && $0 < \delta < 1$]

Out[$\circ$]=

$$\left\{\left\{p_1 \to \frac{2\,(-2+\delta)\,q_o}{-6+\delta},\ \lambda \to -\frac{\left(24-4\,\delta-158\,\delta^2+29\,\delta^3\right)q_o}{32\,t\,(-6+\delta)^3}\right\},\right.$$
$$\left\{p_1 \to \frac{\left(100-64\,\delta+\delta^2-\sqrt{10000-13592\,\delta+5088\,\delta^2-326\,\delta^3+\delta^4}\right)q_o}{6\,\delta},\ \lambda \to 0\right\},$$
$$\left.\left\{p_1 \to \frac{\left(100-64\,\delta+\delta^2+\sqrt{10000-13592\,\delta+5088\,\delta^2-326\,\delta^3+\delta^4}\right)q_o}{6\,\delta},\ \lambda \to 0\right\}\right\}$$

```
(*There are 3 solutions, we check each solution if it satisfies conditions*)

(*Solution 1, boundary solution*)
```

In[$\circ$]:=

$$p_1 = \frac{2\,(-2+\delta)\,q_o}{-6+\delta};$$
$$\lambda = -\frac{\left(24-4\,\delta-158\,\delta^2+29\,\delta^3\right)q_o}{32\,t\,(-6+\delta)^3};$$

```
Reduce[λ > 0 && D1 > 0 && p1 == 2 t D1 && t > 2 qo > 0 && 0 < δ < 1, Reals]
```

Out[$\circ$]=

$q_o > 0 \,\&\&\, t > 2\,q_o \,\&\&\, 0 < \delta < 0.391\ldots$

```
(*Solution 2, interior solution*)
```

In[$\circ$]:=

$$p_1 = \frac{\left(100-64\,\delta+\delta^2-\sqrt{10000-13592\,\delta+5088\,\delta^2-326\,\delta^3+\delta^4}\right)q_o}{6\,\delta};$$
$$\lambda = 0;$$

```
Reduce[D1 > 0 && p1 > 2 t D1 && t > 2 qo > 0 && 0 < δ < 1, Reals]
```

Out[$\circ$]=

$q_o > 0 \,\&\&\, t > 2\,q_o \,\&\&\, 0.391\ldots < \delta < 1$

```
(*Solution 3, interior solution*)
```

In[$\circ$]:=

$$p_1 = \frac{\left(100-64\,\delta+\delta^2+\sqrt{10000-13592\,\delta+5088\,\delta^2-326\,\delta^3+\delta^4}\right)q_o}{6\,\delta};$$
$$\lambda = 0;$$

```
Reduce[D1 > 0 && p1 > 2 t D1 && t > 2 qo > 0 && 0 < δ < 1, Reals]
```

Out[$\circ$]=

False

(*Therefore, when $0<\delta<0.391\ldots$, $p_1=\frac{2\,(-2+\delta)\,q_o}{-6+\delta}$;

when $0.391\ldots<\delta<1$, $p_1=\frac{\left(100-64\,\delta+\delta^2-\sqrt{10000-13592\,\delta+5088\,\delta^2-326\,\delta^3+\delta^4}\right)q_o}{6\,\delta}$ *)

```
(***Profit comparison***)

(***Based on the above 3 scenarios,
we then compare the firm's profits across (0,1) of δ***)

(*Scenario 1, 0<δ<1*)
```

$$p_1 = \frac{2\, q_o \left(-6 + 2\,\delta + \sqrt{36 - 18\,\delta + \delta^2}\right)}{3\,\delta};$$

$$D_1 = \frac{\sqrt{8\, q_o^2\, (2-\delta) - 8\, p_1\, q_o\, \delta + p_1^2\, \delta^2} - 2\, q_o\, (2-\delta)}{t\,\delta};$$

$$p_{2\,P} = \frac{2\, q_o + p_1 - t\, D_1}{4};$$

$$D_{2\,P} = \frac{2\, q_o + p_1 - t\, D_1}{4\, t};$$

$$\Pi_1 = \text{Simplify}\left[p_1\, D_1 + \frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2\,P}\, D_{2\,P}\right];$$

```
(*Scenario 2(i), 0<δ≤ 0.119… *)
```

$$p_1 = \text{Root}\Big[9\,\delta^3\, \#1^5 + 7424\, q_o^5 - 11\,136\,\delta\, q_o^5 + 5568\,\delta^2\, q_o^5 - 928\,\delta^3\, q_o^5 + \#1^4\left(12\,\delta^2\, q_o + 70\,\delta^3\, q_o\right) + \#1^3\left(-960\,\delta\, q_o^2 + 864\,\delta^2\, q_o^2 - 520\,\delta^3\, q_o^2\right) + \#1^2\left(-6912\, q_o^3 + 1920\,\delta\, q_o^3 - 672\,\delta^2\, q_o^3 + 720\,\delta^3\, q_o^3\right) + \#1\left(-6656\, q_o^4 + 11\,712\,\delta\, q_o^4 - 4992\,\delta^2\, q_o^4 + 400\,\delta^3\, q_o^4\right)\ \&,\ 2\Big];$$

$$D_1 = \frac{8\, p_1\, q_o + 4\, q_o^2\, (-2+\delta) - p_1^2\,\delta}{4\, t\, q_o\, (-2+\delta) - 2\, t\, p_1\,\delta};$$

$$p_{2\,P} = \frac{2\, q_o + p_1 - t\, D_1}{4};$$

$$D_{2\,P} = \frac{2\, q_o + p_1 - t\, D_1}{4\, t};$$

$$p_{2\,M} = \frac{2\, p_1 - t\, D_1}{4};$$

$$D_{2\,M} = \frac{2\, p_1 - t\, D_1}{4\, t};$$

$$\Pi_{21} = \text{Simplify}\left[p_1\, D_1 + \frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2\,P}\, D_{2\,P} + \frac{t\, D_1}{2\, q_o}\, p_{2\,M}\, D_{2\,M}\right];$$

```
(*Scenario 2(ii), 0.119… <δ<1*)
```

$$p_1 = \frac{2\,q_o\,(2-\delta)}{6-\delta};$$
$$D_1 = \frac{8\,p_1\,q_o + 4\,q_o^2\,(-2+\delta) - p_1^2\,\delta}{4\,t\,q_o\,(-2+\delta) - 2\,t\,p_1\,\delta};$$
$$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4};$$
$$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t};$$
$$p_{2M} = \frac{2\,p_1 - t\,D_1}{4};$$
$$D_{2M} = \frac{2\,p_1 - t\,D_1}{4\,t};$$
$$\Pi_{22} = \text{Simplify}\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P} + \frac{t\,D_1}{2\,q_o}\,p_{2M}\,D_{2M}\right];$$

```
(*Scenario 3(i), 0<δ<0.391...*)
```

$$p_1 = \frac{2\,q_o\,(2-\delta)}{6-\delta};$$
$$D_1 = \frac{q_o\,(2-\delta) - 2\,p_1}{t\,(2-\delta)};$$
$$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4};$$
$$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t};$$
$$p_{2M} = \frac{2\,p_1 - t\,D_1}{4};$$
$$D_{2M} = \frac{2\,p_1 - t\,D_1}{4\,t};$$
$$p_{2N} = \frac{p_1 - 2\,t\,D_1}{4};$$
$$D_{2N} = \frac{p_1 - 2\,t\,D_1}{4\,t};$$
$$\Pi_{31} = \text{Simplify}\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P} + \frac{t\,D_1}{2\,q_o}\,p_{2M}\,D_{2M} + \frac{p_1}{2\,q_o}\,p_{2N}\,D_{2N}\right];$$

```
(*Scenario 3(ii), 0.391...<δ<1*)
```

$$p_1 = \frac{q_o \left(100 - 64\,\delta + \delta^2 - \sqrt{10\,000 - 13\,592\,\delta + 5088\,\delta^2 - 326\,\delta^3 + \delta^4}\right)}{6\,\delta};$$

$$D_1 = \frac{q_o\,(2-\delta) - 2\,p_1}{t\,(2-\delta)};$$

$$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4};$$

$$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t};$$

$$p_{2M} = \frac{2\,p_1 - t\,D_1}{4};$$

$$D_{2M} = \frac{2\,p_1 - t\,D_1}{4\,t};$$

$$p_{2N} = \frac{p_1 - 2\,t\,D_1}{4};$$

$$D_{2N} = \frac{p_1 - 2\,t\,D_1}{4\,t};$$

$$\Pi_{32} = \text{Simplify}\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P} + \frac{t\,D_1}{2\,q_o}\,p_{2M}\,D_{2M} + \frac{p_1}{2\,q_o}\,p_{2N}\,D_{2N}\right];$$

(*Comparison 1: when $0<\delta<$ 0.119…,

we have to compare profits under scenarios 1, 2(i), and 3(i)*)

$\text{Reduce}\left[\Pi_1 > \Pi_{21}\ \&\&\ 0 < \delta < 0.119\ldots\ \&\&\ t > 2\,q_o > 0\right]$

$\text{Reduce}\left[\Pi_{31} > \Pi_{21}\ \&\&\ 0 < \delta < 0.119\ldots\ \&\&\ t > 2\,q_o > 0\right]$

Out[∘]=

False

Out[∘]=

False

(*Hence, when $0<\delta<$ 0.119…,

$p_1=\text{Root}[9\,\delta^3\ \#1^5+7424\,q_o^5-11136\,\delta\,q_o^5+5568\,\delta^2\,q_o^5-928\,\delta^3\,q_o^5+\#1^4\,(12\,\delta^2\,q_o+70\,\delta^3\,q_o)+$
$\#1^3\,(-960\,\delta\,q_o^2+864\,\delta^2\,q_o^2-520\,\delta^3\,q_o^2)+\#1^2\,(-6912\,q_o^3+1920\,\delta\,q_o^3-672\,\delta^2\,q_o^3+720\,\delta^3\,q_o^3)+$
$\#1\,(-6656\,q_o^4+11712\,\delta\,q_o^4-4992\,\delta^2\,q_o^4+400\,\delta^3\,q_o^4)\&,2]$*)

(*Comparison 2: when 0.119… $<\delta<$ 0.391…,

we have to compare profits under scenarios 1, 2(ii), and 3(i)*)

$\text{Reduce}\left[\Pi_1 > \Pi_{22}\ \&\&\ 0.119\ldots < \delta < 0.391\ldots\ \&\&\ t > 2\,q_o > 0\right]$

$\text{Reduce}\left[\Pi_{31} == \Pi_{22}\ \&\&\ 0.119\ldots < \delta < 0.391\ldots\ \&\&\ t > 2\,q_o > 0\right]$

Out[∘]=

False

Out[∘]=

$0.119\ldots < \delta < 0.391\ldots\ \&\&\ t > 0\ \&\&\ 0 < q_o < \frac{t}{2}$

(*Hence, when 0.119… $<\delta<$ 0.391…, $p_1=\frac{2\,q_o\,(2-\delta)}{6-\delta}$*)

```
(*Comparison 3: when 0.391…<δ<1,
we have to compare profits under scenarios 1, 2(ii), and 3(ii)*)
```

Reduce$[\Pi_1 > \Pi_{32}$ && $0.391\ldots < \delta < 1$ && $t > 2\, q_o > 0]$

Reduce$[\Pi_{22} > \Pi_{32}$ && $0.391\ldots < \delta < 1$ && $t > 2\, q_o > 0]$

Out[•]=

False

Out[•]=

False

(*Hence, when 0.391…<δ<1, $p_1 = \frac{q_o\ \left(100-64\ \delta+\delta^2-\sqrt{10000-13592\ \delta+5088\ \delta^2-326\ \delta^3+\delta^4}\right)}{6\ \delta}$ *)

```
(***The unique contingent pricing strategy,
firm profit, and consumer surplus in euqilibrium***)

(*(i) 0<δ≤ 0.119... *)

In[ ]:= pCL11 = Root[7424 qo^5 - 11136 qo^5 δ + 5568 qo^5 δ^2 + 9 #1^5 δ^3 - 928 qo^5 δ^3 + #1^4 (12 qo δ^2 + 70 qo δ^3) +
          #1^3 (-960 qo^2 δ + 864 qo^2 δ^2 - 520 qo^2 δ^3) + #1^2 (-6912 qo^3 + 1920 qo^3 δ - 672 qo^3 δ^2 + 720 qo^3 δ^3) +
          #1 (-6656 qo^4 + 11712 qo^4 δ - 4992 qo^4 δ^2 + 400 qo^4 δ^3) &, 2];
  (*The first-period price in equilibrium*)
DCL11 = (8 pCL11 qo + 4 qo^2 (-2 + δ) - pCL11^2 δ) / (4 t qo (-2 + δ) - 2 t pCL11 δ); (*The first-period demand*)
pCL2P1 = (2 qo + pCL11 - t DCL11) / 4;
(*The second-period price under completely positive reviews*)
DCL2P1 = (2 qo + pCL11 - t DCL11) / (4 t);
(*The second-period demand under completely positive reviews*)
pCL2M1 = (2 pCL11 - t DCL11) / 4; (*The second-period price under mixed reviews*)
DCL2M1 = (2 pCL11 - t DCL11) / (4 t); (*The second-period demand under mixed reviews*)
ΠCL1 = pCL11 DCL11 + (2 qo - pCL11 - t DCL11) / (2 qo) pCL2P1 DCL2P1 + (t DCL11) / (2 qo) pCL2M1 DCL2M1;
(*The firm's total profit*)
CSCL1 = Integrate[Integrate[Q - pCL11 - t x, {x, 0, DCL11}] / (2 qo), {Q, 0, 2 qo}] + Integrate[
     Integrate[δ (Q - pCL2P1 - t x), {x, DCL11, DCL11 + DCL2P1}] / (2 qo), {Q, pCL11 + t DCL11, 2 qo}] +
    Integrate[Integrate[δ (Q - pCL2M1 - t x), {x, DCL11, DCL11 + DCL2M1}] / (2 qo),
     {Q, pCL11, pCL11 + t DCL11}]; (*Consumer surplus*)

(*(ii)  0.119... <δ≤ 0.391... *)
```

$$p_{CL12} = \frac{2\,q_o\,(-2+\delta)}{-6+\delta};$$

$$D_{CL12} = \frac{8\,p_{CL12}\,q_o + 4\,q_o^2\,(-2+\delta) - p_{CL12}{}^2\,\delta}{4\,t\,q_o\,(-2+\delta) - 2\,t\,p_{CL12}\,\delta};$$

$$p_{CL2P2} = \frac{2\,q_o + p_{CL12} - t\,D_{CL12}}{4};$$

$$D_{CL2P2} = \frac{2\,q_o + p_{CL12} - t\,D_{CL12}}{4\,t};$$

$$p_{CL2M2} = \frac{2\,p_{CL12} - t\,D_{CL12}}{4};$$

$$D_{CL2M2} = \frac{2\,p_{CL12} - t\,D_{CL12}}{4\,t};$$

$$\Pi_{CL2} = p_{CL12}\,D_{CL12} + \frac{2\,q_o - p_{CL12} - t\,D_{CL12}}{2\,q_o}\,p_{CL2P2}\,D_{CL2P2} + \frac{t\,D_{CL12}}{2\,q_o}\,p_{CL2M2}\,D_{CL2M2};$$

```
CS_CL2 = Integrate[Integrate[Q - p_CL12 - t x, {x, 0, D_CL12}] / (2 q_o), {Q, 0, 2 q_o}] +
   Integrate[Integrate[δ (Q - p_CL2P2 - t x), {x, D_CL12, D_CL12 + D_CL2P2}] / (2 q_o),
    {Q, p_CL12 + t D_CL12, 2 q_o}] + Integrate[
    Integrate[δ (Q - p_CL2M2 - t x), {x, D_CL12, D_CL12 + D_CL2M2}] / (2 q_o), {Q, p_CL12, p_CL12 + t D_CL12}];

(*(iii) 0.391... <δ<1*)
```

$$p_{CL13} = \frac{q_o\left(100 - 64\,\delta + \delta^2 - \sqrt{10\,000 - 13\,592\,\delta + 5088\,\delta^2 - 326\,\delta^3 + \delta^4}\right)}{6\,\delta};$$

$$D_{CL13} = \frac{q_o\,(2-\delta) - 2\,p_{CL13}}{t\,(2-\delta)};$$

$$p_{CL2P3} = \frac{2\,q_o + p_{CL13} - t\,D_{CL13}}{4};$$

$$D_{CL2P3} = \frac{2\,q_o + p_{CL13} - t\,D_{CL13}}{4\,t};$$

$$p_{CL2M3} = \frac{2\,p_{CL13} - t\,D_{CL13}}{4};$$

$$D_{CL2M3} = \frac{2\,p_{CL13} - t\,D_{CL13}}{4\,t};$$

$$p_{CL2N3} = \frac{p_{CL13} - 2\,t\,D_{CL13}}{4};$$

$$D_{CL2N3} = \frac{p_{CL13} - 2\,t\,D_{CL13}}{4\,t};$$

$$\Pi_{CL3} = p_{CL13}\,D_{CL13} + \frac{2\,q_o - p_{CL13} - t\,D_{CL13}}{2\,q_o}\,p_{CL2P3}\,D_{CL2P3} + \frac{t\,D_{CL13}}{2\,q_o}\,p_{CL2M3}\,D_{CL2M3} + \frac{p_{CL13}}{2\,q_o}\,p_{CL2N3}\,D_{CL2N3};$$

```
CS_CL3 = Integrate[Integrate[Q - p_CL13 - t x, {x, 0, D_CL13}] / (2 q_o), {Q, 0, 2 q_o}] + Integrate[
    Integrate[δ (Q - p_CL2P3 - t x), {x, D_CL13, D_CL13 + D_CL2P3}] / (2 q_o), {Q, p_CL13 + t D_CL13, 2 q_o}] +
   Integrate[Integrate[δ (Q - p_CL2M3 - t x), {x, D_CL13, D_CL13 + D_CL2M3}] / (2 q_o),
    {Q, p_CL13, p_CL13 + t D_CL13}] +
   Integrate[Integrate[δ (Q - p_CL2N3 - t x), {x, D_CL13, D_CL13 + D_CL2N3}] / (2 q_o), {Q, 0, p_CL13}];
```

```
(***Price comparison***)

(*(i) 0<δ≤ 0.119… *)

In[•]:= Reduce[pCL11 ≤ pCL2P1 && qo > 0 && 0 < δ ≤ 0.119…]
        (*Comparison of the first-period price and the second-
         period price under completely positive reviews*)
        Reduce[pCL11 ≤ pCL2M1 && qo > 0 && 0 < δ ≤ 0.119…] (*Comparison of the first-
         period price and the second-period price under mixed reviews*)

Out[•]= False

Out[•]= False

(*(ii)  0.119… <δ≤ 0.391… *)

In[•]:= Reduce[pCL12 ≤ pCL2P2 && qo > 0 && 0.119… < δ ≤ 0.391…]
        Reduce[pCL12 ≤ pCL2M2 && qo > 0 && 0.119… < δ ≤ 0.391…]

Out[•]= False

Out[•]= False

(*(iii)  0.391… <δ<1*)

In[•]:= Reduce[pCL13 < pCL2P3 && qo > 0 && 0.391… < δ < 1]
        Reduce[pCL13 ≤ pCL2M3 && qo > 0 && 0.391… < δ < 1]
        Reduce[pCL13 ≤ pCL2N3 && qo > 0 && 0.391… < δ < 1]

Out[•]= 0.401… < δ < 1 && qo > 0

Out[•]= False

Out[•]= False

(*Hence, when 0<δ< 0.401… , pCL1>pCL2P; when  0.401… <δ<1,
pCL1<pCL2P. pCL1>pCL2M and pCL1>pCL2N always hold true*)
```

```
(***Proof of Proposition 2(ii): Price with respect to δc***)

(*(i) 0<δ≤ 0.119… *)

In[•]:= Reduce[D[pCL11, δ] ≥ 0 && qo > 0 && 0 < δ ≤ 0.119…] (*Determine the sign of ∂p1CL*/∂δc *)
Reduce[D[pCL2P1, δ] ≥ 0 && qo > 0 && 0 < δ ≤ 0.119…] (*Determine the sign of ∂p2PCL*/∂δc *)
Reduce[D[pCL2M1, δ] ≥ 0 && qo > 0 && 0 < δ ≤ 0.119…] (*Determine the sign of ∂p2MCL*/∂δc *)

Out[•]= False
Out[•]= False
Out[•]= False

(*(ii) 0.119… <δ≤ 0.391… *)

In[•]:= Reduce[D[pCL12, δ] ≥ 0 && qo > 0 && 0.119… < δ ≤ 0.391…]
Reduce[D[pCL2P2, δ] ≥ 0 && qo > 0 && 0.119… < δ ≤ 0.391…]
Reduce[D[pCL2M2, δ] ≥ 0 && qo > 0 && 0.119… < δ ≤ 0.391…]

Out[•]= False
Out[•]= False
Out[•]= False

(*(iii) 0.391… <δ<1*)

In[•]:= Reduce[D[pCL13, δ] ≥ 0 && qo > 0 && 0.391… < δ < 1]
Reduce[D[pCL2P3, δ] > 0 && qo > 0 && 0.391… < δ < 1]
Reduce[D[pCL2M3, δ] > 0 && qo > 0 && 0.391… < δ < 1]
Reduce[D[pCL2N3, δ] ≤ 0 && qo > 0 && 0.391… < δ < 1]

Out[•]= False
Out[•]= 0.494… < δ < 1 && qo > 0
Out[•]= 0.762… < δ < 1 && qo > 0
Out[•]= False

(*Hence, ∂p1CL*/∂δc <0. When 0<δ< 0.494…, ∂p2PCL*/∂δc <0; when 0.494… <δ<1,
∂p2PCL*/∂δc >0. When 0<δ< 0.762…, ∂p2MCL*/∂δc <0; when 0.762… <δ<1,
∂p2MCL*/∂δc >0. When 0<δ< 0.391…, ∂p2NCL*/∂δc =0; when 0.391… <δ<1, ∂p2NCL*/∂δc >0*)
```

```
(***Proof of Proposition 2(iii): Profit and demand with respect to δc*)
```

(*Determine the sign of $\frac{\partial D_1^{CL*}}{\partial \delta_c}$ *)

In[•]:=
```
Reduce[D[DCL11, δ] ≥ 0 && t > 2 qo > 0 && 0 < δ < 0.119…]
Reduce[D[DCL12, δ] ≥ 0 && t > 2 qo > 0 && 0.119… < δ < 0.391…]
Reduce[D[DCL13, δ] ≥ 0 && t > 2 qo > 0 && 0.391… < δ < 1]
```

Out[•]=
```
False
```

Out[•]=
```
False
```

Out[•]=
```
False
```

(*Hence, $\frac{\partial D_1^{CL*}}{\partial \delta_c}$ <0*)

(*Determine the sign of $\frac{\partial \Pi^{CL}}{\partial \delta_c}$ *)

In[•]:=
```
Reduce[D[ΠCL1, δ] ≥ 0 && t > 2 qo > 0 && 0 < δ < 0.119…]
Reduce[D[ΠCL2, δ] ≥ 0 && t > 2 qo > 0 && 0.119… < δ < 0.391…]
Reduce[D[ΠCL3, δ] ≥ 0 && t > 2 qo > 0 && 0.391… < δ < 1]
```

Out[•]=
```
False
```

Out[•]=
```
False
```

Out[•]=
```
False
```

(*Hence, $\frac{\partial \Pi^{CL}}{\partial \delta_c}$ <0*)

```
(***Proof of Proposition 3: Quality beliefs***)
```

(* i. Result of q_R^{CL} and Pr_R^{CL} when $0<\delta\leq$ 0.119… *)

In[∘]:= $q_{CLP1} = \frac{2\, q_o + p_{CL11} + t\, D_{CL11}}{2};$

$q_{CLM1} = \frac{2\, p_{CL11} + t\, D_{CL11}}{2};$

$q_{CLN1} = \frac{p_{CL11}}{2};$

$Pr_{CLP1} = \frac{2\, q_o - p_{CL11} - t\, D_{CL11}}{2\, q_o};$

$Pr_{CLM1} = \frac{t\, D_{CL11}}{2\, q_o};$

$Pr_{CLN1} = \frac{p_{CL11}}{2\, q_o};$

(* ii. Result of q_R^{CL} and Pr_R^{CL} when 0.119… $<\delta\leq$ 0.391… *)

In[∘]:= $q_{CLP2} = \frac{2\, q_o + p_{CL12} + t\, D_{CL12}}{2};$

$q_{CLM2} = \frac{2\, p_{CL12} + t\, D_{CL12}}{2};$

$q_{CLN2} = \frac{p_{CL12}}{2};$

$Pr_{CLP2} = \frac{2\, q_o - p_{CL12} - t\, D_{CL12}}{2\, q_o};$

$Pr_{CLM2} = \frac{t\, D_{CL12}}{2\, q_o};$

$Pr_{CLN2} = \frac{p_{CL12}}{2\, q_o};$

(* iii. Result of q_R^{CL} and Pr_R^{CL} when 0.391… $<\delta<1$ *)

In[∘]:= $q_{CLP3} = \frac{2\, q_o + p_{CL13} + t\, D_{CL13}}{2};$

$q_{CLM3} = \frac{2\, p_{CL13} + t\, D_{CL13}}{2};$

$q_{CLN3} = \frac{p_{CL13}}{2};$

$Pr_{CLP3} = \frac{2\, q_o - p_{CL13} - t\, D_{CL13}}{2\, q_o};$

$Pr_{CLM3} = \frac{t\, D_{CL13}}{2\, q_o};$

$Pr_{CLN3} = \frac{p_{CL13}}{2\, q_o};$

```
(*Part (i)*)
```

In[∘]:=

```
Reduce[qCLP1 ≤ qo && qo > 0 && 0 < δ ≤ 0.119…]
Reduce[qCLM1 ≥ qo && qo > 0 && 0 < δ ≤ 0.119…]
Reduce[qCLN1 ≥ qo && qo > 0 && 0 < δ ≤ 0.119…]
Reduce[qCLP2 ≤ qo && qo > 0 && 0.119… < δ ≤ 0.391…]
Reduce[qCLM2 ≥ qo && qo > 0 && 0.119… < δ ≤ 0.391…]
Reduce[qCLN2 ≥ qo && qo > 0 && 0.119… < δ ≤ 0.391…]
Reduce[qCLP3 ≤ qo && qo > 0 && 0.391… < δ < 1]
Reduce[qCLM3 ≥ qo && qo > 0 && 0.391… < δ < 1]
Reduce[qCLN3 ≥ qo && qo > 0 && 0.391… < δ < 1]
```

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

(*Hence, $q_P^{CL}>q_o$, $q_M^{CL}<q_o$, $q_N^{CL}<q_o$*)

(*Part (ii)*)

In[∘]:=

```
Reduce[D[qCLP1, δ] ≥ 0 && qo > 0 && 0 < δ ≤ 0.119…]
Reduce[D[qCLM1, δ] ≥ 0 && qo > 0 && 0 < δ ≤ 0.119…]
Reduce[D[qCLN1, δ] ≥ 0 && qo > 0 && 0 < δ ≤ 0.119…]
Reduce[D[qCLP2, δ] ≥ 0 && qo > 0 && 0.119… < δ ≤ 0.391…]
Reduce[D[qCLM2, δ] ≥ 0 && qo > 0 && 0.119… < δ ≤ 0.391…]
Reduce[D[qCLN2, δ] ≥ 0 && qo > 0 && 0.119… < δ ≤ 0.391…]
Reduce[D[qCLP3, δ] ≥ 0 && qo > 0 && 0.391… < δ < 1]
Reduce[D[qCLM3, δ] ≥ 0 && qo > 0 && 0.391… < δ < 1]
Reduce[D[qCLN3, δ] ≥ 0 && qo > 0 && 0.391… < δ < 1]
```

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

(*Hence, $\frac{\partial q_P^{CL}}{\partial \delta}<0$, $\frac{\partial q_M^{CL}}{\partial \delta}<0$, $\frac{\partial q_N^{CL}}{\partial \delta}<0$*)

(*Part (iii)*)

In[ ]:=
```
Reduce[D[Pr_CLP1, δ] ≤ 0 && q_o > 0 && 0 < δ ≤ 0.119…]
Reduce[D[Pr_CLM1, δ] ≥ 0 && q_o > 0 && 0 < δ ≤ 0.119…]
Reduce[D[Pr_CLN1, δ] ≥ 0 && q_o > 0 && 0 < δ ≤ 0.119…]
Reduce[D[Pr_CLP2, δ] ≤ 0 && q_o > 0 && 0.119… < δ ≤ 0.391…]
Reduce[D[Pr_CLM2, δ] ≥ 0 && q_o > 0 && 0.119… < δ ≤ 0.391…]
Reduce[D[Pr_CLN2, δ] ≥ 0 && q_o > 0 && 0.119… < δ ≤ 0.391…]
Reduce[D[Pr_CLP3, δ] ≤ 0 && q_o > 0 && 0.391… < δ < 1]
Reduce[D[Pr_CLM3, δ] ≥ 0 && q_o > 0 && 0.391… < δ < 1]
Reduce[D[Pr_CLN3, δ] ≥ 0 && q_o > 0 && 0.391… < δ < 1]
```

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=
False

Out[ ]=

```
False
```

Out[ ]=

```
False
```

Out[ ]=

```
False
```

(*Hence, $\frac{\partial Pr_P^{CL}}{\partial\delta}>0$, $\frac{\partial Pr_M^{CL}}{\partial\delta}<0$, $\frac{\partial Pr_N^{CL}}{\partial\delta}<0$*)

Comparisons between CN and CL

```
(*Results of case CN*)
```

In[ ]:=
$$p_{CN1} = \frac{q_o\ (2-\delta)^2}{6-4\,\delta};$$
$$D_{CN1} = \frac{q_o\ (1-\delta)}{t\ (3-2\,\delta)};$$
$$p_{CN2} = \frac{q_o\ (2-\delta)}{6-4\,\delta};$$
$$D_{CN2} = \frac{q_o\ (2-\delta)}{2\,t\ (3-2\,\delta)};$$
$$\Pi_{CN} = \frac{q_o^2\ (2-\delta)^2}{4\,t\ (3-2\,\delta)};$$
$$CS_{CN} = \frac{q_o^2\ (4+\delta\ (-2+\delta)\ (-2+5\,\delta))}{8\,t\ (3-2\,\delta)^2};$$

```
(*Results of case CL*)
```

(*(i) 0<δ≤ 0.119… *)

$$p_{CL11} = \text{Root}\big[7424\,q_o^5 - 11\,136\,q_o^5\,\delta + 5568\,q_o^5\,\delta^2 + 9\,\#1^5\,\delta^3 - 928\,q_o^5\,\delta^3 + \#1^4\,\left(12\,q_o\,\delta^2 + 70\,q_o\,\delta^3\right) + \#1^3\,\left(-960\,q_o^2\,\delta + 864\,q_o^2\,\delta^2 - 520\,q_o^2\,\delta^3\right) + \#1^2\,\left(-6912\,q_o^3 + 1920\,q_o^3\,\delta - 672\,q_o^3\,\delta^2 + 720\,q_o^3\,\delta^3\right) + \#1\,\left(-6656\,q_o^4 + 11\,712\,q_o^4\,\delta - 4992\,q_o^4\,\delta^2 + 400\,q_o^4\,\delta^3\right)\ \&,\ 2\big];$$

$$D_{CL11} = \frac{8\,p_{CL11}\,q_o + 4\,q_o^2\ (-2+\delta) - p_{CL11}{}^2\,\delta}{4\,t\,q_o\ (-2+\delta) - 2\,t\,p_{CL11}\,\delta};$$
$$p_{CL2P1} = \frac{2\,q_o + p_{CL11} - t\,D_{CL11}}{4};$$
$$D_{CL2P1} = \frac{2\,q_o + p_{CL11} - t\,D_{CL11}}{4\,t};$$
$$p_{CL2M1} = \frac{2\,p_{CL11} - t\,D_{CL11}}{4};$$
$$D_{CL2M1} = \frac{2\,p_{CL11} - t\,D_{CL11}}{4\,t};$$
$$\Pi_{CL1} = p_{CL11}\,D_{CL11} + \frac{2\,q_o - p_{CL11} - t\,D_{CL11}}{2\,q_o}\,p_{CL2P1}\,D_{CL2P1} + \frac{t\,D_{CL11}}{2\,q_o}\,p_{CL2M1}\,D_{CL2M1};$$

$$CS_{CL1} = \text{Integrate}[\text{Integrate}[Q - p_{CL11} - t\,x,\ \{x,\ 0,\ D_{CL11}\}]\ /\ (2\,q_o),\ \{Q,\ 0,\ 2\,q_o\}] + \text{Integrate}[\text{Integrate}[\delta\ (Q - p_{CL2P1} - t\,x),\ \{x,\ D_{CL11},\ D_{CL11} + D_{CL2P1}\}]\ /\ (2\,q_o),\ \{Q,\ p_{CL11} + t\,D_{CL11},\ 2\,q_o\}] + \text{Integrate}[\text{Integrate}[\delta\ (Q - p_{CL2M1} - t\,x),\ \{x,\ D_{CL11},\ D_{CL11} + D_{CL2M1}\}]\ /\ (2\,q_o),\ \{Q,\ p_{CL11},\ p_{CL11} + t\,D_{CL11}\}];$$

(*(ii) 0.119… <δ≤ 0.391… *)

$$p_{CL12} = \frac{2\,q_o\,(-2+\delta)}{-6+\delta};$$

$$D_{CL12} = \frac{8\,p_{CL12}\,q_o + 4\,q_o^2\,(-2+\delta) - p_{CL12}{}^2\,\delta}{4\,t\,q_o\,(-2+\delta) - 2\,t\,p_{CL12}\,\delta};$$

$$p_{CL2P2} = \frac{2\,q_o + p_{CL12} - t\,D_{CL12}}{4};$$

$$D_{CL2P2} = \frac{2\,q_o + p_{CL12} - t\,D_{CL12}}{4\,t};$$

$$p_{CL2M2} = \frac{2\,p_{CL12} - t\,D_{CL12}}{4};$$

$$D_{CL2M2} = \frac{2\,p_{CL12} - t\,D_{CL12}}{4\,t};$$

$$\Pi_{CL2} = p_{CL12}\,D_{CL12} + \frac{2\,q_o - p_{CL12} - t\,D_{CL12}}{2\,q_o}\,p_{CL2P2}\,D_{CL2P2} + \frac{t\,D_{CL12}}{2\,q_o}\,p_{CL2M2}\,D_{CL2M2};$$

$$\begin{aligned}CS_{CL2} = {} & \text{Integrate}[\text{Integrate}[Q - p_{CL12} - t\,x,\ \{x,\ 0,\ D_{CL12}\}]\,/\,(2\,q_o),\ \{Q,\ 0,\ 2\,q_o\}] + \\ & \text{Integrate}[\text{Integrate}[\delta\,(Q - p_{CL2P2} - t\,x),\ \{x,\ D_{CL12},\ D_{CL12} + D_{CL2P2}\}]\,/\,(2\,q_o), \\ & \{Q,\ p_{CL12} + t\,D_{CL12},\ 2\,q_o\}] + \text{Integrate}[\\ & \text{Integrate}[\delta\,(Q - p_{CL2M2} - t\,x),\ \{x,\ D_{CL12},\ D_{CL12} + D_{CL2M2}\}]\,/\,(2\,q_o),\ \{Q,\ p_{CL12},\ p_{CL12} + t\,D_{CL12}\}];\end{aligned}$$

(*(iii) 0.391... <δ<1*)

$$p_{CL13} = \frac{q_o\left(100 - 64\,\delta + \delta^2 - \sqrt{10\,000 - 13\,592\,\delta + 5088\,\delta^2 - 326\,\delta^3 + \delta^4}\right)}{6\,\delta};$$

$$D_{CL13} = \frac{q_o\,(2-\delta) - 2\,p_{CL13}}{t\,(2-\delta)};$$

$$p_{CL2P3} = \frac{2\,q_o + p_{CL13} - t\,D_{CL13}}{4};$$

$$D_{CL2P3} = \frac{2\,q_o + p_{CL13} - t\,D_{CL13}}{4\,t};$$

$$p_{CL2M3} = \frac{2\,p_{CL13} - t\,D_{CL13}}{4};$$

$$D_{CL2M3} = \frac{2\,p_{CL13} - t\,D_{CL13}}{4\,t};$$

$$p_{CL2N3} = \frac{p_{CL13} - 2\,t\,D_{CL13}}{4};$$

$$D_{CL2N3} = \frac{p_{CL13} - 2\,t\,D_{CL13}}{4\,t};$$

$$\Pi_{CL3} = p_{CL13}\,D_{CL13} + \frac{2\,q_o - p_{CL13} - t\,D_{CL13}}{2\,q_o}\,p_{CL2P3}\,D_{CL2P3} + \frac{t\,D_{CL13}}{2\,q_o}\,p_{CL2M3}\,D_{CL2M3} + \frac{p_{CL13}}{2\,q_o}\,p_{CL2N3}\,D_{CL2N3};$$

$$\begin{aligned}CS_{CL3} = {} & \text{Integrate}[\text{Integrate}[Q - p_{CL13} - t\,x,\ \{x,\ 0,\ D_{CL13}\}]\,/\,(2\,q_o),\ \{Q,\ 0,\ 2\,q_o\}] + \text{Integrate}[\\ & \text{Integrate}[\delta\,(Q - p_{CL2P3} - t\,x),\ \{x,\ D_{CL13},\ D_{CL13} + D_{CL2P3}\}]\,/\,(2\,q_o),\ \{Q,\ p_{CL13} + t\,D_{CL13},\ 2\,q_o\}] + \\ & \text{Integrate}[\text{Integrate}[\delta\,(Q - p_{CL2M3} - t\,x),\ \{x,\ D_{CL13},\ D_{CL13} + D_{CL2M3}\}]\,/\,(2\,q_o), \\ & \{Q,\ p_{CL13},\ p_{CL13} + t\,D_{CL13}\}] + \\ & \text{Integrate}[\text{Integrate}[\delta\,(Q - p_{CL2N3} - t\,x),\ \{x,\ D_{CL13},\ D_{CL13} + D_{CL2N3}\}]\,/\,(2\,q_o),\ \{Q,\ 0,\ p_{CL13}\}];\end{aligned}$$

(***Proof of Proposition 4: Price comparison***)

(*Part i*)

```
In[∘]:= Reduce[pCN1 ≤ pCL11 && qo > 0 && 0 < δ ≤ 0.119…]
        Reduce[pCN1 ≤ pCL12 && qo > 0 && 0.119… < δ ≤ 0.391…]
        Reduce[pCN1 ≤ pCL13 && qo > 0 && 0.391… < δ < 1]
Out[∘]= False
Out[∘]= False
Out[∘]= False
```

(*Hence, $p_1^{CN*}>p_1^{CL*}$*)

(*Part ii*)

(*(i) 0<δ<0.119…*)

```
In[∘]:= Reduce[pCN2 ≥ pCL2P1 && qo > 0 && 0 < δ ≤ 0.119…]
        Reduce[pCN2 ≤ pCL2M1 && qo > 0 && 0 < δ ≤ 0.119…]
Out[∘]= False
Out[∘]= False
```

(*(ii) 0.119…<δ<0.391…)

```
In[∘]:= Reduce[pCN2 ≥ pCL2P2 && qo > 0 && 0.119… < δ ≤ 0.391…]
        Reduce[pCN2 ≤ pCL2M2 && qo > 0 && 0.119… < δ ≤ 0.391…]
Out[∘]= False
Out[∘]= False
```

(*(ii) 0.391…<δ<1)

```
In[∘]:= Reduce[pCN2 ≥ pCL2P3 && qo > 0 && 0.391… < δ < 1]
        Reduce[pCN2 ≤ pCL2M3 && qo > 0 && 0.391… < δ < 1]
        Reduce[pCN2 ≤ pCL2N3 && qo > 0 && 0.391… < δ < 1]
Out[∘]= False
Out[∘]= False
Out[∘]= False
```

(*Hence, $p_2^{CN*}<p_{2P}^{CL*}$, $p_2^{CN*}>p_{2M}^{CL*}$, $p_2^{CN*}>p_{2N}^{CL*}$*)

```
(***Proof of Proposition 5: Profit and consumer surplus comparisons***)
(*Part i*) (*Compare profits Π^CN and Π^CL*)
```

$\text{Reduce}\left[\Pi_{CN} \geq \Pi_{CL1} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0 < \delta \leq 0.119\ldots\right]$

$\text{Reduce}\left[\Pi_{CN} \geq \Pi_{CL2} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0.119\ldots < \delta \leq 0.391\ldots\right]$

$\text{Reduce}\left[\Pi_{CN} \geq \Pi_{CL3} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0.391\ldots < \delta < 1\right]$

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

```
(*Hence, Π^CN<Π^CL*)
(*Part ii*) (*Compare consumer surplus CS^CN and CS^CL*)
```

In[∘]:= $\text{Reduce}\left[CS_{CN} \geq CS_{CL1} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0 < \delta \leq 0.119\ldots\right]$

$\text{Reduce}\left[CS_{CN} \geq CS_{CL2} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0.119\ldots < \delta \leq 0.391\ldots\right]$

$\text{Reduce}\left[CS_{CN} \geq CS_{CL3} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0.391\ldots < \delta < 1\right]$

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

```
(*Hence, CS^CN<CS^CL*)
```

Case 3. Price guarantee without social learning (GN)

```
(***Results of case GN***)
```

$$p_{GN1} = \frac{q_o}{2};$$

$$D_{GN1} = \frac{q_o}{2\,t};$$

$$p_{GN2} = \frac{q_o}{2};$$

$$D_{GN2} = 0;$$

$$\Pi_{GN} = \frac{q_o^2}{4\,t};$$

$$CS_{GN} = \frac{q_o^2}{8\,t};$$

Case 4. Price guarantee with social learning (GL)

(***Proof of Proposition 7(i) Step 1: Profit-maximizing first-period prices***)

(*Combination 1. The conditions are $0<p_1<\frac{2q_o-tD_1}{3}$, $0<p_1\leq\frac{(3+2\sqrt{2})tD_1}{2}$, and $0<p_1\leq 16tD_1$*)

In[ ]:=
$$p_{2P} = \frac{2q_o + p_1 - tD_1}{4};$$ (*The second-period price under completely positive reviews*)

$p_{2M} = p_1$; (*The second-period price under mixed reviews*)

$p_{2N} = p_1$; (*The second-period price under completely negative reviews*)

$$D_{2P} = \frac{2q_o + p_1 - tD_1}{4t};$$ (*The second-period demand under completely positive reviews*)

$D_{2M} = 0$; (*The second-period demand under mixed reviews*)

$D_{2N} = 0$; (*The second-period demand under completely negative reviews*)

In[ ]:=
$U_1 = q_o - p_1 - tD_1;$

$$U_2 = \delta\left(\frac{2q_o - p_1 - tD_1}{2q_o}\left(\frac{2q_o + p_1 + tD_1}{2} - p_{2P} - tD_1\right)\right);$$

Simplify[Solve[$U_1 == U_2$, D_1]] (*We obtain two solutions for D_1, then, we check each solution if it satisfies conditions*)

Out[ ]=
$$\left\{\left\{D_1 \to \frac{2(-2+\delta)q_o - \sqrt{\delta^2 p_1^2 - 8\delta p_1 q_o - 8(-2+\delta)q_o^2}}{t\delta}\right\},\left\{D_1 \to \frac{2(-2+\delta)q_o + \sqrt{\delta^2 p_1^2 - 8\delta p_1 q_o - 8(-2+\delta)q_o^2}}{t\delta}\right\}\right\}$$

$$D_1 = \frac{2(-2+\delta)q_o - \sqrt{\delta^2 p_1^2 - 8\delta p_1 q_o - 8(-2+\delta)q_o^2}}{t\delta};$$

$$\text{Reduce}\left[0 < p_1 < \frac{2q_o - tD_1}{3} \&\& 0 < p_1 \leq \frac{(3+2\sqrt{2})tD_1}{2} \&\& 0 < p_1 \leq 16tD_1 \&\& D_1 > 0 \&\& t > 2q_o > 0 \&\& 0 < \delta < 1\right]$$

Out[ ]=
False

(*Hence, the first solution does not satisfy conditions*)

$$D_1 = \frac{2(-2+\delta)q_o + \sqrt{\delta^2 p_1^2 - 8\delta p_1 q_o - 8(-2+\delta)q_o^2}}{t\delta};$$

$$\text{Reduce}\left[0 < p_1 < \frac{2q_o - tD_1}{3} \&\& 0 < p_1 \leq \frac{(3+2\sqrt{2})tD_1}{2} \&\& 0 < p_1 \leq 16tD_1 \&\& D_1 > 0 \&\& t > 2q_o > 0 \&\& 0 < \delta < 1\right]$$ (*Determine the sign of $\frac{\partial^2\Pi}{\partial p_1^2}$*)

Out[∘]=

$$p_1 > 0 \,\&\&\, \Big(\Big(\text{Root}\big[(68 + 48\sqrt{2})\,\#1^3 - 158\,p_1^3 - 104\sqrt{2}\,p_1^3 + \#1^2\,(-160\,p_1 - 112\sqrt{2}\,p_1) + \#1\,(171\,p_1^2 + 116\sqrt{2}\,p_1^2)\,\&,\,1\big] < q_o \le 2\,p_1 \,\&\&\, t > 2\,q_o \,\&\&\, \frac{2\,p_1\,q_o - q_o^2}{p_1^2} < \delta \le \frac{-184\,p_1\,q_o - 128\sqrt{2}\,p_1\,q_o + 136\,q_o^2 + 96\sqrt{2}\,q_o^2}{-13\,p_1^2 - 12\sqrt{2}\,p_1^2 - 24\,p_1\,q_o - 16\sqrt{2}\,p_1\,q_o + 68\,q_o^2 + 48\sqrt{2}\,q_o^2} \,\&\&\, tD_1 \ge \frac{p_1}{16}\Big) \,||$$

$$\Big(2\,p_1 < q_o < \frac{2\,(10\,p_1 + 7\sqrt{2}\,p_1)}{17 + 12\sqrt{2}} + \frac{\sqrt{2659\,p_1^2 + 1880\sqrt{2}\,p_1^2}}{2\,(17 + 12\sqrt{2})} \,\&\&\, t > 2\,q_o \,\&\&\, 0 < \delta \le \frac{-184\,p_1\,q_o - 128\sqrt{2}\,p_1\,q_o + 136\,q_o^2 + 96\sqrt{2}\,q_o^2}{-13\,p_1^2 - 12\sqrt{2}\,p_1^2 - 24\,p_1\,q_o - 16\sqrt{2}\,p_1\,q_o + 68\,q_o^2 + 48\sqrt{2}\,q_o^2} \,\&\&\, tD_1 \ge \frac{p_1}{16}\Big) \,||$$

$$\Big(q_o == \frac{2\,(10\,p_1 + 7\sqrt{2}\,p_1)}{17 + 12\sqrt{2}} + \frac{\sqrt{2659\,p_1^2 + 1880\sqrt{2}\,p_1^2}}{2\,(17 + 12\sqrt{2})} \,\&\&\, t > 2\,q_o \,\&\&\, 0 < \delta < \frac{-184\,p_1\,q_o - 128\sqrt{2}\,p_1\,q_o + 136\,q_o^2 + 96\sqrt{2}\,q_o^2}{-13\,p_1^2 - 12\sqrt{2}\,p_1^2 - 24\,p_1\,q_o - 16\sqrt{2}\,p_1\,q_o + 68\,q_o^2 + 48\sqrt{2}\,q_o^2} \,\&\&\, tD_1 \ge \frac{p_1}{16}\Big) \,||$$

$$\Big(0 < \delta < 1 \,\&\&\, q_o > \frac{2\,(10\,p_1 + 7\sqrt{2}\,p_1)}{17 + 12\sqrt{2}} + \frac{\sqrt{2659\,p_1^2 + 1880\sqrt{2}\,p_1^2}}{2\,(17 + 12\sqrt{2})} \,\&\&\, t > 2\,q_o \,\&\&\, tD_1 \ge \frac{p_1}{16}\Big)\Big)$$

(*Hence, the second solution satisfies conditions*)

In[∘]:=

$$D_1 = \frac{2\,(-2 + \delta)\,q_o + \sqrt{\delta^2\,p_1^2 - 8\,\delta\,p_1\,q_o - 8\,(-2 + \delta)\,q_o^2}}{t\,\delta};$$

$$\Pi = \text{Simplify}\Big[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P}\Big];\text{(*The firm's total profit*)}$$

$$\text{Reduce}\Big[\text{D}[\text{D}[\Pi, p_1], p_1] \ge 0 \,\&\&\, 0 < p_1 < \frac{2\,q_o - t\,D_1}{3} \,\&\&\, 0 < p_1 \le \frac{(3 + 2\sqrt{2})\,t\,D_1}{2} \,\&\&\, 0 < p_1 \le 16\,tD_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2\,q_o > 0 \,\&\&\, 0 < \delta < 1\Big]$$

Out[∘]=

False

(*$\frac{\partial^2\Pi}{\partial p_1^2}$<0, meaning Π is concave and it has a maximum value at point where $\frac{\partial\Pi}{\partial p_1}$=0*)

(*Construct KKT conditions*)

$$g_1 = \frac{2\,q_o - t\,D_1}{3} - p_1;$$

$$g_2 = \frac{(3 + 2\sqrt{2})\,t\,D_1}{2} - p_1;$$

$$g_3 = 16\,t\,D_1 - p_1;$$

$$L = \text{Simplify}[-\Pi - \lambda_1\,g_1 - \lambda_2\,g_2 - \lambda_3\,g_3];$$

```
In[*]:= Simplify[Solve[{D[L, p1] == 0, λ1 g1 == 0, λ2 g2 == 0, λ3 g3 == 0}, {p1, λ1, λ2, λ3}, Reals],
         0 < δ < 1 && t > 2 qo > 0]

Out[*]=
{{p1 → ConditionalExpression[
    Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2
       (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &,
     1], 8/9 < δ < 0.967… || 0.543… < δ < 8/9 || 0.339… < δ < 0.442… ||
    δ < 0.339… || 0.442… < δ < 0.543… || δ > 0.967…],
  λ1 → ConditionalExpression[(3 (8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
             #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
              (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 + (32 + 60 δ - 32 δ^2) qo^2 +
         qo (-δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] +
             (-8 - 17 δ + 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                    (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                   #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 -
                8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                    (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                    (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] qo - 8 (-2 + δ) qo^2)))) /
     (4 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
             #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
             #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] -
         4 qo + 3 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                 #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                 #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 -
             8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                 #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                 #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] qo - 8 (-2 + δ) qo^2))),
    8/9 < δ < 0.967… || 0.543… < δ < 8/9 || 0.339… <
      δ <
      0.442… || δ <
      0.339… ||
     0.442… <
      δ <
      0.543… || δ >
      0.967…],
  λ2 → ConditionalExpression[
    0,
```

```
8/9 <
  δ <
  0.967… || 0.543… <
  δ <
  8/9 || 0.339… <
  δ <
  0.442… ||
 δ < 0.339… || 0.442… <
  δ <
  0.543… || δ >
  0.967…],
λ3 → ConditionalExpression[
 0,
 8/9 <
  δ < 0.967… ||
 0.543… < δ < 8/9 || 0.339… < δ <
  0.442… ||
 δ < 0.339… || 0.442… < δ < 0.543… ||
 δ > 0.967…]},
{p1 → ConditionalExpression[
  Root[
   8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
    #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
    #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2],
 8/9 < δ < 0.967… || 0.543… < δ < 8/9 ||
 0.339… < δ < 0.442… ||
 δ < 0.339… ||
 0.442… < δ < 0.543… ||
 δ > 0.967…],
λ1 → ConditionalExpression[
 (3 (8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
          (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
          (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2]^2 + (32 + 60 δ - 32 δ^2) qo^2 +
    qo (-δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
          #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
```

```
#1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2] +
(-8 - 17 δ + 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 + #1^3
   (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
   #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2]^2 -
 8 δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 + #1^3
   (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) + #1
   (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2] q_o - 8 (-2 + δ) q_o^2)))) /
(4 t δ^2 (δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 + #1^3
   (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
   #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2] -
 4 q_o + 3 √(δ^2 Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
     #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
     #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2]^2 -
   8 δ Root[8 δ^4 #1^4 + 144 q_o^4 + 225 δ q_o^4 - 272 δ^2 q_o^4 + 64 δ^3 q_o^4 +
     #1^3 (-16 δ^2 q_o - 94 δ^3 q_o) + #1^2 (124 δ q_o^2 + 376 δ^2 q_o^2 - 30 δ^3 q_o^2 - 8 δ^4 q_o^2) +
     #1 (-224 q_o^3 - 480 δ q_o^3 + 104 δ^2 q_o^3 + 64 δ^3 q_o^3) &, 2] q_o - 8 (-2 + δ) q_o^2))),
8/9 < δ < 0.967… || 0.543… < δ < 8/9 || 0.339… < δ < 0.442… ||
 δ <
  0.339… || 0.442… <
  δ <
  0.543… || δ >
  0.967…],
λ2 → ConditionalExpression[
 0,
 8/9 <
  δ <
  0.967… ||
 0.543… < δ < 8/9 || 0.339… < δ <
  0.442… || δ <
  0.339… ||
 0.442… < δ < 0.543… || δ >
  0.967…],
λ3 → ConditionalExpression[0,
 8/9 < δ <
  0.967… ||
 0.543… < δ < 8/9 || 0.339… < δ <
```

```
      0.442... ||
     δ < 0.339... || 0.442... < δ < 0.543... ||
     δ > 0.967... ]},
{p1 → ConditionalExpression[
   Root[
    8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 +
      64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
      #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
      #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3],
   0.339... < δ < 0.442... || 0.442... < δ < 0.543... ],
 λ1 → ConditionalExpression[
   (3 (8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
              (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
              (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 + (32 + 60 δ - 32 δ^2) qo^2 +
         qo (-δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
               #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
               #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] +
            (-8 - 17 δ + 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                      (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                     #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 -
                 8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                      (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                      (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] qo - 8 (-2 + δ) qo^2)))) /
    (4 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
              (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
             #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] -
         4 qo + 3 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 -
              8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] qo - 8 (-2 + δ) qo^2))),
   0.339... < δ < 0.442... || 0.442... < δ < 0.543... ], λ2 →
  ConditionalExpression[
   0,
   0.339... <
     δ <
      0.442... ||
    0.442... < δ < 0.543... ], λ3 →
```

```
   ConditionalExpression[
    0,
    0.339… <
     δ <
      0.442… ||
     0.442… < δ < 0.543…]},
 {p1 → ConditionalExpression[
    Root[
     8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 +
       64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
       #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
       #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &,
     4], 0.339… < δ < 0.442… ||
     δ < 0.339… ||
     0.442… <
      δ < 0.543…],
  λ1 → ConditionalExpression[
    (3 (8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 + (32 + 60 δ - 32 δ^2) qo^2 +
         qo (-δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] +
              (-8 - 17 δ + 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                        (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                        #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 -
                  8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                        (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                        (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] qo - 8 (-2 + δ) qo^2)))) /
     (4 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
               (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
             #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] -
         4 qo + 3 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 -
             8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] qo - 8 (-2 + δ) qo^2))),
    0.339… < δ < 0.442… || δ < 0.339… || 0.442… < δ <
```

```
      ⓡ 0.543… ],
 λ2 → ConditionalExpression[
   0,
   ⓡ 0.339… <
     δ <
      ⓡ 0.442… ||
    δ < ⓡ 0.339… || ⓡ 0.442… <
     δ <
      ⓡ 0.543… ],
 λ3 → ConditionalExpression[
   0,
   ⓡ 0.339… <
     δ <
      ⓡ 0.442… || δ <
      ⓡ 0.339… ||
    ⓡ 0.442… < δ < ⓡ 0.543… ]]}, {p1 →
  ConditionalExpression[
```

$$-\frac{\left(-1+\sqrt{1-\delta}\right) q_o}{\delta},$$

```
   8/9 <
     δ <
      ⓡ 0.967… ||
    ⓡ 0.543… < δ < 8/9 || ⓡ 0.339… <
     δ <
      ⓡ 0.442… ||
    δ < ⓡ 0.339… || ⓡ 0.442… < δ <
      ⓡ 0.543… ||
    δ > ⓡ 0.967… ], λ1 → ConditionalExpression[
```

$$-\left(\left(3\left(8\left(3-\sqrt{10+6\sqrt{1-\delta}-9\delta}+\sqrt{1-\delta}\right)+\left(29-17\sqrt{10+6\sqrt{1-\delta}-9\delta}+31\sqrt{1-\delta}\right)\delta+8\left(-5+\sqrt{10+6\sqrt{1-\delta}-9\delta}\right)\delta^2\right)q_o\right)\Big/\left(4t\left(3-3\sqrt{10+6\sqrt{1-\delta}-9\delta}+\sqrt{1-\delta}\right)\delta^2\right)\right),$$

```
   8/9 < δ < ⓡ 0.967… || ⓡ 0.543… < δ < 8/9 ||
    ⓡ 0.339… <
     δ < ⓡ 0.442… ||
    δ < ⓡ 0.339… || ⓡ 0.442… < δ <
```

```
      0.543… || δ >
      0.967…],
  λ2 → ConditionalExpression[0,
     8/9 < δ <
      0.967… ||
     0.543… < δ < 8/9 || 0.339… < δ <
      0.442… ||
     δ < 0.339… || 0.442… < δ < 0.543… ||
     δ > 0.967…],
  λ3 → ConditionalExpression[
    0,
    8/9 < δ <
      0.967… ||
     0.543… < δ < 8/9 || 0.339… < δ <
      0.442… ||
     δ < 0.339… || 0.442… < δ < 0.543… ||
     δ > 0.967…]},
 {p1 → ConditionalExpression[
    (1 + √(1 - δ)) qo / δ,
    8/9 < δ < 0.967… ||
     δ > 0.967…],
  λ1 → ConditionalExpression[
    -((3 (8 (-3 + √(10 - 6 √(1 - δ) - 9 δ) + √(1 - δ)) +
            (-29 + 17 √(10 - 6 √(1 - δ) - 9 δ) + 31 √(1 - δ)) δ - 8 (-5 + √(10 - 6 √(1 - δ) - 9 δ)) δ^2)
         qo) / (4 t (-3 + 3 √(10 - 6 √(1 - δ) - 9 δ) + √(1 - δ)) δ^2)),
    8/9 < δ < 0.967… || δ > 0.967…], λ2 → ConditionalExpression[
    0,
    8/9 < δ < 0.967… ||
     δ > 0.967…],
  λ3 → ConditionalExpression[
    0,
```

$$\frac{8}{9} < \delta < \boxed{0.967\ldots} \;||$$

$$\delta > \boxed{0.967\ldots}\Big]\Big\},$$

$$\Big\{p_1 \to -\frac{32\left(-34+\delta+\sqrt{1156-578\,\delta+256\,\delta^2}\right) q_o}{255\,\delta},$$

$\lambda_1 \to$
0,
$\lambda_2 \to$
0,
$\lambda_3 \to$

$$\Big(\Big(8\,487\,936\,\sqrt{2}\,\delta^4 - 8\,\delta^3\left(111\,733\,809\,\sqrt{2} - 6\,234\,624\,\sqrt{578-289\,\delta+128\,\delta^2} + 119\,344\sqrt{\left(148\,546+32\,896\,\delta^2 - 544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)}\right) - 8160\left(-152\,881\,\sqrt{2} + 5168\,\sqrt{578-289\,\delta+128\,\delta^2} - 289\sqrt{\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)} + 16\,\sqrt{2}\sqrt{\left(\left(578-289\,\delta+128\,\delta^2\right)\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right) - 136\,\delta\left(38\,751\,279\,\sqrt{2} - 2\,441\,592\,\sqrt{578-289\,\delta+128\,\delta^2} + 1232\sqrt{\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)} + 892\,\sqrt{2}\sqrt{\left(\left(578-289\,\delta+128\,\delta^2\right)\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right) + \delta^2\left(2\,869\,482\,513\,\sqrt{2} - 176\,660\,736\,\sqrt{578-289\,\delta+128\,\delta^2} + 997\,696\sqrt{\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)} + 89\,728\,\sqrt{2}\sqrt{\left(\left(578-289\,\delta+128\,\delta^2\right)\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right) q_o\Big) \Big/ \Big(1020\,t\,\delta^2\Big(7\,259\,102\,\sqrt{2} + 1\,546\,112\,\sqrt{2}\,\delta^2 - 312\,256\,\sqrt{578-289\,\delta+128\,\delta^2} - 578\sqrt{\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)} - 208\,\sqrt{2}\sqrt{\left(\left(578-289\,\delta+128\,\delta^2\right)\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)} + \delta\left(-3\,568\,351\,\sqrt{2} + 24\,064\,\sqrt{578-289\,\delta+128\,\delta^2} - 208\sqrt{\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2}+\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)}\right)\Big)\Big)\Big\},$$

$$\Big\{p_1 \to \frac{32\left(34-\delta+\sqrt{1156-578\,\delta+256\,\delta^2}\right) q_o}{255\,\delta},$$

$\lambda_1 \to$
0,
$\lambda_2 \to$
0,

$$\lambda_3 \to \Big(\Big(8\,487\,936\,\sqrt{2}\,\delta^4 - 8\,\delta^3\,\Big(111\,733\,809\,\sqrt{2} + 6\,234\,624\,\sqrt{578 - 289\,\delta + 128\,\delta^2} + 119\,344\,\sqrt{\Big(148\,546 - 139\,553\,\delta + 32\,896\,\delta^2 + 544\,\sqrt{1156 - 578\,\delta + 256\,\delta^2} - 256\,\delta\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)}\Big) + \delta^2\,\Big(2\,869\,482\,513\,\sqrt{2} + 176\,660\,736\,\sqrt{578 - 289\,\delta + 128\,\delta^2} + 997\,696\,\sqrt{\Big(148\,546 - 139\,553\,\delta + 32\,896\,\delta^2 + 544\,\sqrt{1156 - 578\,\delta + 256\,\delta^2} - 256\,\delta\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)} - 89\,728\,\sqrt{2}\,\sqrt{\Big(-\big(578 - 289\,\delta + 128\,\delta^2\big)\,\Big(-32\,896\,\delta^2 - 34\,\Big(4369 + 16\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big) + \delta\,\Big(139\,553 + 256\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)\Big)\Big)}\Big) + 8160\,\Big(152\,881\,\sqrt{2} + 5168\,\sqrt{578 - 289\,\delta + 128\,\delta^2} + 289\,\sqrt{\Big(148\,546 - 139\,553\,\delta + 32\,896\,\delta^2 + 544\,\sqrt{1156 - 578\,\delta + 256\,\delta^2} - 256\,\delta\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)} + 16\,\sqrt{2}\,\sqrt{\Big(-\big(578 - 289\,\delta + 128\,\delta^2\big)\,\Big(-32\,896\,\delta^2 - 34\,\Big(4369 + 16\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big) + \delta\,\Big(139\,553 + 256\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)\Big)\Big)}\Big) + 136\,\delta\,\Big(-38\,751\,279\,\sqrt{2} - 2\,441\,592\,\sqrt{578 - 289\,\delta + 128\,\delta^2} - 1232\,\sqrt{\Big(148\,546 - 139\,553\,\delta + 32\,896\,\delta^2 + 544\,\sqrt{1156 - 578\,\delta + 256\,\delta^2} - 256\,\delta\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)} + 892\,\sqrt{2}\,\sqrt{\Big(-\big(578 - 289\,\delta + 128\,\delta^2\big)\,\Big(-32\,896\,\delta^2 - 34\,\Big(4369 + 16\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big) + \delta\,\Big(139\,553 + 256\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)\Big)\Big)}\Big)\Big)\,q_o\Big) \Big/ \Big(1020\,t\,\delta^2\,\Big(7\,259\,102\,\sqrt{2} + 1\,546\,112\,\sqrt{2}\,\delta^2 + 312\,256\,\sqrt{578 - 289\,\delta + 128\,\delta^2} - 578\,\sqrt{\Big(148\,546 - 139\,553\,\delta + 32\,896\,\delta^2 + 544\,\sqrt{1156 - 578\,\delta + 256\,\delta^2} - 256\,\delta\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)} + 208\,\sqrt{2}\,\sqrt{\Big(-\big(578 - 289\,\delta + 128\,\delta^2\big)\,\Big(-32\,896\,\delta^2 - 34\,\Big(4369 + 16\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big) + \delta\,\Big(139\,553 + 256\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)\Big)\Big)} - \delta\,\Big(3\,568\,351\,\sqrt{2} + 24\,064\,\sqrt{578 - 289\,\delta + 128\,\delta^2} + 208\,\sqrt{\Big(148\,546 - 139\,553\,\delta + 32\,896\,\delta^2 + 544\,\sqrt{1156 - 578\,\delta + 256\,\delta^2} - 256\,\delta\,\sqrt{1156 - 578\,\delta + 256\,\delta^2}\Big)}\Big)\Big)\Big)\Big\},$$

$$\Big\{p_1 \to -\frac{1}{\big(13 + 12\,\sqrt{2}\big)\,\delta}\,2\,\Big(-46 - 32\,\sqrt{2} + \big(6 + 4\,\sqrt{2}\big)\,\delta + \sqrt{4164 + 2944\,\sqrt{2} - 2\,\big(1041 + 736\,\sqrt{2}\big)\,\delta + \big(577 + 408\,\sqrt{2}\big)\,\delta^2}\Big)\,q_o,\ \lambda_1 \to 0,$$

$$\lambda_2 \to \Big(\Big(66\,137\,065\,856 + 46\,765\,851\,008\,\sqrt{2} - 496\,241\,983\,524\,\delta - 350\,896\,298\,928\,\sqrt{2}\,\delta + 340\,867\,291\,589\,\delta^2 + 241\,029\,775\,600\,\sqrt{2}\,\delta^2 - 104\,266\,872\,389\,\delta^3 - 73\,727\,847\,384\,\sqrt{2}\,\delta^3 + 11\,467\,166\,556\,\delta^4 + 8\,108\,511\,360\,\sqrt{2}\,\delta^4 - 597\,709\,904\,\sqrt{4164 + 2944\,\sqrt{2} - 2\,\big(1041 + 736\,\sqrt{2}\big)\,\delta + \big(577 + 408\,\sqrt{2}\big)\,\delta^2} + 5\,439\,895\,586\,\delta\,\sqrt{4164 + 2944\,\sqrt{2} - 2\,\big(1041 + 736\,\sqrt{2}\big)\,\delta + \big(577 + 408\,\sqrt{2}\big)\,\delta^2} - 2\,702\,520\,212\,\delta^2\,\sqrt{4164 + 2944\,\sqrt{2} - 2\,\big(1041 + 736\,\sqrt{2}\big)\,\delta + \big(577 + 408\,\sqrt{2}\big)\,\delta^2} + 466\,864\,488\,\delta^3\,\sqrt{4164 + 2944\,\sqrt{2} - 2\,\big(1041 + 736\,\sqrt{2}\big)\,\delta + \big(577 + 408\,\sqrt{2}\big)\,\delta^2} - 422\,607\,040\,\sqrt{8328 + 5888\,\sqrt{2} - 4\,\big(1041 + 736\,\sqrt{2}\big)\,\delta + 2\,\big(577 + 408\,\sqrt{2}\big)\,\delta^2} + 3\,846\,700\,296\,\delta\,\sqrt{8328 + 5888\,\sqrt{2} - 4\,\big(1041 + 736\,\sqrt{2}\big)\,\delta + 2\,\big(577 + 408\,\sqrt{2}\big)\,\delta^2} -$$

$$1911012712\,\delta^2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+$$
$$330123904\,\delta^3\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+$$
$$395575056\sqrt{\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-\right.}$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta$$
$$\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\right.$$
$$\left.\left.2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)-$$
$$451700242\,\delta\sqrt{\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-\right.}$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta$$
$$\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\right.$$
$$\left.\left.2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)+$$
$$198705458\,\delta^2\sqrt{\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-\right.}$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta$$
$$\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\right.$$
$$\left.\left.2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)-$$
$$82466688\,\delta^3\sqrt{\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-\right.}$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta$$
$$\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\right.$$
$$\left.\left.2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)-$$
$$6456592\sqrt{\left(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)\right.}$$
$$\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-\right.$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta\left(-841-\right.$$
$$588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$
$$\left.\left.\left.2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)\right)-$$
$$4562688\sqrt{2}\sqrt{\left(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)\right.}$$
$$\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-\right.$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta\left(-841-\right.$$
$$588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$
$$\left.\left.\left.2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)\right)-$$

$$120\,609\,\delta\sqrt{\left(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\delta\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)\right)}\,-$$

$$76\,144\sqrt{2}\,\delta\sqrt{\left(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\delta\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)\right)}\,+$$

$$2\,332\,140\,\delta^2\sqrt{\left(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\delta\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)\right)}\,+$$

$$1\,646\,720\sqrt{2}\,\delta^2\sqrt{\left(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)\left(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\delta\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)\right)}\,+$$

$$279\,700\,544\sqrt{\left(9384+6528\sqrt{2}+6\left(215+152\sqrt{2}\right)\delta^2-80\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-32\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+8\delta\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)}\,-$$

$$319\,428\,480\,\delta\sqrt{\left(9384+6528\sqrt{2}+6\left(215+152\sqrt{2}\right)\delta^2-80\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-32\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+8\delta\left(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\right)\right)}\,+$$

$$140\,521\,948\,\delta^2 \surd\Big(\Big(9384+6528\sqrt{2}+6\left(215+152\sqrt{2}\right)\delta^2-$$
$$80\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$32\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+8\,\delta$$
$$\Big(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)-$$
$$58\,313\,520\,\delta^3 \surd\Big(\Big(9384+6528\sqrt{2}+6\left(215+152\sqrt{2}\right)\delta^2-$$
$$80\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$32\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+8\,\delta$$
$$\Big(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)\Big)\,q_o\Big)\Big/$$
$$\Big(\left(13+12\sqrt{2}\right)t\,\delta^2\Big(2\,234\,636\,780+1\,580\,120\,776\sqrt{2}-1\,081\,312\,010\,\delta-$$
$$764\,599\,812\sqrt{2}\,\delta+268\,858\,383\,\delta^2+190\,111\,578\sqrt{2}\,\delta^2-$$
$$22\,459\,780\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$
$$4\,859\,460\,\delta\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$15\,880\,200\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+$$
$$3\,436\,112\,\delta\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}-$$
$$3\,299\,028\surd\Big(\Big(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta$$
$$\Big(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)+$$
$$26\,858\,\delta\surd\Big(\Big(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta$$
$$\Big(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)+$$
$$2127\surd\Big(\Big(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)$$
$$\Big(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-$$
$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta\Big(-841-$$
$$588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)\Big)+$$
$$1762\sqrt{2}\surd\Big(\Big(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)$$
$$\Big(4692+3264\sqrt{2}+\left(645+456\sqrt{2}\right)\delta^2-$$

$$40\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;-$$
$$16\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+4\,\delta\,\Big(-841-$$
$$588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)\Big)-$$
$$2\,333\,600\sqrt{\Big(9384+6528\sqrt{2}+6\left(215+152\sqrt{2}\right)\delta^2-}$$
$$80\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;-$$
$$32\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+8\,\delta$$
$$\Big(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)+$$
$$19\,080\,\delta\sqrt{\Big(9384+6528\sqrt{2}+6\left(215+152\sqrt{2}\right)\delta^2-}$$
$$80\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;-$$
$$32\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}+8\,\delta$$
$$\Big(-841-588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)\Big)\Big),$$
$$\lambda_3\to 0\Big\},\ \Big\{p_1\to-\frac{1}{\left(13+12\sqrt{2}\right)\delta}\,2\,\Big(-46-32\sqrt{2}+\left(6+4\sqrt{2}\right)\delta-$$
$$\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\Big)\,q_o,$$
$$\lambda_1\to 0,\ \lambda_2\to\Big(\Big(66\,137\,065\,856+46\,765\,851\,008\sqrt{2}-496\,241\,983\,524\,\delta-$$
$$350\,896\,298\,928\sqrt{2}\,\delta+340\,867\,291\,589\,\delta^2+$$
$$241\,029\,775\,600\sqrt{2}\,\delta^2-104\,266\,872\,389\,\delta^3-$$
$$73\,727\,847\,384\sqrt{2}\,\delta^3+11\,467\,166\,556\,\delta^4+8\,108\,511\,360\sqrt{2}\,\delta^4+$$
$$597\,709\,904\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;-$$
$$5\,439\,895\,586\,\delta\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;+$$
$$2\,702\,520\,212\,\delta^2\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;-$$
$$466\,864\,488\,\delta^3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;+$$
$$422\,607\,040\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\;-$$
$$3\,846\,700\,296\,\delta\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\;+$$
$$1\,911\,012\,712\,\delta^2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\;-$$
$$330\,123\,904\,\delta^3\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\;+$$
$$395\,575\,056\sqrt{\Big(\left(645+456\sqrt{2}\right)\delta^2-4\,\delta}$$
$$\Big(841+588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}\;+$$
$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)+4\,\Big(1173+$$
$$816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+4$$
$$\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)-$$
$$451\,700\,242\,\delta\sqrt{\Big(\left(645+456\sqrt{2}\right)\delta^2-4\,\delta\,\Big(841+588\sqrt{2}+3}$$

$\sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} + 2$

$\sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,) + 4\, (1173 +$

$816 \sqrt{2} + 10 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$4 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,)\,) +$

$198\,705\,458\, \delta^2 \sqrt{}\, (\,((645 + 456 \sqrt{2}) \delta^2 - 4 \delta\, (841 + 588 \sqrt{2} + 3$

$\sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} + 2$

$\sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,) + 4\, (1173 +$

$816 \sqrt{2} + 10 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$4 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,)\,) -$

$82\,466\,688\, \delta^3 \sqrt{}\, (\,((645 + 456 \sqrt{2}) \delta^2 - 4 \delta\, (841 + 588 \sqrt{2} + 3$

$\sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} + 2$

$\sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,) + 4\, (1173 +$

$816 \sqrt{2} + 10 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$4 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,)\,) +$

$6\,456\,592 \sqrt{}\, (\,(4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2)$

$(\,(645 + 456 \sqrt{2}) \delta^2 - 4 \delta\, (841 + 588 \sqrt{2} +$

$3 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$2 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,) + 4\, (1173 +$

$816 \sqrt{2} + 10 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$4 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,)\,)\,) +$

$4\,562\,688 \sqrt{2} \sqrt{}\, (\,(4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2)$

$(\,(645 + 456 \sqrt{2}) \delta^2 - 4 \delta\, (841 + 588 \sqrt{2} +$

$3 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$2 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,) + 4\, (1173 +$

$816 \sqrt{2} + 10 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$4 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,)\,)\,) +$

$120\,609\, \delta \sqrt{}\, (\,(4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2)$

$(\,(645 + 456 \sqrt{2}) \delta^2 - 4 \delta\, (841 + 588 \sqrt{2} +$

$3 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$2 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,) + 4\, (1173 +$

$816 \sqrt{2} + 10 \sqrt{4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2} +$

$4 \sqrt{8328 + 5888 \sqrt{2} - 4 (1041 + 736 \sqrt{2}) \delta + 2 (577 + 408 \sqrt{2}) \delta^2}\,)\,)\,) +$

$76\,144 \sqrt{2}\, \delta \sqrt{}\, (\,(4164 + 2944 \sqrt{2} - 2 (1041 + 736 \sqrt{2}) \delta + (577 + 408 \sqrt{2}) \delta^2)$

$(\,(645 + 456 \sqrt{2}) \delta^2 - 4 \delta\, (841 + 588 \sqrt{2} +$

$$
\begin{aligned}
&3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)+4\Big(1173+\\
&816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)\Big)-\\
&2332140\,\delta^2\surd\Big(\Big(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)\\
&\Big(\left(645+456\sqrt{2}\right)\delta^2-4\delta\Big(841+588\sqrt{2}+\\
&3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)+4\Big(1173+\\
&816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)\Big)-\\
&1646720\sqrt{2}\,\delta^2\surd\Big(\Big(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)\\
&\Big(\left(645+456\sqrt{2}\right)\delta^2-4\delta\Big(841+588\sqrt{2}+\\
&3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)+4\Big(1173+\\
&816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)\Big)+\\
&279700544\surd\Big(6\left(215+152\sqrt{2}\right)\delta^2-8\delta\Big(841+588\sqrt{2}+3\\
&\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\\
&\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)+8\Big(1173+\\
&816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)-\\
&319428480\,\delta\surd\Big(6\left(215+152\sqrt{2}\right)\delta^2-8\delta\Big(841+588\sqrt{2}+3\\
&\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\\
&\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)+8\Big(1173+\\
&816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)+\\
&140521948\,\delta^2\surd\Big(6\left(215+152\sqrt{2}\right)\delta^2-8\delta\Big(841+588\sqrt{2}+3\\
&\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2\\
&\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)+8\Big(1173+\\
&816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+\\
&4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\Big)\Big)-\\
&58313520\,\delta^3\surd\Big(6\left(215+152\sqrt{2}\right)\delta^2-8\delta\Big(841+588\sqrt{2}+3\\
&\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2
\end{aligned}
$$

$$\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)+8\,\Big(1173+$$

$$816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$

$$4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)\Big)\Big)\,q_o\Big)\Big/$$

$$\Big(\left(13+12\sqrt{2}\right)t\,\delta^2\,\Big(2\,234\,636\,780+1\,580\,120\,776\sqrt{2}-1\,081\,312\,010\,\delta-$$

$$764\,599\,812\sqrt{2}\,\delta+268\,858\,383\,\delta^2+190\,111\,578\sqrt{2}\,\delta^2+$$

$$22\,459\,780\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}-$$

$$4\,859\,460\,\delta\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$

$$15\,880\,200\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}-$$

$$3\,436\,112\,\delta\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}-$$

$$3\,299\,028\,\surd\Big(\Big(\left(645+456\sqrt{2}\right)\delta^2-4\,\delta$$

$$\Big(841+588\sqrt{2}+3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$

$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)+4\,\Big(1173+$$

$$816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+4$$

$$\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)\Big)+$$

$$26\,858\,\delta\,\surd\Big(\Big(\left(645+456\sqrt{2}\right)\delta^2-4\,\delta\,\Big(841+588\sqrt{2}+3$$

$$\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2$$

$$\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)+4\,\Big(1173+$$

$$816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$

$$4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)\Big)-$$

$$2127\,\surd\Big(\Big(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)$$

$$\Big(\left(645+456\sqrt{2}\right)\delta^2-4\,\delta\,\Big(841+588\sqrt{2}+$$

$$3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$

$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)+4\,\Big(1173+$$

$$816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$

$$4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)\Big)\Big)-$$

$$1762\sqrt{2}\,\surd\Big(\Big(\left(4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2\right)$$

$$\Big(\left(645+456\sqrt{2}\right)\delta^2-4\,\delta\,\Big(841+588\sqrt{2}+$$

$$3\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$

$$2\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)+4\,\Big(1173+$$

$$816\sqrt{2}+10\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+$$

$$4\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)\Big)\Big)-$$

$$2\,333\,600\,\surd\Big(6\left(215+152\sqrt{2}\right)\delta^2-8\,\delta\,\Big(841+588\sqrt{2}+3$$

$$\sqrt{4164+2944\sqrt{2}-2\left(1041+736\sqrt{2}\right)\delta+\left(577+408\sqrt{2}\right)\delta^2}+2$$

$$\sqrt{8328+5888\sqrt{2}-4\left(1041+736\sqrt{2}\right)\delta+2\left(577+408\sqrt{2}\right)\delta^2}\,\Big)+8\,\Big(1173+$$

```
816 √2 + 10 √(4164 + 2944 √2 - 2 (1041 + 736 √2) δ + (577 + 408 √2) δ^2 +
4 √(8328 + 5888 √2 - 4 (1041 + 736 √2) δ + 2 (577 + 408 √2) δ^2))) +
19080 δ √((6 (215 + 152 √2) δ^2 - 8 δ (841 + 588 √2 + 3
√(4164 + 2944 √2 - 2 (1041 + 736 √2) δ + (577 + 408 √2) δ^2) + 2
√(8328 + 5888 √2 - 4 (1041 + 736 √2) δ + 2 (577 + 408 √2) δ^2)) + 8 (1173 +
816 √2 + 10 √(4164 + 2944 √2 - 2 (1041 + 736 √2) δ + (577 + 408 √2) δ^2 +
4 √(8328 + 5888 √2 - 4 (1041 + 736 √2) δ + 2 (577 + 408 √2) δ^2))))), λ3 → 0}}
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(*There are 10 solutions, we then check each solution if it satisfies condisions*)

(*Solution 1, interior solution*)

p1 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
     #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
     #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1];
λ1 = (3 (8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
            #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
            #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 + (32 + 60 δ - 32 δ^2) qo^2 +
       qo (-δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
              #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] +
          (-8 - 17 δ + 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 -
              8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                  (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] qo - 8 (-2 + δ) qo^2)))) /
  (4 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
          #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
          #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] -
       4 qo + 3 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
              #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1]^2 -
          8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
              (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
              (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1] qo - 8 (-2 + δ) qo^2)));
λ2 = 0;
λ3 = 0;
Reduce[λ1 == 0 && p1 < (2 qo - t D1)/3 &&
  0 < p1 < ((3 + 2 √2) t D1)/2 && 0 < p1 < 16 t D1 && t > 2 qo > 0 && 0 < δ < 1]
```

Out[∘]=

```
0.442… < δ < 0.864… && qo > 0 && t > 2 qo
```

```
(*Hence, when 0.442…<δ<0.864…, solution 1 satisfies conditions of λ1==0,
p1<(2qo-tD1)/3, 0<p1<((3+2 √2) tD1)/2, and 0<p1<16tD1*)

(*We define P1^GL(qo,δ)=
 Root[8 δ^4 #1^4+144 qo^4+225 δ qo^4-272 δ^2 qo^4+64 δ^3 qo^4+#1^3 (-16 δ^2 qo-94 δ^3 qo)+
    #1^2 (124 δ qo^2+376 δ^2 qo^2-30 δ^3 qo^2-8 δ^4 qo^2)+
    #1 (-224 qo^3-480 δ qo^3+104 δ^2 qo^3+64 δ^3 qo^3)&,1],
which is a product of qo and a polynomial of δ*)

(*Solution 2, interior solution*)
```

In[∘]:=

```
p1 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
      #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
      #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2];
λ1 = (3 (8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
              #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2]^2 + (32 + 60 δ - 32 δ^2) qo^2 +
        qo (-δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2] +
           (-8 - 17 δ + 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                      #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                      #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2]^2 -
                 8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                      (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                     #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2] qo - 8 (-2 + δ) qo^2)))) /
    (4 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
              #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2] -
        4 qo + 3 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                  #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2]^2 -
             8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                  (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                  (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 2] qo - 8 (-2 + δ) qo^2)));
λ2 = 0;
λ3 = 0;
Reduce[λ1 == 0 && p1 < (2 qo - t D1)/3 &&
  0 < p1 < ((3 + 2 √2) t D1)/2 && 0 < p1 < 16 t D1 && t > 2 qo > 0 && 0 < δ < 1]
```

Out[ ]=

```
False
```

```
(*Hence, solution 2 does not satisfy conditions of λ1==0,
p1<(2qo-tD1)/3, 0<p1<((3+2 √2) tD1)/2, and 0<p1<16tD1*)
```

```
(*Solution 3, interior solution*)
```

In[ ]:=

```
p1 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
     #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
     #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3];
λ1 = (3 (8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
             #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
             #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 + (32 + 60 δ - 32 δ^2) qo^2 +
        qo (-δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
              #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] +
           (-8 - 17 δ + 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                    #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                    #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 -
               8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                    (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                   #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] qo - 8 (-2 + δ) qo^2)))) /
    (4 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
            #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
            #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] -
        4 qo + 3 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3]^2 -
            8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 3] qo - 8 (-2 + δ) qo^2)));
λ2 = 0;
λ3 = 0;
Reduce[λ1 == 0 && p1 < (2 qo - t D1)/3 &&
  0 < p1 < ((3 + 2 √2) t D1)/2 && 0 < p1 < 16 t D1 && t > 2 qo > 0 && 0 < δ < 1, Reals]
```

Out[ ]=

```
False
```

```
(*Hence, solution 3 does not satisfy conditions of λ1==0,
p1<(2qo-tD1)/3, 0<p1<((3+2 √2) tD1)/2, and 0<p1<16tD1*)
```

```
(*Solution 4, interior solution*)
```

In[•]:=
```
p1 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
     #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
     #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4];
λ1 = (3 (8 δ^3 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
            #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
            #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 + (32 + 60 δ - 32 δ^2) qo^2 +
        qo (-δ (8 + 47 δ) Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
              #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
              #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] +
           (-8 - 17 δ + 8 δ^2) √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                   #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                   #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 -
               8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                   (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                  #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] qo - 8 (-2 + δ) qo^2)))) /
    (4 t δ^2 (δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3 (-16 δ^2 qo - 94 δ^3 qo) +
            #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
            #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] -
        4 qo + 3 √(δ^2 Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
                #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
                #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4]^2 -
            8 δ Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 + #1^3
                (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) + #1
                (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 4] qo - 8 (-2 + δ) qo^2)));
λ2 = 0;
λ3 = 0;
Reduce[λ1 == 0 && p1 < (2 qo - t D1)/3 &&
   0 < p1 < ((3 + 2 √2) t D1)/2 && 0 < p1 < 16 t D1 && t > 2 qo > 0 && 0 < δ < 1]
```

Out[•]=
```
False
```

```
(*Hence, solution 4 does not satisfy conditions of λ1==0,
p1<(2qo-tD1)/3, 0<p1<((3+2 √2) tD1)/2, and 0<p1<16tD1*)

(*Solution 5, boundary solution, which is the solution of p1=(2qo-tD1)/3*)
```

In[∘]:=
$$p_1 = -\frac{\left(-1+\sqrt{1-\delta}\right) q_o}{\delta};$$
$$\lambda_1 = -\left(\left(3\left(8\left(3-\sqrt{10+6\sqrt{1-\delta}-9\delta}+\sqrt{1-\delta}\right)+\left(29-17\sqrt{10+6\sqrt{1-\delta}-9\delta}+31\sqrt{1-\delta}\right)\delta+8\left(-5+\sqrt{10+6\sqrt{1-\delta}-9\delta}\right)\delta^2\right)q_o\right)\Big/\left(4t\left(3-3\sqrt{10+6\sqrt{1-\delta}-9\delta}+\sqrt{1-\delta}\right)\delta^2\right)\right);$$
$$\lambda_2 = 0;$$
$$\lambda_3 = 0;$$
$$\text{Reduce}\left[\lambda_1 > 0 \,\&\&\, p_1 == \frac{2q_o - tD_1}{3} \,\&\&\, 0 < p_1 < \frac{\left(3+2\sqrt{2}\right)tD_1}{2} \,\&\&\, 0 < p_1 < 16tD_1 \,\&\&\, t > 2q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

Out[∘]=
$$0 < \delta < 0.442\ldots \,\&\&\, q_o > 0 \,\&\&\, t > 2q_o$$

(*Hence, when $0<\delta<0.442\ldots$, solution 5 satisfies conditions of $\lambda_1>0$, $p_1=\frac{2q_o-tD_1}{3}$, $0<p_1<\frac{(3+2\sqrt{2})tD_1}{2}$, and $0<p_1<16tD_1$*)

(*Solution 6, boundary solution, which is the solution of $p_1=\frac{2q_o-tD_1}{3}$*)

In[∘]:=
$$p_1 = \frac{\left(1+\sqrt{1-\delta}\right) q_o}{\delta};$$
$$\lambda_1 = -\left(\left(3\left(8\left(-3+\sqrt{10-6\sqrt{1-\delta}-9\delta}+\sqrt{1-\delta}\right)+\left(-29+17\sqrt{10-6\sqrt{1-\delta}-9\delta}+31\sqrt{1-\delta}\right)\delta-8\left(-5+\sqrt{10-6\sqrt{1-\delta}-9\delta}\right)\delta^2\right)q_o\right)\Big/\left(4t\left(-3+3\sqrt{10-6\sqrt{1-\delta}-9\delta}+\sqrt{1-\delta}\right)\delta^2\right)\right);$$
$$\lambda_2 = 0;$$
$$\lambda_3 = 0;$$
$$\text{Reduce}\left[\lambda_1 > 0 \,\&\&\, p_1 == \frac{2q_o - tD_1}{3} \,\&\&\, 0 < p_1 < \frac{\left(3+2\sqrt{2}\right)tD_1}{2} \,\&\&\, 0 < p_1 < 16tD_1 \,\&\&\, t > 2q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

Out[∘]=
False

(*Hence, solution 6 does not satisfy conditions of $\lambda_1>0$, $p_1=\frac{2q_o-tD_1}{3}$, $0<p_1<\frac{(3+2\sqrt{2})tD_1}{2}$, and $0<p_1<16tD_1$*)

(*Solution 7, boundary solution, which is the solution of $p_1=16tD_1$*)

In[∘]:=

$$p_1 = -\frac{32\left(-34+\delta+\sqrt{1156-578\,\delta+256\,\delta^2}\right)q_o}{255\,\delta};$$

$$\lambda_1 = 0;$$

$$\lambda_2 = 0;$$

$$\begin{aligned}
\lambda_3 = \Bigg(\Bigg(&8\,487\,936\,\sqrt{2}\,\delta^4 - \\
&8\,\delta^3\left(111\,733\,809\,\sqrt{2} - 6\,234\,624\,\sqrt{578-289\,\delta+128\,\delta^2} + 119\,344\sqrt{\left(148\,546+32\,896\,\delta^2 - \right.}\right.\\
&\quad\left.\left.\overline{544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)}\right) - \\
&8160\left(-152\,881\,\sqrt{2} + 5168\,\sqrt{578-289\,\delta+128\,\delta^2} - 289\sqrt{\left(148\,546+32\,896\,\delta^2 - \right.}\right.\\
&\quad\overline{\left.544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)} + \\
&\quad 16\,\sqrt{2}\,\sqrt{\left(\left(578-289\,\delta+128\,\delta^2\right)\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \right.\right.}\\
&\quad\quad\left.\left.\left.\overline{\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right) - \\
&136\,\delta\left(38\,751\,279\,\sqrt{2} - 2\,441\,592\,\sqrt{578-289\,\delta+128\,\delta^2} + 1232\sqrt{\left(148\,546+32\,896\,\delta^2 - \right.}\right.\\
&\quad\overline{\left.544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)} + \\
&\quad 892\,\sqrt{2}\,\sqrt{\left(\left(578-289\,\delta+128\,\delta^2\right)\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \right.\right.}\\
&\quad\quad\left.\left.\left.\overline{\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right) + \\
&\delta^2\left(2\,869\,482\,513\,\sqrt{2} - 176\,660\,736\,\sqrt{578-289\,\delta+128\,\delta^2} + 997\,696\right.\\
&\quad\sqrt{\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \right.}\\
&\quad\quad\overline{\left.\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)} + 89\,728\,\sqrt{2}\\
&\quad\sqrt{\left(\left(578-289\,\delta+128\,\delta^2\right)\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \right.\right.}\\
&\quad\quad\left.\left.\left.\left.\overline{\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right)\right)q_o\Bigg)\Big/\\
\Bigg(&1020\,t\,\delta^2\left(7\,259\,102\,\sqrt{2} + 1\,546\,112\,\sqrt{2}\,\delta^2 - 312\,256\,\sqrt{578-289\,\delta+128\,\delta^2} - \right.\\
&\quad 578\sqrt{\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \right.}\\
&\quad\quad\overline{\left.\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)} - \\
&\quad 208\,\sqrt{2}\,\sqrt{\left(\left(578-289\,\delta+128\,\delta^2\right)\left(148\,546+32\,896\,\delta^2-544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \right.\right.}\\
&\quad\quad\left.\left.\overline{\delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)} + \\
&\quad\delta\left(-3\,568\,351\,\sqrt{2} + 24\,064\,\sqrt{578-289\,\delta+128\,\delta^2} - 208\sqrt{\left(148\,546+32\,896\,\delta^2 - \right.}\right.\\
&\quad\quad\left.\left.\overline{544\,\sqrt{1156-578\,\delta+256\,\delta^2} + \delta\left(-139\,553+256\,\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)}\right)\right)\Bigg);
\end{aligned}$$

$$\text{Reduce}\left[\lambda_3 > 0\ \&\&\ p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ 0 < p_1 < \frac{\left(3+2\,\sqrt{2}\right)t\,D_1}{2}\ \&\&\ 0 < p_1 == 16\,t\,D_1\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

Out[∘]=

False

(*Hence, solution 7 does not satisfy conditions of λ_3>0, $p_1<\frac{2q_o-tD_1}{3}$, $0<p_1<\frac{(3+2\sqrt{2})tD_1}{2}$, and $0<p_1=16tD_1$*)

```
(*Solution 8, boundary solution, which is the solution of p1=16tD1*)
```

In[•]:=

$$p_1 = \frac{32\left(34-\delta+\sqrt{1156-578\,\delta+256\,\delta^2}\right) q_o}{255\,\delta};$$

$\lambda_1 = 0;$

$\lambda_2 = 0;$

$$\lambda_3 = \Bigg(\Bigg(8\,487\,936\sqrt{2}\,\delta^4 - 8\,\delta^3\left(111\,733\,809\sqrt{2} + 6\,234\,624\sqrt{578-289\,\delta+128\,\delta^2} + 119\,344\sqrt{\left(148\,546-139\,553\,\delta + 32\,896\,\delta^2 + 544\sqrt{1156-578\,\delta+256\,\delta^2} - 256\,\delta\sqrt{1156-578\,\delta+256\,\delta^2}\right)}\right) + \delta^2\left(2\,869\,482\,513\sqrt{2} + 176\,660\,736\sqrt{578-289\,\delta+128\,\delta^2} + 997\,696\sqrt{\left(148\,546-139\,553\,\delta + 32\,896\,\delta^2 + 544\sqrt{1156-578\,\delta+256\,\delta^2} - 256\,\delta\sqrt{1156-578\,\delta+256\,\delta^2}\right)} - 89\,728\sqrt{2}\sqrt{\left(-\left(578-289\,\delta+128\,\delta^2\right)\left(-32\,896\,\delta^2 - 34\left(4369+16\sqrt{1156-578\,\delta+256\,\delta^2}\right) + \delta\left(139\,553+256\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right) + 8160\left(152\,881\sqrt{2} + 5168\sqrt{578-289\,\delta+128\,\delta^2} + 289\sqrt{\left(148\,546-139\,553\,\delta + 32\,896\,\delta^2 + 544\sqrt{1156-578\,\delta+256\,\delta^2} - 256\,\delta\sqrt{1156-578\,\delta+256\,\delta^2}\right)} + 16\sqrt{2}\sqrt{\left(-\left(578-289\,\delta+128\,\delta^2\right)\left(-32\,896\,\delta^2 - 34\left(4369+16\sqrt{1156-578\,\delta+256\,\delta^2}\right) + \delta\left(139\,553+256\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right) + 136\,\delta\left(-38\,751\,279\sqrt{2} - 2\,441\,592\sqrt{578-289\,\delta+128\,\delta^2} - 1232\sqrt{\left(148\,546-139\,553\,\delta + 32\,896\,\delta^2 + 544\sqrt{1156-578\,\delta+256\,\delta^2} - 256\,\delta\sqrt{1156-578\,\delta+256\,\delta^2}\right)} + 892\sqrt{2}\sqrt{\left(-\left(578-289\,\delta+128\,\delta^2\right)\left(-32\,896\,\delta^2 - 34\left(4369+16\sqrt{1156-578\,\delta+256\,\delta^2}\right) + \delta\left(139\,553+256\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)}\right)\Bigg) q_o\Bigg) \Big/ \left(1020\,t\,\delta^2\left(7\,259\,102\sqrt{2} + 1\,546\,112\sqrt{2}\,\delta^2 + 312\,256\sqrt{578-289\,\delta+128\,\delta^2} - 578\sqrt{\left(148\,546-139\,553\,\delta+32\,896\,\delta^2 + 544\sqrt{1156-578\,\delta+256\,\delta^2} - 256\,\delta\sqrt{1156-578\,\delta+256\,\delta^2}\right)} + 208\sqrt{2}\sqrt{\left(-\left(578-289\,\delta+128\,\delta^2\right)\left(-32\,896\,\delta^2 - 34\left(4369+16\sqrt{1156-578\,\delta+256\,\delta^2}\right) + \delta\left(139\,553+256\sqrt{1156-578\,\delta+256\,\delta^2}\right)\right)\right)} - \delta\left(3\,568\,351\sqrt{2} + 24\,064\sqrt{578-289\,\delta+128\,\delta^2} + 208\sqrt{\left(148\,546-139\,553\,\delta + 32\,896\,\delta^2 + 544\sqrt{1156-578\,\delta+256\,\delta^2} - 256\,\delta\sqrt{1156-578\,\delta+256\,\delta^2}\right)}\right)\right)\right);$$

$$\text{Reduce}\left[\lambda_3 > 0 \,\&\&\, p_1 < \frac{2\,q_o - t\,D_1}{3} \,\&\&\, 0 < p_1 < \frac{\left(3+2\sqrt{2}\right)t\,D_1}{2} \,\&\&\, 0 < p_1 == 16\,t\,D_1 \,\&\&\, t > 2\,q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

Out[•]=

```
False
```

(*Hence, solution 8 does not satisfy conditions of $\lambda_3>0$, $p_1<\frac{2q_o-tD_1}{3}$, $0<p_1<\frac{(3+2\sqrt{2})tD_1}{2}$, and $0<p_1=16tD_1$*)

```
(*Given the complexity of solutions 9 and 10, we refine them as follows*)
```

(*Solution 9, boundary solution, which is the solution of $p_1=\frac{(3+2\sqrt{2})tD_1}{2}$*)

In[∘]:=
$$p_1 = -\frac{1}{119\,\delta}2\left(-170-136\sqrt{2}+2\left(9+10\sqrt{2}\right)\delta+\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}\right)q_o;$$

$$\lambda_1 = 0;$$

$$\lambda_2 = \frac{1}{28322\,t\,\delta^2\left(324-224\sqrt{2}+2\left(-81+56\sqrt{2}\right)\delta+\delta^2\right)}\Bigg(-16\left(-2671+708\sqrt{2}\right)\delta^4+1904\Big(-10370+7412\sqrt{2}-467\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}+330\sqrt{2}\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}\Big)+\delta^3\Big(-9598965+6632482\sqrt{2}-8280\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}+6032\sqrt{2}\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}\Big)+34\,\delta\Big(-1170994+807364\sqrt{2}-58277\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}+41262\sqrt{2}\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}\Big)-17\,\delta^2\Big(-2573426+1792004\sqrt{2}-61847\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}+43782\sqrt{2}\sqrt{1156\left(57+40\sqrt{2}\right)-578\left(57+40\sqrt{2}\right)\delta+\left(9097+6432\sqrt{2}\right)\delta^2}\Big)\Bigg)q_o;$$

$$\lambda_3 = 0;$$

$$\text{Reduce}\left[\lambda_2>0\;\&\&\;p_1<\frac{2q_o-tD_1}{3}\;\&\&\;0<p_1==\frac{\left(3+2\sqrt{2}\right)tD_1}{2}\;\&\&\;0<p_1<16\,t\,D_1\;\&\&\;t>2q_o>0\;\&\&\;0<\delta<1\right]$$

Out[∘]=
$$0.864\ldots<\delta<1\;\&\&\;q_o>0\;\&\&\;t>2q_o$$

(*Hence, when $0.864\ldots<\delta<1$, solution 9 satisfies conditions of $\lambda_2>0$, $p_1<\frac{2q_o-tD_1}{3}$, $0<p_1=\frac{(3+2\sqrt{2})tD_1}{2}$, and $0<p_1<16tD_1$*)

(*Solution 10, boundary solution, which is the solution of $p_1=\frac{(3+2\sqrt{2})tD_1}{2}$*

In[∘]:=
$$p_1 = \frac{1}{119\,\delta} 2 \left(170 + 136 \sqrt{2} - 2 \left(9 + 10 \sqrt{2}\right) \delta + \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2}\right) q_o;$$

$$\lambda_1 = 0;$$

$$\lambda_2 = -\frac{1}{28\,322\, t\, \delta^2 \left(324 - 224 \sqrt{2} + 2 \left(-81 + 56 \sqrt{2}\right) \delta + \delta^2\right)} \Bigg(16 \left(-2671 + 708 \sqrt{2}\right) \delta^4 + 1904 \Big(10\,370 - 7412 \sqrt{2} - 467 \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2} + 330 \sqrt{2} \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2}\Big) + \delta^3 \Big(9\,598\,965 - 6\,632\,482 \sqrt{2} - 8280 \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2} + 6032 \sqrt{2} \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2}\Big) + 34\,\delta \Big(1\,170\,994 - 807\,364 \sqrt{2} - 58\,277 \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2} + 41\,262 \sqrt{2} \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2}\Big) - 17\,\delta^2 \Big(2\,573\,426 - 1\,792\,004 \sqrt{2} - 61\,847 \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2} + 43\,782 \sqrt{2} \sqrt{1156 \left(57 + 40 \sqrt{2}\right) - 578 \left(57 + 40 \sqrt{2}\right) \delta + \left(9097 + 6432 \sqrt{2}\right) \delta^2}\Big)\Bigg) q_o;$$

$$\lambda_3 = 0;$$

$$\text{Reduce}\left[\lambda_2 > 0 \,\&\&\, p_1 < \frac{2 q_o - t D_1}{3} \,\&\&\, 0 < p_1 == \frac{\left(3 + 2 \sqrt{2}\right) t D_1}{2} \,\&\&\, 0 < p_1 < 16\, t\, D_1 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

Out[∘]=

False

(*Hence, solution 10 does not satisfy conditions of $\lambda_2>0$, $p_1<\frac{2q_o-tD_1}{3}$, $0<p_1=\frac{(3+2\sqrt{2})tD_1}{2}$, and $0<p_1<16tD_1$*)

(*Overall, when $0<\delta<$ 0.442…, $p_1=-\frac{(-1+\sqrt{1-\delta})\ q_o}{\delta}$; when 0.442…$<\delta<$0.864…, $p_1=P_1^{GL}(q_o,\delta)=$Root[$8\ \delta^4\ \#1^4+144\ q_o^4+225\ \delta\ q_o^4-272\ \delta^2\ q_o^4+64\ \delta^3\ q_o^4+\#1^3\ (-16\ \delta^2\ q_o-94\ \delta^3\ q_o)+\#1^2\ (124\ \delta\ q_o^2+376\ \delta^2\ q_o^2-30\ \delta^3\ q_o^2-8\ \delta^4\ q_o^2)+\#1\ (-224\ q_o^3-480\ \delta\ q_o^3+104\ \delta^2\ q_o^3+64\ \delta^3\ q_o^3)$&,1]

(i.e., solution 1 as defined before); when 0.864…$<\delta<1$,

$p_1=-\frac{2\ (-170-136\ \sqrt{2}+2\ (9+10\ \sqrt{2})\ \delta+\sqrt{1156\ (57+40\ \sqrt{2})-578\ (57+40\ \sqrt{2})\ \delta+(9097+6432\ \sqrt{2})\ \delta^2})\ q_o}{119\ \delta}$ *)

(*Combination 2. There is no intersection between conditions of $0<p_1<\frac{2q_o-tD_1}{3}$, $p_1\le\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1>16tD_1$*)

(*Combination 3. The conditions are $0<p_1<\frac{2q_o-tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $0<p_1\le16tD_1$*)

In[∘]:=
$$p_{2P} = \frac{2 q_o + p_1 - t D_1}{4};$$
$$p_{2M} = \frac{2 p_1 + t D_1}{4};$$
$$p_{2N} = p_1;$$
$$D_{2P} = \frac{2 q_o + p_1 - t D_1}{4 t};$$
$$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t};$$
$$D_{2N} = 0;$$

In[∘]:=
$$U_1 = q_o - p_1 - t D_1 + \delta \frac{t D_1}{2 q_o} (p_1 - p_{2M});$$

(*Consumers' expected utility purchasing in the first period*)

$$U_2 = \delta \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right)\right);$$

(*Consumers' expected utility purchasing in the second period*)

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1], $p_1 > 0$ && $t > 2 q_o > 0$ && $0 < \delta < 1$]

Out[∘]=
$$\left\{\left\{D_1 \to -\frac{2 (-2+\delta) q_o + \sqrt{-\delta^2 p_1^2 + 8 \delta p_1 q_o + 8 \left(2 - 3 \delta + \delta^2\right) q_o^2}}{t \delta}\right\}, \left\{D_1 \to \frac{-2 (-2+\delta) q_o + \sqrt{-\delta^2 p_1^2 + 8 \delta p_1 q_o + 8 \left(2 - 3 \delta + \delta^2\right) q_o^2}}{t \delta}\right\}\right\}$$

(*There are two solutions of D_1,
we then check each solution if it satisfies conditions*)

In[∘]:=
$$D_1 = -\frac{2 (-2+\delta) q_o + \sqrt{-\delta^2 p_1^2 + 8 \delta p_1 q_o + 8 \left(2 - 3 \delta + \delta^2\right) q_o^2}}{t \delta};$$

Reduce[
$0 < p_1 < \frac{2 q_o - t D_1}{3}$ && $p_1 > \frac{\left(3 + 2\sqrt{2}\right) t D_1}{2}$ && $0 < p_1 \le 16 t D_1$ && $D_1 > 0$ && $t > 2 q_o > 0$ && $0 < \delta < 1$]

Out[∘]=

$$p_1 > 0 \;\&\&\; \left(\left(\frac{49\,p_1}{32} < q_o \le \text{Root}\left[\left(68 + 48\sqrt{2}\right)\#1^3 - 272\,p_1^3 - 184\sqrt{2}\,p_1^3 + \#1^2\left(-432\,p_1 - 304\sqrt{2}\,p_1\right) + \#1\left(695\,p_1^2 + 484\sqrt{2}\,p_1^2\right)\&, 2\right] \;\&\&\; t > 2\,q_o \;\&\&\; \frac{-8\,p_1\,q_o + 4\,q_o^2}{5\,p_1^2 - 12\,p_1\,q_o + 4\,q_o^2} < \delta \le \frac{-2176\,p_1\,q_o + 2048\,q_o^2}{-257\,p_1^2 - 64\,p_1\,q_o + 1024\,q_o^2}\right) \;||\right.$$

$$\left(\text{Root}\left[\left(68 + 48\sqrt{2}\right)\#1^3 - 272\,p_1^3 - 184\sqrt{2}\,p_1^3 + \#1^2\left(-432\,p_1 - 304\sqrt{2}\,p_1\right) + \#1\left(695\,p_1^2 + 484\sqrt{2}\,p_1^2\right)\&, 2\right] < q_o < \frac{33\,p_1}{32} + \frac{1}{4}\sqrt{13}\sqrt{p_1^2} \;\&\&\; t > 2\,q_o \;\&\&\; \frac{-184\,p_1\,q_o - 128\sqrt{2}\,p_1\,q_o + 136\,q_o^2 + 96\sqrt{2}\,q_o^2}{-21\,p_1^2 - 12\sqrt{2}\,p_1^2 - 24\,p_1\,q_o - 16\sqrt{2}\,p_1\,q_o + 68\,q_o^2 + 48\sqrt{2}\,q_o^2} < \delta \le \frac{-2176\,p_1\,q_o + 2048\,q_o^2}{-257\,p_1^2 - 64\,p_1\,q_o + 1024\,q_o^2}\right) \;||$$

$$\left(q_o == \frac{33\,p_1}{32} + \frac{1}{4}\sqrt{13}\sqrt{p_1^2} \;\&\&\; t > 2\,q_o \;\&\&\; \frac{-184\,p_1\,q_o - 128\sqrt{2}\,p_1\,q_o + 136\,q_o^2 + 96\sqrt{2}\,q_o^2}{-21\,p_1^2 - 12\sqrt{2}\,p_1^2 - 24\,p_1\,q_o - 16\sqrt{2}\,p_1\,q_o + 68\,q_o^2 + 48\sqrt{2}\,q_o^2} < \delta < \frac{-2176\,p_1\,q_o + 2048\,q_o^2}{-257\,p_1^2 - 64\,p_1\,q_o + 1024\,q_o^2}\right) \;||$$

$$\left.\left(\frac{33\,p_1}{32} + \frac{1}{4}\sqrt{13}\sqrt{p_1^2} < q_o < \frac{2\left(10\,p_1 + 7\sqrt{2}\,p_1\right)}{17 + 12\sqrt{2}} + \frac{\sqrt{2523\,p_1^2 + 1784\sqrt{2}\,p_1^2}}{2\left(17 + 12\sqrt{2}\right)} \;\&\&\; t > 2\,q_o \;\&\&\; \frac{-184\,p_1\,q_o - 128\sqrt{2}\,p_1\,q_o + 136\,q_o^2 + 96\sqrt{2}\,q_o^2}{-21\,p_1^2 - 12\sqrt{2}\,p_1^2 - 24\,p_1\,q_o - 16\sqrt{2}\,p_1\,q_o + 68\,q_o^2 + 48\sqrt{2}\,q_o^2} < \delta < 1\right)\right)$$

(*Hence, the first solution satisfies conditions*)

In[∘]:=

$$D_1 = \frac{-2\,(-2 + \delta)\,q_o + \sqrt{-\delta^2\,p_1^2 + 8\,\delta\,p_1\,q_o + 8\left(2 - 3\,\delta + \delta^2\right)q_o^2}}{t\,\delta};$$

$$\text{Reduce}\left[0 < p_1 < \frac{2\,q_o - t\,D_1}{3} \;\&\&\; p_1 > \frac{\left(3 + 2\sqrt{2}\right)t\,D_1}{2} \;\&\&\; 0 < p_1 \le 16\,t\,D_1 \;\&\&\; D_1 > 0 \;\&\&\; t > 2\,q_o > 0 \;\&\&\; 0 < \delta < 1\right]$$

Out[∘]=

False

(*Hence, the second solution does not satisfy conditions*)

In[∘]:=

$$D_1 = -\frac{2\,(-2 + \delta)\,q_o + \sqrt{-\delta^2\,p_1^2 + 8\,\delta\,p_1\,q_o + 8\left(2 - 3\,\delta + \delta^2\right)q_o^2}}{t\,\delta};$$

(*The first-period demand function*)

$$\Pi = \text{Simplify}\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P} + \frac{t\,D_1}{2\,q_o}\left(p_{2M}\,D_{2M} - D_1\,(p_1 - p_{2M})\right)\right];$$

```
Reduce[D[D[Π, p1], p1] ≥ 0 && 0 < p1 < (2 qo - t D1)/3 && p1 > ((3 + 2 √2) t D1)/2 &&
  0 < p1 ≤ 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1] (*Determine the sign of ∂²Π/∂p1² *)
```

Out[ ]=

False

```
(*∂²Π/∂p1²<0, meaning Π is concave and it has a maximum value at point where ∂Π/∂p1 =0*)

(*KKT conditions*)

g1 = (2 qo - t D1)/3 - p1;
g2 = p1 - ((3 + 2 √2) t D1)/2;
g3 = 16 t D1 - p1;
L = -Π - λ1 g1 - λ2 g2 - λ3 g3;

Simplify[D[L, p1], p1 > 0 && qo > 0 && 0 < δ < 1] (*Calculate the first order of L for p1*)
```

Out[ ]=

$$\frac{1}{96\, t\, \delta^2\, q_o \sqrt{-\delta^2 p_1^2 + 8\,\delta\, p_1 q_o + 8\left(2 - 3\,\delta + \delta^2\right) q_o^2}} \Bigg(-45\,\delta^3 p_1^3 - 18\,\delta^2 p_1^2 \left((-46 + 24\,\delta)\, q_o + 5\sqrt{-\delta^2 p_1^2 + 8\,\delta\, p_1 q_o + 8\left(2 - 3\,\delta + \delta^2\right) q_o^2}\right) + 4\,\delta\, p_1 q_o \Big(\left(-672 + 432\,\delta + 87\,\delta^2\right) q_o + 102\sqrt{-\delta^2 p_1^2 + 8\,\delta\, p_1 q_o + 8\left(2 - 3\,\delta + \delta^2\right) q_o^2} + 27\,\delta\sqrt{-\delta^2 p_1^2 + 8\,\delta\, p_1 q_o + 8\left(2 - 3\,\delta + \delta^2\right) q_o^2} + 8\, t\, \delta^2 \lambda_1 + 36\, t\, \delta^2 \lambda_2 + 24\sqrt{2}\, t\, \delta^2 \lambda_2 - 384\, t\, \delta^2 \lambda_3\Big) + 8\, q_o \Big(6\left(-88 + 216\,\delta - 161\,\delta^2 + 36\,\delta^3\right) q_o^2 + 12\, t\, \delta^2 \sqrt{-\delta^2 p_1^2 + 8\,\delta\, p_1 q_o + 8\left(2 - 3\,\delta + \delta^2\right) q_o^2}\, (\lambda_1 - \lambda_2 + \lambda_3) + q_o \Big(132\sqrt{-\delta^2 p_1^2 + 8\,\delta\, p_1 q_o + 8\left(2 - 3\,\delta + \delta^2\right) q_o^2} - 225\,\delta \sqrt{-\delta^2 p_1^2 + 8\,\delta\, p_1 q_o + 8\left(2 - 3\,\delta + \delta^2\right) q_o^2} + 69\,\delta^2 \sqrt{-\delta^2 p_1^2 + 8\,\delta\, p_1 q_o + 8\left(2 - 3\,\delta + \delta^2\right) q_o^2} - 16\, t\, \delta^2 \lambda_1 - 72\, t\, \delta^2 \lambda_2 - 48\sqrt{2}\, t\, \delta^2 \lambda_2 + 768\, t\, \delta^2 \lambda_3\Big)\Big)\Bigg)$$

```
(*We only take the molecular part of the results*)
```

```
In[]:= firstorderL = -45 δ^3 p1^3 - 18 δ^2 p1^2 ((-46 + 24 δ) qo + 5 Sqrt[-δ^2 p1^2 + 8 δ p1 qo + 8 (2 - 3 δ + δ^2) qo^2]) +
   4 δ p1 qo ((-672 + 432 δ + 87 δ^2) qo + 102 Sqrt[-δ^2 p1^2 + 8 δ p1 qo + 8 (2 - 3 δ + δ^2) qo^2] +
      27 δ Sqrt[-δ^2 p1^2 + 8 δ p1 qo + 8 (2 - 3 δ + δ^2) qo^2] +
      8 t δ^2 λ1 + 36 t δ^2 λ2 + 24 Sqrt[2] t δ^2 λ2 - 384 t δ^2 λ3) +
   8 qo (6 (-88 + 216 δ - 161 δ^2 + 36 δ^3) qo^2 + 12 t δ^2 Sqrt[-δ^2 p1^2 + 8 δ p1 qo + 8 (2 - 3 δ + δ^2) qo^2]
      (λ1 - λ2 + λ3) + qo (132 Sqrt[-δ^2 p1^2 + 8 δ p1 qo + 8 (2 - 3 δ + δ^2) qo^2] -
         225 δ Sqrt[-δ^2 p1^2 + 8 δ p1 qo + 8 (2 - 3 δ + δ^2) qo^2] +
         69 δ^2 Sqrt[-δ^2 p1^2 + 8 δ p1 qo + 8 (2 - 3 δ + δ^2) qo^2] -
         16 t δ^2 λ1 - 72 t δ^2 λ2 - 48 Sqrt[2] t δ^2 λ2 + 768 t δ^2 λ3));
Simplify[Solve[{firstorderL == 0, λ1 g1 == 0, λ2 g2 == 0, λ3 g3 == 0}, {p1, λ1, λ2, λ3}],
  p1 > 0 && t > 2 qo > 0 && 0 < δ < 1] (*Solve the KKT equations*)
Out[]=
```

$$\Bigg\{\Bigg\{p_1 \to \frac{2\left(-2+3\delta+\sqrt{4-7\delta+4\delta^2}\right) q_o}{5\delta},\ \lambda_1 \to -\frac{1}{400\, t\, \delta^2 \left(4-7\delta+4\delta^2\right)} 3\Big(1024\delta^4+\delta\left(5676-1816\sqrt{4-7\delta+4\delta^2}\right)+1168\left(-2+\sqrt{4-7\delta+4\delta^2}\right)+4\delta^3\left(-51+172\sqrt{4-7\delta+4\delta^2}\right)+\delta^2\left(-4091+179\sqrt{4-7\delta+4\delta^2}\right)\Big) q_o,\ \lambda_2 \to 0,\ \lambda_3 \to 0\Bigg\},$$

$$\Bigg\{p_1 \to -\frac{2\left(2-3\delta+\sqrt{4-7\delta+4\delta^2}\right) q_o}{5\delta},\ \lambda_1 \to \frac{1}{400\, t\, \delta^2 \left(4-7\delta+4\delta^2\right)} 3\Big(-1024\delta^4+1168\left(2+\sqrt{4-7\delta+4\delta^2}\right)+4\delta^3\left(51+172\sqrt{4-7\delta+4\delta^2}\right)+\delta^2\left(4091+179\sqrt{4-7\delta+4\delta^2}\right)-4\delta\left(1419+454\sqrt{4-7\delta+4\delta^2}\right)\Big) q_o,\ \lambda_2 \to 0,\ \lambda_3 \to 0\Bigg\},$$

$$\Big\{p_1 \to \mathrm{Root}\Big[1125\,\#1^6+60928\,q_o^6+\frac{236544\,q_o^6}{\delta^4}-\frac{766464\,q_o^6}{\delta^3}+\frac{859904\,q_o^6}{\delta^2}-\frac{388608\,q_o^6}{\delta}+\#1^5\left(2160\,q_o-\frac{23640\,q_o}{\delta}\right)+\#1^4\left(312\,q_o^2+\frac{151312\,q_o^2}{\delta^2}-\frac{12096\,q_o^2}{\delta}\right)+\#1^3\left(-20160\,q_o^3-\frac{205056\,q_o^3}{\delta^3}-\frac{193152\,q_o^3}{\delta^2}+\frac{73664\,q_o^3}{\delta}\right)+\#1^2\left(-40624\,q_o^4-\frac{574464\,q_o^4}{\delta^4}+\frac{1165824\,q_o^4}{\delta^3}-\frac{582720\,q_o^4}{\delta^2}+\frac{266496\,q_o^4}{\delta}\right)+\#1\left(27648\,q_o^5+\frac{67584\,q_o^5}{\delta^4}+\frac{170496\,q_o^5}{\delta^3}-\frac{376320\,q_o^5}{\delta^2}+\frac{58240\,q_o^5}{\delta}\right)\ \&,\ 1\Big],\ \lambda_1 \to 0,\ \lambda_2 \to 0,\ \lambda_3 \to 0\Big\},$$

$$\Big\{p_1 \to \mathrm{Root}\Big[1125\,\#1^6+60928\,q_o^6+\frac{236544\,q_o^6}{\delta^4}-\frac{766464\,q_o^6}{\delta^3}+\frac{859904\,q_o^6}{\delta^2}-\frac{388608\,q_o^6}{\delta}+\#1^5\left(2160\,q_o-\frac{23640\,q_o}{\delta}\right)+\#1^4\left(312\,q_o^2+\frac{151312\,q_o^2}{\delta^2}-\frac{12096\,q_o^2}{\delta}\right)+\#1^3\left(-20160\,q_o^3-\frac{205056\,q_o^3}{\delta^3}-\frac{193152\,q_o^3}{\delta^2}+\frac{73664\,q_o^3}{\delta}\right)+\#1^2\left(-40624\,q_o^4-\frac{574464\,q_o^4}{\delta^4}+\frac{1165824\,q_o^4}{\delta^3}-\frac{582720\,q_o^4}{\delta^2}+\frac{266496\,q_o^4}{\delta}\right)+$$

$\#1\left(27648\,q_o^5+\frac{67584\,q_o^5}{\delta^4}+\frac{170496\,q_o^5}{\delta^3}-\frac{376320\,q_o^5}{\delta^2}+\frac{58240\,q_o^5}{\delta}\right)\&,2\Big]$, $\lambda_1\to 0$, $\lambda_2\to 0$,

$\lambda_3\to 0\Big\}$, $\Big\{p_1\to\mathrm{Root}\Big[1125\,\#1^6+60928\,q_o^6+\frac{236544\,q_o^6}{\delta^4}-\frac{766464\,q_o^6}{\delta^3}+\frac{859904\,q_o^6}{\delta^2}-$

$\frac{388608\,q_o^6}{\delta}+\#1^5\left(2160\,q_o-\frac{23640\,q_o}{\delta}\right)+\#1^4\left(312\,q_o^2+\frac{151312\,q_o^2}{\delta^2}-\frac{12096\,q_o^2}{\delta}\right)+$

$\#1^3\left(-20160\,q_o^3-\frac{205056\,q_o^3}{\delta^3}-\frac{193152\,q_o^3}{\delta^2}+\frac{73664\,q_o^3}{\delta}\right)+$

$\#1^2\left(-40624\,q_o^4-\frac{574464\,q_o^4}{\delta^4}+\frac{1165824\,q_o^4}{\delta^3}-\frac{582720\,q_o^4}{\delta^2}+\frac{266496\,q_o^4}{\delta}\right)+$

$\#1\left(27648\,q_o^5+\frac{67584\,q_o^5}{\delta^4}+\frac{170496\,q_o^5}{\delta^3}-\frac{376320\,q_o^5}{\delta^2}+\frac{58240\,q_o^5}{\delta}\right)\&,3\Big]$, $\lambda_1\to 0$, $\lambda_2\to 0$,

$\lambda_3\to 0\Big\}$, $\Big\{p_1\to\mathrm{Root}\Big[1125\,\#1^6+60928\,q_o^6+\frac{236544\,q_o^6}{\delta^4}-\frac{766464\,q_o^6}{\delta^3}+\frac{859904\,q_o^6}{\delta^2}-$

$\frac{388608\,q_o^6}{\delta}+\#1^5\left(2160\,q_o-\frac{23640\,q_o}{\delta}\right)+\#1^4\left(312\,q_o^2+\frac{151312\,q_o^2}{\delta^2}-\frac{12096\,q_o^2}{\delta}\right)+$

$\#1^3\left(-20160\,q_o^3-\frac{205056\,q_o^3}{\delta^3}-\frac{193152\,q_o^3}{\delta^2}+\frac{73664\,q_o^3}{\delta}\right)+$

$\#1^2\left(-40624\,q_o^4-\frac{574464\,q_o^4}{\delta^4}+\frac{1165824\,q_o^4}{\delta^3}-\frac{582720\,q_o^4}{\delta^2}+\frac{266496\,q_o^4}{\delta}\right)+$

$\#1\left(27648\,q_o^5+\frac{67584\,q_o^5}{\delta^4}+\frac{170496\,q_o^5}{\delta^3}-\frac{376320\,q_o^5}{\delta^2}+\frac{58240\,q_o^5}{\delta}\right)\&,4\Big]$, $\lambda_1\to 0$, $\lambda_2\to 0$,

$\lambda_3\to 0\Big\}$, $\Big\{p_1\to\mathrm{Root}\Big[1125\,\#1^6+60928\,q_o^6+\frac{236544\,q_o^6}{\delta^4}-\frac{766464\,q_o^6}{\delta^3}+\frac{859904\,q_o^6}{\delta^2}-$

$\frac{388608\,q_o^6}{\delta}+\#1^5\left(2160\,q_o-\frac{23640\,q_o}{\delta}\right)+\#1^4\left(312\,q_o^2+\frac{151312\,q_o^2}{\delta^2}-\frac{12096\,q_o^2}{\delta}\right)+$

$\#1^3\left(-20160\,q_o^3-\frac{205056\,q_o^3}{\delta^3}-\frac{193152\,q_o^3}{\delta^2}+\frac{73664\,q_o^3}{\delta}\right)+$

$\#1^2\left(-40624\,q_o^4-\frac{574464\,q_o^4}{\delta^4}+\frac{1165824\,q_o^4}{\delta^3}-\frac{582720\,q_o^4}{\delta^2}+\frac{266496\,q_o^4}{\delta}\right)+$

$\#1\left(27648\,q_o^5+\frac{67584\,q_o^5}{\delta^4}+\frac{170496\,q_o^5}{\delta^3}-\frac{376320\,q_o^5}{\delta^2}+\frac{58240\,q_o^5}{\delta}\right)\&,5\Big]$, $\lambda_1\to 0$, $\lambda_2\to 0$,

$\lambda_3\to 0\Big\}$, $\Big\{p_1\to\mathrm{Root}\Big[1125\,\#1^6+60928\,q_o^6+\frac{236544\,q_o^6}{\delta^4}-\frac{766464\,q_o^6}{\delta^3}+\frac{859904\,q_o^6}{\delta^2}-$

$\frac{388608\,q_o^6}{\delta}+\#1^5\left(2160\,q_o-\frac{23640\,q_o}{\delta}\right)+\#1^4\left(312\,q_o^2+\frac{151312\,q_o^2}{\delta^2}-\frac{12096\,q_o^2}{\delta}\right)+$

$\#1^3\left(-20160\,q_o^3-\frac{205056\,q_o^3}{\delta^3}-\frac{193152\,q_o^3}{\delta^2}+\frac{73664\,q_o^3}{\delta}\right)+$

$\#1^2\left(-40624\,q_o^4-\frac{574464\,q_o^4}{\delta^4}+\frac{1165824\,q_o^4}{\delta^3}-\frac{582720\,q_o^4}{\delta^2}+\frac{266496\,q_o^4}{\delta}\right)+$

$$\sharp 1 \left(27\,648\, q_o^5 + \frac{67\,584\, q_o^5}{\delta^4} + \frac{170\,496\, q_o^5}{\delta^3} - \frac{376\,320\, q_o^5}{\delta^2} + \frac{58\,240\, q_o^5}{\delta}\right) \&,\ 6\Big],$$

$$\lambda_1 \to 0,\ \lambda_2 \to 0,\ \lambda_3 \to 0\Big\},\ \Big\{p_1 \to -\frac{1}{51\,\delta} 2 \Big(-66 - 40\sqrt{2} + 2\left(5 + 2\sqrt{2}\right)\delta +$$
$$\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big)\, q_o,$$

$$\lambda_1 \to 0,\ \lambda_2 \to -\frac{1}{88\,434\, t\, \delta^2 \left(1380 + 416\sqrt{2} - 2\left(545 + 8\sqrt{2}\right)\delta + 337\,\delta^2\right)}$$
$$\Big(2696\left(-23\,673 + 22\,382\sqrt{2}\right)\delta^4 + \delta^2 \Big(-1\,105\,071\,510 + 1\,074\,027\,996\sqrt{2} +$$
$$68\,063\,987\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} -$$
$$49\,651\,982\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big) +$$
$$16\Big(-23\,547\,734 + 27\,232\,444\sqrt{2} + 3\,229\,019$$
$$\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} -$$
$$2\,369\,594\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big) +$$
$$\delta^3\Big(455\,454\,255 - 425\,220\,206\sqrt{2} - 13\,397\,152$$
$$\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} +$$
$$9\,798\,584\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big) +$$
$$2\,\delta\Big(534\,736\,262 - 580\,570\,684\sqrt{2} - 53\,102\,861$$
$$\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} +$$
$$39\,005\,958\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big)\Big)\, q_o,$$

$$\lambda_3 \to 0\Big\},\ \Big\{p_1 \to \frac{1}{51\,\delta} 2 \Big(66 + 40\sqrt{2} - 2\left(5 + 2\sqrt{2}\right)\delta +$$
$$\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big)\, q_o,$$

$$\lambda_1 \to 0,\ \lambda_2 \to \frac{1}{88\,434\, t\, \delta^2 \left(1380 + 416\sqrt{2} - 2\left(545 + 8\sqrt{2}\right)\delta + 337\,\delta^2\right)}$$
$$\Big(-2696\left(-23\,673 + 22\,382\sqrt{2}\right)\delta^4 + \delta^2 \Big(1\,105\,071\,510 - 1\,074\,027\,996\sqrt{2} +$$
$$68\,063\,987\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} -$$
$$49\,651\,982\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big) +$$
$$16\Big(23\,547\,734 - 27\,232\,444\sqrt{2} + 3\,229\,019$$
$$\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} -$$
$$2\,369\,594\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big) +$$
$$\delta^3\Big(-455\,454\,255 + 425\,220\,206\sqrt{2} - 13\,397\,152$$
$$\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} +$$
$$9\,798\,584\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big) +$$
$$2\,\delta\Big(-534\,736\,262 + 580\,570\,684\sqrt{2} - 53\,102\,861$$
$$\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} +$$
$$39\,005\,958\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Big)\Big)\, q_o,$$

$$\lambda_3 \to 0\}, \left\{p_1 \to \frac{32\left(34-\delta+\sqrt{1156-582\,\delta+258\,\delta^2}\right) q_o}{257\,\delta},\right.$$

$$\lambda_1 \to 0,\ \lambda_2 \to 0,$$

$$\lambda_3 \to -\frac{1}{135\,796\,744\,t\,\delta^2\left(578-291\,\delta+129\,\delta^2\right)}\left(-29\,614\,530\,\delta^4+296\,627\,288\left(34+\sqrt{1156-582\,\delta+258\,\delta^2}\right)+204\,\delta\left(66\,107\,713+2\,310\,374\sqrt{1156-582\,\delta+258\,\delta^2}\right)+12\,\delta^3\left(350\,823\,124+10\,851\,603\sqrt{1156-582\,\delta+258\,\delta^2}\right)-\delta^2\left(7\,227\,814\,826+196\,273\,391\sqrt{1156-582\,\delta+258\,\delta^2}\right)\right) q_o\},$$

$$\left\{p_1 \to -\frac{32\left(-34+\delta+\sqrt{1156-582\,\delta+258\,\delta^2}\right) q_o}{257\,\delta},\right.$$

$$\lambda_1 \to 0,\ \lambda_2 \to 0,$$

$$\lambda_3 \to \frac{1}{135\,796\,744\,t\,\delta^2\left(578-291\,\delta+129\,\delta^2\right)}\left(29\,614\,530\,\delta^4+\delta^2\left(7\,227\,814\,826-196\,273\,391\sqrt{1156-582\,\delta+258\,\delta^2}\right)+296\,627\,288\left(-34+\sqrt{1156-582\,\delta+258\,\delta^2}\right)+204\,\delta\left(-66\,107\,713+2\,310\,374\sqrt{1156-582\,\delta+258\,\delta^2}\right)+12\,\delta^3\left(-350\,823\,124+10\,851\,603\sqrt{1156-582\,\delta+258\,\delta^2}\right)\right) q_o\}\}$$

(*Hence, there are 12 solutions. We then check each solution if it satisfies conditions*)

(*Solution 1, boundary solution, which is the solution of $p_1 = \frac{2q_o - t\ D_1}{3}$*)

In[∘]:=

$$p_1 = \frac{2\left(-2+3\,\delta+\sqrt{4-7\,\delta+4\,\delta^2}\right) q_o}{5\,\delta};$$

$$\lambda_1 = -\frac{1}{400\,t\,\delta^2\left(4-7\,\delta+4\,\delta^2\right)} 3\left(1024\,\delta^4+\delta\left(5676-1816\sqrt{4-7\,\delta+4\,\delta^2}\right)+1168\left(-2+\sqrt{4-7\,\delta+4\,\delta^2}\right)+4\,\delta^3\left(-51+172\sqrt{4-7\,\delta+4\,\delta^2}\right)+\delta^2\left(-4091+179\sqrt{4-7\,\delta+4\,\delta^2}\right)\right) q_o;$$

$$\lambda_2 = 0;$$

$$\lambda_3 = 0;$$

$$\text{Reduce}\left[\lambda_1 > 0\ \&\&\ p_1 == \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3+2\sqrt{2}\right) t\,D_1}{2}\ \&\&\ 0 < p_1 < 16\,t\,D_1\ \&\&\ 0 < D_1 < 1\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

Out[∘]=

False

```
(*Hence, solution 1 does not satisfy conditions of p1=(2qo-tD1)/3 &&p1>((3+2√2)tD1)/2 && 0<p1≤16tD1*)

(*Solution 2, boundary solution, which is the solution of p1=(2qo-t D1)/3 *)
```

In[◦]:=

$$p_1 = -\frac{2\left(2-3\,\delta+\sqrt{4-7\,\delta+4\,\delta^2}\right) q_o}{5\,\delta};$$

$$\lambda_1 = \frac{1}{400\,t\,\delta^2\left(4-7\,\delta+4\,\delta^2\right)} 3\left(-1024\,\delta^4+1168\left(2+\sqrt{4-7\,\delta+4\,\delta^2}\right)+4\,\delta^3\left(51+172\sqrt{4-7\,\delta+4\,\delta^2}\right)+\delta^2\left(4091+179\sqrt{4-7\,\delta+4\,\delta^2}\right)-4\,\delta\left(1419+454\sqrt{4-7\,\delta+4\,\delta^2}\right)\right) q_o;$$

$$\lambda_2 = 0;$$

$$\lambda_3 = 0;$$

$$\text{Reduce}\left[\lambda_1 > 0\,\&\&\,p_1 == \frac{2\,q_o - t\,D_1}{3}\,\&\&\,p_1 > \frac{\left(3+2\sqrt{2}\right) t\,D_1}{2}\,\&\&\,0 < p_1 < 16\,t\,D_1\,\&\&\,0 < D_1 < 1\,\&\&\,t > 2\,q_o > 0\,\&\&\,0 < \delta < 1\right]$$

Out[◦]=

```
False
```

```
(*Hence, solution 2 does not satisfy conditions of p1=(2qo-tD1)/3 &&p1>((3+2√2)tD1)/2 && 0<p1≤16tD1*)

(*Solution 3, interior solution*)
```

In[◦]:=

$$p_1 = \text{Root}\Big[1125\,\#1^6 + 60928\,q_o^6 + \frac{236544\,q_o^6}{\delta^4} - \frac{766464\,q_o^6}{\delta^3} + \frac{859904\,q_o^6}{\delta^2} - \frac{388608\,q_o^6}{\delta} + \#1^5\left(2160\,q_o - \frac{23640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151312\,q_o^2}{\delta^2} - \frac{12096\,q_o^2}{\delta}\right) + \#1^3\left(-20160\,q_o^3 - \frac{205056\,q_o^3}{\delta^3} - \frac{193152\,q_o^3}{\delta^2} + \frac{73664\,q_o^3}{\delta}\right) + \#1^2\left(-40624\,q_o^4 - \frac{574464\,q_o^4}{\delta^4} + \frac{1165824\,q_o^4}{\delta^3} - \frac{582720\,q_o^4}{\delta^2} + \frac{266496\,q_o^4}{\delta}\right) + \#1\left(27648\,q_o^5 + \frac{67584\,q_o^5}{\delta^4} + \frac{170496\,q_o^5}{\delta^3} - \frac{376320\,q_o^5}{\delta^2} + \frac{58240\,q_o^5}{\delta}\right)\&,\,1\Big];$$

$$\lambda_1 = 0;$$

$$\lambda_2 = 0;$$

$$\lambda_3 = 0;$$

$$\text{Reduce}\left[p_1 < \frac{2\,q_o - t\,D_1}{3}\,\&\&\,p_1 > \frac{\left(3+2\sqrt{2}\right) t\,D_1}{2}\,\&\&\,p_1 < 16\,t\,D_1\,\&\&\,D_1 > 0\,\&\&\,t > 2\,q_o > 0\,\&\&\,0 < \delta < 1\right]$$

Out[◦]=

```
False
```

```
(*Hence, solution 3 does not satisfy conditions of p1<(2qo-tD1)/3,
p1>((3+2√2)tD1)/2, and p1<16tD1*)
```

```
(*Solution 4, interior solution*)
```

In[∘]:= $p_1 = \text{Root}\Big[1125\,\#1^6 + 60\,928\,q_o^6 + \frac{236\,544\,q_o^6}{\delta^4} - \frac{766\,464\,q_o^6}{\delta^3} + \frac{859\,904\,q_o^6}{\delta^2} - \frac{388\,608\,q_o^6}{\delta} + \#1^5\left(2160\,q_o - \frac{23\,640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151\,312\,q_o^2}{\delta^2} - \frac{12\,096\,q_o^2}{\delta}\right) + \#1^3\left(-20\,160\,q_o^3 - \frac{205\,056\,q_o^3}{\delta^3} - \frac{193\,152\,q_o^3}{\delta^2} + \frac{73\,664\,q_o^3}{\delta}\right) + \#1^2\left(-40\,624\,q_o^4 - \frac{574\,464\,q_o^4}{\delta^4} + \frac{1\,165\,824\,q_o^4}{\delta^3} - \frac{582\,720\,q_o^4}{\delta^2} + \frac{266\,496\,q_o^4}{\delta}\right) + \#1\left(27\,648\,q_o^5 + \frac{67\,584\,q_o^5}{\delta^4} + \frac{170\,496\,q_o^5}{\delta^3} - \frac{376\,320\,q_o^5}{\delta^2} + \frac{58\,240\,q_o^5}{\delta}\right)\&,\,2\Big];$

$\lambda_1 = 0;$
$\lambda_2 = 0;$
$\lambda_3 = 0;$

$\text{Reduce}\left[p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3+2\sqrt{2}\right)t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[∘]=

False

(*Hence, solution 4 does not satisfy conditions of $p_1 < \frac{2q_o - tD_1}{3}$, $p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 < 16tD_1$*)

```
(*Solution 5, interior solution*)
```

In[∘]:= $p_1 = \text{Root}\Big[1125\,\#1^6 + 60\,928\,q_o^6 + \frac{236\,544\,q_o^6}{\delta^4} - \frac{766\,464\,q_o^6}{\delta^3} + \frac{859\,904\,q_o^6}{\delta^2} - \frac{388\,608\,q_o^6}{\delta} + \#1^5\left(2160\,q_o - \frac{23\,640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151\,312\,q_o^2}{\delta^2} - \frac{12\,096\,q_o^2}{\delta}\right) + \#1^3\left(-20\,160\,q_o^3 - \frac{205\,056\,q_o^3}{\delta^3} - \frac{193\,152\,q_o^3}{\delta^2} + \frac{73\,664\,q_o^3}{\delta}\right) + \#1^2\left(-40\,624\,q_o^4 - \frac{574\,464\,q_o^4}{\delta^4} + \frac{1\,165\,824\,q_o^4}{\delta^3} - \frac{582\,720\,q_o^4}{\delta^2} + \frac{266\,496\,q_o^4}{\delta}\right) + \#1\left(27\,648\,q_o^5 + \frac{67\,584\,q_o^5}{\delta^4} + \frac{170\,496\,q_o^5}{\delta^3} - \frac{376\,320\,q_o^5}{\delta^2} + \frac{58\,240\,q_o^5}{\delta}\right)\&,\,3\Big];$

$\lambda_1 = 0;$
$\lambda_2 = 0;$
$\lambda_3 = 0;$

$\text{Reduce}\left[p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3+2\sqrt{2}\right)t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[∘]=

$0.777\ldots < \delta < 1\ \&\&\ q_o > 0\ \&\&\ t > 2\,q_o$

(*Hence, when $0.777\ldots < \delta < 1$,

solution 5 satisfies conditions of $p_1 < \frac{2q_o - tD_1}{3}$, $p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 < 16tD_1$*)

```
(*For convenience, we define solution 5 as P2GL(qo,δ)*)

(*Solution 6, interior solution*)
```

In[∘]:=

$$p_1 = \text{Root}\Big[1125\,\#1^6 + 60\,928\,q_o^6 + \frac{236\,544\,q_o^6}{\delta^4} - \frac{766\,464\,q_o^6}{\delta^3} + \frac{859\,904\,q_o^6}{\delta^2} - \frac{388\,608\,q_o^6}{\delta} + \#1^5\left(2160\,q_o - \frac{23\,640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151\,312\,q_o^2}{\delta^2} - \frac{12\,096\,q_o^2}{\delta}\right) + \#1^3\left(-20\,160\,q_o^3 - \frac{205\,056\,q_o^3}{\delta^3} - \frac{193\,152\,q_o^3}{\delta^2} + \frac{73\,664\,q_o^3}{\delta}\right) + \#1^2\left(-40\,624\,q_o^4 - \frac{574\,464\,q_o^4}{\delta^4} + \frac{1\,165\,824\,q_o^4}{\delta^3} - \frac{582\,720\,q_o^4}{\delta^2} + \frac{266\,496\,q_o^4}{\delta}\right) + \#1\left(27\,648\,q_o^5 + \frac{67\,584\,q_o^5}{\delta^4} + \frac{170\,496\,q_o^5}{\delta^3} - \frac{376\,320\,q_o^5}{\delta^2} + \frac{58\,240\,q_o^5}{\delta}\right)\&,\,4\Big];$$

$\lambda_1 = 0;$
$\lambda_2 = 0;$
$\lambda_3 = 0;$

$$\text{Reduce}\Big[p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\sqrt{2}\right)t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\Big]$$

Out[∘]=

```
False
```

(*Hence, solution 6 does not satisfy conditions of $p_1<\frac{2q_o-tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

```
(*Solution 7, interior solution*)
```

In[∘]:=

$$p_1 = \text{Root}\Big[1125\,\#1^6 + 60\,928\,q_o^6 + \frac{236\,544\,q_o^6}{\delta^4} - \frac{766\,464\,q_o^6}{\delta^3} + \frac{859\,904\,q_o^6}{\delta^2} - \frac{388\,608\,q_o^6}{\delta} + \#1^5\left(2160\,q_o - \frac{23\,640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151\,312\,q_o^2}{\delta^2} - \frac{12\,096\,q_o^2}{\delta}\right) + \#1^3\left(-20\,160\,q_o^3 - \frac{205\,056\,q_o^3}{\delta^3} - \frac{193\,152\,q_o^3}{\delta^2} + \frac{73\,664\,q_o^3}{\delta}\right) + \#1^2\left(-40\,624\,q_o^4 - \frac{574\,464\,q_o^4}{\delta^4} + \frac{1\,165\,824\,q_o^4}{\delta^3} - \frac{582\,720\,q_o^4}{\delta^2} + \frac{266\,496\,q_o^4}{\delta}\right) + \#1\left(27\,648\,q_o^5 + \frac{67\,584\,q_o^5}{\delta^4} + \frac{170\,496\,q_o^5}{\delta^3} - \frac{376\,320\,q_o^5}{\delta^2} + \frac{58\,240\,q_o^5}{\delta}\right)\&,\,5\Big];$$

$\lambda_1 = 0;$
$\lambda_2 = 0;$
$\lambda_3 = 0;$

$$\text{Reduce}\Big[p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\sqrt{2}\right)t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\Big]$$

Out[∘]=

```
False
```

```
(*Hence, solution 7 does not satisfy conditions of p1<(2qo-tD1)/3,
p1>((3+2 √2) tD1)/2, and p1<16tD1*)

(*Solution 8, interior solution*)
```

In[]:= $p_1 = \text{Root}\Big[1125\,\#1^6 + 60928\,q_o^6 + \frac{236544\,q_o^6}{\delta^4} - \frac{766464\,q_o^6}{\delta^3} + \frac{859904\,q_o^6}{\delta^2} - \frac{388608\,q_o^6}{\delta} + \#1^5\left(2160\,q_o - \frac{23640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151312\,q_o^2}{\delta^2} - \frac{12096\,q_o^2}{\delta}\right) + \#1^3\left(-20160\,q_o^3 - \frac{205056\,q_o^3}{\delta^3} - \frac{193152\,q_o^3}{\delta^2} + \frac{73664\,q_o^3}{\delta}\right) + \#1^2\left(-40624\,q_o^4 - \frac{574464\,q_o^4}{\delta^4} + \frac{1165824\,q_o^4}{\delta^3} - \frac{582720\,q_o^4}{\delta^2} + \frac{266496\,q_o^4}{\delta}\right) + \#1\left(27648\,q_o^5 + \frac{67584\,q_o^5}{\delta^4} + \frac{170496\,q_o^5}{\delta^3} - \frac{376320\,q_o^5}{\delta^2} + \frac{58240\,q_o^5}{\delta}\right)\&,\ 6\Big];$

$\lambda_1 = 0;$
$\lambda_2 = 0;$
$\lambda_3 = 0;$

$\text{Reduce}\Big[p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\sqrt{2}\right)t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\Big]$

Out[]=
False

```
(*Hence, solution 8 does not satisfy conditions of p1<(2qo-tD1)/3,
p1>((3+2 √2) tD1)/2, and p1<16tD1*)

(*Solution 9, boundary solution, which is the solution of p1=((3+2 √2) tD1)/2*)
```

In[∘]:=
```
p1 = -1/(51 δ) 2 (-66 - 40 √2 + 2 (5 + 2 √2) δ +
      √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2)) qo;
λ1 = 0;
λ2 = -1/(88434 t δ^2 (1380 + 416 √2 - 2 (545 + 8 √2) δ + 337 δ^2))
   (2696 (-23673 + 22382 √2) δ^4 + δ^2 (-1105071510 + 1074027996 √2 +
       68063987 √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2) -
       49651982 √2 √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2)) +
     16 (-23547734 + 27232444 √2 + 3229019
         √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2) -
       2369594 √2 √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2)) +
     δ^3 (455454255 - 425220206 √2 - 13397152
         √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2) +
       9798584 √2 √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2)) +
     2 δ (534736262 - 580570684 √2 - 53102861
         √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2) +
       39005958 √2 √(7556 + 5280 √2 - 2 (2153 + 1480 √2) δ + (1305 + 896 √2) δ^2))) qo;
λ3 = 0;
Reduce[λ2 > 0 && p1 < (2 qo - t D1)/3 &&
  p1 == ((3 + 2 √2) t D1)/2 && p1 < 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1]
```

Out[∘]=

```
0.566… < δ < 0.777… && qo > 0 && t > 2 qo

(*Hence, when 0.566…<δ<0.777…,
solution 9 satisfies conditions of p1<(2qo-tD1)/3, p1=((3+2√2)tD1)/2, and p1<16tD1*)

(*Solution 10, boundary solution, which is the solution of p1=((3+2√2)tD1)/2*)
```

In[ ]:=
$$p_1 = \frac{1}{51\,\delta} 2 \left(66 + 40\sqrt{2} - 2\left(5 + 2\sqrt{2}\right)\delta + \sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\right) q_o;$$

$$\lambda_1 = 0;$$

$$\lambda_2 = \frac{1}{88434\, t\, \delta^2 \left(1380 + 416\sqrt{2} - 2\left(545 + 8\sqrt{2}\right)\delta + 337\,\delta^2\right)} \Bigg(-2696\left(-23673 + 22382\sqrt{2}\right)\delta^4 + \delta^2 \Bigg(1105071510 - 1074027996\sqrt{2} + 68063987\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} - 49651982\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Bigg) + 16\Bigg(23547734 - 27232444\sqrt{2} + 3229019\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} - 2369594\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Bigg) + \delta^3\Bigg(-455454255 + 425220206\sqrt{2} - 13397152\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} + 9798584\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Bigg) + 2\,\delta\Bigg(-534736262 + 580570684\sqrt{2} - 53102861\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2} + 39005958\sqrt{2}\sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\Bigg)\Bigg) q_o;$$

$$\lambda_3 = 0;$$

$$\text{Reduce}\left[\lambda_2 > 0\ \&\&\ p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 == \frac{\left(3 + 2\sqrt{2}\right) t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

Out[ ]=

False

(*Hence, solution 10 does not satisfy conditions of $p_1 < \frac{2q_o - tD_1}{3}$, $p_1 = \frac{\left(3+2\sqrt{2}\right)tD_1}{2}$, and $p_1 < 16tD_1$*)

(*Solution 11, boundary solution, which is the solution of $p_1 = 16tD_1$*)

In[•]:=
$$p_1 = \frac{32\left(34 - \delta + \sqrt{1156 - 582\,\delta + 258\,\delta^2}\right) q_o}{257\,\delta};$$
$$\lambda_1 = 0;$$
$$\lambda_2 = 0;$$
$$\lambda_3 = -\frac{1}{135\,796\,744\,t\,\delta^2\left(578 - 291\,\delta + 129\,\delta^2\right)}\left(-29\,614\,530\,\delta^4 + 296\,627\,288\left(34 + \sqrt{1156 - 582\,\delta + 258\,\delta^2}\right) + 204\,\delta\left(66\,107\,713 + 2\,310\,374\sqrt{1156 - 582\,\delta + 258\,\delta^2}\right) + 12\,\delta^3\left(350\,823\,124 + 10\,851\,603\sqrt{1156 - 582\,\delta + 258\,\delta^2}\right) - \delta^2\left(7\,227\,814\,826 + 196\,273\,391\sqrt{1156 - 582\,\delta + 258\,\delta^2}\right)\right) q_o;$$
$$\text{Reduce}\left[\lambda_3 > 0\;\&\&\; p_1 < \frac{2\,q_o - t\,D_1}{3}\;\&\&\; p_1 > \frac{\left(3 + 2\sqrt{2}\right) t\,D_1}{2}\;\&\&\; p_1 == 16\,t\,D_1\;\&\&\; D_1 > 0\;\&\&\; t > 2\,q_o > 0\;\&\&\; 0 < \delta < 1\right]$$

Out[•]=
False

(*Hence, solution 11 does not satisfy conditions of $p_1 < \frac{2q_o - tD_1}{3}$, $p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 = 16tD_1$*)

(*Solution 12, boundary solution, which is the solution of $p_1 = 16tD_1$*)

In[•]:=
$$p_1 = -\frac{32\left(-34 + \delta + \sqrt{1156 - 582\,\delta + 258\,\delta^2}\right) q_o}{257\,\delta};$$
$$\lambda_1 = 0;$$
$$\lambda_2 = 0;$$
$$\lambda_3 = \frac{1}{135\,796\,744\,t\,\delta^2\left(578 - 291\,\delta + 129\,\delta^2\right)}\left(29\,614\,530\,\delta^4 + \delta^2\left(7\,227\,814\,826 - 196\,273\,391\sqrt{1156 - 582\,\delta + 258\,\delta^2}\right) + 296\,627\,288\left(-34 + \sqrt{1156 - 582\,\delta + 258\,\delta^2}\right) + 204\,\delta\left(-66\,107\,713 + 2\,310\,374\sqrt{1156 - 582\,\delta + 258\,\delta^2}\right) + 12\,\delta^3\left(-350\,823\,124 + 10\,851\,603\sqrt{1156 - 582\,\delta + 258\,\delta^2}\right)\right) q_o;$$
$$\text{Reduce}\left[\lambda_3 > 0\;\&\&\; p_1 < \frac{2\,q_o - t\,D_1}{3}\;\&\&\; p_1 > \frac{\left(3 + 2\sqrt{2}\right) t\,D_1}{2}\;\&\&\; p_1 == 16\,t\,D_1\;\&\&\; D_1 > 0\;\&\&\; t > 2\,q_o > 0\;\&\&\; 0 < \delta < 1\right]$$

Out[•]=
False

(*Hence, solution 12 does not satisfy conditions of $p_1 < \frac{2q_o - tD_1}{3}$, $p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 = 16tD_1$*)

```
(*Overall, when 0.566...<δ<0.777...,
```

$$p_1=-\frac{2\left(-66-40\sqrt{2}+2\left(5+2\sqrt{2}\right)\delta+\sqrt{7556+5280\sqrt{2}-2\left(2153+1480\sqrt{2}\right)\delta+\left(1305+896\sqrt{2}\right)\delta^2}\right)q_o}{51\,\delta};$$

```
when 0.777...<δ<1, p1=P2^GL(qo,δ) (i.e., solution 5 as defined before*)
```

```
(*Combination 4. The conditions are 0<p1<(2qo-tD1)/3, p1>(3+2√2)tD1/2, and p1>16tD1*)
```

$$p_{2P}=\frac{2q_o+p_1-tD_1}{4};\ \text{(*The second-period price under completely positive reviews*)}$$

$$p_{2M}=\frac{2p_1+tD_1}{4};\ \text{(*The second-period price under mixed reviews*)}$$

$$p_{2N}=\frac{p_1}{4};\ \text{(*The second-period price under completely negative reviews*)}$$

$$D_{2P}=\frac{2q_o+p_1-tD_1}{4t};\ \text{(*The second-period demand under completely positive reviews*)}$$

$$D_{2M}=\frac{2p_1-3tD_1}{4t};\ \text{(*The second-period demand under mixed reviews*)}$$

$$D_{2N}=\frac{p_1-4tD_1}{4t};\ \text{(*The second-period demand under completely negative reviews*)}$$

$$U_1=q_o-p_1-tD_1+\delta\left(\frac{tD_1}{2q_o}(p_1-p_{2M})+\frac{p_1}{2q_o}(p_1-p_{2N})\right);$$

```
(*Consumers' expected utility purchasing in the first period*)
```

$$U_2=\delta\left(\frac{2q_o-p_1-tD_1}{2q_o}\left(\frac{2q_o+p_1+tD_1}{2}-p_{2P}-tD_1\right)+\frac{tD_1}{2q_o}\left(\frac{2p_1+tD_1}{2}-p_{2M}-tD_1\right)+\frac{p_1}{2q_o}\left(\frac{p_1}{2}-p_{2N}-tD_1\right)\right);$$

```
(*Consumers' expected utility purchasing in the second period*)
```

In[•]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[•]=

$$\left\{\left\{D_1\to-\frac{2\delta p_1-4q_o+2\delta q_o+\sqrt{\delta^2p_1^2+8(-1+\delta)\delta p_1q_o+8\left(2-3\delta+\delta^2\right)q_o^2}}{t\delta}\right\},\left\{D_1\to\frac{-2\delta p_1+4q_o-2\delta q_o+\sqrt{\delta^2p_1^2+8(-1+\delta)\delta p_1q_o+8\left(2-3\delta+\delta^2\right)q_o^2}}{t\delta}\right\}\right\}$$

```
(*Hence, there are two solutions of D1. We
 then check each solution if it satisfies conditions*)
```

In[•]:=

$$D_1=-\frac{2\delta p_1-4q_o+2\delta q_o+\sqrt{\delta^2p_1^2+8(-1+\delta)\delta p_1q_o+8\left(2-3\delta+\delta^2\right)q_o^2}}{t\delta};$$

$$\text{Reduce}\left[0<p_1<\frac{2q_o-tD_1}{3}\ \&\&\ p_1>\frac{\left(3+2\sqrt{2}\right)tD_1}{2}\ \&\&\ p_1>16\,tD_1\ \&\&\ D_1>0\ \&\&\ t>2q_o>0\ \&\&\ 0<\delta<1\right]$$

Out[•]=

False

```
(*The first solution does not satisfy conditions*)
```

In[∘]:= $$D_1 = \frac{-2\,\delta\,p_1 + 4\,q_o - 2\,\delta\,q_o + \sqrt{\delta^2\,p_1^2 + 8\,(-1+\delta)\,\delta\,p_1\,q_o + 8\,\left(2 - 3\,\delta + \delta^2\right)\,q_o^2}}{t\,\delta};$$

$$\text{Reduce}\left[0 < p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\ \&\&\ p_1 > 16\,tD_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

Out[∘]=
False

(*The second solution does not satisfy conditions*)

(*Therefore, there are no feasible solutions for combination 4*)

(*Combination 5. The conditions are $\frac{2q_o-tD_1}{3} \le p_1 \le \frac{2q_o+tD_1}{3}$, $p_1 \le \frac{(3+2\sqrt{2})\,tD_1}{2}$, and $p_1 \le 16tD_1$*)

$p_{2\,P} = p_1$; (*The second-period price under completely positive reviews*)
$p_{2\,M} = p_1$; (*The second-period price under mixed reviews*)
$p_{2\,N} = p_1$; (*The second-period price under completely negative reviews*)
$D_{2\,P} = \frac{2\,q_o - p_1 - t\,D_1}{2\,t}$; (*The second-period demand under completely positive reviews*)
$D_{2\,M} = 0$; (*The second-period demand under mixed reviews*)
$D_{2\,N} = 0$; (*The second-period demand under completely negative reviews*)

$U_1 = q_o - p_1 - t\,D_1$; (*Consumers' expected utility purchasing in the first period*)
$$U_2 = \delta\,\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2\,P} - t\,D_1\right);$$
(*Consumers' expected utility purchasing in the second period*)

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1], $t > 0$]

Out[∘]=
$$\left\{\left\{D_1 \to -\frac{\delta\,p_1 + 2\,\left(1 + \sqrt{1-\delta} - \delta\right)\,q_o}{t\,\delta}\right\},\ \left\{D_1 \to \frac{-\delta\,p_1 + 2\,\left(-1 + \sqrt{1-\delta} + \delta\right)\,q_o}{t\,\delta}\right\}\right\}$$

(*There are two solution of D_1. We then
check each solution if it satisfies conditions*)

In[∘]:= $$D_1 = -\frac{\delta\,p_1 + 2\,\left(1 + \sqrt{1-\delta} - \delta\right)\,q_o}{t\,\delta};$$

$$\text{Reduce}\left[0 < \frac{2\,q_o - t\,D_1}{3} \le p_1 \le \frac{2\,q_o + t\,D_1}{3}\ \&\&\ 0 < p_1 \le \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\ \&\&\ p_1 \le 16\,tD_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

Out[∘]=
False

(*The first solution does not satisfy conditions*)

In[∘]:= $D_1 = \dfrac{-\delta p_1 + 2\left(-1+\sqrt{1-\delta}+\delta\right) q_o}{t\,\delta};$

$\text{Reduce}\Big[0 < \dfrac{2 q_o - t D_1}{3} \le p_1 \le \dfrac{2 q_o + t D_1}{3}$ &&

$0 < p_1 \le \dfrac{\left(3+2\sqrt{2}\right) t D_1}{2}$ && $p_1 \le 16\, tD_1$ && $D_1 > 0$ && $t > 2 q_o > 0$ && $0 < \delta < 1\Big]$

Out[∘]=

$p_1 > 0$ && $\Bigg(\bigg(\dfrac{23 p_1 + 16\sqrt{2}\, p_1}{17+12\sqrt{2}} < q_o \le \text{Root}\Big[\left(68+48\sqrt{2}\right) \#1^3 - 158 p_1^3 -$

$104\sqrt{2}\, p_1^3 + \#1^2\left(-228 p_1 - 160\sqrt{2}\, p_1\right) + \#1\left(285 p_1^2 + 196\sqrt{2}\, p_1^2\right)\ \&,\ 1\Big]$ &&

$0 < \delta \le \dfrac{-92 p_1 q_o - 64\sqrt{2}\, p_1 q_o + 68 q_o^2 + 48\sqrt{2}\, q_o^2}{33 p_1^2 + 20\sqrt{2}\, p_1^2 - 92 p_1 q_o - 64\sqrt{2}\, p_1 q_o + 68 q_o^2 + 48\sqrt{2}\, q_o^2}$ && $t > 2 q_o$ && $tD_1 \ge \dfrac{p_1}{16}\bigg)$ ||

$\bigg(\text{Root}\Big[\left(68+48\sqrt{2}\right) \#1^3 - 158 p_1^3 - 104\sqrt{2}\, p_1^3 + \#1^2\left(-228 p_1 - 160\sqrt{2}\, p_1\right) +$

$\#1\left(285 p_1^2 + 196\sqrt{2}\, p_1^2\right)\ \&,\ 1\Big] < q_o < 2 p_1$ && $0 < \delta \le \dfrac{2 p_1 q_o - q_o^2}{p_1^2}$ && $t > 2 q_o$ && $tD_1 \ge \dfrac{p_1}{16}\bigg)\Bigg)$

(*The second solution satisfies conditions*)

(*Hence, the response function of D_1 is given by*)

$D_1 = \dfrac{-\delta p_1 + 2\left(-1+\sqrt{1-\delta}+\delta\right) q_o}{t\,\delta};$

$\Pi = \text{Simplify}\Big[p_1 D_1 + \dfrac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P}\Big];$ (*The firm's total profit*)

$\text{Reduce}\Big[D[D[\Pi, p_1], p_1] \ge 0$ && $0 < \dfrac{2 q_o - t D_1}{3} \le p_1 \le \dfrac{2 q_o + t D_1}{3}$ && $0 < p_1 \le \dfrac{\left(3+2\sqrt{2}\right) t D_1}{2}$ &&

$p_1 \le 16\, tD_1$ && $D_1 > 0$ && $t > 2 q_o > 0$ && $0 < \delta < 1\Big]$ (*Determine the sign of $\frac{\partial^2\Pi}{\partial p_1^2}$*)

Out[∘]=

False

(*$\frac{\partial^2\Pi}{\partial p_1^2}$<0, meaning Π is concave and it has a maximum value at point where $\frac{\partial\Pi}{\partial p_1}$=0*)

(*KKT conditions*)

$g_1 = p_1 - \dfrac{2 q_o - t D_1}{3};$

$g_2 = \dfrac{2 q_o + t D_1}{3} - p_1;$

$g_3 = \dfrac{\left(3+2\sqrt{2}\right) t D_1}{2} - p_1;$

$g_4 = 16\, t\, D_1 - p_1;$

$L = -\Pi - \lambda_1 g_1 - \lambda_2 g_2 - \lambda_3 g_3 - \lambda_4 g_4;$

In[∘]:= Simplify[Solve[{D[L, p_1] == 0, $\lambda_1 g_1$ == 0, $\lambda_2 g_2$ == 0, $\lambda_3 g_3$ == 0, $\lambda_4 g_4$ == 0},

{p_1, λ_1, λ_2, λ_3, λ_4}], 0 < δ < 1]

Out[ ]=

$$\left\{\left\{p_1 \to \frac{\left(2-2\sqrt{1-\delta}+\left(-3+2\sqrt{1-\delta}\right)\delta+2\delta^2\right)q_o}{2\delta^2},\ \lambda_1\to 0,\ \lambda_2\to 0,\ \lambda_3\to 0,\ \lambda_4\to 0\right\},\right.$$

$$\left\{p_1 \to -\frac{\left(-1+\sqrt{1-\delta}\right)q_o}{\delta},\ \lambda_1\to -\frac{3\left(-2+2\sqrt{1-\delta}+\delta\right)(-1+2\delta)\,q_o}{2\,t\,\delta^2},\ \lambda_2\to 0,\ \lambda_3\to 0,\ \lambda_4\to 0\right\},$$

$$\left\{p_1 \to \frac{\left(-1+\sqrt{1-\delta}+2\delta\right)q_o}{2\delta},\ \lambda_1\to 0,\ \lambda_2\to \frac{3\left(2-2\sqrt{1-\delta}+\left(-2+\sqrt{1-\delta}\right)\delta\right)q_o}{4\,t\,\delta^2},\right.$$

$$\left.\lambda_3\to 0,\ \lambda_4\to 0\right\},\ \left\{p_1\to \frac{2\left(3+2\sqrt{2}\right)\left(-1+\sqrt{1-\delta}+\delta\right)q_o}{\left(5+2\sqrt{2}\right)\delta},\ \lambda_1\to 0,\ \lambda_2\to 0,\ \lambda_3\to\right.$$

$$-\frac{2\left(2\left(5+2\sqrt{2}\right)\left(-1+\sqrt{1-\delta}\right)+\left(3-2\sqrt{2}+4\sqrt{2-2\delta}+2\sqrt{1-\delta}\right)\delta+\left(2+4\sqrt{2}\right)\delta^2\right)q_o}{\left(33+20\sqrt{2}\right)t\,\delta^2},$$

$$\left.\lambda_4\to 0\right\},\ \left\{p_1\to \frac{32\left(-1+\sqrt{1-\delta}+\delta\right)q_o}{17\,\delta},\ \lambda_1\to 0,\ \lambda_2\to 0,\ \lambda_3\to 0,\right.$$

$$\left.\left.\lambda_4\to -\frac{\left(34\left(-1+\sqrt{1-\delta}\right)+\left(-13+30\sqrt{1-\delta}\right)\delta+30\delta^2\right)q_o}{289\,t\,\delta^2}\right\}\right\}$$

(*There are 5 solutions, we then check each solution if it satisfies conditions*)

(*Solution 1, interior solution*)

In[ ]:=

$$p_1 = \frac{\left(2-2\sqrt{1-\delta}+\left(-3+2\sqrt{1-\delta}\right)\delta+2\delta^2\right)q_o}{2\delta^2};$$

$$\text{Reduce}\left[0<\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}\ \&\&\ 0<p_1<\frac{\left(3+2\sqrt{2}\right)tD_1}{2}\ \&\&\ p_1<16\,t\,D_1\ \&\&\ D_1>0\ \&\&\ t>2q_o>0\ \&\&\ 0<\delta<1\right]$$

Out[ ]=

$$0<\delta<\frac{1}{2}\ \&\&\ q_o>0\ \&\&\ t>2q_o$$

(*Hence, when $0<\delta<\frac{1}{2}$, solution 1 satisfies conditions of $0<\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $0<p_1<\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 2, boundary solution, which is the solution of $p_1=\frac{2q_o-tD_1}{3}$*)

In[∘]:= $p_1 = -\frac{\left(-1+\sqrt{1-\delta}\right) q_o}{\delta}$;

$\lambda_1 = -\frac{3\left(-2+2\sqrt{1-\delta}+\delta\right)(-1+2\delta)\, q_o}{2\, t\, \delta^2}$;

$\lambda_2 = 0$;

$\lambda_3 = 0$;

$\lambda_4 = 0$;

Reduce$\left[\lambda_1 > 0\ \&\&\ 0 < \frac{2 q_o - t D_1}{3} == p_1 < \frac{2 q_o + t D_1}{3}\ \&\&\right.$

$\left. 0 < p_1 < \frac{\left(3+2\sqrt{2}\right) t D_1}{2}\ \&\&\ p_1 < 16\, t\, D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2 q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[∘]=

$q_o > 0\ \&\&\ \frac{1}{2} < \delta < \frac{35+28\sqrt{2}}{68+48\sqrt{2}}\ \&\&\ t > 2 q_o$

(*Hence, when $\frac{1}{2}<\delta<\frac{35+28\sqrt{2}}{68+48\sqrt{2}}$, solution 2 satisfies conditions of $0<\frac{2q_o-tD_1}{3}=p_1<\frac{2q_o+tD_1}{3}$, $0<p_1<\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 3, boundary solution, which is the solution of $p_1=\frac{(3+2\sqrt{2})tD_1}{2}$*)

In[∘]:= $p_1 = \frac{\left(-1+\sqrt{1-\delta}+2\delta\right) q_o}{2\delta}$;

$\lambda_1 = 0$;

$\lambda_2 = \frac{3\left(2-2\sqrt{1-\delta}+\left(-2+\sqrt{1-\delta}\right)\delta\right) q_o}{4\, t\, \delta^2}$;

$\lambda_3 = 0$;

$\lambda_4 = 0$;

Reduce$\left[\lambda_2 > 0\ \&\&\ 0 < \frac{2 q_o - t D_1}{3} < p_1 == \frac{2 q_o + t D_1}{3}\ \&\&\right.$

$\left. 0 < p_1 < \frac{\left(3+2\sqrt{2}\right) t D_1}{2}\ \&\&\ p_1 < 16\, t\, D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2 q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[∘]=

False

(*Hence, solution 3 does not satisfy conditions of $0<\frac{2q_o-tD_1}{3}<p_1=\frac{2q_o+tD_1}{3}$, $0<p_1<\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 4, boundary solution, which is the solution of $p_1=\frac{(3+2\sqrt{2})tD_1}{2}$*)

In[ ]:= $p_1 = \frac{2\left(3+2\sqrt{2}\right)\left(-1+\sqrt{1-\delta}+\delta\right)q_o}{\left(5+2\sqrt{2}\right)\delta}$;

$\lambda_1 = 0$;

$\lambda_2 = 0$;

$\lambda_3 = -\frac{2\left(2\left(5+2\sqrt{2}\right)\left(-1+\sqrt{1-\delta}\right)+\left(3-2\sqrt{2}+4\sqrt{2-2\delta}+2\sqrt{1-\delta}\right)\delta+\left(2+4\sqrt{2}\right)\delta^2\right)q_o}{\left(33+20\sqrt{2}\right)t\,\delta^2}$;

$\lambda_4 = 0$;

Reduce$\left[\lambda_3 > 0 \,\&\&\, 0 < \frac{2q_o - tD_1}{3} < p_1 < \frac{2q_o + tD_1}{3} \,\&\&\, 0 < p_1 == \frac{\left(3+2\sqrt{2}\right)tD_1}{2} \,\&\&\, p_1 < 16\,t\,D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2q_o > 0 \,\&\&\, 0 < \delta < 1\right]$

Out[ ]=

False

(*Hence, solution 4 does not satisfy conditions of $0<\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $0<p_1=\frac{\left(3+2\sqrt{2}\right)tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 5, boundary solution, which is the solution of p_1=16tD_1*)

In[ ]:= $p_1 = \frac{32\left(-1+\sqrt{1-\delta}+\delta\right)q_o}{17\,\delta}$;

$\lambda_1 = 0$;

$\lambda_2 = 0$;

$\lambda_3 = 0$;

$\lambda_4 = -\frac{\left(34\left(-1+\sqrt{1-\delta}\right)+\left(-13+30\sqrt{1-\delta}\right)\delta+30\,\delta^2\right)q_o}{289\,t\,\delta^2}$;

Reduce$\left[\lambda_4 > 0 \,\&\&\, 0 < \frac{2q_o - tD_1}{3} < p_1 < \frac{2q_o + tD_1}{3} \,\&\&\, 0 < p_1 < \frac{\left(3+2\sqrt{2}\right)tD_1}{2} \,\&\&\, p_1 == 16\,t\,D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2q_o > 0 \,\&\&\, 0 < \delta < 1\right]$

Out[ ]=

False

(*Hence, solution 5 does not satisfy conditions of $0<\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $0<p_1<\frac{\left(3+2\sqrt{2}\right)tD_1}{2}$, and $p_1=16tD_1$*)

(*Overall, when $0<\delta<\frac{1}{2}$, $p_1=\frac{\left(2-2\sqrt{1-\delta}+\left(-3+2\sqrt{1-\delta}\right)\delta+2\delta^2\right)q_o}{2\delta^2}$;
when $\frac{1}{2}<\delta<\frac{35+28\sqrt{2}}{68+48\sqrt{2}}$, $p_1=-\frac{\left(-1+\sqrt{1-\delta}\right)q_o}{\delta}$*)

(*Combination 6. There is no intersection between conditions of $\frac{2q_o-tD_1}{3}\le p_1\le\frac{2q_o+tD_1}{3}$, $p_1\le\frac{\left(3+2\sqrt{2}\right)tD_1}{2}$, and $p_1>16tD_1$*)

(*Combination 7. The conditions are $\frac{2q_o-tD_1}{3}\le p_1\le\frac{2q_o+tD_1}{3}$, $p_1>\frac{\left(3+2\sqrt{2}\right)tD_1}{2}$, and $p_1\le16tD_1$*)

In[*]:=

$p_{2P} = p_1;$ (*The second-period price under completely positive reviews*)

$p_{2M} = \frac{2 p_1 + t D_1}{4};$ (*The second-period price under mixed reviews*)

$p_{2N} = p_1;$ (*The second-period price under completely negative reviews*)

$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2 t};$ (*The second-period demand under completely positive reviews*)

$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t};$ (*The second-period demand under mixed reviews*)

$D_{2N} = 0;$ (*The second-period demand under completely negative reviews*)

$U_1 = q_o - p_1 - t D_1 + \delta \frac{t D_1}{2 q_o} (p_1 - p_{2M});$

(*Consumers' expected utility purchasing in the first period*)

$U_2 = \delta \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right)\right);$

(*Consumers' expected utility purchasing in the second period*)

In[*]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[*]=

$$\left\{\left\{D_1 \to -\frac{\delta p_1^2 - 4 (-1 + \delta) p_1 q_o + 4 (-1 + \delta) q_o^2}{2 t (\delta p_1 - 2 (-1 + \delta) q_o)}\right\}\right\}$$

$D_1 = -\frac{\delta p_1^2 - 4 (-1 + \delta) p_1 q_o + 4 (-1 + \delta) q_o^2}{2 t (\delta p_1 - 2 (-1 + \delta) q_o)};$ (*The response function of D_1*)

Reduce$\left[\frac{2 q_o - t D_1}{3} \le p_1 \le \frac{2 q_o + t D_1}{3} \;\&\&\; p_1 > \frac{\left(3 + 2 \sqrt{2}\right) t D_1}{2} \;\&\&\; p_1 \le 16 t D_1 \;\&\&\; D_1 > 0 \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta < 1\right]$ (*Check if D_1 satisfies conditions*)

Out[*]=

$$p_1 > 0 \;\&\&\; \left(\left(\frac{4 p_1}{3} < q_o \le \frac{5 p_1 + 2 \sqrt{2} p_1}{3 + 2 \sqrt{2}} \;\&\&\; 0 < \delta \le \frac{-16 p_1 q_o + 12 q_o^2}{7 p_1^2 - 20 p_1 q_o + 12 q_o^2} \;\&\&\; t > 2 q_o\right) \;||\right.$$

$$\left(\frac{5 p_1 + 2 \sqrt{2} p_1}{3 + 2 \sqrt{2}} < q_o \le \frac{47 p_1}{32} \;\&\&\; \frac{-20 p_1 q_o - 8 \sqrt{2} p_1 q_o + 12 q_o^2 + 8 \sqrt{2} q_o^2}{7 p_1^2 + 2 \sqrt{2} p_1^2 - 20 p_1 q_o - 8 \sqrt{2} p_1 q_o + 12 q_o^2 + 8 \sqrt{2} q_o^2} < \delta \le \frac{-16 p_1 q_o + 12 q_o^2}{7 p_1^2 - 20 p_1 q_o + 12 q_o^2} \;\&\&\; t > 2 q_o\right) \;||$$

$$\left(\frac{47 p_1}{32} < q_o \le \frac{49 p_1}{32} \;\&\&\; \frac{-20 p_1 q_o - 8 \sqrt{2} p_1 q_o + 12 q_o^2 + 8 \sqrt{2} q_o^2}{7 p_1^2 + 2 \sqrt{2} p_1^2 - 20 p_1 q_o - 8 \sqrt{2} p_1 q_o + 12 q_o^2 + 8 \sqrt{2} q_o^2} < \delta \le \frac{-34 p_1 q_o + 32 q_o^2}{9 p_1^2 - 34 p_1 q_o + 32 q_o^2} \;\&\&\; t > 2 q_o\right) \;|| \left(\frac{49 p_1}{32} < q_o < \frac{11 p_1 + 6 \sqrt{2} p_1}{6 + 4 \sqrt{2}} \;\&\&\right.$$

$$\left.\left.\frac{-20 p_1 q_o - 8 \sqrt{2} p_1 q_o + 12 q_o^2 + 8 \sqrt{2} q_o^2}{7 p_1^2 + 2 \sqrt{2} p_1^2 - 20 p_1 q_o - 8 \sqrt{2} p_1 q_o + 12 q_o^2 + 8 \sqrt{2} q_o^2} < \delta \le \frac{-8 p_1 q_o + 4 q_o^2}{5 p_1^2 - 12 p_1 q_o + 4 q_o^2} \;\&\&\; t > 2 q_o\right)\right)$$

(*Hence, the response function of D_1 satisfies conditionsis and is given by*)

$D_1 = -\frac{\delta p_1^2 - 4 (-1 + \delta) p_1 q_o + 4 (-1 + \delta) q_o^2}{2 t (\delta p_1 - 2 (-1 + \delta) q_o)};$

$$\Pi = \text{Simplify}\left[p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P} + \frac{t D_1}{2 q_o} (p_{2M} D_{2M} - D_1 (p_1 - p_{2M}))\right];$$

(*The firm's total profit function*)

$$\text{Reduce}\left[D[D[\Pi, p_1], p_1] \geq 0 \,\&\&\, \frac{2 q_o - t D_1}{3} \leq p_1 \leq \frac{2 q_o + t D_1}{3} \,\&\&\, p_1 > \frac{\left(3 + 2\sqrt{2}\right) t D_1}{2} \,\&\&\, p_1 \leq 16 t D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta < 1\right] \text{(*Determine the sign of } \frac{\partial^2 \Pi}{\partial p_1^2} \text{*)}$$

Out[◦]=

False

(* $\frac{\partial^2 \Pi}{\partial p_1^2} < 0$, meaning Π is concave and it has a maximum value at point where $\frac{\partial \Pi}{\partial p_1} = 0$ *)

(*KKT conditions*)

$$g_1 = p_1 - \frac{2 q_o - t D_1}{3};$$

$$g_2 = \frac{2 q_o + t D_1}{3} - p_1;$$

$$g_3 = p_1 - \frac{\left(3 + 2\sqrt{2}\right) t D_1}{2};$$

$$g_4 = 16 t D_1 - p_1;$$

$$L = -\Pi - \lambda_1 g_1 - \lambda_2 g_2 - \lambda_3 g_3 - \lambda_4 g_4;$$

In[◦]:=

$$\text{Simplify}[\text{Solve}[\{D[L, p_1] == 0, \lambda_1 g_1 == 0, \lambda_2 g_2 == 0, \lambda_3 g_3 == 0, \lambda_4 g_4 == 0\}, \{p_1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}], t > 2 q_o > 0 \,\&\&\, 0 < \delta < 1]$$

Out[◦]=

$$\{\{p_1 \to \text{Root}[75 \delta^4 \#1^6 - 3200 q_o^6 + 12992 \delta q_o^6 - 19776 \delta^2 q_o^6 + 13376 \delta^3 q_o^6 - 3392 \delta^4 q_o^6 + \#1^5 (996 \delta^3 q_o - 484 \delta^4 q_o) + \#1^4 (4616 \delta^2 q_o^2 - 5532 \delta^3 q_o^2 + 660 \delta^4 q_o^2) + \#1^3 (9088 \delta q_o^3 - 18144 \delta^2 q_o^3 + 6976 \delta^3 q_o^3 + 2080 \delta^4 q_o^3) + \#1^2 (6528 q_o^4 - 18304 \delta q_o^4 + 9424 \delta^2 q_o^4 + 9952 \delta^3 q_o^4 - 7600 \delta^4 q_o^4) + \#1 (256 q_o^5 - 9408 \delta q_o^5 + 26688 \delta^2 q_o^5 - 26176 \delta^3 q_o^5 + 8640 \delta^4 q_o^5) \,\&, 1], \lambda_1 \to 0, \lambda_2 \to 0,$$
$$\lambda_3 \to 0, \lambda_4 \to 0\}, \{p_1 \to \text{Root}[75 \delta^4 \#1^6 - 3200 q_o^6 + 12992 \delta q_o^6 - 19776 \delta^2 q_o^6 + 13376 \delta^3 q_o^6 - 3392 \delta^4 q_o^6 + \#1^5 (996 \delta^3 q_o - 484 \delta^4 q_o) + \#1^4 (4616 \delta^2 q_o^2 - 5532 \delta^3 q_o^2 + 660 \delta^4 q_o^2) + \#1^3 (9088 \delta q_o^3 - 18144 \delta^2 q_o^3 + 6976 \delta^3 q_o^3 + 2080 \delta^4 q_o^3) + \#1^2 (6528 q_o^4 - 18304 \delta q_o^4 + 9424 \delta^2 q_o^4 + 9952 \delta^3 q_o^4 - 7600 \delta^4 q_o^4) + \#1 (256 q_o^5 - 9408 \delta q_o^5 + 26688 \delta^2 q_o^5 - 26176 \delta^3 q_o^5 + 8640 \delta^4 q_o^5) \,\&, 2], \lambda_1 \to 0, \lambda_2 \to 0,$$
$$\lambda_3 \to 0, \lambda_4 \to 0\}, \{p_1 \to \text{Root}[75 \delta^4 \#1^6 - 3200 q_o^6 + 12992 \delta q_o^6 - 19776 \delta^2 q_o^6 + 13376 \delta^3 q_o^6 - 3392 \delta^4 q_o^6 + \#1^5 (996 \delta^3 q_o - 484 \delta^4 q_o) + \#1^4 (4616 \delta^2 q_o^2 - 5532 \delta^3 q_o^2 + 660 \delta^4 q_o^2) + \#1^3 (9088 \delta q_o^3 - 18144 \delta^2 q_o^3 + 6976 \delta^3 q_o^3 + 2080 \delta^4 q_o^3) + \#1^2 (6528 q_o^4 - 18304 \delta q_o^4 + 9424 \delta^2 q_o^4 + 9952 \delta^3 q_o^4 - 7600 \delta^4 q_o^4) + \#1 (256 q_o^5 - 9408 \delta q_o^5 + 26688 \delta^2 q_o^5 - 26176 \delta^3 q_o^5 + 8640 \delta^4 q_o^5) \,\&, 3], \lambda_1 \to 0, \lambda_2 \to 0,$$
$$\lambda_3 \to 0, \lambda_4 \to 0\}, \{p_1 \to \text{Root}[75 \delta^4 \#1^6 - 3200 q_o^6 + 12992 \delta q_o^6 - 19776 \delta^2 q_o^6 + 13376 \delta^3 q_o^6 - 3392 \delta^4 q_o^6 + \#1^5 (996 \delta^3 q_o - 484 \delta^4 q_o) + \#1^4 (4616 \delta^2 q_o^2 - 5532 \delta^3 q_o^2 + 660 \delta^4 q_o^2) + \#1^3 (9088 \delta q_o^3 - 18144 \delta^2 q_o^3 + 6976 \delta^3 q_o^3 + 2080 \delta^4 q_o^3) + \#1^2 (6528 q_o^4 - 18304 \delta q_o^4 + 9424 \delta^2 q_o^4 + 9952 \delta^3 q_o^4 - 7600 \delta^4 q_o^4) + \#1 (256 q_o^5 - 9408 \delta q_o^5 + 26688 \delta^2 q_o^5 - 26176 \delta^3 q_o^5 + 8640 \delta^4 q_o^5) \,\&, 4], \lambda_1 \to 0, \lambda_2 \to 0,$$
$$\lambda_3 \to 0, \lambda_4 \to 0\}, \{p_1 \to \text{Root}[75 \delta^4 \#1^6 - 3200 q_o^6 + 12992 \delta q_o^6 - 19776 \delta^2 q_o^6 + 13376 \delta^3 q_o^6 -$$

$$3392\,\delta^4\,q_o^6+\sharp 1^5\,\left(996\,\delta^3\,q_o-484\,\delta^4\,q_o\right)+\sharp 1^4\,\left(4616\,\delta^2\,q_o^2-5532\,\delta^3\,q_o^2+660\,\delta^4\,q_o^2\right)+$$
$$\sharp 1^3\,\left(9088\,\delta\,q_o^3-18\,144\,\delta^2\,q_o^3+6976\,\delta^3\,q_o^3+2080\,\delta^4\,q_o^3\right)+$$
$$\sharp 1^2\,\left(6528\,q_o^4-18\,304\,\delta\,q_o^4+9424\,\delta^2\,q_o^4+9952\,\delta^3\,q_o^4-7600\,\delta^4\,q_o^4\right)+$$
$$\sharp 1\,\left(256\,q_o^5-9408\,\delta\,q_o^5+26\,688\,\delta^2\,q_o^5-26\,176\,\delta^3\,q_o^5+8640\,\delta^4\,q_o^5\right)\,\&,\,5\Big],\,\lambda_1\to 0,\,\lambda_2\to 0,$$
$$\lambda_3\to 0,\,\lambda_4\to 0\Big\},\,\Big\{p_1\to \text{Root}\Big[75\,\delta^4\,\sharp 1^6-3200\,q_o^6+12\,992\,\delta\,q_o^6-19\,776\,\delta^2\,q_o^6+13\,376\,\delta^3\,q_o^6-$$
$$3392\,\delta^4\,q_o^6+\sharp 1^5\,\left(996\,\delta^3\,q_o-484\,\delta^4\,q_o\right)+\sharp 1^4\,\left(4616\,\delta^2\,q_o^2-5532\,\delta^3\,q_o^2+660\,\delta^4\,q_o^2\right)+$$
$$\sharp 1^3\,\left(9088\,\delta\,q_o^3-18\,144\,\delta^2\,q_o^3+6976\,\delta^3\,q_o^3+2080\,\delta^4\,q_o^3\right)+$$
$$\sharp 1^2\,\left(6528\,q_o^4-18\,304\,\delta\,q_o^4+9424\,\delta^2\,q_o^4+9952\,\delta^3\,q_o^4-7600\,\delta^4\,q_o^4\right)+$$
$$\sharp 1\,\left(256\,q_o^5-9408\,\delta\,q_o^5+26\,688\,\delta^2\,q_o^5-26\,176\,\delta^3\,q_o^5+8640\,\delta^4\,q_o^5\right)\,\&,\,6\Big],$$
$$\lambda_1\to 0,\,\lambda_2\to 0,\,\lambda_3\to 0,\,\lambda_4\to 0\Big\},\,\Big\{p_1\to -\frac{\left(17-17\,\delta+\sqrt{289-290\,\delta+\delta^2}\right)\,q_o}{9\,\delta},$$
$$\lambda_1\to 0,$$
$$\lambda_2\to 0,$$
$$\lambda_3\to 0,$$
$$\lambda_4\to \Big(\Big(-2\,013\,017\,\delta^3-13\,830\,095\,\left(17+\sqrt{289-290\,\delta+\delta^2}\right)+$$
$$\delta^2\,\left(583\,846\,131+414\,553\,\sqrt{289-290\,\delta+\delta^2}\right)-$$
$$17\,\delta\,\left(35\,383\,851+2\,489\,578\,\sqrt{289-290\,\delta+\delta^2}\right)\Big)\,q_o\Big)\Big/\left(15\,925\,248\,t\,(-289+\delta)\,\delta^2\right)\Big\},$$
$$\Big\{p_1\to \frac{\left(-17+17\,\delta+\sqrt{289-290\,\delta+\delta^2}\right)\,q_o}{9\,\delta},\,\lambda_1\to 0,\,\lambda_2\to 0,$$
$$\lambda_3\to 0,$$
$$\lambda_4\to \Big(\Big(-2\,013\,017\,\delta^3+\delta^2\,\left(583\,846\,131-414\,553\,\sqrt{289-290\,\delta+\delta^2}\right)+13\,830\,095$$
$$\left(-17+\sqrt{289-290\,\delta+\delta^2}\right)+17\,\delta\,\left(-35\,383\,851+2\,489\,578\,\sqrt{289-290\,\delta+\delta^2}\right)\Big)\,q_o\Big)\Big/$$
$$\left(15\,925\,248\,t\,(-289+\delta)\,\delta^2\right)\Big\},\,\Big\{p_1\to -\frac{2\,\left(4-5\,\delta+\sqrt{16-19\,\delta+4\,\delta^2}\right)\,q_o}{7\,\delta},\,\lambda_1\to 0,$$
$$\lambda_2\to \Big(3\,\Big(-10\,624\,\delta^4-14\,944\,\left(4+\sqrt{16-19\,\delta+4\,\delta^2}\right)+32\,\delta^3\,\left(2300+177\,\sqrt{16-19\,\delta+4\,\delta^2}\right)-$$
$$2\,\delta^2\,\left(83\,668+11\,575\,\sqrt{16-19\,\delta+4\,\delta^2}\right)+\delta\,\left(163\,528+32\,009\,\sqrt{16-19\,\delta+4\,\delta^2}\right)\Big)\,q_o\Big)\Big/$$
$$\left(2744\,t\,\delta^2\,\left(16-19\,\delta+4\,\delta^2\right)\right),\,\lambda_3\to 0,\,\lambda_4\to 0\Big\},$$
$$\Big\{p_1\to \frac{2\,\left(-4+5\,\delta+\sqrt{16-19\,\delta+4\,\delta^2}\right)\,q_o}{7\,\delta},\,\lambda_1\to 0,$$
$$\lambda_2\to -\Big(\Big(3\,\Big(10\,624\,\delta^4+\delta^2\,\left(167\,336-23\,150\,\sqrt{16-19\,\delta+4\,\delta^2}\right)-14\,944\,\left(-4+\sqrt{16-19\,\delta+4\,\delta^2}\right)+$$
$$32\,\delta^3\,\left(-2300+177\,\sqrt{16-19\,\delta+4\,\delta^2}\right)+\delta\,\left(-163\,528+32\,009\,\sqrt{16-19\,\delta+4\,\delta^2}\right)\Big)\,q_o\Big)\Big/$$
$$\left(2744\,t\,\delta^2\,\left(16-19\,\delta+4\,\delta^2\right)\right)\Big),\,\lambda_3\to 0,\,\lambda_4\to 0\Big\},$$
$$\Big\{p_1\to -\frac{2\,\left(2-3\,\delta+\sqrt{4-7\,\delta+4\,\delta^2}\right)\,q_o}{5\,\delta},\,\lambda_1\to \frac{1}{200\,t\,\delta^2\,\left(4-7\,\delta+4\,\delta^2\right)}$$
$$3\,\Big(-2496\,\delta^4-1688\,\left(2+\sqrt{4-7\,\delta+4\,\delta^2}\right)+8\,\delta^3\,\left(1277+144\,\sqrt{4-7\,\delta+4\,\delta^2}\right)-$$
$$2\,\delta^2\,\left(8053+1967\,\sqrt{4-7\,\delta+4\,\delta^2}\right)+\delta\,\left(11\,756+4401\,\sqrt{4-7\,\delta+4\,\delta^2}\right)\Big)\,q_o,$$
$$\lambda_2\to 0,\,\lambda_3\to 0,\,\lambda_4\to 0\Big\},\,\Big\{p_1\to \frac{2\,\left(-2+3\,\delta+\sqrt{4-7\,\delta+4\,\delta^2}\right)\,q_o}{5\,\delta},$$

$$\lambda_1 \to -\frac{1}{200\, t\, \delta^2 \left(4 - 7\delta + 4\delta^2\right)} 3\left(2496\,\delta^4 + \delta^2\left(16\,106 - 3934\sqrt{4 - 7\delta + 4\delta^2}\right) - 1688\left(-2 + \sqrt{4 - 7\delta + 4\delta^2}\right) + 8\,\delta^3\left(-1277 + 144\sqrt{4 - 7\delta + 4\delta^2}\right) + \delta\left(-11\,756 + 4401\sqrt{4 - 7\delta + 4\delta^2}\right)\right) q_o,\ \lambda_2 \to 0,$$

$$\lambda_3 \to 0,\ \lambda_4 \to 0\Big\},\ \Big\{p_1 \to -\frac{2\left(5 + 2\sqrt{2} + \sqrt{-\left(33 + 20\sqrt{2} - 4\delta\right)(-1+\delta)} - \left(5 + 2\sqrt{2}\right)\delta\right) q_o}{\left(7 + 2\sqrt{2}\right)\delta},$$

$$\lambda_1 \to 0,$$
$$\lambda_2 \to 0,$$
$$\lambda_3 \to \Big(2\Big(2\Big(943\,832\,313\,601\,170\,011 + 667\,390\,238\,251\,698\,994\sqrt{2} + 120\,564\,742\,058\,767\,887\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)} + 85\,252\,150\,826\,832\,644\sqrt{2}\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)}\Big) + \Big(11\,921\,942\,456\,398\,320\,797 + 8\,430\,086\,305\,301\,959\,302\sqrt{2} + 1\,651\,337\,897\,258\,726\,964\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)} + 1\,167\,672\,209\,572\,729\,224\sqrt{2}\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)}\Big)\delta - 2\Big(5\,019\,564\,885\,908\,409\,461 + 3\,549\,368\,345\,863\,217\,261\sqrt{2} + 132\,441\,590\,019\,058\,464\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)} + 93\,650\,342\,749\,320\,960\sqrt{2}\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)}\Big)\delta^2 + 8\left(75\,492\,027\,485\,593\,603 + 53\,380\,922\,734\,122\,349\sqrt{2}\right)\delta^3\Big) q_o\Big) \Big/ \Big(t\,\delta^2\Big(-45\,995\,727\,438\,280\,404\,979 - 32\,523\,890\,777\,074\,585\,314\sqrt{2} + 4\left(750\,530\,707\,198\,165\,923 + 530\,705\,352\,518\,523\,238\sqrt{2}\right)\delta\Big)\Big),\ \lambda_4 \to 0\Big\},$$

$$\Big\{p_1 \to \frac{2\left(-5 - 2\sqrt{2} + \sqrt{-\left(33 + 20\sqrt{2} - 4\delta\right)(-1+\delta)} + \left(5 + 2\sqrt{2}\right)\delta\right) q_o}{\left(7 + 2\sqrt{2}\right)\delta},$$

$$\lambda_1 \to 0,$$
$$\lambda_2 \to 0,$$
$$\lambda_3 \to \Big(2\Big(2\Big(943\,832\,313\,601\,170\,011 + 667\,390\,238\,251\,698\,994\sqrt{2} - 120\,564\,742\,058\,767\,887\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)} - 85\,252\,150\,826\,832\,644\sqrt{2}\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)}\Big) + \Big(11\,921\,942\,456\,398\,320\,797 + 8\,430\,086\,305\,301\,959\,302\sqrt{2} - 1\,651\,337\,897\,258\,726\,964\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)} - 1\,167\,672\,209\,572\,729\,224\sqrt{2}\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)}\Big)\delta + 2\Big(-5\,019\,564\,885\,908\,409\,461 - 3\,549\,368\,345\,863\,217\,261\sqrt{2} + 132\,441\,590\,019\,058\,464\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)} + 93\,650\,342\,749\,320\,960\sqrt{2}\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)}\Big)\delta^2 + 8\left(75\,492\,027\,485\,593\,603 + 53\,380\,922\,734\,122\,349\sqrt{2}\right)\delta^3\Big) q_o\Big) \Big/ \Big(t\,\delta^2\Big(-45\,995\,727\,438\,280\,404\,979 - 32\,523\,890\,777\,074\,585\,314\sqrt{2} + 4\left(750\,530\,707\,198\,165\,923 + 530\,705\,352\,518\,523\,238\sqrt{2}\right)\delta\Big)\Big),\ \lambda_4 \to 0\Big\}\Big\}$$

```
(*There are 14 solutions, we then check each solution if it satisfies conditions*)

(*Solution 1, interior solution*)
```

In[•]:= $p_1 = \text{Root}\big[75\,\delta^4\,\#1^6 - 3200\,q_o^6 + 12992\,\delta\,q_o^6 - 19776\,\delta^2\,q_o^6 + 13376\,\delta^3\,q_o^6 - 3392\,\delta^4\,q_o^6 + \#1^5\,(996\,\delta^3\,q_o - 484\,\delta^4\,q_o) + \#1^4\,(4616\,\delta^2\,q_o^2 - 5532\,\delta^3\,q_o^2 + 660\,\delta^4\,q_o^2) + \#1^3\,(9088\,\delta\,q_o^3 - 18144\,\delta^2\,q_o^3 + 6976\,\delta^3\,q_o^3 + 2080\,\delta^4\,q_o^3) + \#1^2\,(6528\,q_o^4 - 18304\,\delta\,q_o^4 + 9424\,\delta^2\,q_o^4 + 9952\,\delta^3\,q_o^4 - 7600\,\delta^4\,q_o^4) + \#1\,(256\,q_o^5 - 9408\,\delta\,q_o^5 + 26688\,\delta^2\,q_o^5 - 26176\,\delta^3\,q_o^5 + 8640\,\delta^4\,q_o^5)\ \&,\ 1\big];$

$\lambda_1 = 0;$
$\lambda_2 = 0;$
$\lambda_3 = 0;$
$\lambda_4 = 0;$

$\text{Reduce}\Big[\frac{2\,q_o - t\,D_1}{3} < p_1 < \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{(3 + 2\,\sqrt{2})\,t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1,\ \text{Reals}\Big]$

Out[•]=

```
False
```

(*Hence, solution 1 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

```
(*Solution 2, interior solution*)
```

In[•]:= $p_1 = \text{Root}\big[75\,\delta^4\,\#1^6 - 3200\,q_o^6 + 12992\,\delta\,q_o^6 - 19776\,\delta^2\,q_o^6 + 13376\,\delta^3\,q_o^6 - 3392\,\delta^4\,q_o^6 + \#1^5\,(996\,\delta^3\,q_o - 484\,\delta^4\,q_o) + \#1^4\,(4616\,\delta^2\,q_o^2 - 5532\,\delta^3\,q_o^2 + 660\,\delta^4\,q_o^2) + \#1^3\,(9088\,\delta\,q_o^3 - 18144\,\delta^2\,q_o^3 + 6976\,\delta^3\,q_o^3 + 2080\,\delta^4\,q_o^3) + \#1^2\,(6528\,q_o^4 - 18304\,\delta\,q_o^4 + 9424\,\delta^2\,q_o^4 + 9952\,\delta^3\,q_o^4 - 7600\,\delta^4\,q_o^4) + \#1\,(256\,q_o^5 - 9408\,\delta\,q_o^5 + 26688\,\delta^2\,q_o^5 - 26176\,\delta^3\,q_o^5 + 8640\,\delta^4\,q_o^5)\ \&,\ 2\big];$

$\lambda_1 = 0;$
$\lambda_2 = 0;$
$\lambda_3 = 0;$
$\lambda_4 = 0;$

$\text{Reduce}\Big[\frac{2\,q_o - t\,D_1}{3} < p_1 < \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{(3 + 2\,\sqrt{2})\,t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1,\ \text{Reals}\Big]$

Out[•]=

```
False
```

(*Hence, solution 2 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

```
(*Solution 3, interior solution*)
```

In[ ]:=
```
p1 = Root[75 δ^4 #1^6 - 3200 qo^6 + 12992 δ qo^6 - 19776 δ^2 qo^6 + 13376 δ^3 qo^6 -
    3392 δ^4 qo^6 + #1^5 (996 δ^3 qo - 484 δ^4 qo) + #1^4 (4616 δ^2 qo^2 - 5532 δ^3 qo^2 + 660 δ^4 qo^2) +
    #1^3 (9088 δ qo^3 - 18144 δ^2 qo^3 + 6976 δ^3 qo^3 + 2080 δ^4 qo^3) +
    #1^2 (6528 qo^4 - 18304 δ qo^4 + 9424 δ^2 qo^4 + 9952 δ^3 qo^4 - 7600 δ^4 qo^4) +
    #1 (256 qo^5 - 9408 δ qo^5 + 26688 δ^2 qo^5 - 26176 δ^3 qo^5 + 8640 δ^4 qo^5) &, 3];
λ1 = 0;
λ2 = 0;
λ3 = 0;
λ4 = 0;
Reduce[(2 qo - t D1)/3 < p1 < (2 qo + t D1)/3 && p1 > ((3 + 2 Sqrt[2]) t D1)/2 &&
  p1 < 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1, Reals]
```

Out[ ]=
```
False
```

(*Hence, solution 3 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 4, interior solution*)

In[ ]:=
```
p1 = Root[75 δ^4 #1^6 - 3200 qo^6 + 12992 δ qo^6 - 19776 δ^2 qo^6 + 13376 δ^3 qo^6 -
    3392 δ^4 qo^6 + #1^5 (996 δ^3 qo - 484 δ^4 qo) + #1^4 (4616 δ^2 qo^2 - 5532 δ^3 qo^2 + 660 δ^4 qo^2) +
    #1^3 (9088 δ qo^3 - 18144 δ^2 qo^3 + 6976 δ^3 qo^3 + 2080 δ^4 qo^3) +
    #1^2 (6528 qo^4 - 18304 δ qo^4 + 9424 δ^2 qo^4 + 9952 δ^3 qo^4 - 7600 δ^4 qo^4) +
    #1 (256 qo^5 - 9408 δ qo^5 + 26688 δ^2 qo^5 - 26176 δ^3 qo^5 + 8640 δ^4 qo^5) &, 4];
λ1 = 0;
λ2 = 0;
λ3 = 0;
λ4 = 0;
Reduce[(2 qo - t D1)/3 < p1 < (2 qo + t D1)/3 && p1 > ((3 + 2 Sqrt[2]) t D1)/2 &&
  p1 < 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1, Reals]
```

Out[ ]=
```
0.557… < δ < 0.569… && qo > 0 && t > 2 qo
```

(*Hence, when 0.557…<δ<0.569…,
solution 4 satisfies conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*For convenience, we define solution 4 as $P_3^{GL}(q_o,\delta)$*)

(*Solution 5, interior solution*)

In[◦]:=
```
p1 = Root[75 δ^4 #1^6 - 3200 qo^6 + 12 992 δ qo^6 - 19 776 δ^2 qo^6 + 13 376 δ^3 qo^6 -
    3392 δ^4 qo^6 + #1^5 (996 δ^3 qo - 484 δ^4 qo) + #1^4 (4616 δ^2 qo^2 - 5532 δ^3 qo^2 + 660 δ^4 qo^2) +
    #1^3 (9088 δ qo^3 - 18 144 δ^2 qo^3 + 6976 δ^3 qo^3 + 2080 δ^4 qo^3) +
    #1^2 (6528 qo^4 - 18 304 δ qo^4 + 9424 δ^2 qo^4 + 9952 δ^3 qo^4 - 7600 δ^4 qo^4) +
    #1 (256 qo^5 - 9408 δ qo^5 + 26 688 δ^2 qo^5 - 26 176 δ^3 qo^5 + 8640 δ^4 qo^5) &, 5];
λ1 = 0;
λ2 = 0;
λ3 = 0;
λ4 = 0;
Reduce[(2 qo - t D1)/3 < p1 < (2 qo + t D1)/3 && p1 > ((3 + 2 √2) t D1)/2 &&
  p1 < 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1, Reals]
```
Out[◦]=
```
False
```

(*Hence, solution 5 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 6, interior solution*)

In[◦]:=
```
p1 = Root[75 δ^4 #1^6 - 3200 qo^6 + 12 992 δ qo^6 - 19 776 δ^2 qo^6 + 13 376 δ^3 qo^6 -
    3392 δ^4 qo^6 + #1^5 (996 δ^3 qo - 484 δ^4 qo) + #1^4 (4616 δ^2 qo^2 - 5532 δ^3 qo^2 + 660 δ^4 qo^2) +
    #1^3 (9088 δ qo^3 - 18 144 δ^2 qo^3 + 6976 δ^3 qo^3 + 2080 δ^4 qo^3) +
    #1^2 (6528 qo^4 - 18 304 δ qo^4 + 9424 δ^2 qo^4 + 9952 δ^3 qo^4 - 7600 δ^4 qo^4) +
    #1 (256 qo^5 - 9408 δ qo^5 + 26 688 δ^2 qo^5 - 26 176 δ^3 qo^5 + 8640 δ^4 qo^5) &, 6];
λ1 = 0;
λ2 = 0;
λ3 = 0;
λ4 = 0;
Reduce[(2 qo - t D1)/3 < p1 < (2 qo + t D1)/3 && p1 > ((3 + 2 √2) t D1)/2 &&
  p1 < 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1, Reals]
```
Out[◦]=
```
False
```

(*Hence, solution 6 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 7, boundary solution, which is the solution of $p_1=16tD_1$*)

In[∘]:=
$$p_1 = -\frac{\left(17 - 17\,\delta + \sqrt{289 - 290\,\delta + \delta^2}\right) q_o}{9\,\delta};$$
$$\lambda_1 = 0;$$
$$\lambda_2 = 0;$$
$$\lambda_3 = 0;$$
$$\lambda_4 = \left(\left(-2\,013\,017\,\delta^3 - 13\,830\,095\left(17 + \sqrt{289 - 290\,\delta + \delta^2}\right) + \delta^2\left(583\,846\,131 + 414\,553\sqrt{289 - 290\,\delta + \delta^2}\right) - 17\,\delta\left(35\,383\,851 + 2\,489\,578\sqrt{289 - 290\,\delta + \delta^2}\right)\right) q_o\right) \Big/ \left(15\,925\,248\,t\,(-289 + \delta)\,\delta^2\right);$$
$$\text{Reduce}\left[\lambda_4 > 0\ \&\&\ \frac{2\,q_o - t\,D_1}{3} < p_1 < \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\sqrt{2}\right) t\,D_1}{2}\ \&\&\ p_1 == 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

Out[∘]=
False

(*Hence, solution 7 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and p_1=16tD_1*)

(*Solution 8, boundary solution, which is the solution of p_1=16tD_1*)

In[∘]:=
$$p_1 = \frac{\left(-17 + 17\,\delta + \sqrt{289 - 290\,\delta + \delta^2}\right) q_o}{9\,\delta};$$
$$\lambda_1 = 0;$$
$$\lambda_2 = 0;$$
$$\lambda_3 = 0;$$
$$\lambda_4 = \left(\left(-2\,013\,017\,\delta^3 + \delta^2\left(583\,846\,131 - 414\,553\sqrt{289 - 290\,\delta + \delta^2}\right) + 13\,830\,095\left(-17 + \sqrt{289 - 290\,\delta + \delta^2}\right) + 17\,\delta\left(-35\,383\,851 + 2\,489\,578\sqrt{289 - 290\,\delta + \delta^2}\right)\right) q_o\right) \Big/ \left(15\,925\,248\,t\,(-289 + \delta)\,\delta^2\right);$$
$$\text{Reduce}\left[\lambda_4 > 0\ \&\&\ \frac{2\,q_o - t\,D_1}{3} < p_1 < \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\sqrt{2}\right) t\,D_1}{2}\ \&\&\ p_1 == 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

Out[∘]=
False

(*Hence, solution 8 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and p_1=16tD_1*)

(*Solution 9, boundary solution, which is the solution of $p_1=\frac{2q_o+tD_1}{3}$*)

In[∘]:= $p_1 = -\frac{2\left(4-5\delta+\sqrt{16-19\delta+4\delta^2}\right)q_o}{7\delta};$
$\lambda_1 = 0;$
$\lambda_2 = \left(3\left(-10624\delta^4 - 14944\left(4+\sqrt{16-19\delta+4\delta^2}\right) + 32\delta^3\left(2300+177\sqrt{16-19\delta+4\delta^2}\right) - 2\delta^2\left(83668+11575\sqrt{16-19\delta+4\delta^2}\right) + \delta\left(163528+32009\sqrt{16-19\delta+4\delta^2}\right)\right)q_o\right) \Big/ \left(2744\,t\,\delta^2\left(16-19\delta+4\delta^2\right)\right);$
$\lambda_3 = 0;$
$\lambda_4 = 0;$
$\text{Reduce}\left[\lambda_2 > 0\ \&\&\ \frac{2q_o - tD_1}{3} < p_1 == \frac{2q_o + tD_1}{3}\ \&\&\ p_1 > \frac{\left(3+2\sqrt{2}\right)tD_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[∘]=
False

(*Hence, solution 9 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1=\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 10, boundary solution, which is the solution of $p_1=16tD_1$*)

In[∘]:= $p_1 = \frac{2\left(-4+5\delta+\sqrt{16-19\delta+4\delta^2}\right)q_o}{7\delta};$
$\lambda_1 = 0;$
$\lambda_2 = -\left(\left(3\left(10624\delta^4 + \delta^2\left(167336-23150\sqrt{16-19\delta+4\delta^2}\right) - 14944\left(-4+\sqrt{16-19\delta+4\delta^2}\right) + 32\delta^3\left(-2300+177\sqrt{16-19\delta+4\delta^2}\right) + \delta\left(-163528+32009\sqrt{16-19\delta+4\delta^2}\right)\right)q_o\right) \Big/ \left(2744\,t\,\delta^2\left(16-19\delta+4\delta^2\right)\right)\right);$
$\lambda_3 = 0;$
$\lambda_4 = 0;$
$\text{Reduce}\left[\lambda_2 > 0\ \&\&\ \frac{2q_o - tD_1}{3} < p_1 == \frac{2q_o + tD_1}{3}\ \&\&\ p_1 > \frac{\left(3+2\sqrt{2}\right)tD_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[∘]=
False

(*Hence, solution 10 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1=\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 11, boundary solution, which is the solution of $p_1=\frac{2q_o-tD_1}{3}$*)

In[∘]:= $p_1 = -\frac{2\left(2 - 3\delta + \sqrt{4 - 7\delta + 4\delta^2}\right) q_o}{5\delta};$

$\lambda_1 = \frac{1}{200\, t\, \delta^2 \left(4 - 7\delta + 4\delta^2\right)} 3\left(-2496\delta^4 - 1688\left(2 + \sqrt{4 - 7\delta + 4\delta^2}\right) + 8\delta^3\left(1277 + 144\sqrt{4 - 7\delta + 4\delta^2}\right) - 2\delta^2\left(8053 + 1967\sqrt{4 - 7\delta + 4\delta^2}\right) + \delta\left(11756 + 4401\sqrt{4 - 7\delta + 4\delta^2}\right)\right) q_o;$

$\lambda_2 = 0;$

$\lambda_3 = 0;$

$\lambda_4 = 0;$

$\text{Reduce}\left[\lambda_1 > 0 \,\&\&\, \frac{2 q_o - t D_1}{3} == p_1 < \frac{2 q_o + t D_1}{3} \,\&\&\, p_1 > \frac{\left(3 + 2\sqrt{2}\right) t D_1}{2} \,\&\&\, p_1 < 16\, t\, D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta < 1\right]$

Out[∘]=

False

(*Hence, solution 11 does not satisfy conditions of $\frac{2q_o - tD_1}{3} = p_1 < \frac{2q_o + tD_1}{3}$, $p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 < 16tD_1$*)

(*Solution 12, boundary solution, which is the solution of $p_1 = \frac{2q_o - tD_1}{3}$*)

In[∘]:= $p_1 = \frac{2\left(-2 + 3\delta + \sqrt{4 - 7\delta + 4\delta^2}\right) q_o}{5\delta};$

$\lambda_1 = -\frac{1}{200\, t\, \delta^2 \left(4 - 7\delta + 4\delta^2\right)} 3\left(2496\delta^4 + \delta^2\left(16106 - 3934\sqrt{4 - 7\delta + 4\delta^2}\right) - 1688\left(-2 + \sqrt{4 - 7\delta + 4\delta^2}\right) + 8\delta^3\left(-1277 + 144\sqrt{4 - 7\delta + 4\delta^2}\right) + \delta\left(-11756 + 4401\sqrt{4 - 7\delta + 4\delta^2}\right)\right) q_o;$

$\lambda_2 = 0;$

$\lambda_3 = 0;$

$\lambda_4 = 0;$

$\text{Reduce}\left[\lambda_1 > 0 \,\&\&\, \frac{2 q_o - t D_1}{3} == p_1 < \frac{2 q_o + t D_1}{3} \,\&\&\, p_1 > \frac{\left(3 + 2\sqrt{2}\right) t D_1}{2} \,\&\&\, p_1 < 16\, t\, D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta < 1\right]$

Out[∘]=

$q_o > 0 \,\&\&\, \boxed{0.569\ldots} < \delta < \frac{245}{341} \,\&\&\, t > 2 q_o$

(*Hence, when $\boxed{0.569\ldots} < \delta < \frac{245}{341}$,

solution 12 satisfies conditions of $\frac{2q_o - tD_1}{3} = p_1 < \frac{2q_o + tD_1}{3}$, $p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 < 16tD_1$*)

(*Solution 13, boundary solution, which is the solution of $p_1 = \frac{(3+2\sqrt{2})tD_1}{2}$*)

In[ ]:=

$$p_1 = -\frac{2\left(5+2\sqrt{2}+\sqrt{-\left(33+20\sqrt{2}-4\delta\right)(-1+\delta)}-\left(5+2\sqrt{2}\right)\delta\right)q_o}{\left(7+2\sqrt{2}\right)\delta};$$

$$\lambda_1 = 0;$$

$$\lambda_2 = 0;$$

$$\begin{aligned}\lambda_3 = \Big(2\Big(2\Big(&943\,832\,313\,601\,170\,011 + 667\,390\,238\,251\,698\,994\sqrt{2} + \\ &120\,564\,742\,058\,767\,887\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} + \\ &85\,252\,150\,826\,832\,644\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big) + \\ \Big(&11\,921\,942\,456\,398\,320\,797 + 8\,430\,086\,305\,301\,959\,302\sqrt{2} + \\ &1\,651\,337\,897\,258\,726\,964\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} + \\ &1\,167\,672\,209\,572\,729\,224\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big)\delta - \\ 2\Big(&5\,019\,564\,885\,908\,409\,461 + 3\,549\,368\,345\,863\,217\,261\sqrt{2} + \\ &132\,441\,590\,019\,058\,464\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} + \\ &93\,650\,342\,749\,320\,960\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big)\delta^2 + \\ 8&\left(75\,492\,027\,485\,593\,603 + 53\,380\,922\,734\,122\,349\sqrt{2}\right)\delta^3\Big)q_o\Big) \Big/ \\ \Big(t\,\delta^2\Big(&-45\,995\,727\,438\,280\,404\,979 - 32\,523\,890\,777\,074\,585\,314\sqrt{2} + \\ 4&\left(750\,530\,707\,198\,165\,923 + 530\,705\,352\,518\,523\,238\sqrt{2}\right)\delta\Big)\Big);\end{aligned}$$

$$\lambda_4 = 0;$$

$$\text{Reduce}\left[\lambda_3 > 0 \,\&\&\, \frac{2q_o - t\,D_1}{3} < p_1 < \frac{2q_o + t\,D_1}{3} \,\&\&\, p_1 == \frac{\left(3+2\sqrt{2}\right)t\,D_1}{2} \,\&\&\, p_1 < 16\,t\,D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

Out[ ]=

False

(*Hence, solution 13 does not satisfy conditions of $\frac{2q_o-tD_1}{3}<p_1<\frac{2q_o+tD_1}{3}$, $p_1=\frac{\left(3+2\sqrt{2}\right)tD_1}{2}$, and $p_1<16tD_1$*)

(*Solution 14, boundary solution, which is the solution of $p_1=\frac{\left(3+2\sqrt{2}\right)tD_1}{2}$*)

In[∘]:=
$$p_1 = \frac{2\left(-5-2\sqrt{2}+\sqrt{-\left(33+20\sqrt{2}-4\delta\right)(-1+\delta)}+\left(5+2\sqrt{2}\right)\delta\right)q_o}{\left(7+2\sqrt{2}\right)\delta};$$

$\lambda_1 = 0;$

$\lambda_2 = 0;$

$$\lambda_3 = \Big(2\Big(2\Big(943\,832\,313\,601\,170\,011 + 667\,390\,238\,251\,698\,994\sqrt{2} - 120\,564\,742\,058\,767\,887\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} - 85\,252\,150\,826\,832\,644\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big) + \Big(11\,921\,942\,456\,398\,320\,797 + 8\,430\,086\,305\,301\,959\,302\sqrt{2} - 1\,651\,337\,897\,258\,726\,964\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} - 1\,167\,672\,209\,572\,729\,224\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big)\delta + 2\Big(-5\,019\,564\,885\,908\,409\,461 - 3\,549\,368\,345\,863\,217\,261\sqrt{2} + 132\,441\,590\,019\,058\,464\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} + 93\,650\,342\,749\,320\,960\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big)\delta^2 + 8\left(75\,492\,027\,485\,593\,603 + 53\,380\,922\,734\,122\,349\sqrt{2}\right)\delta^3\Big)q_o\Big) \Big/ \Big(t\,\delta^2\Big(-45\,995\,727\,438\,280\,404\,979 - 32\,523\,890\,777\,074\,585\,314\sqrt{2} + 4\left(750\,530\,707\,198\,165\,923 + 530\,705\,352\,518\,523\,238\sqrt{2}\right)\delta\Big)\Big);$$

$\lambda_4 = 0;$

$$\text{Reduce}\left[\lambda_3 > 0 \,\&\&\, \frac{2q_o - tD_1}{3} < p_1 < \frac{2q_o + tD_1}{3} \,\&\&\, p_1 = \frac{\left(3+2\sqrt{2}\right)tD_1}{2} \,\&\&\, p_1 < 16\,t\,D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

Out[∘]=

$0 < \delta < 0.557\ldots \,\&\&\, q_o > 0 \,\&\&\, t > 2q_o$

```
(*Hence, when 0<δ<0.557…,
solution 14 satisfies conditions of (2qo-tD1)/3<p1<(2qo+tD1)/3, p1=(3+2√2)tD1/2, and p1<16tD1*)
```

(*Overall, when $0<\delta<0.557\ldots$, $p_1=\frac{2\left(-5-2\sqrt{2}+\sqrt{-\left(33+20\sqrt{2}-4\delta\right)(-1+\delta)}+\left(5+2\sqrt{2}\right)\delta\right)q_o}{\left(7+2\sqrt{2}\right)\delta}$;

when $0.557\ldots<\delta<0.569\ldots$, $p_1=P_3^{GL}(q_o,\delta)$ (i.e., solution 4 as defined before);

when $0.569\ldots<\delta<\frac{245}{341}$, $p_1=\frac{2\left(-2+3\delta+\sqrt{4-7\delta+4\delta^2}\right)q_o}{5\delta}$ *)

```
(*Combination 8. The conditions are (2qo-tD1)/3 ≤p1≤ (2qo+tD1)/3, p1>((3+2√2)tD1)/2, and p1>16tD1*)
```

$p_{2P} = p_1$; (*The second-period price under completely positive reviews*)

$p_{2M} = \frac{2 p_1 + t D_1}{4}$; (*The second-period price under mixed reviews*)

$p_{2N} = \frac{p_1}{4}$; (*The second-period price under completely negative reviews*)

$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2 t}$; (*The second-period demand under completely positive reviews*)

$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t}$; (*The second-period demand under mixed reviews*)

$D_{2N} = \frac{p_1 - 4 t D_1}{4 t}$; (*The second-period demand under completely negative reviews*)

$U_1 = q_o - p_1 - t D_1 + \delta \left(\frac{t D_1}{2 q_o} (p_1 - p_{2M}) + \frac{p_1}{2 q_o} (p_1 - p_{2N})\right)$;

(*Consumers' expected utility purchasing in the first period*)

$U_2 = \delta \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right) + \frac{p_1}{2 q_o} \left(\frac{p_1}{2} - p_{2N} - t D_1\right)\right)$;

(*Consumers' expected utility purchasing in the second period*)

In[ ]:= Simplify[Solve[$U_1 == U_2, D_1$], $t > 0$ && $q_o > 0$ && $0 < \delta < 1$]

Out[ ]=

$\left\{\left\{D_1 \to \frac{-p_1 + q_o}{t}\right\}\right\}$

$D_1 = \frac{-p_1 + q_o}{t}$; (*The response function of D_1*)

Reduce[$\frac{2 q_o - t D_1}{3} \le p_1 \le \frac{2 q_o + t D_1}{3}$ && $p_1 > \frac{(3 + 2 \sqrt{2}) t D_1}{2}$ && $p_1 > 16 t D_1$ && $D_1 > 0$ && $t > 2 q_o > 0$ && $0 < \delta < 1$] (*We check if D_1 satisfies conditions*)

Out[ ]=

False

(*Hence, there are no feasible solutions for combination 8*)

```
(*Combination 9. The conditions are p1>(2qo+tD1)/3, p1≤((3+2√2)tD1)/2, and p1≤16tD1*)
```

$p_{2P} = \frac{2\,q_o + p_1 + t\,D_1}{4}$; (*The second-period price under completely positive reviews*)

$p_{2M} = p_1$; (*The second-period price under mixed reviews*)

$p_{2N} = p_1$; (*The second-period price under completely negative reviews*)

$D_{2P} = \frac{2\,q_o + p_1 - 3\,t\,D_1}{4\,t}$; (*The second-period demand under completely positive reviews*)

$D_{2M} = 0$; (*The second-period demand under mixed reviews*)

$D_{2N} = 0$; (*The second-period demand under completely negative reviews*)

$U_1 = q_o - p_1 - t\,D_1 + \delta\,\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,(p_1 - p_{2P})$;

(*Consumers' expected utility purchasing in the first period*)

$U_2 = \delta\,\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2P} - t\,D_1\right)$;

(*Consumers' expected utility purchasing in the second period*)

In[ ]:= Simplify[Solve[$U_1 == U_2$, D_1], t > 0]

Out[ ]=

$$\left\{\left\{D_1 \to -\frac{\delta\,p_1 + 2\left(1 + \sqrt{1-\delta} - \delta\right)q_o}{t\,\delta}\right\},\ \left\{D_1 \to \frac{-\delta\,p_1 + 2\left(-1 + \sqrt{1-\delta} + \delta\right)q_o}{t\,\delta}\right\}\right\}$$

(*There are 2 solutions of response function D_1,
we then check each solution if it satisfies conditions*)

In[ ]:= $D_1 = -\frac{\delta\,p_1 + 2\left(1 + \sqrt{1-\delta} - \delta\right)q_o}{t\,\delta}$;

Reduce$\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 \le \frac{\left(3 + 2\,\sqrt{2}\right)t\,D_1}{2}\ \&\&\ p_1 \le 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[ ]=

False

(*The first solution does not satisfy conditions*)

In[ ]:= $D_1 = \frac{-\delta\,p_1 + 2\left(-1 + \sqrt{1-\delta} + \delta\right)q_o}{t\,\delta}$;

Reduce$\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 \le \frac{\left(3 + 2\,\sqrt{2}\right)t\,D_1}{2}\ \&\&\ p_1 \le 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$

Out[ ]=

False

(*The second solution does not satisfy conditions*)

(*Therefore, there are no feasible solutions for combination 9*)

(*Combination 10. There is no intersection between conditions of $p_1 > \frac{2q_o + tD_1}{3}$, $p_1 \le \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 > 16tD_1$*)

(*Combination 11. The conditions are $p_1 > \frac{2q_o + tD_1}{3}$, $p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 \le 16tD_1$*)

$$p_{2P} = \frac{2 q_o + p_1 + t D_1}{4};$$ (*The second-period price under completely positive reviews*)

$$p_{2M} = \frac{2 p_1 + t D_1}{4};$$ (*The second-period price under mixed reviews*)

$p_{2N} = p_1;$ (*The second-period price under completely negative reviews*)

$$D_{2P} = \frac{2 q_o + p_1 - 3 t D_1}{4 t};$$ (*The second-period demand under completely positive reviews*)

$$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t};$$ (*The second-period demand under mixed reviews*)

$D_{2N} = 0;$ (*The second-period demand under completely negative reviews*)

$$U_1 = q_o - p_1 - t D_1 + \delta \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} (p_1 - p_{2P}) + \frac{t D_1}{2 q_o} (p_1 - p_{2M})\right);$$

(*Consumers' expected utility purchasing in the first period*)

$$U_2 = \delta \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right)\right);$$

(*Consumers' expected utility purchasing in the second period*)

In[ ]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[ ]=

$$\left\{\left\{D_1 \to -\frac{\delta p_1^2 - 4 (-1 + \delta) p_1 q_o + 4 (-1 + \delta) q_o^2}{2 t (\delta p_1 - 2 (-1 + \delta) q_o)}\right\}\right\}$$

$$D_1 = -\frac{\delta p_1^2 - 4 (-1 + \delta) p_1 q_o + 4 (-1 + \delta) q_o^2}{2 t (\delta p_1 - 2 (-1 + \delta) q_o)};$$ (*The response function of D_1*)

$$\text{Reduce}\left[p_1 > \frac{2 q_o + t D_1}{3} \,\&\&\, p_1 > \frac{\left(3 + 2 \sqrt{2}\right) t D_1}{2} \,\&\&\, p_1 \le 16 t D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

(*We check if D_1 satisfies conditions*)

Out[ ]=

$$p_1 > 0 \,\&\&\, \left(\left(\frac{17 p_1}{16} < q_o \le \frac{4 p_1}{3} \,\&\&\, 0 < \delta \le \frac{-34 p_1 q_o + 32 q_o^2}{9 p_1^2 - 34 p_1 q_o + 32 q_o^2} \,\&\&\, t > 2 q_o\right) \,||\, \left(\frac{4 p_1}{3} < q_o < \frac{47 p_1}{32} \,\&\&\, \frac{-16 p_1 q_o + 12 q_o^2}{7 p_1^2 - 20 p_1 q_o + 12 q_o^2} < \delta \le \frac{-34 p_1 q_o + 32 q_o^2}{9 p_1^2 - 34 p_1 q_o + 32 q_o^2} \,\&\&\, t > 2 q_o\right)\right)$$

(*Hence, the response function of D_1 satisfies conditionsis and is given by*)

In[ ]:= $$D_1 = -\frac{\delta p_1^2 - 4 (-1 + \delta) p_1 q_o + 4 (-1 + \delta) q_o^2}{2 t (\delta p_1 - 2 (-1 + \delta) q_o)};$$

$$\Pi = \text{Simplify}\left[p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} (p_{2P} D_{2P} - D_1 (p_1 - p_{2P})) + \frac{t D_1}{2 q_o} (p_{2M} D_{2M} - D_1 (p_1 - p_{2M}))\right];$$

(*The firm's total profit function*)

$$\text{Reduce}\left[D[D[\Pi, p_1], p_1] \ge 0 \,\&\&\, p_1 > \frac{2 q_o + t D_1}{3} \,\&\&\, p_1 > \frac{\left(3 + 2 \sqrt{2}\right) t D_1}{2} \,\&\&\, p_1 \le 16 t D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$ (*Determine the sign of $\frac{\partial^2 \Pi}{\partial p_1^2}$*)

Out[ ]=

False

```
(*∂²Π/∂p1² <0, meaning Π is concave and it has a maximum value at point where ∂Π/∂p1 =0*)

(*KKT conditions*)

g1 = p1 - (2 qo + t D1)/3;
g2 = p1 - ((3 + 2 √2) t D1)/2;
g3 = 16 t D1 - p1;
L = -Π - λ1 g1 - λ2 g2 - λ3 g3;
```

In[∘]:=
```
Simplify[Solve[{D[L, p1] == 0, λ1 g1 == 0, λ2 g2 == 0, λ3 g3 == 0}, {p1, λ1, λ2, λ3}],
 t > 2 qo > 0 && 0 < δ < 1]
```

Out[∘]=

$$\{\{p_1 \to \text{Root}[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\,(794\,\delta^2\,q_o - 790\,\delta^3\,q_o) + \#1^3\,(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2) + \#1^2\,(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3) + \#1\,(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4)\,\&,\,1],\ \lambda_1 \to 0,\ \lambda_2 \to 0,\ \lambda_3 \to 0\},$$

$$\{p_1 \to \text{Root}[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\,(794\,\delta^2\,q_o - 790\,\delta^3\,q_o) + \#1^3\,(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2) + \#1^2\,(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3) + \#1\,(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4)\,\&,\,2],\ \lambda_1 \to 0,\ \lambda_2 \to 0,\ \lambda_3 \to 0\},$$

$$\{p_1 \to \text{Root}[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\,(794\,\delta^2\,q_o - 790\,\delta^3\,q_o) + \#1^3\,(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2) + \#1^2\,(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3) + \#1\,(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4)\,\&,\,3],\ \lambda_1 \to 0,\ \lambda_2 \to 0,\ \lambda_3 \to 0\},$$

$$\{p_1 \to \text{Root}[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\,(794\,\delta^2\,q_o - 790\,\delta^3\,q_o) + \#1^3\,(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2) + \#1^2\,(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3) + \#1\,(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4)\,\&,\,4],\ \lambda_1 \to 0,\ \lambda_2 \to 0,\ \lambda_3 \to 0\},$$

$$\{p_1 \to \text{Root}[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\,(794\,\delta^2\,q_o - 790\,\delta^3\,q_o) + \#1^3\,(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2) + \#1^2\,(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3) + \#1\,(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4)\,\&,\,5],$$

$$\lambda_1 \to 0,\ \lambda_2 \to 0,\ \lambda_3 \to 0\},\ \{p_1 \to -\frac{\left(17 - 17\,\delta + \sqrt{289 - 290\,\delta + \delta^2}\right) q_o}{9\,\delta},$$

$$\lambda_1 \to 0,$$

$$\lambda_2 \to 0,$$

$$\lambda_3 \to \frac{1}{497\,664\,t\,(-289 + \delta)\,\delta^2}\left(-58\,400\,\delta^3 + 1\,999\,591\left(17 + \sqrt{289 - 290\,\delta + \delta^2}\right) + 7\,\delta^2\left(2\,436\,201 + 544\,\sqrt{289 - 290\,\delta + \delta^2}\right) - 17\,\delta\left(2\,995\,638 + 117\,199\,\sqrt{289 - 290\,\delta + \delta^2}\right)\right) q_o\},$$

$$\{p_1 \to \frac{\left(-17 + 17\,\delta + \sqrt{289 - 290\,\delta + \delta^2}\right) q_o}{9\,\delta},\ \lambda_1 \to 0,\ \lambda_2 \to 0,$$

$$\lambda_3 \to -\frac{1}{497\,664\,t\,(-289 + \delta)\,\delta^2}\left(58\,400\,\delta^3 + \delta\left(50\,925\,846 - 1\,992\,383\,\sqrt{289 - 290\,\delta + \delta^2}\right) + 1\,999\,591\left(-17 + \sqrt{289 - 290\,\delta + \delta^2}\right) + 7\,\delta^2\left(-2\,436\,201 + 544\,\sqrt{289 - 290\,\delta + \delta^2}\right)\right) q_o\},$$

$$\{p_1 \to -\frac{2\left(4 - 5\,\delta + \sqrt{16 - 19\,\delta + 4\,\delta^2}\right) q_o}{7\,\delta},\ \lambda_1 \to$$

$$\Big(3\,\Big(10624\,\delta^4+14944\,\Big(4+\sqrt{16-19\,\delta+4\,\delta^2}\Big)-32\,\delta^3\,\Big(2300+177\,\sqrt{16-19\,\delta+4\,\delta^2}\Big)+2\,\delta^2\,\Big(83668+11575\,\sqrt{16-19\,\delta+4\,\delta^2}\Big)-\delta\,\Big(163528+32009\,\sqrt{16-19\,\delta+4\,\delta^2}\Big)\Big)\,q_o\Big)\Big/\Big(2744\,t\,\delta^2\,\big(16-19\,\delta+4\,\delta^2\big)\Big),\ \lambda_2\to 0,\ \lambda_3\to 0\Big\},\ \Big\{p_1\to\frac{2\,\Big(-4+5\,\delta+\sqrt{16-19\,\delta+4\,\delta^2}\Big)\,q_o}{7\,\delta},$$

$$\lambda_1\to\Big(3\,\Big(10624\,\delta^4+\delta^2\,\Big(167336-23150\,\sqrt{16-19\,\delta+4\,\delta^2}\Big)-14944\,\Big(-4+\sqrt{16-19\,\delta+4\,\delta^2}\Big)+32\,\delta^3\,\Big(-2300+177\,\sqrt{16-19\,\delta+4\,\delta^2}\Big)+\delta\,\Big(-163528+32009\,\sqrt{16-19\,\delta+4\,\delta^2}\Big)\Big)\,q_o\Big)\Big/\Big(2744\,t\,\delta^2\,\big(16-19\,\delta+4\,\delta^2\big)\Big),\ \lambda_2\to 0,\ \lambda_3\to 0\Big\},$$

$$\Big\{p_1\to-\frac{2\,\Big(5+2\,\sqrt{2}+\sqrt{-\big(33+20\,\sqrt{2}-4\,\delta\big)\,(-1+\delta)}-\big(5+2\,\sqrt{2}\big)\,\delta\Big)\,q_o}{\big(7+2\,\sqrt{2}\big)\,\delta},$$

$$\lambda_1\to 0,$$

$$\lambda_2\to-\Big(\Big(\Big(4\,\Big(2517144475568043+1779889761461026\,\sqrt{2}+321539019529183\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}+227362344480532\,\sqrt{2}\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}\Big)-\Big(16419286962987025+11610187675221514\,\sqrt{2}+1412341611519143\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}+998675933302184\,\sqrt{2}\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}\Big)\,\delta+4\,\Big(1716856959487607+1214000938601780\,\sqrt{2}+45448007246920\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}+32136565884196\,\sqrt{2}\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}\Big)\,\delta^2-96\,\big(3969644595753+2806960280177\,\sqrt{2}\big)\,\delta^3\Big)\,q_o\Big)\Big/\Big(2\,t\,\delta^2\,\Big(-8312163094737191-5877586879245434\,\sqrt{2}+4\,\big(135632896043287+95906938132718\,\sqrt{2}\big)\,\delta\Big)\Big)\Big),$$

$$\lambda_3\to 0\Big\},\ \Big\{p_1\to\frac{2\,\Big(-5-2\,\sqrt{2}+\sqrt{-\big(33+20\,\sqrt{2}-4\,\delta\big)\,(-1+\delta)}+\big(5+2\,\sqrt{2}\big)\,\delta\Big)\,q_o}{\big(7+2\,\sqrt{2}\big)\,\delta},$$

$$\lambda_1\to 0,$$

$$\lambda_2\to\Big(\Big(4\,\Big(-2517144475568043-1779889761461026\,\sqrt{2}+321539019529183\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}+227362344480532\,\sqrt{2}\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}\Big)+\Big(16419286962987025+11610187675221514\,\sqrt{2}-1412341611519143\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}-998675933302184\,\sqrt{2}\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}\Big)\,\delta+4\,\Big(-1716856959487607-1214000938601780\,\sqrt{2}+45448007246920\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}+32136565884196\,\sqrt{2}\,\sqrt{\big(33+20\,\sqrt{2}-4\,\delta\big)\,(1-\delta)}\Big)\,\delta^2+96\,\big(3969644595753+2806960280177\,\sqrt{2}\big)\,\delta^3\Big)\,q_o\Big)\Big/\Big(2\,t\,\delta^2\,\Big(-8312163094737191-5877586879245434\,\sqrt{2}+4\,\big(135632896043287+95906938132718\,\sqrt{2}\big)\,\delta\Big)\Big),\ \lambda_3\to 0\Big\}\Big\}$$

(*There are 11 solutions, we then check each solution if it satisfies conditions*)

```
(*Solution 1, interior solution*)
```

In[∘]:= p_1 = Root$\left[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\,\left(794\,\delta^2\,q_o - 790\,\delta^3\,q_o\right) + \#1^3\,\left(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2\right) + \#1^2\,\left(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3\right) + \#1\,\left(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4\right)\ \&,\ 1\right]$;

$\lambda_1 = 0;$

$\lambda_2 = 0;$

$\lambda_3 = 0;$

Reduce$\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1,\ \text{Reals}\right]$

Out[∘]=

False

(*Hence, solution 1 does not satisfy conditions of $p_1 > \frac{2q_o+tD_1}{3}\&\&p_1 > \frac{(3+2\sqrt{2})tD_1}{2}\&\&p_1 < 16tD_1$*)

```
(*Solution 2, interior solution*)
```

In[∘]:= p_1 = Root$\left[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\,\left(794\,\delta^2\,q_o - 790\,\delta^3\,q_o\right) + \#1^3\,\left(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2\right) + \#1^2\,\left(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3\right) + \#1\,\left(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4\right)\ \&,\ 2\right]$;

$\lambda_1 = 0;$

$\lambda_2 = 0;$

$\lambda_3 = 0;$

Reduce$\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1,\ \text{Reals}\right]$

Out[∘]=

False

(*Hence, solution 2 does not satisfy conditions of $p_1 > \frac{2q_o+tD_1}{3}\&\&p_1 > \frac{(3+2\sqrt{2})tD_1}{2}\&\&p_1 < 16tD_1$*)

```
(*Solution 3, interior solution*)
```

In[∘]:= p_1 = Root$\left[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\,\left(794\,\delta^2\,q_o - 790\,\delta^3\,q_o\right) + \#1^3\,\left(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2\right) + \#1^2\,\left(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3\right) + \#1\,\left(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4\right)\ \&,\ 3\right]$;

$\lambda_1 = 0;$

$\lambda_2 = 0;$

$\lambda_3 = 0;$

Reduce$\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\ \&\&\ p_1 < 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1,\ \text{Reals}\right]$

Out[∘]=

False

(*Hence, solution 3 does not satisfy conditions of $p_1 > \frac{2q_o+tD_1}{3}\&\&p_1 > \frac{(3+2\sqrt{2})tD_1}{2}\&\&p_1 < 16tD_1$*)

```
(*Solution 4, interior solution*)
```

In[•]:= $p_1 = \text{Root}\left[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\left(794\,\delta^2\,q_o - 790\,\delta^3\,q_o\right) + \#1^3\left(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2\right) + \#1^2\left(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3\right) + \#1\left(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4\right)\,\&,\,4\right];$

$\lambda_1 = 0;$

$\lambda_2 = 0;$

$\lambda_3 = 0;$

$\text{Reduce}\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\;\&\&\;p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\;\&\&\;p_1 < 16\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta < 1,\,\text{Reals}\right]$

Out[•]=

False

(*Hence, solution 4 does not satisfy conditions of $p_1 > \frac{2q_o+tD_1}{3}$ &&$p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$ &&$p_1 < 16tD_1$*)

```
(*Solution 5, interior solution*)
```

In[•]:= $p_1 = \text{Root}\left[111\,\delta^3\,\#1^5 - 32\,q_o^5 + 96\,\delta\,q_o^5 - 96\,\delta^2\,q_o^5 + 32\,\delta^3\,q_o^5 + \#1^4\left(794\,\delta^2\,q_o - 790\,\delta^3\,q_o\right) + \#1^3\left(1968\,\delta\,q_o^2 - 4048\,\delta^2\,q_o^2 + 2072\,\delta^3\,q_o^2\right) + \#1^2\left(1632\,q_o^3 - 5664\,\delta\,q_o^3 + 6384\,\delta^2\,q_o^3 - 2352\,\delta^3\,q_o^3\right) + \#1\left(-960\,q_o^4 + 2896\,\delta\,q_o^4 - 2880\,\delta^2\,q_o^4 + 944\,\delta^3\,q_o^4\right)\,\&,\,5\right];$

$\lambda_1 = 0;$

$\lambda_2 = 0;$

$\lambda_3 = 0;$

$\text{Reduce}\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\;\&\&\;p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\;\&\&\;p_1 < 16\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta < 1,\,\text{Reals}\right]$

Out[•]=

False

(*Hence, solution 5 does not satisfy conditions of $p_1 > \frac{2q_o+tD_1}{3}$ &&$p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$ &&$p_1 < 16tD_1$*)

(*Solution 6, boundary solution, which is the solution of $p_1 = 16tD_1$*)

In[•]:= $p_1 = -\frac{\left(17 - 17\,\delta + \sqrt{289 - 290\,\delta + \delta^2}\right)\,q_o}{9\,\delta};$

$\lambda_1 = 0;$

$\lambda_2 = 0;$

$\lambda_3 = \frac{1}{497\,664\,t\,(-289 + \delta)\,\delta^2}\left(-58\,400\,\delta^3 + 1\,999\,591\left(17 + \sqrt{289 - 290\,\delta + \delta^2}\right) + 7\,\delta^2\left(2\,436\,201 + 544\,\sqrt{289 - 290\,\delta + \delta^2}\right) - 17\,\delta\left(2\,995\,638 + 117\,199\,\sqrt{289 - 290\,\delta + \delta^2}\right)\right)\,q_o;$

$\text{Reduce}\left[\lambda_3 > 0\;\&\&\;p_1 > \frac{2\,q_o + t\,D_1}{3}\;\&\&\;p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\;\&\&\;p_1 == 16\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta < 1,\,\text{Reals}\right]$

Out[=]=

False

(*Hence, solution 6 does not satisfy conditions of $p_1>\frac{2q_o+tD_1}{3}$&&$p_1>\frac{(3+2\sqrt{2})tD_1}{2}$&&$p_1$=16t$D_1$*)

(*Solution 7, boundary solution, which is the solution of p_1=16tD_1*)

In[=]:=

$$p_1 = \frac{\left(-17 + 17\,\delta + \sqrt{289 - 290\,\delta + \delta^2}\right) q_o}{9\,\delta};$$

$$\lambda_1 = 0;$$

$$\lambda_2 = 0;$$

$$\lambda_3 = -\frac{1}{497\,664\,t\,(-289+\delta)\,\delta^2}\left(58\,400\,\delta^3 + \delta\left(50\,925\,846 - 1\,992\,383\,\sqrt{289 - 290\,\delta + \delta^2}\right) + 1\,999\,591\left(-17 + \sqrt{289 - 290\,\delta + \delta^2}\right) + 7\,\delta^2\left(-2\,436\,201 + 544\,\sqrt{289 - 290\,\delta + \delta^2}\right)\right) q_o;$$

$$\text{Reduce}\left[\lambda_3 > 0\;\&\&\;p_1 > \frac{2\,q_o + t\,D_1}{3}\;\&\&\;p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\;\&\&\;p_1 == 16\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta < 1,\ \text{Reals}\right]$$

Out[=]=

False

(*Hence, solution 7 does not satisfy conditions of $p_1>\frac{2q_o+tD_1}{3}$&&$p_1>\frac{(3+2\sqrt{2})tD_1}{2}$&&$p_1$=16t$D_1$*)

(*Solution 8, boundary solution, which is the solution of $p_1=\frac{2q_o+tD_1}{3}$*)

In[=]:=

$$p_1 = -\frac{2\left(4 - 5\,\delta + \sqrt{16 - 19\,\delta + 4\,\delta^2}\right) q_o}{7\,\delta};$$

$$\lambda_1 = \left(3\left(10\,624\,\delta^4 + 14\,944\left(4 + \sqrt{16 - 19\,\delta + 4\,\delta^2}\right) - 32\,\delta^3\left(2300 + 177\,\sqrt{16 - 19\,\delta + 4\,\delta^2}\right) + 2\,\delta^2\left(83\,668 + 11\,575\,\sqrt{16 - 19\,\delta + 4\,\delta^2}\right) - \delta\left(163\,528 + 32\,009\,\sqrt{16 - 19\,\delta + 4\,\delta^2}\right)\right) q_o\right) \Big/ \left(2744\,t\,\delta^2\left(16 - 19\,\delta + 4\,\delta^2\right)\right);$$

$$\lambda_2 = 0;$$

$$\lambda_3 = 0;$$

$$\text{Reduce}\left[\lambda_1 > 0\;\&\&\;p_1 == \frac{2\,q_o + t\,D_1}{3}\;\&\&\;p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\;\&\&\;p_1 < 16\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta < 1,\ \text{Reals}\right]$$

Out[=]=

False

(*Hence, solution 8 does not satisfy conditions of $p_1=\frac{2q_o+tD_1}{3}$&&$p_1>\frac{(3+2\sqrt{2})tD_1}{2}$&&$p_1$<16t$D_1$*)

(*Solution 9, boundary solution, which is the solution of $p_1=\frac{2q_o+tD_1}{3}$*)

In[]:=
$$p_1 = \frac{2\left(-4+5\,\delta+\sqrt{16-19\,\delta+4\,\delta^2}\right)q_o}{7\,\delta};$$

$$\lambda_1 = \left(3\left(10624\,\delta^4+\delta^2\left(167336-23150\,\sqrt{16-19\,\delta+4\,\delta^2}\right)-14944\left(-4+\sqrt{16-19\,\delta+4\,\delta^2}\right)+32\,\delta^3\left(-2300+177\,\sqrt{16-19\,\delta+4\,\delta^2}\right)+\delta\left(-163528+32009\,\sqrt{16-19\,\delta+4\,\delta^2}\right)\right)q_o\right)\Big/\left(2744\,t\,\delta^2\left(16-19\,\delta+4\,\delta^2\right)\right);$$

$\lambda_2 = 0;$

$\lambda_3 = 0;$

$$\text{Reduce}\left[\lambda_1>0\;\&\&\;p_1==\frac{2\,q_o+t\,D_1}{3}\;\&\&\;p_1>\frac{\left(3+2\,\sqrt{2}\right)t\,D_1}{2}\;\&\&\;p_1<16\,t\,D_1\;\&\&\;D_1>0\;\&\&\;t>2\,q_o>0\;\&\&\;0<\delta<1,\text{Reals}\right]$$

Out[]=
$$q_o>0\;\&\&\;0<\delta<\frac{611}{899}\;\&\&\;t>2\,q_o$$

(*Hence, when $0<\delta<\frac{611}{899}$,
solution 9 satisfies conditions of $p_1=\frac{2q_o+tD_1}{3}$&&$p_1>\frac{(3+2\sqrt{2})tD_1}{2}$&&$p_1<16tD_1$*)

(*Solution 10, boundary solution, which is the solution of $p_1=\frac{(3+2\sqrt{2})tD_1}{2}$*)

In[]:=
$$p_1 = -\frac{2\left(5+2\,\sqrt{2}+\sqrt{-\left(33+20\,\sqrt{2}-4\,\delta\right)(-1+\delta)}-\left(5+2\,\sqrt{2}\right)\delta\right)q_o}{\left(7+2\,\sqrt{2}\right)\delta};$$

$\lambda_1 = 0;$

$$\lambda_2 = -\left(\left(\left(4\left(2517144475568043+1779889761461026\,\sqrt{2}+321539019529183\,\sqrt{\left(33+20\,\sqrt{2}-4\,\delta\right)(1-\delta)}+227362344480532\,\sqrt{2}\,\sqrt{\left(33+20\,\sqrt{2}-4\,\delta\right)(1-\delta)}\right)-\left(16419286962987025+11610187675221514\,\sqrt{2}+1412341611519143\,\sqrt{\left(33+20\,\sqrt{2}-4\,\delta\right)(1-\delta)}+998675933302184\,\sqrt{2}\,\sqrt{\left(33+20\,\sqrt{2}-4\,\delta\right)(1-\delta)}\right)\delta+4\left(1716856959487607+1214000938601780\,\sqrt{2}+45448007246920\,\sqrt{\left(33+20\,\sqrt{2}-4\,\delta\right)(1-\delta)}+32136565884196\,\sqrt{2}\,\sqrt{\left(33+20\,\sqrt{2}-4\,\delta\right)(1-\delta)}\right)\delta^2-96\left(3969644595753+2806960280177\,\sqrt{2}\right)\delta^3\right)q_o\right)\Big/\left(2\,t\,\delta^2\left(-8312163094737191-5877586879245434\,\sqrt{2}+4\left(135632896043287+95906938132718\,\sqrt{2}\right)\delta\right)\right)\right);$$

$\lambda_3 = 0;$

$$\text{Reduce}\left[\lambda_2>0\;\&\&\;p_1>\frac{2\,q_o+t\,D_1}{3}\;\&\&\;p_1==\frac{\left(3+2\,\sqrt{2}\right)t\,D_1}{2}\;\&\&\;p_1<16\,t\,D_1\;\&\&\;D_1>0\;\&\&\;t>2\,q_o>0\;\&\&\;0<\delta<1,\text{Reals}\right]$$

Out[]=
False

```
(*Hence,
solution 10 does not satisfy conditions of p1>(2qo+tD1)/3 &&p1=((3+2 √2) tD1)/2 &&p1<16tD1*)
(*Solution 11, boundary solution, which is the solution of p1=((3+2 √2) tD1)/2 *)
```

In[]:=

$$p_1 = \frac{2\left(-5-2\sqrt{2}+\sqrt{-\left(33+20\sqrt{2}-4\delta\right)(-1+\delta)}+\left(5+2\sqrt{2}\right)\delta\right)q_o}{\left(7+2\sqrt{2}\right)\delta};$$

$\lambda_1 = 0;$

$$\lambda_2 = \Big(\Big(4\Big(-2517144475568043 - 1779889761461026\sqrt{2} + 321539019529183\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} + 227362344480532\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big) + \Big(16419286962987025 + 11610187675221514\sqrt{2} - 1412341611519143\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} - 998675933302184\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big)\delta + 4\Big(-1716856959487607 - 1214000938601780\sqrt{2} + 45448007246920\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)} + 32136565884196\sqrt{2}\sqrt{\left(33+20\sqrt{2}-4\delta\right)(1-\delta)}\Big)\delta^2 + 96\left(3969644595753 + 2806960280177\sqrt{2}\right)\delta^3\Big)q_o\Big) \Big/ \Big(2\,t\,\delta^2\Big(-8312163094737191 - 5877586879245434\sqrt{2} + 4\left(135632896043287 + 95906938132718\sqrt{2}\right)\delta\Big)\Big);$$

$\lambda_3 = 0;$

$$\text{Reduce}\left[\lambda_2 > 0 \,\&\&\, p_1 > \frac{2q_o + t D_1}{3} \,\&\&\, p_1 == \frac{\left(3+2\sqrt{2}\right)t D_1}{2} \,\&\&\, p_1 < 16\,t D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2q_o > 0 \,\&\&\, 0 < \delta < 1, \text{Reals}\right]$$

Out[]=

False

```
(*Hence,
solution 11 does not satisfy conditions of p1>(2qo+tD1)/3 &&p1=((3+2 √2) tD1)/2 &&p1<16tD1*)
(*Overall, when 0<δ<611/899, p1=(2 (-4+5 δ+ √(16-19 δ+4 δ^2)) qo)/(7 δ) *)
(*Combination 12. The conditions are p1>(2qo+tD1)/3, p1>((3+2 √2) tD1)/2, and p1>16tD1*)
```

```
p2P = (2 qo + p1 + t D1)/4;(*The second-period price under compltely positive reviews*)
p2M = (2 p1 + t D1)/4;(*The second-period price under mixed reviews*)
p2N = p1/4;(*The second-period price under compltely negative reviews*)
D2P = (2 qo + p1 - 3 t D1)/(4 t);(*The second-period demand under compltely positive reviews*)
D2M = (2 p1 - 3 t D1)/(4 t);(*The second-period demand under mixed reviews*)
D2N = (p1 - 4 t D1)/(4 t);(*The second-period demand under compltely negative reviews*)
```

In[]:=

$$U_1 = \text{Simplify}\left[q_o - p_1 - t\,D_1 + \delta\left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,(p_1 - p_{2P}) + \frac{t\,D_1}{2\,q_o}\,(p_1 - p_{2M}) + \frac{p_1}{2\,q_o}\,(p_1 - p_{2N})\right)\right];$$

(*Consumers' expected utility purchasing in the first period*)

$$U_2 = \text{Simplify}\left[\delta\left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2P} - t\,D_1\right) + \frac{t\,D_1}{2\,q_o}\left(\frac{2\,p_1 + t\,D_1}{2} - p_{2M} - t\,D_1\right) + \frac{p_1}{2\,q_o}\left(\frac{p_1}{2} - p_{2N} - t\,D_1\right)\right)\right];$$

(*Consumers' expected utility when purchasing in the second period*)

In[]:= Simplify[Solve[U_1 == U_2, D_1]]

Out[]=

$$\left\{\left\{D_1 \to \frac{-p_1 + q_o}{t}\right\}\right\}$$

$$D_1 = \frac{-p_1 + q_o}{t};$$ (*The response function of D_1*)

$$\text{Reduce}\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\sqrt{2}\right)t\,D_1}{2}\ \&\&\ p_1 > 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta < 1\right]$$

(*We check if D_1 satisfies conditions*)

Out[]=

$$p_1 > 0\ \&\&\ p_1 < q_o < \frac{17\,p_1}{16}\ \&\&\ t > 2\,q_o\ \&\&\ 0 < \delta < 1$$

(*Hence, the response function of D_1 satisfies conditionsis and is given by*)

In[]:=

$$D_1 = \frac{-p_1 + q_o}{t};$$

$$\Pi = \text{Simplify}\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,(p_{2P}\,D_{2P} - D_1\,(p_1 - p_{2P})) + \frac{t\,D_1}{2\,q_o}\,(p_{2M}\,D_{2M} - D_1\,(p_1 - p_{2M})) + \frac{p_1}{2\,q_o}\,(p_{2N}\,D_{2N} - D_1\,(p_1 - p_{2N}))\right];$$ (*The firm's total profit function*)

Simplify[D[D[Π, p_1], p_1]] (*Calculate $\frac{\partial^2\Pi}{\partial p_1^2}$*)

Out[]=

$$-\frac{1}{16\,t}$$

(*$\frac{\partial^2\Pi}{\partial p_1^2}$<0, meaning Π is concave and it has a maximum value at point where $\frac{\partial\Pi}{\partial p_1}$=0*)

```
(*KKT conditions*)
```

$$g_1 = p_1 - \frac{2\,q_o + t\,D_1}{3};$$

$$g_2 = p_1 - \frac{\left(3 + 2\sqrt{2}\right) t\,D_1}{2};$$

$$g_3 = p_1 - 16\,t\,D_1;$$

$$L = -\Pi - \lambda_1\,g_1 - \lambda_2\,g_2 - \lambda_3\,g_3;$$

In[∘]:= $\text{Simplify}[\text{Solve}[\{D[L, p_1] == 0, \lambda_1 g_1 == 0, \lambda_2 g_2 == 0, \lambda_3 g_3 == 0\}, \{p_1, \lambda_1, \lambda_2, \lambda_3\}]]$

Out[∘]=

$$\left\{\left\{p_1 \to \frac{\left(3 + 2\sqrt{2}\right) q_o}{5 + 2\sqrt{2}}, \lambda_1 \to 0, \lambda_2 \to \frac{\left(1 + 2\sqrt{2}\right) q_o}{16\left(5 + 2\sqrt{2}\right)^2 t}, \lambda_3 \to 0\right\},\right.$$
$$\left\{p_1 \to \frac{3\,q_o}{4}, \lambda_1 \to \frac{3\,q_o}{256\,t}, \lambda_2 \to 0, \lambda_3 \to 0\right\},$$
$$\left.\left\{p_1 \to \frac{16\,q_o}{17}, \lambda_1 \to 0, \lambda_2 \to 0, \lambda_3 \to \frac{15\,q_o}{9248\,t}\right\}, \left\{p_1 \to \frac{q_o}{2}, \lambda_1 \to 0, \lambda_2 \to 0, \lambda_3 \to 0\right\}\right\}$$

```
(*There are 4 solutions, we then check each solution if it satisfies conditions*)
```

(*Solution 1, boundary solution, which is the solution of $p_1 = \frac{\left(3+2\sqrt{2}\right) tD_1}{2}$ *)

$$p_1 = \frac{\left(3 + 2\sqrt{2}\right) q_o}{5 + 2\sqrt{2}};$$

$$\lambda_1 = 0;$$

$$\lambda_2 = \frac{\left(1 + 2\sqrt{2}\right) q_o}{16\left(5 + 2\sqrt{2}\right)^2 t};$$

$$\lambda_3 = 0;$$

$$\text{Reduce}\left[\lambda_2 > 0 \,\&\&\, p_1 > \frac{2\,q_o + t\,D_1}{3} \,\&\&\, p_1 == \frac{\left(3 + 2\sqrt{2}\right) t\,D_1}{2} \,\&\&\, p_1 > 16\,t\,D_1 \,\&\&\, D_1 > 0 \,\&\&\, t > 2\,q_o > 0 \,\&\&\, 0 < \delta < 1\right]$$

Out[∘]=

```
False
```

(*Hence, solution 1 does not satisfy conditions of $p_1 > \frac{2q_o + tD_1}{3}$, $p_1 = \frac{\left(3+2\sqrt{2}\right) tD_1}{2}$, and $p_1 > 16tD_1$*)

(*Solution 2, boundary solution, which is the solution of $p_1 = \frac{2q_o + tD_1}{3}$ *)

In[∘]:=
```
p1 = 3 qo / 4;
λ1 = 3 qo / (256 t);
λ2 = 0;
λ3 = 0;
Reduce[λ1 > 0 && p1 == (2 qo + t D1)/3 &&
  p1 > ((3 + 2 Sqrt[2]) t D1)/2 && p1 > 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1]
```

Out[∘]=

False

(*Hence, solution 2 does not satisfy conditions of $p_1=\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1>16tD_1$*)

(*Solution 3, boundary solution, which is the solution of $p_1=16tD_1$*)

In[∘]:=
```
p1 = 16 qo / 17;
λ1 = 0;
λ2 = 0;
λ3 = 15 qo / (9248 t);
Reduce[λ3 > 0 && p1 > (2 qo + t D1)/3 &&
  p1 > ((3 + 2 Sqrt[2]) t D1)/2 && p1 == 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1]
```

Out[∘]=

$q_o > 0$ && $t > 2 q_o$ && $0 < \delta < 1$

(*Hence, when $0<\delta<1$, solution 3 satisfies conditions of $p_1>\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1=16tD_1$*)

(*Solution 4, interior solution*)

In[∘]:=
```
p1 = qo / 2;
λ1 = 0;
λ2 = 0;
λ3 = 0;
Reduce[p1 > (2 qo + t D1)/3 && p1 > ((3 + 2 Sqrt[2]) t D1)/2 && p1 > 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δ < 1]
```

Out[∘]=

False

(*Hence, solution 4 does not satisfy conditions of $p_1>\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1>16tD_1$*)

```
(*Overall, when 0<δ<1, p1=16 qo/17 *)

(*Proof of Proposition 7(i) Step 2: Optimal price through profit comparison*)

(*The results of the 12 price combinations are presented below. For convenience,
we use πyz to denote the profit of the zth scenario under the yth price conbination*)

(*Results of price combination 1(i) 0<δ< 0.442… *)
```

In[∘]:=
$$p_1 = -\frac{q_o\left(-1+\sqrt{1-\delta}\right)}{\delta};$$
$$D_1 = \frac{2\,q_o\,(-2+\delta) + \sqrt{-8\,q_o^2\,(-2+\delta) - 8\,p_1\,q_o\,\delta + p_1^2\,\delta^2}}{t\,\delta};$$
$$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4};$$
$$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t};$$
$$\Pi_{11} = \text{Simplify}\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P}\right];$$

```
(*Results of price combination 1(ii) 0.442…<δ<0.864… *)
```

In[∘]:=
$$p_1 = \text{Root}\left[8\,\delta^4\,\#1^4 + 144\,q_o^4 + 225\,\delta\,q_o^4 - 272\,\delta^2\,q_o^4 + 64\,\delta^3\,q_o^4 + \#1^3\left(-16\,\delta^2\,q_o - 94\,\delta^3\,q_o\right) + \#1^2\left(124\,\delta\,q_o^2 + 376\,\delta^2\,q_o^2 - 30\,\delta^3\,q_o^2 - 8\,\delta^4\,q_o^2\right) + \#1\left(-224\,q_o^3 - 480\,\delta\,q_o^3 + 104\,\delta^2\,q_o^3 + 64\,\delta^3\,q_o^3\right)\,\&,\,1\right];$$
$$D_1 = \frac{2\,q_o\,(-2+\delta) + \sqrt{-8\,q_o^2\,(-2+\delta) - 8\,p_1\,q_o\,\delta + p_1^2\,\delta^2}}{t\,\delta};$$
$$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4};$$
$$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t};$$
$$\Pi_{12} = p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P};$$

```
(*Results of price combination 1(iii) 0.864…<δ<1)
```

In[∘]:=
$$p_1 = -\frac{1}{119\,\delta}\,2\left(-170 - 136\sqrt{2} + 2\left(9 + 10\sqrt{2}\right)\delta + \sqrt{1156\left(57+40\sqrt{2}\right) - 578\left(57+40\sqrt{2}\right)\delta + \left(9097 + 6432\sqrt{2}\right)\delta^2}\right)q_o;$$
$$D_1 = \frac{2\,q_o\,(-2+\delta) + \sqrt{-8\,q_o^2\,(-2+\delta) - 8\,p_1\,q_o\,\delta + p_1^2\,\delta^2}}{t\,\delta};$$
$$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4};$$
$$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t};$$
$$\Pi_{13} = \text{Simplify}\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P}\right];$$

```
(*Results of price combination 3(i) 0.566…<δ<0.777…*)
```

In[∘]:= $p_1 = -\frac{1}{51\,\delta}\,2\left(-66-40\sqrt{2}+2\left(5+2\sqrt{2}\right)\delta+\sqrt{7556+5280\sqrt{2}-2\left(2153+1480\sqrt{2}\right)\delta+\left(1305+896\sqrt{2}\right)\delta^2}\right)q_o;$

$D_1 = -\frac{2\,(-2+\delta)\,q_o+\sqrt{-\delta^2\,p_1^2+8\,\delta\,p_1\,q_o+8\left(2-3\,\delta+\delta^2\right)q_o^2}}{t\,\delta};$

$p_{2P} = \frac{2\,q_o+p_1-t\,D_1}{4};$

$p_{2M} = \frac{2\,p_1+t\,D_1}{4};$

$D_{2P} = \frac{2\,q_o+p_1-t\,D_1}{4\,t};$

$D_{2M} = \frac{2\,p_1-3\,t\,D_1}{4\,t};$

$\Pi_{31} = \text{Simplify}\left[p_1\,D_1+\frac{2\,q_o-p_1-t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P}+\frac{t\,D_1}{2\,q_o}\,(p_{2M}\,D_{2M}-D_1\,(p_1-p_{2M}))\right];$

```
(*Results of price combination 3(ii) 0.777…<δ<1*)
```

In[∘]:= $p_1 = \text{Root}\Big[1125\,\#1^6+60\,928\,q_o^6+\frac{236\,544\,q_o^6}{\delta^4}-\frac{766\,464\,q_o^6}{\delta^3}+\frac{859\,904\,q_o^6}{\delta^2}-\frac{388\,608\,q_o^6}{\delta}+\#1^5\left(2160\,q_o-\frac{23\,640\,q_o}{\delta}\right)+\#1^4\left(312\,q_o^2+\frac{151\,312\,q_o^2}{\delta^2}-\frac{12\,096\,q_o^2}{\delta}\right)+\#1^3\left(-20\,160\,q_o^3-\frac{205\,056\,q_o^3}{\delta^3}-\frac{193\,152\,q_o^3}{\delta^2}+\frac{73\,664\,q_o^3}{\delta}\right)+\#1^2\left(-40\,624\,q_o^4-\frac{574\,464\,q_o^4}{\delta^4}+\frac{1\,165\,824\,q_o^4}{\delta^3}-\frac{582\,720\,q_o^4}{\delta^2}+\frac{266\,496\,q_o^4}{\delta}\right)+\#1\left(27\,648\,q_o^5+\frac{67\,584\,q_o^5}{\delta^4}+\frac{170\,496\,q_o^5}{\delta^3}-\frac{376\,320\,q_o^5}{\delta^2}+\frac{58\,240\,q_o^5}{\delta}\right)\&,3\Big];$

$D_1 = -\frac{2\,(-2+\delta)\,q_o+\sqrt{-\delta^2\,p_1^2+8\,\delta\,p_1\,q_o+8\left(2-3\,\delta+\delta^2\right)q_o^2}}{t\,\delta};$

$p_{2P} = \frac{2\,q_o+p_1-t\,D_1}{4};$

$p_{2M} = \frac{2\,p_1+t\,D_1}{4};$

$D_{2P} = \frac{2\,q_o+p_1-t\,D_1}{4\,t};$

$D_{2M} = \frac{2\,p_1-3\,t\,D_1}{4\,t};$

$\Pi_{32} = \text{Simplify}\left[p_1\,D_1+\frac{2\,q_o-p_1-t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P}+\frac{t\,D_1}{2\,q_o}\,(p_{2M}\,D_{2M}-D_1\,(p_1-p_{2M}))\right];$

```
(*Results of price combination 5(i) 0<δ<1/2*)
```

In[∘]:= $p_1 = \frac{\left(2 - 2\sqrt{1-\delta} + \left(-3 + 2\sqrt{1-\delta}\right)\delta + 2\delta^2\right) q_o}{2\delta^2}$;

$D_1 = \frac{-\delta\, p_1 + 2\left(-1 + \sqrt{1-\delta} + \delta\right) q_o}{t\,\delta}$;

$p_{2P} = p_1$;

$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2t}$;

$\Pi_{51} = \text{Simplify}\left[p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P}\right]$;

(*Results of price combination 5(ii) $\frac{1}{2}<\delta<\frac{35+28\sqrt{2}}{68+48\sqrt{2}}$ *)

In[∘]:= $p_1 = -\frac{q_o\left(-1 + \sqrt{1-\delta}\right)}{\delta}$;

$D_1 = \frac{-\delta\, p_1 + 2\left(-1 + \sqrt{1-\delta} + \delta\right) q_o}{t\,\delta}$;

$p_{2P} = p_1$;

$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2t}$;

$\Pi_{52} = \text{Simplify}\left[p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P}\right]$;

(*Results of price combination 7(i) 0<δ< 0.557... *)

In[∘]:= $p_1 = \frac{2\left(-5 - 2\sqrt{2} + \sqrt{-\left(33 + 20\sqrt{2} - 4\delta\right)(-1+\delta)} + \left(5 + 2\sqrt{2}\right)\delta\right) q_o}{\left(7 + 2\sqrt{2}\right)\delta}$;

$D_1 = \frac{-4 p_1 q_o (-1+\delta) + 4 q_o^2 (-1+\delta) + p_1^2 \delta}{4 t q_o (-1+\delta) - 2 t p_1 \delta}$;

$p_{2P} = p_1$;

$p_{2M} = \frac{2 p_1 + t D_1}{4}$;

$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2t}$;

$D_{2M} = \frac{2 p_1 - 3 t D_1}{4t}$;

$\Pi_{71} = \text{Simplify}\left[p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P} + \frac{t D_1}{2 q_o} (p_{2M} D_{2M} - D_1 (p_1 - p_{2M}))\right]$;

(*Results of price combination 7(ii) 0.557... <δ< 0.569... *)

In[ ]:= $p_1 = \text{Root}\Big[-3200\, q_o^6 + 12\,992\, q_o^6\, \delta - 19\,776\, q_o^6\, \delta^2 + 13\,376\, q_o^6\, \delta^3 + 75\, \#1^6\, \delta^4 - 3392\, q_o^6\, \delta^4 + \#1^5\, \left(996\, q_o\, \delta^3 - 484\, q_o\, \delta^4\right) + \#1^4\, \left(4616\, q_o^2\, \delta^2 - 5532\, q_o^2\, \delta^3 + 660\, q_o^2\, \delta^4\right) + \#1^3\, \left(9088\, q_o^3\, \delta - 18\,144\, q_o^3\, \delta^2 + 6976\, q_o^3\, \delta^3 + 2080\, q_o^3\, \delta^4\right) + \#1^2\, \left(6528\, q_o^4 - 18\,304\, q_o^4\, \delta + 9424\, q_o^4\, \delta^2 + 9952\, q_o^4\, \delta^3 - 7600\, q_o^4\, \delta^4\right) + \#1\, \left(256\, q_o^5 - 9408\, q_o^5\, \delta + 26\,688\, q_o^5\, \delta^2 - 26\,176\, q_o^5\, \delta^3 + 8640\, q_o^5\, \delta^4\right)\, \&,\, 4\Big];$

$D_1 = \dfrac{-4\, p_1\, q_o\, (-1+\delta) + 4\, q_o^2\, (-1+\delta) + p_1^2\, \delta}{4\, t\, q_o\, (-1+\delta) - 2\, t\, p_1\, \delta};$

$p_{2\,P} = p_1;$

$p_{2\,M} = \dfrac{2\, p_1 + t\, D_1}{4};$

$D_{2\,P} = \dfrac{2\, q_o - p_1 - t\, D_1}{2\, t};$

$D_{2\,M} = \dfrac{2\, p_1 - 3\, t\, D_1}{4\, t};$

$\Pi_{72} = \text{Simplify}\Big[p_1\, D_1 + \dfrac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2\,P}\, D_{2\,P} + \dfrac{t\, D_1}{2\, q_o}\, (p_{2\,M}\, D_{2\,M} - D_1\, (p_1 - p_{2\,M}))\Big];$

(*Results of price combination 7(iii) $0.569\ldots < \delta < \frac{245}{341}$ *)

In[ ]:= $p_1 = \dfrac{2\, q_o\, \left(-2 + 3\, \delta + \sqrt{4 - 7\, \delta + 4\, \delta^2}\right)}{5\, \delta};$

$D_1 = \dfrac{-4\, p_1\, q_o\, (-1+\delta) + 4\, q_o^2\, (-1+\delta) + p_1^2\, \delta}{4\, t\, q_o\, (-1+\delta) - 2\, t\, p_1\, \delta};$

$p_{2\,P} = p_1;$

$p_{2\,M} = \dfrac{2\, p_1 + t\, D_1}{4};$

$D_{2\,P} = \dfrac{2\, q_o - p_1 - t\, D_1}{2\, t};$

$D_{2\,M} = \dfrac{2\, p_1 - 3\, t\, D_1}{4\, t};$

$\Pi_{73} = \text{Simplify}\Big[p_1\, D_1 + \dfrac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2\,P}\, D_{2\,P} + \dfrac{t\, D_1}{2\, q_o}\, (p_{2\,M}\, D_{2\,M} - D_1\, (p_1 - p_{2\,M}))\Big];$

(*Results of price combination 11 $0 < \delta < \frac{611}{899}$ *)

In[ ]:= $p_1 = \frac{2\left(-4+5\,\delta+\sqrt{16-19\,\delta+4\,\delta^2}\right)q_o}{7\,\delta}$;

$D_1 = \frac{-4\,p_1\,q_o\,(-1+\delta)+4\,q_o^2\,(-1+\delta)+p_1^2\,\delta}{4\,t\,q_o\,(-1+\delta)-2\,t\,p_1\,\delta}$;

$p_{2P} = \frac{2\,q_o+p_1+t\,D_1}{4}$;

$p_{2M} = \frac{2\,p_1+t\,D_1}{4}$;

$D_{2P} = \frac{2\,q_o+p_1-3\,t\,D_1}{4\,t}$;

$D_{2M} = \frac{2\,p_1-3\,t\,D_1}{4\,t}$;

$\Pi_{111} =$

$\text{Simplify}\left[p_1\,D_1+\frac{2\,q_o-p_1-t\,D_1}{2\,q_o}\,(p_{2P}\,D_{2P}-D_1\,(p_1-p_{2P}))+\frac{t\,D_1}{2\,q_o}\,(p_{2M}\,D_{2M}-D_1\,(p_1-p_{2M}))\right]$;

(*Results of price combination 12 0<δ<1*)

In[ ]:= $p_1 = \frac{16\,q_o}{17}$;

$D_1 = \frac{-p_1+q_o}{t}$;

$p_{2P} = \frac{2\,q_o+p_1+t\,D_1}{4}$;

$p_{2M} = \frac{2\,p_1+t\,D_1}{4}$;

$p_{2N} = \frac{p_1}{4}$;

$D_{2P} = \frac{2\,q_o+p_1-3\,t\,D_1}{4\,t}$;

$D_{2M} = \frac{2\,p_1-3\,t\,D_1}{4\,t}$;

$D_{2N} = \frac{p_1-4\,t\,D_1}{4\,t}$;

$\Pi_{121} = \text{Simplify}\left[p_1\,D_1+\frac{2\,q_o-p_1-t\,D_1}{2\,q_o}\,(p_{2P}\,D_{2P}-D_1\,(p_1-p_{2P}))+\frac{t\,D_1}{2\,q_o}\,(p_{2M}\,D_{2M}-D_1\,(p_1-p_{2M}))+\frac{p_1}{2\,q_o}\,(p_{2N}\,D_{2N}-D_1\,(p_1-p_{2N}))\right]$;

(*According to the thresholds of δ among above combinations, including 0.442..., 0.864..., 0.566..., 0.777..., $\frac{1}{2}$, $\frac{35+28\sqrt{2}}{68+48\sqrt{2}}$, 0.557..., 0.569..., $\frac{245}{341}$, $\frac{611}{899}$, we can obtain that 0.442...<$\frac{1}{2}$<$\frac{35+28\sqrt{2}}{68+48\sqrt{2}}$<0.557...<0.566...<0.569...<$\frac{611}{899}$<$\frac{245}{341}$<0.777...<0.864.... Therefore, there are 11 scenarios of comparison*)

(*Comparison scenario 1. 0<δ<0.442...,
compare profits under combination 1, 5, 7, 11, 12*)

```
In[∘]:= Reduce[Π11 ≥ Π51 && t > 2 qo > 0 && 0 < δ < 0.442...]
        Reduce[Π71 ≥ Π51 && t > 2 qo > 0 && 0 < δ < 0.442...]
        Reduce[Π111 ≥ Π51 && t > 2 qo > 0 && 0 < δ < 0.442...]
        Reduce[Π121 ≥ Π51 && t > 2 qo > 0 && 0 < δ < 0.442...]

Out[∘]= False

Out[∘]= False

Out[∘]= False

Out[∘]= False

        (*Hence, Π51 dominates over other scenarios*)

        (*Comparison scenario 2. 0.442...<δ<1/2,
        compare profits under combination 1, 5, 7, 11, 12*)

        Reduce[Π12 ≥ Π51 && t > 2 qo > 0 && 0.442... < δ < 1/2]
        Reduce[Π71 ≥ Π51 && t > 2 qo > 0 && 0.442... < δ < 1/2]
        Reduce[Π111 ≥ Π51 && t > 2 qo > 0 && 0.442... < δ < 1/2]
        Reduce[Π121 ≥ Π51 && t > 2 qo > 0 && 0.442... < δ < 1/2]
        Reduce[Π12 ≥ Π71 && t > 2 qo > 0 && 0.469... < δ < 1/2]

Out[∘]= 23 0.469... ≤ δ < 1/2 && qo > 0 && t > 2 qo

Out[∘]= 0.467... ≤ δ < 1/2 && qo > 0 && t > 2 qo

Out[∘]= False

Out[∘]= False

Out[∘]= False

        (* Hence, when 0.442...<δ<0.467..., Π51 dominates over other scenarios;
        when 0.467...<δ<1/2, Π71 dominates over other scenarios*)

        (*Comparison scenario 3. 1/2<δ<(35+28 √2)/(68+48 √2),
        compare profits under combination 1, 5, 7, 11, 12*)
```

$$\text{Reduce}\left[\Pi_{12} \leq \Pi_{51} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{1}{2} < \delta < \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}}\right]$$

$$\text{Reduce}\left[\Pi_{12} \leq \Pi_{111} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{1}{2} < \delta < \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}}\right]$$

$$\text{Reduce}\left[\Pi_{12} \leq \Pi_{121} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{1}{2} < \delta < \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}}\right]$$

$$\text{Reduce}\left[\Pi_{71} \leq \Pi_{51} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{1}{2} < \delta < \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}}\right]$$

$$\text{Reduce}\left[\Pi_{71} \leq \Pi_{111} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{1}{2} < \delta < \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}}\right]$$

$$\text{Reduce}\left[\Pi_{71} \leq \Pi_{121} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{1}{2} < \delta < \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}}\right]$$

$$\text{Reduce}\left[\Pi_{12} \leq \Pi_{71} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{1}{2} < \delta < \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}}\right]$$

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

False

Out[∘]=

$$\frac{1}{2} < \delta \leq 0.541\ldots \,\&\&\, q_o > 0 \,\&\&\, t > 2\,q_o$$

(* Hence, when $\frac{1}{2}<\delta<0.540855$, Π_{71} dominates over other scenarios;

when $0.540855<\delta<\frac{35+28\sqrt{2}}{68+48\sqrt{2}}$, Π_{12} dominates over other scenarios*)

(*Comparison scenario 4. $\frac{35+28\sqrt{2}}{68+48\sqrt{2}}<\delta<0.557\ldots$,

compare profits under combination 1, 7, 11, 12*)

$$\text{Reduce}\left[\Pi_{12} \leq \Pi_{71} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}} < \delta < 0.557\ldots\right]$$

$$\text{Reduce}\left[\Pi_{12} \leq \Pi_{111} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}} < \delta < 0.557\ldots\right]$$

$$\text{Reduce}\left[\Pi_{12} \leq \Pi_{121} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{35 + 28\,\sqrt{2}}{68 + 48\,\sqrt{2}} < \delta < 0.557\ldots\right]$$

```
Out[•]=
False
Out[•]=
False
Out[•]=
False
```

(*Hence, when $\frac{35+28\sqrt{2}}{68+48\sqrt{2}}<\delta<$ 0.557…, Π_{12} dominates over other scenarios*)

(*Comparison scenario 5. 0.557…<δ<0.566…,
compare profits under combination 1, 7, 11, 12*)

Reduce$[\Pi_{12} \le \Pi_{72}$ && $t > 2\, q_o > 0$ && 0.557… $< \delta <$ 0.566…]
Reduce$[\Pi_{12} \le \Pi_{111}$ && $t > 2\, q_o > 0$ && 0.557… $< \delta <$ 0.566…]
Reduce$[\Pi_{12} \le \Pi_{121}$ && $t > 2\, q_o > 0$ && 0.557… $< \delta <$ 0.566…]

```
Out[•]=
False
Out[•]=
False
Out[•]=
False
```

(* Hence, when 0.557…<δ<0.566…, Π_{12} dominates over other scenarios*)

(*Comparison scenario 6. 0.566…<δ<0.569…,
compare profits under combination 1, 3, 7, 11, 12*)

Reduce$[\Pi_{12} \le \Pi_{111}$ && $t > 2\, q_o > 0$ && 0.566… $< \delta <$ 0.569…]
Reduce$[\Pi_{12} \le \Pi_{121}$ && $t > 2\, q_o > 0$ && 0.566… $< \delta <$ 0.569…]
Reduce$[\Pi_{72} \le \Pi_{111}$ && $t > 2\, q_o > 0$ && 0.566… $< \delta <$ 0.569…]
Reduce$[\Pi_{72} \le \Pi_{121}$ && $t > 2\, q_o > 0$ && 0.566… $< \delta <$ 0.569…]
Reduce$[\Pi_{12} \le \Pi_{72}$ && $t > 2\, q_o > 0$ && 0.566… $< \delta <$ 0.569…]
Reduce$[\Pi_{12} \le \Pi_{31}$ && $t > 2\, q_o > 0$ && 0.566… $< \delta <$ 0.569…]

```
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
$Aborted
```

```
(*It is difficult to obtain the comparative
 result of "Π12<=Π31" using the "Reduce" function directly,
we then compare them by derive the monotonicity of Π12 and Π31 with respect to δ*)

Reduce[D[Π12, δ] > 0 && t > 2 qo > 0 && 0.566… < δ < 0.569…]
Reduce[D[Π31, δ] > 0 && t > 2 qo > 0 && 0.566… < δ < 0.569…]
```

Out[∘]=

```
False
```

Out[∘]=

```
False
```

```
(*Both Π12 and Π31 are decreasing in δ,
we then compare Π12 and Π31 at the endpoints of δ*)

Reduce[(Π12 /. {δ → 0.566…}) ≤ (Π31 /. {δ → 0.566…}) && t > 2 qo > 0]
Reduce[(Π12 /. {δ → 0.570}) ≤ (Π31 /. {δ → 0.570}) && t > 2 qo > 0]
```

Out[∘]=

```
False
```

Out[∘]=

```
False
```

```
(*Given that Π12>Π31 always holds true at the endpoints of δ,
we construct a linear function Y by taking two points (0.565,Π31(δ=0.565))
 and (0.570,Π31(δ=0.570) 1000001/1000000), where Π31(δ=0.570) 1000001/1000000<Π12(δ=0.570),
and then compare the magnitude relationship between Π12 and Y,
Π31 and Y,respectively. The linear function Y is given by *)

(*Check if Π31(δ=0.570) 1000001/1000000<Π12(δ=0.570)*)

Reduce[((Π31 /. {δ → 0.570}) 1000001/1000000 ≥ (Π12 /. {δ → 0.570})) && t > 2 qo > 0]
```

Out[∘]=

```
False
```

```
(*Derive the gradient*)

grad = ((Π31 /. {δ → 0.570}) 1000001/1000000 - (Π31 /. {δ → 0.565}))/ (0.570 - 0.565);

(*Construct linear function Y with point (0.566,Π31/.{δ→0.566})*)

Y = grad (δ - 0.565) + (Π31 /. {δ → 0.565});

Reduce[Π12 ≤ Y && t > 2 qo > 0 && 0.565 < δ < 0.570]
Reduce[Y ≤ Π31 && t > 2 qo > 0 && 0.565 < δ < 0.570]
```

Out[∘]=

```
False
```

Out[∘]=

```
False
```

```
(*Hence, Π31<Y<Π12, Π12 dominates over other scenarios*)
```

```
(*Comparison scenario 7. 0.569…<δ<611/899,
compare profits under combination 1, 3, 7, 11, 12*)
Reduce[Π12 ≤ Π73 && t > 2 qo > 0 && 0.569… < δ < 611/899]
Reduce[Π12 ≤ Π111 && t > 2 qo > 0 && 0.569… < δ < 611/899]
Reduce[Π12 ≤ Π121 && t > 2 qo > 0 && 0.569… < δ < 611/899]
Reduce[Π31 ≤ Π73 && t > 2 qo > 0 && 0.569… < δ < 611/899]
Reduce[Π31 ≤ Π111 && t > 2 qo > 0 && 0.569… < δ < 611/899]
Reduce[Π31 ≤ Π121 && t > 2 qo > 0 && 0.569… < δ < 611/899]
```

Out[•]=
```
False
```

Out[•]=
```
False
```

Out[•]=
```
False
```

Out[•]=
```
False
```

Out[•]=
```
False
```

Out[•]=
```
False
```

```
(*It is difficult to derive the comparative result of Π12≤Π31,
we then compare their gradient by constructing a linear function Y*)
Reduce[D[Π12, δ] > 0 && t > 2 qo > 0 && 0.569 < δ < 611/899]
Reduce[D[Π31, δ] > 0 && t > 2 qo > 0 && 0.569 < δ < 611/899]
```

Out[•]=
```
False
```

Out[•]=
```
False
```

```
Reduce[(Π12 /. {δ → 0.569}) ≤ (Π31 /. {δ → 0.569}) && t > 2 qo > 0]
Reduce[(Π12 /. {δ → 611/899}) ≥ (Π31 /. {δ → 611/899}) && t > 2 qo > 0]
```

Out[•]=
```
False
```

Out[•]=
```
False
```

```
(*Both Π12 and Π31 are decreasing in δ and Π12>Π31 at the endpoint of δ=
 0.569… and Π12<Π31 at the endpoint of δ=611/899. We construct a linear function
   Y by taking two points (0.569,Π31(δ=0.569)) and (611/899,Π31(δ=611/899) 1000000/1001000),
where Π31(δ=611/899) 1000000/1001000>Π12(δ=611/899), and then compare the relationship
 between ∂Π12/∂δ and ∂Y/∂δ, ∂Π31/∂δ and ∂Y/∂δ, respectively.*)
(*Check if Π31(δ=611/899) 1000000/1001000>Π12(δ=611/899)*)
Reduce[(Π31 /. {δ → 611/899}) 1000000/1001000 ≤ (Π12 /. {δ → 611/899}) && t > 2 qo > 0]
```

Out[∘]=

```
False
```

```
(*Derive the gradient*)
gradi =
 Simplify[((Π31 /. {δ → 611/899}) 1000000/1001000 - (Π31 /. {δ → 0.569})) / (611/899 - 0.569), qo > 0]
```

Out[∘]=

$$-\frac{0.0899837\, q_o^2}{t}$$

```
(*Compare ∂Π12/∂δ, gradi, and ∂Π31/∂δ*)
Reduce[D[Π12, δ] ≥ -0.08998374142304977` qo^2/t && t > 2 qo > 0 && 0.569 < δ < 611/899]
Reduce[-0.08998374142304977` qo^2/t ≥ D[Π31, δ] && t > 2 qo > 0 && 0.569 < δ < 611/899]
```

Out[∘]=

```
False
```

Out[∘]=

```
False
```

```
(*Therefore, ∂Π12/∂δ<∂Y/∂δ<∂Π31/∂δ. Given that both Π12 and Π31 are decreasing in δ and Π12>
  Π31 at the endpoint of δ= 0.569… and Π12<Π31 at the endpoint of δ=611/899,
we can derive that there is only one crosspoint when  0.569… <δ<611/899*)
(*Comparison scenario 8. 611/899<δ<245/341, compare profits under combination 1, 3, 7, 12*)
Reduce[Π31 ≤ Π12 && t > 2 qo > 0 && 611/899 < δ < 245/341]
Reduce[Π31 ≤ Π73 && t > 2 qo > 0 && 611/899 < δ < 245/341]
Reduce[Π31 ≤ Π121 && t > 2 qo > 0 && 611/899 < δ < 245/341]
```

Out[∘]=

```
$Aborted
```

Out[∘]=

```
False
```

Out[•]=
False

```
(*We cannot directly derive the comparative result of Π31≤Π12,
we then compare them by constructing a linear function Y*)
```

Reduce$\left[D[\Pi_{12},\delta] > 0 \,\&\&\, t > 2 q_o > 0 \,\&\&\, \frac{611}{899} < \delta < \frac{245}{341}\right]$

Reduce$\left[D[\Pi_{31},\delta] > 0 \,\&\&\, t > 2 q_o > 0 \,\&\&\, \frac{611}{899} < \delta < \frac{245}{341}\right]$

Out[•]=
False

Out[•]=
False

Reduce$\left[\left(\Pi_{12} \,/.\, \left\{\delta \to \frac{611}{899}\right\}\right) \geq \left(\Pi_{31} \,/.\, \left\{\delta \to \frac{611}{899}\right\}\right) \,\&\&\, t > 2 q_o > 0\right]$

Reduce$\left[\left(\Pi_{12} \,/.\, \left\{\delta \to \frac{245}{341}\right\}\right) \geq \left(\Pi_{31} \,/.\, \left\{\delta \to \frac{245}{341}\right\}\right) \,\&\&\, t > 2 q_o > 0\right]$

Out[•]=
False

Out[•]=
False

(*Both Π_{12} and Π_{31} are decreasing in δ,
$\Pi_{31}>\Pi_{12}$ always holds true at the two endpoints. The linear function Y is constructed
with two points of $\left(\frac{611}{899},\Pi_{31}/.\left\{\delta\to\frac{611}{899}\right\}\right)$ and $\left(\frac{245}{341},\Pi_{31}/.\left\{\delta\to\frac{245}{341}\right\}\frac{1000000}{1000010}\right)$,
where $\Pi_{31}/.\left\{\delta\to\frac{245}{341}\right\}\frac{1000000}{1000010}>\Pi_{12}/.\left\{\delta\to\frac{245}{341}\right\}$*)

(*Derive the gradient*)

In[•]:= gradi =
Simplify$\left[\left(\frac{0.3236092180950003\text{`}\, q_o^2}{t}\,\frac{1\,000\,000}{1\,000\,010} - \frac{0.32699090282255494\text{`}\, q_o^2}{t}\right)\Big/\left(\frac{245}{341}-\frac{611}{899}\right)\right]$

Out[•]=
$$-\frac{0.0871705\, q_o^2}{t}$$

(*Construct linear function Y with point $\left(\frac{611}{899},\Pi_{31}/.\left\{\delta\to\frac{611}{899}\right\}\right)$*)

Y = Simplify$\left[\text{gradi}\left(\delta-\frac{611}{899}\right) + \left(\Pi_{31} \,/.\, \left\{\delta\to\frac{611}{899}\right\}\right),\ q_o > 0\right]$

Out[•]=
$$\frac{(0.386236 - 0.0871705\,\delta)\, q_o^2}{t}$$

In[•]:= $Y = \frac{(0.38623583149949287\text{`} - 0.08717052517278892\text{`}\,\delta)\, q_o^2}{t};$

Reduce$\left[\Pi_{12} \geq Y \,\&\&\, t > 2 q_o > 0 \,\&\&\, \frac{611}{899} < \delta < \frac{245}{341}\right]$

Reduce$\left[\Pi_{31} \leq Y \,\&\&\, t > 2 q_o > 0 \,\&\&\, \frac{611}{899} < \delta < \frac{245}{341}\right]$

Out[∘]=

False

Out[∘]=

False

(*We obtain that Π_{12}<Y<Π_{31} always holds true,
hence, Π_{31} dominates over other scenarios*)

(*Comparison scenario 9. $\frac{245}{341}<\delta<$ 0.777…,
compare profits under combination 1, 3, 12*)

$$\text{Reduce}\left[\Pi_{31} \le \Pi_{12} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{245}{341} < \delta < 0.777\ldots\right]$$

$$\text{Reduce}\left[\Pi_{31} \le \Pi_{121} \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{245}{341} < \delta < 0.777\ldots\right]$$

Out[∘]=

$Aborted

Out[∘]=

False

(*We cannot directly derive the comparative result of $\Pi_{31}\le\Pi_{12}$,
we then compare them by constructing a linear function Y*)

$$\text{Reduce}\left[D[\Pi_{12}, \delta] > 0 \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{245}{341} < \delta < 0.777\ldots\right]$$

$$\text{Reduce}\left[D[\Pi_{31}, \delta] > 0 \,\&\&\, t > 2\,q_o > 0 \,\&\&\, \frac{245}{341} < \delta < 0.777\ldots\right]$$

Out[∘]=

False

Out[∘]=

False

$$\text{Reduce}\left[\left(\Pi_{12} \,/.\, \left\{\delta \to \frac{245}{341}\right\}\right) \ge \left(\Pi_{31} \,/.\, \left\{\delta \to \frac{245}{341}\right\}\right) \,\&\&\, t > 2\,q_o > 0\right]$$

$$\text{Reduce}[(\Pi_{12} \,/.\, \{\delta \to 0.778\}) \ge (\Pi_{31} \,/.\, \{\delta \to 0.778\}) \,\&\&\, t > 2\,q_o > 0]$$

Out[∘]=

False

Out[∘]=

False

(*Both Π_{12} and Π_{31} are decreasing in δ,
Π_{31}>Π_{12} always holds true at the two endpoints. The linear function Y is constructed
with two points of $\left(\frac{245}{341},\Pi_{31}/.\left\{\delta\to\frac{245}{341}\right\}\right)$ and $\left(0.778,\Pi_{31}/.\{\delta\to0.778\}\frac{1000000}{1001000}\right)$,
where $\Pi_{31}/.\{\delta\to0.778\}\frac{1000000}{1001000}>\Pi_{12}/.\{\delta\to0.778\}$*)

(*Check if $\Pi_{31}/.\{\delta\to0.778\}\frac{1000000}{1001000}>\Pi_{12}/.\{\delta\to0.778\}$*)

$$\text{Reduce}\left[(\Pi_{31} \,/.\, \{\delta \to 0.778\}) \frac{1\,000\,000}{1\,001\,000} \le (\Pi_{12} \,/.\, \{\delta \to 0.778\}) \,\&\&\, t > 2\,q_o > 0\right]$$

Out[∘]=

False

(*Derive the gradient*)

```
gradi =
 Simplify[((Π31 /. {δ → 0.778}) 1000000/1001000 - (Π31 /. {δ → 245/341}))/(0.778 - 245/341), qo > 0]
```

Out[•]=

$$-\frac{0.0922372\, q_o^2}{t}$$

```
(*Construct the linear function Y with point (245/341,Π31/.{δ→245/341})*)

Y = Simplify[-0.09223720171617801` qo^2/t (δ - 245/341) + (Π31 /. {δ → 245/341}), qo > 0]
```

Out[•]=

$$\frac{(0.389879 - 0.0922372\, \delta)\, q_o^2}{t}$$

```
Y = ((0.3898793483602876` - 0.09223720171617801` δ) qo^2)/t;

Reduce[Π12 ≥ Y && t > 2 qo > 0 && 245/341 < δ < 0.778]
Reduce[Π31 ≤ Y && t > 2 qo > 0 && 245/341 < δ < 0.778]
```

Out[•]=

False

Out[•]=

$0.718475 < \delta \le 0.718475$ && $q_o > 0$ && $t > 2.\, q_o$

```
(*We obtain that Π12<Y<Π31 always holds true,
hence, Π31 dominates over other scenarios*)

(* Hence, when 245/341<δ<0.777…, Π31 dominates over other scenarios*)

(*Comparison scenario 10. 0.777…<δ<0.864…,
compare profits under combination 1, 3, 12*)

Reduce[Π32 ≤ Π12 && t > 2 qo > 0 && 0.777… < δ < 0.864…]
Reduce[Π32 ≤ Π121 && t > 2 qo > 0 && 0.777… < δ < 0.864…]
```

Out[•]=

False

Out[•]=

$0.827\ldots \le \delta < 0.864\ldots$ && $q_o > 0$ && $t > 2\, q_o$

```
(*Hence, when 0.777…<δ<0.827…, Π32 dominates over other scenarios;
when 0.827…<δ<0.864…, Π121 dominates over other scenarios*)

(*Comparison scenario 11. 0.864…<δ<1,
compare profits under combination 1, 3, 12*)

Reduce[Π121 ≤ Π13 && t > 2 qo > 0 && 0.864… < δ < 1]
Reduce[Π121 ≤ Π31 && t > 2 qo > 0 && 0.864… < δ < 1]
```

Out[ ]=
False

Out[ ]=
False

```
(*Hence, when 0.864...<δ<1, Π121 dominates over other scenarios*)

(*Overall, based on the above 11 comparison scenarios,
we obtain the following 6 pricing strategies*)

(* (i). 0<δ≤0.467...*)
```

$$p_{GL11} = \frac{q_o \left(2 - 2\sqrt{1-\delta} + \left(-3 + 2\sqrt{1-\delta}\right)\delta + 2\delta^2\right)}{2\delta^2};$$

$$D_{GL11} = \frac{-p_{GL11}\,\delta + 2 q_o \left(-1 + \sqrt{1-\delta} + \delta\right)}{t\,\delta};$$

$p_{GL2P1} = p_{GL11};$
$p_{GL2M1} = p_{GL11};$
$p_{GL2N1} = p_{GL11};$

$$D_{GL2P1} = \frac{2 q_o - p_{GL11} - t\,D_{GL11}}{2\,t};$$

$$\Pi_{GL1} = p_{GL11}\,D_{GL11} + \frac{2 q_o - p_{GL11} - t\,D_{GL11}}{2 q_o}\,p_{GL2P1}\,D_{GL2P1};$$

```
(* (ii). 0.467...<δ≤0.541...*)
```

In[ ]:=

$$p_{GL12} = \frac{2 q_o \left(\sqrt{\left(33 + 20\sqrt{2} - 4\delta\right)(1-\delta)} + \left(5 + 2\sqrt{2}\right)\delta - 5 - 2\sqrt{2}\right)}{\left(7 + 2\sqrt{2}\right)\delta};$$

$$D_{GL12} = \frac{-4\,p_{GL12}\,q_o\,(-1+\delta) + 4 q_o^2\,(-1+\delta) + (p_{GL12})^2\,\delta}{4\,t\,q_o\,(-1+\delta) - 2\,t\,p_{GL12}\,\delta};$$

$p_{GL2P2} = p_{GL12};$

$$p_{GL2M2} = \frac{2\,p_{GL12} + t\,D_{GL12}}{4};$$

$p_{GL2N2} = p_{GL12};$

$$D_{GL2P2} = \frac{2 q_o - p_{GL12} - t\,D_{GL12}}{2\,t};$$

$$D_{GL2M2} = \frac{2\,p_{GL12} - 3\,t\,D_{GL12}}{4\,t};$$

$$\Pi_{GL2} = p_{GL12}\,D_{GL12} + \frac{2 q_o - p_{GL12} - t\,D_{GL12}}{2 q_o}\,p_{GL2P2}\,D_{GL2P2} + \frac{t\,D_{GL12}}{2 q_o}\,\left(p_{GL2M2}\,D_{GL2M2} - D_{GL12}\,(p_{GL12} - p_{GL2M2})\right);$$

```
(* (iii). 0.541...<δ≤0.5984*)
```

```
In[•]:= pGL13 = Root[8 δ^4 #1^4 + 144 qo^4 + 225 δ qo^4 - 272 δ^2 qo^4 + 64 δ^3 qo^4 +
          #1^3 (-16 δ^2 qo - 94 δ^3 qo) + #1^2 (124 δ qo^2 + 376 δ^2 qo^2 - 30 δ^3 qo^2 - 8 δ^4 qo^2) +
          #1 (-224 qo^3 - 480 δ qo^3 + 104 δ^2 qo^3 + 64 δ^3 qo^3) &, 1];
```

$$D_{GL13} = \frac{2\, q_o\, (-2+\delta) + \sqrt{-8\, q_o^2\, (-2+\delta) - 8\, p_{GL13}\, q_o\, \delta + (p_{GL13})^2\, \delta^2}}{t\, \delta};$$

$$p_{GL2P3} = \frac{2\, q_o + p_{GL13} - t\, D_{GL13}}{4};$$

$$p_{GL2M3} = p_{GL13};$$

$$p_{GL2N3} = p_{GL13};$$

$$D_{GL2P3} = \frac{2\, q_o + p_{GL13} - t\, D_{GL13}}{4\, t};$$

$$\Pi_{GL3} = p_{GL13}\, D_{GL13} + \frac{2\, q_o - p_{GL13} - t\, D_{GL13}}{2\, q_o}\, p_{GL2P3}\, D_{GL2P3};$$

```
(* (iv). 0.5984<δ≤ 0.777… *)
```

In[•]:=

$$p_{GL14} = \frac{1}{51\,\delta}\, 2\, q_o \left(66 + 40\sqrt{2} - 2\left(5 + 2\sqrt{2}\right)\delta - \sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\right);$$

$$D_{GL14} = -\frac{2\, q_o\, (-2+\delta) + \sqrt{8\, p_{GL14}\, q_o\, \delta - (p_{GL14})^2\, \delta^2 + 8\, q_o^2\, (2 - 3\,\delta + \delta^2)}}{t\, \delta};$$

$$p_{GL2P4} = \frac{2\, q_o + p_{GL14} - t\, D_{GL14}}{4};$$

$$p_{GL2M4} = \frac{2\, p_{GL14} + t\, D_{GL14}}{4};$$

$$p_{GL2N4} = p_{GL14};$$

$$D_{GL2P4} = \frac{2\, q_o + p_{GL14} - t\, D_{GL14}}{4\, t};$$

$$D_{GL2M4} = \frac{2\, p_{GL14} - 3\, t\, D_{GL14}}{4\, t};$$

$$\Pi_{GL4} = p_{GL14}\, D_{GL14} + \frac{2\, q_o - p_{GL14} - t\, D_{GL14}}{2\, q_o}\, p_{GL2P4}\, D_{GL2P4} + \frac{t\, D_{GL14}}{2\, q_o}\, (p_{GL2M4}\, D_{GL2M4} - D_{GL14}\, (p_{GL14} - p_{GL2M4}));$$

```
(* (v). 0.777…<δ≤ 0.827… *)
```

In[ ]:=

$$p_{GL15} = \mathrm{Root}\Big[1125\,\#1^6 + 60928\,q_o^6 + \#1^5\left(2160\,q_o - \frac{23640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151312\,q_o^2}{\delta^2} - \frac{12096\,q_o^2}{\delta}\right) + \#1^3\left(-20160\,q_o^3 - \frac{205056\,q_o^3}{\delta^3} - \frac{193152\,q_o^3}{\delta^2} + \frac{73664\,q_o^3}{\delta}\right) + \#1^2\left(-40624\,q_o^4 - \frac{574464\,q_o^4}{\delta^4} + \frac{1165824\,q_o^4}{\delta^3} - \frac{582720\,q_o^4}{\delta^2} + \frac{266496\,q_o^4}{\delta}\right) + \#1\left(27648\,q_o^5 + \frac{67584\,q_o^5}{\delta^4} + \frac{170496\,q_o^5}{\delta^3} - \frac{376320\,q_o^5}{\delta^2} + \frac{58240\,q_o^5}{\delta}\right) + \frac{236544\,q_o^6}{\delta^4} - \frac{766464\,q_o^6}{\delta^3} + \frac{859904\,q_o^6}{\delta^2} - \frac{388608\,q_o^6}{\delta}\ \&,\ 3\Big];$$

$$D_{GL15} = -\frac{2\,q_o\,(-2+\delta) + \sqrt{8\,p_{GL15}\,q_o\,\delta - (p_{GL15})^2\,\delta^2 + 8\,q_o^2\,(2 - 3\,\delta + \delta^2)}}{t\,\delta};$$

$$p_{GL2P5} = \frac{2\,q_o + p_{GL15} - t\,D_{GL15}}{4};$$

$$p_{GL2M5} = \frac{2\,p_{GL15} + t\,D_{GL15}}{4};$$

$$p_{GL2N5} = p_{GL15};$$

$$D_{GL2P5} = \frac{2\,q_o + p_{GL15} - t\,D_{GL15}}{4\,t};$$

$$D_{GL2M5} = \frac{2\,p_{GL15} - 3\,t\,D_{GL15}}{4\,t};$$

$$\Pi_{GL5} = p_{GL15}\,D_{GL15} + \frac{2\,q_o - p_{GL15} - t\,D_{GL15}}{2\,q_o}\,p_{GL2P5}\,D_{GL2P5} + \frac{t\,D_{GL15}}{2\,q_o}\,(p_{GL2M5}\,D_{GL2M5} - D_{GL15}\,(p_{GL15} - p_{GL2M5}));$$

```
(* (vi). 0.827...<δ<1*)
```

In[∘]:= $p_{GL16} = \frac{16\, q_o}{17}$;

$D_{GL16} = \frac{-p_{GL16} + q_o}{t}$;

$p_{GL2P6} = \frac{2\, q_o + p_{GL16} + t\, D_{GL16}}{4}$;

$p_{GL2M6} = \frac{2\, p_{GL16} + t\, D_{GL16}}{4}$;

$p_{GL2N6} = \frac{p_{GL16}}{4}$;

$D_{GL2P6} = \frac{2\, q_o + p_{GL16} - 3\, t\, D_{GL16}}{4\, t}$;

$D_{GL2M6} = \frac{2\, p_{GL16} - 3\, t\, D_{GL16}}{4\, t}$;

$D_{GL2N6} = \frac{p_{GL16} - 4\, t\, D_{GL16}}{4\, t}$;

$\Pi_{GL6} = p_{GL16}\, D_{GL16} + \frac{2\, q_o - p_{GL16} - t\, D_{GL16}}{2\, q_o} (p_{GL2P6}\, D_{GL2P6} - D_{GL16}\, (p_{GL16} - p_{GL2P6})) + \frac{t\, D_{GL16}}{2\, q_o} (p_{GL2M6}\, D_{GL2M6} - D_{GL16}\, (p_{GL16} - p_{GL2M6})) + \frac{p_{GL16}}{2\, q_o} (p_{GL2N6}\, D_{GL2N6} - D_{GL16}\, (p_{GL16} - p_{GL2N6}))$;

(***Proof of Proposition 7 (ii): Optimal price with respect to δ_c***)

(*Determine the sign of $\frac{\partial p_1^{GL*}}{\partial \delta}$ *)

In[∘]:= Reduce[D[p_{GL11}, δ] ≥ 0 && 0 < δ ≤ 0.467… && q_o > 0] (*Determine if $\frac{\partial p_{GL11}}{\partial \delta}$ ≥0*)

Reduce[D[p_{GL12}, δ] ≥ 0 && 0.467… < δ ≤ 0.540855 && q_o > 0] (*Determine if $\frac{\partial p_{GL12}}{\partial \delta}$ ≥0*)

Reduce[D[p_{GL13}, δ] ≥ 0 && 0.540855 < δ ≤ 0.5984 && q_o > 0] (*Determine if $\frac{\partial p_{GL13}}{\partial \delta}$ ≥0*)

Reduce[D[p_{GL14}, δ] ≥ 0 && 0.5984 < δ ≤ 0.777… && q_o > 0] (*Determine if $\frac{\partial p_{GL14}}{\partial \delta}$ ≥0*)

Reduce[D[p_{GL15}, δ] ≥ 0 && 0.777… < δ ≤ 0.827… && q_o > 0] (*Determine if $\frac{\partial p_{GL15}}{\partial \delta}$ ≥0*)

Reduce[D[p_{GL16}, δ] == 0 && 0.827… < δ < 1 && q_o > 0] (*Determine if $\frac{\partial p_{GL16}}{\partial \delta}$ =0*)

Out[∘]= False

Out[∘]= False

Out[∘]= False

Out[∘]= False

Out[∘]= False

Out[∘]= 0.827… < δ < 1 && q_o > 0

(*Hence, when 0<δ≤0.827…, $\frac{\partial p_1^{GL*}}{\partial \delta}$ <0; when 0.827…<δ<1, $\frac{\partial p_1^{GL*}}{\partial \delta}$ =0*)

```
(*Determine the sign of ∂p2P^GL*/∂δ *)
```

```
In[•]:= Reduce[D[pGL2P1, δ] ≥ 0 && 0 < δ ≤ 0.467... && qo > 0] (*Determine if ∂pGL2P1/∂δ ≥0*)
Reduce[D[pGL2P2, δ] ≥ 0 && 0.467... < δ ≤ 0.540855 && qo > 0] (*Determine if ∂pGL2P2/∂δ ≥0*)
Reduce[D[pGL2P3, δ] ≤ 0 && 0.540855 < δ ≤ 0.5984 && qo > 0] (*Determine if ∂pGL2P3/∂δ ≤0*)
Reduce[D[pGL2P4, δ] ≥ 0 && 0.5984 < δ ≤ 0.777... && qo > 0] (*Determine if ∂pGL2P4/∂δ ≥0*)
Reduce[D[pGL2P5, δ] ≤ 0 && 0.777... < δ ≤ 0.827... && qo > 0] (*Determine if ∂pGL2P5/∂δ ≤0*)
Reduce[D[pGL2P6, δ] == 0 && 0.827... < δ < 1 && qo > 0] (*Determine if ∂pGL2P6/∂δ ==0*)
```

```
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= 0.827... < δ < 1 && qo > 0
```

```
(*Hence, when 0<δ≤0.540855 and 0.5984<δ≤0.777...,
∂p2P^GL*/∂δ <0; when 0.540855<δ≤0.5984 and 0.777...<δ≤0.827...,
∂p2P^GL*/∂δ >0; when 0.827...<δ<1, ∂p2P^GL*/∂δ =0*)

(*Determine the sign of ∂p2M^GL*/∂δ *)
```

```
In[•]:= Reduce[D[pGL2M1, δ] ≥ 0 && 0 < δ ≤ 0.467... && qo > 0] (*Determine if ∂pGL2M1/∂δ ≥0*)
Reduce[D[pGL2M2, δ] ≥ 0 && 0.467... < δ ≤ 0.540855 && qo > 0] (*Determine if ∂pGL2M2/∂δ ≥0*)
Reduce[D[pGL2M3, δ] ≥ 0 && 0.540855 < δ ≤ 0.5984 && qo > 0] (*Determine if ∂pGL2M3/∂δ ≥0*)
Reduce[D[pGL2M4, δ] ≥ 0 && 0.5984 < δ ≤ 0.777... && qo > 0] (*Determine if ∂pGL2M4/∂δ ≥0*)
Reduce[D[pGL2M5, δ] ≥ 0 && 0.777... < δ ≤ 0.827... && qo > 0] (*Determine if ∂pGL2M5/∂δ ≥0*)
Reduce[D[pGL2M6, δ] == 0 && 0.827... < δ < 1 && qo > 0] (*Determine if ∂pGL2M6/∂δ ==0*)
```

```
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
```

Out[∘]=

```
False
```

Out[∘]=

```
0.827… < δ < 1 && q_o > 0

(*Hence, when 0<δ≤ 0.827…, ∂p_2M^GL*/∂δ <0; when 0.827…<δ<1, ∂p_2M^GL*/∂δ =0*)

(*Determine the sign of ∂p_2N^GL*/∂δ *)
```

In[∘]:=

```
Reduce[D[p_GL2N1, δ] ≥ 0 && 0 < δ ≤ 0.467… && q_o > 0] (*Determine if ∂p_GL2N1/∂δ ≥0*)
Reduce[D[p_GL2N2, δ] ≥ 0 && 0.467… < δ ≤ 0.540855 && q_o > 0] (*Determine if ∂p_GL2N2/∂δ ≥0*)
Reduce[D[p_GL2N3, δ] ≥ 0 && 0.540855 < δ ≤ 0.5984 && q_o > 0] (*Determine if ∂p_GL2N3/∂δ ≥0*)
Reduce[D[p_GL2N4, δ] ≥ 0 && 0.5984 < δ ≤ 0.777… && q_o > 0] (*Determine if ∂p_GL2N4/∂δ ≥0*)
Reduce[D[p_GL2N5, δ] ≥ 0 && 0.777… < δ ≤ 0.827… && q_o > 0] (*Determine if ∂p_GL2N5/∂δ ≥0*)
Reduce[D[p_GL2N6, δ] == 0 && 0.827… < δ < 1 && q_o > 0] (*Determine if ∂p_GL2N6/∂δ ==0*)
```

Out[∘]=

```
False
```

Out[∘]=

```
False
```

Out[∘]=

```
False
```

Out[∘]=

```
False
```

Out[∘]=

```
False
```

Out[∘]=

```
0.827… < δ < 1 && q_o > 0

(*Hence, when 0<δ≤ 0.827…, ∂p_2N^GL*/∂δ <0; when 0.827…<δ<1, ∂p_2N^GL*/∂δ =0*)

(***Proof of Proposition 7 (iii): Profit and first-period demand
with respect to δ_c***)

(*Determine the sign of ∂Π^GL/∂δ *)

Reduce[D[Π_GL1, δ] ≥ 0 && 0 < δ ≤ 0.467… && t > 2 q_o > 0] (*Determine if ∂Π_GL1/∂δ ≥0*)
Reduce[D[Π_GL2, δ] ≥ 0 && 0.467… < δ ≤ 0.540855 && t > 2 q_o > 0] (*Determine if ∂Π_GL2/∂δ ≥0*)
Reduce[D[Π_GL3, δ] ≥ 0 && 0.540855 < δ ≤ 0.5984 && t > 2 q_o > 0] (*Determine if ∂Π_GL3/∂δ ≥0*)
Reduce[D[Π_GL4, δ] ≥ 0 && 0.5984 < δ ≤ 0.777… && t > 2 q_o > 0] (*Determine if ∂Π_GL4/∂δ ≥0*)
Reduce[D[Π_GL5, δ] ≥ 0 && 0.777… < δ ≤ 0.827… && t > 2 q_o > 0]
(*Determine if ∂Π_GL5/∂δ ≥0*)
Reduce[D[Π_GL6, δ] == 0 && 0.827… < δ < 1 && t > 2 q_o > 0] (*Determine if ∂Π_GL6/∂δ =0*)
```

Out[∘]=

```
False
```

Out[∘]=
False

Out[∘]=
False

Out[∘]=
False

Out[∘]=
False

Out[∘]=
$0.827\ldots < \delta < 1 \,\&\&\, t > 0 \,\&\&\, 0 < q_o < \frac{t}{2}$

(*Hence, when 0<δ≤ 0.827..., $\frac{\partial\pi^{GL}}{\partial\delta}$<0; when 0.827...<δ<1, $\frac{\partial\pi^{GL}}{\partial\delta}$=0*)

(*Determine the sign of $\frac{\partial D_1^{GL*}}{\partial\delta}$*)

In[∘]:=
Reduce[D[D_{GL11}, δ] ≥ 0 && 0 < δ ≤ 0.467... && t > 2 q_o > 0] (*Determine if $\frac{\partial D_{GL11}}{\partial\delta}$ ≥0*)
Reduce[D[D_{GL12}, δ] ≥ 0 && 0.467... < δ ≤ 0.540855 && t > 2 q_o > 0] (*Determine if $\frac{\partial D_{GL12}}{\partial\delta}$ ≥0*)
Reduce[D[D_{GL13}, δ] ≥ 0 && 0.540855 < δ ≤ 0.5984 && t > 2 q_o > 0] (*Determine if $\frac{\partial D_{GL13}}{\partial\delta}$ ≥0*)
Reduce[D[D_{GL14}, δ] ≥ 0 && 0.5984 < δ ≤ 0.777... && t > 2 q_o > 0] (*Determine if $\frac{\partial D_{GL14}}{\partial\delta}$ ≥0*)
Reduce[D[D_{GL15}, δ] ≥ 0 && 0.777... < δ ≤ 0.827... && t > 2 q_o > 0]
(*Determine if $\frac{\partial D_{GL15}}{\partial\delta}$ ≥0*)
Reduce[D[D_{GL16}, δ] == 0 && 0.827... < δ < 1 && t > 2 q_o > 0] (*Determine if $\frac{\partial D_{GL16}}{\partial\delta}$ =0*)

Out[∘]=
False

Out[∘]=
False

Out[∘]=
False

Out[∘]=
False

Out[∘]=
False

Out[∘]=
^{12} 0.827... < δ < 1

(*Hence, when 0<δ≤ 0.827..., $\frac{\partial D_1^{GL*}}{\partial\delta}$<0; when 0.827...<δ<1, $\frac{\partial D_1^{GL*}}{\partial\delta}$=0*)

(***Proof of Proposition 8: Quality beliefs***)

(* i. Results of q_R^{GL} and Pr_R^{GL} when 0<δ≤ 0.467..., where R∈{P,M,N}*)

In[ ]:= $q_{GLP1} = \frac{2\, q_o + p_{GL11} + t\, D_{GL11}}{2};$

$q_{GLM1} = \frac{2\, p_{GL11} + t\, D_{GL11}}{2};$

$q_{GLN1} = \frac{p_{GL11}}{2};$

$Pr_{GLP1} = \frac{2\, q_o - p_{GL11} - t\, D_{GL11}}{2\, q_o};$

$Pr_{GLM1} = \frac{t\, D_{GL11}}{2\, q_o};$

$Pr_{GLN1} = \frac{p_{GL11}}{2\, q_o};$

(* ii. Results of q_R^{GL} and Pr_R^{GL} when 0.467…<δ≤ 0.541…, where R∈{P,M,N}*)

In[ ]:= $q_{GLP2} = \frac{2\, q_o + p_{GL12} + t\, D_{GL12}}{2};$

$q_{GLM2} = \frac{2\, p_{GL12} + t\, D_{GL12}}{2};$

$q_{GLN2} = \frac{p_{GL12}}{2};$

$Pr_{GLP2} = \frac{2\, q_o - p_{GL12} - t\, D_{GL12}}{2\, q_o};$

$Pr_{GLM2} = \frac{t\, D_{GL12}}{2\, q_o};$

$Pr_{GLN2} = \frac{p_{GL12}}{2\, q_o};$

(* iii. Results of q_R^{GL} and Pr_R^{GL} when 0.541…<δ≤0.5984, where R∈{P,M,N}*)

In[ ]:= $q_{GLP3} = \frac{2\, q_o + p_{GL13} + t\, D_{GL13}}{2};$

$q_{GLM3} = \frac{2\, p_{GL13} + t\, D_{GL13}}{2};$

$q_{GLN3} = \frac{p_{GL13}}{2};$

$Pr_{GLP3} = \frac{2\, q_o - p_{GL13} - t\, D_{GL13}}{2\, q_o};$

$Pr_{GLM3} = \frac{t\, D_{GL13}}{2\, q_o};$

$Pr_{GLN3} = \frac{p_{GL13}}{2\, q_o};$

(* iv. Results of q_R^{GL} and Pr_R^{GL} when 0.5984<δ≤0.777…, where R∈{P,M,N}*)

In[ ]:= $q_{GLP4} = \frac{2\,q_o + p_{GL14} + t\,D_{GL14}}{2};$

$q_{GLM4} = \frac{2\,p_{GL14} + t\,D_{GL14}}{2};$

$q_{GLN4} = \frac{p_{GL14}}{2};$

$Pr_{GLP4} = \frac{2\,q_o - p_{GL14} - t\,D_{GL14}}{2\,q_o};$

$Pr_{GLM4} = \frac{t\,D_{GL14}}{2\,q_o};$

$Pr_{GLN4} = \frac{p_{GL14}}{2\,q_o};$

(* v. Results of q_R^{GL} and Pr_R^{GL} when 0.777…<δ≤0.827…, where R∈{P,M,N}*)

In[ ]:= $q_{GLP5} = \frac{2\,q_o + p_{GL15} + t\,D_{GL15}}{2};$

$q_{GLM5} = \frac{2\,p_{GL15} + t\,D_{GL15}}{2};$

$q_{GLN5} = \frac{p_{GL15}}{2};$

$Pr_{GLP5} = \frac{2\,q_o - p_{GL15} - t\,D_{GL15}}{2\,q_o};$

$Pr_{GLM5} = \frac{t\,D_{GL15}}{2\,q_o};$

$Pr_{GLN5} = \frac{p_{GL15}}{2\,q_o};$

(* vi. Results of q_R^{GL} and Pr_R^{GL} when 0.827…<δ<1, where R∈{P,M,N}*)

In[ ]:= $q_{GLP6} = \frac{2\,q_o + p_{GL16} + t\,D_{GL16}}{2};$

$q_{GLM6} = \frac{2\,p_{GL16} + t\,D_{GL16}}{2};$

$q_{GLN6} = \frac{p_{GL16}}{2};$

$Pr_{GLP6} = \frac{2\,q_o - p_{GL16} - t\,D_{GL16}}{2\,q_o};$

$Pr_{GLM6} = \frac{t\,D_{GL16}}{2\,q_o};$

$Pr_{GLN6} = \frac{p_{GL16}}{2\,q_o};$

(*Part (i)*)

```
Reduce[qGLP1 ≤ qo && qo > 0 && 0 < δ ≤ 0.467…]
Reduce[qGLM1 ≥ qo && qo > 0 && 0 < δ ≤ 0.467…]
Reduce[qGLN1 ≥ qo && qo > 0 && 0 < δ ≤ 0.467…]
Reduce[qGLP2 ≤ qo && qo > 0 && 0.467… < δ ≤ 0.541…]
Reduce[qGLM2 ≥ qo && qo > 0 && 0.467… < δ ≤ 0.541…]
Reduce[qGLN2 ≥ qo && qo > 0 && 0.467… < δ ≤ 0.541…]
Reduce[qGLP3 ≤ qo && qo > 0 && 0.541… < δ ≤ 0.5984]
Reduce[qGLM3 ≥ qo && qo > 0 && 0.541… < δ ≤ 0.5984]
Reduce[qGLN3 ≥ qo && qo > 0 && 0.541… < δ ≤ 0.5984]
Reduce[qGLP4 ≤ qo && qo > 0 && 0.5984 < δ ≤ 0.777…]
Reduce[qGLM4 ≥ qo && qo > 0 && 0.5984 < δ ≤ 0.777…]
Reduce[qGLN4 ≥ qo && qo > 0 && 0.5984 < δ ≤ 0.777…]
Reduce[qGLP5 ≤ qo && qo > 0 && 0.777… < δ ≤ 0.827…]
Reduce[qGLM5 ≥ qo && qo > 0 && 0.777… < δ ≤ 0.827…]
Reduce[qGLN5 ≥ qo && qo > 0 && 0.777… < δ ≤ 0.827…]
Reduce[qGLP6 ≤ qo && qo > 0 && 0.827… < δ < 1]
Reduce[qGLM6 ≥ qo && qo > 0 && 0.827… < δ < 1]
Reduce[qGLN6 ≥ qo && qo > 0 && 0.827… < δ < 1]

Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
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Out[•]=
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False
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Out[•]=
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False
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Out[•]=
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False
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Out[•]=
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False
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Out[•]=
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False
```

Out[•]=
```
False
```

Out[•]=
```
False
```

(*Hence, $q_P^{GL}>q_o, q_M^{GL}<q_o, q_N^{GL}<q_o$*)

(*Part (ii)*)

(*Determine the sign of $\frac{\partial q_R^{GL}}{\partial \delta}$*)

Reduce[$D[q_{GLP1}, \delta] \geq 0$ && $q_o > 0$ && $0 < \delta \leq$ 0.467…]

Reduce[$D[q_{GLM1}, \delta] \geq 0$ && $q_o > 0$ && $0 < \delta \leq$ 0.467…]

Reduce[$D[q_{GLN1}, \delta] \geq 0$ && $q_o > 0$ && $0 < \delta \leq$ 0.467…]

Reduce[$D[q_{GLP2}, \delta] \geq 0$ && $q_o > 0$ && 0.467… $< \delta \leq$ 0.541…]

Reduce[$D[q_{GLM2}, \delta] \geq 0$ && $q_o > 0$ && 0.467… $< \delta \leq$ 0.541…]

Reduce[$D[q_{GLN2}, \delta] \geq 0$ && $q_o > 0$ && 0.467… $< \delta \leq$ 0.541…]

Reduce[$D[q_{GLP3}, \delta] \geq 0$ && $q_o > 0$ && 0.541… $< \delta \leq 0.5984$]

Reduce[$D[q_{GLM3}, \delta] \geq 0$ && $q_o > 0$ && 0.541… $< \delta \leq 0.5984$]

Reduce[$D[q_{GLN3}, \delta] \geq 0$ && $q_o > 0$ && 0.541… $< \delta \leq 0.5984$]

Reduce[$D[q_{GLP4}, \delta] \geq 0$ && $q_o > 0$ && $0.5984 < \delta \leq$ 0.777…]

Reduce[$D[q_{GLM4}, \delta] \geq 0$ && $q_o > 0$ && $0.5984 < \delta \leq$ 0.777…]

Reduce[$D[q_{GLN4}, \delta] \geq 0$ && $q_o > 0$ && $0.5984 < \delta \leq$ 0.777…]

Reduce[$D[q_{GLP5}, \delta] \geq 0$ && $q_o > 0$ && 0.777… $< \delta \leq$ 0.827…]

Reduce[$D[q_{GLM5}, \delta] \geq 0$ && $q_o > 0$ && 0.777… $< \delta \leq$ 0.827…]

Reduce[$D[q_{GLN5}, \delta] \geq 0$ && $q_o > 0$ && 0.777… $< \delta \leq$ 0.827…]

Reduce[$D[q_{GLP6}, \delta] == 0$ && $q_o > 0$ && 0.827… $< \delta < 1$]

Reduce[$D[q_{GLM6}, \delta] == 0$ && $q_o > 0$ && 0.827… $< \delta < 1$]

Reduce[$D[q_{GLN6}, \delta] == 0$ && $q_o > 0$ && 0.827… $< \delta < 1$]

Out[•]=
```
False
```

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
$q_o > 0$ && $0.827\ldots < \delta < 1$

Out[•]=
$q_o > 0$ && $0.827\ldots < \delta < 1$

Out[•]=
$q_o > 0$ && $0.827\ldots < \delta < 1$

(*Hence, when $0<\delta\leq 0.827\ldots$, $\frac{\partial q_P^{GL}}{\partial \delta}<0$, $\frac{\partial q_M^{GL}}{\partial \delta}<0$, and $\frac{\partial q_N^{GL}}{\partial \delta}<0$ in their corresponding scenarios. When $0.827\ldots<\delta<1$, $\frac{\partial q_P^{GL}}{\partial \delta}=0$, $\frac{\partial q_M^{GL}}{\partial \delta}=0$, and $\frac{\partial q_N^{GL}}{\partial \delta}=0$*)

(*Part (iii)*)

(*Determine the sign of $\frac{\partial Pr_R^{GL}}{\partial \delta}$*)

```
In[•]:= Reduce[D[Pr_GLP1, δ] ≤ 0 && q_o > 0 && 0 < δ ≤ 0.467…]
        Reduce[D[Pr_GLM1, δ] ≥ 0 && q_o > 0 && 0 < δ ≤ 0.467…]
        Reduce[D[Pr_GLN1, δ] ≥ 0 && q_o > 0 && 0 < δ ≤ 0.467…]
        Reduce[D[Pr_GLP2, δ] ≤ 0 && q_o > 0 && 0.467… < δ ≤ 0.541…]
        Reduce[D[Pr_GLM2, δ] ≥ 0 && q_o > 0 && 0.467… < δ ≤ 0.541…]
        Reduce[D[Pr_GLN2, δ] ≥ 0 && q_o > 0 && 0.467… < δ ≤ 0.541…]
        Reduce[D[Pr_GLP3, δ] ≤ 0 && q_o > 0 && 0.541… < δ ≤ 0.5984]
        Reduce[D[Pr_GLM3, δ] ≥ 0 && q_o > 0 && 0.541… < δ ≤ 0.5984]
        Reduce[D[Pr_GLN3, δ] ≥ 0 && q_o > 0 && 0.541… < δ ≤ 0.5984]
        Reduce[D[Pr_GLP4, δ] ≤ 0 && q_o > 0 && 0.5984 < δ ≤ 0.777…]
        Reduce[D[Pr_GLM4, δ] ≥ 0 && q_o > 0 && 0.5984 < δ ≤ 0.777…]
        Reduce[D[Pr_GLN4, δ] ≥ 0 && q_o > 0 && 0.5984 < δ ≤ 0.777…]
        Reduce[D[Pr_GLP5, δ] ≤ 0 && q_o > 0 && 0.777… < δ ≤ 0.827…]
        Reduce[D[Pr_GLM5, δ] ≥ 0 && q_o > 0 && 0.777… < δ ≤ 0.827…]
        Reduce[D[Pr_GLN5, δ] ≥ 0 && q_o > 0 && 0.777… < δ ≤ 0.827…]
        Reduce[D[Pr_GLP6, δ] == 0 && q_o > 0 && 0.827… < δ < 1]
        Reduce[D[Pr_GLM6, δ] == 0 && q_o > 0 && 0.827… < δ < 1]
        Reduce[D[Pr_GLN6, δ] == 0 && q_o > 0 && 0.827… < δ < 1]
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
Out[•]=
        False
```

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
$q_o > 0$ && 0.827… $< \delta < 1$

Out[•]=
$q_o > 0$ && 0.827… $< \delta < 1$

Out[•]=
$q_o > 0$ && 0.827… $< \delta < 1$

(*Hence, when 0<δ≤0.827…, $\frac{\partial Pr_P^{GL}}{\partial \delta}>0$, $\frac{\partial Pr_M^{GL}}{\partial \delta}<0$, and $\frac{\partial Pr_N^{GL}}{\partial \delta}<0$ in their corresponding scenarios. When 0.827…<δ<1, $\frac{\partial Pr_P^{GL}}{\partial \delta}=0$, $\frac{\partial Pr_M^{GL}}{\partial \delta}=0$, and $\frac{\partial Pr_N^{GL}}{\partial \delta}=0$*)

Comparisons between GN and GL

```
(*Results of case GN*)
```

In[∘]:= $p_{GN1} = \frac{q_o}{2};$

$D_{GN1} = \frac{q_o}{2\,t};$

$p_{GN2} = \frac{q_o}{2};$

$D_{GN2} = 0;$

$\Pi_{GN} = \frac{q_o^2}{4\,t};$

$CS_{GN} = \frac{q_o^2}{8\,t};$

```
(*Results of case GL*)
```

(* i. $0<\delta\leq$ 0.467… *)

In[∘]:= $p_{GL11} = \frac{q_o\left(2 - 2\sqrt{1-\delta} + \left(-3 + 2\sqrt{1-\delta}\right)\delta + 2\,\delta^2\right)}{2\,\delta^2};;$

$D_{GL11} = \frac{-p_{GL11}\,\delta + 2\,q_o\left(-1 + \sqrt{1-\delta} + \delta\right)}{t\,\delta};$

$p_{GL2P1} = p_{GL11};$

$D_{GL2P1} = \frac{2\,q_o - p_{GL11} - t\,D_{GL11}}{2\,t};$

$\Pi_{GL1} = p_{GL11}\,D_{GL11} + \frac{2\,q_o - p_{GL11} - t\,D_{GL11}}{2\,q_o}\,p_{GL2P1}\,D_{GL2P1};$

CS_{GL1} = Integrate[(Integrate[$Q - p_{GL11} - t\,x$, {x, 0, D_{GL11}}]) / (2 q_o), {Q, 0, 2 q_o}] +
Integrate[(Integrate[$\delta\,(Q - p_{GL2P1} - t\,x)$, {x, D_{GL11}, $D_{GL11} + D_{GL2P1}$}]) / (2 q_o),
{Q, $p_{GL11} + t\,D_{GL11}$, 2 q_o}];

(* ii. 0.467… $<\delta\leq$ 0.541… *)

In[∘]:= $$p_{GL12}=\frac{2\,q_o\left(\sqrt{\left(33+20\sqrt{2}-4\,\delta\right)\,(1-\delta)}+\left(5+2\sqrt{2}\right)\delta-5-2\sqrt{2}\right)}{\left(7+2\sqrt{2}\right)\delta};$$

$$D_{GL12}=\frac{-4\,p_{GL12}\,q_o\,(-1+\delta)+4\,q_o^2\,(-1+\delta)+(p_{GL12})^2\,\delta}{4\,t\,q_o\,(-1+\delta)-2\,t\,p_{GL12}\,\delta};$$

$$p_{GL2P2}=p_{GL12};$$

$$p_{GL2M2}=\frac{2\,p_{GL12}+t\,D_{GL12}}{4};$$

$$D_{GL2P2}=\frac{2\,q_o-p_{GL12}-t\,D_{GL12}}{2\,t};$$

$$D_{GL2M2}=\frac{2\,p_{GL12}-3\,t\,D_{GL12}}{4\,t};$$

$$\Pi_{GL2}=p_{GL12}\,D_{GL12}+\frac{2\,q_o-p_{GL12}-t\,D_{GL12}}{2\,q_o}\,p_{GL2P2}\,D_{GL2P2}+\frac{t\,D_{GL12}}{2\,q_o}\,(p_{GL2M2}\,D_{GL2M2}-D_{GL12}\,(p_{GL12}-p_{GL2M2}));$$

$$CS_{GL2}=\text{Integrate}\left[\left(\text{Integrate}\left[Q-p_{GL12}-t\,x+\frac{t\,D_{GL12}}{2\,q_o}\,\delta\,(p_{GL12}-p_{GL2M2}),\,\{x,0,D_{GL12}\}\right]\right)\Big/\,(2\,q_o),\right.$$
$$\left.\{Q,0,2\,q_o\}\right]+\text{Integrate}[(\text{Integrate}[\delta\,(Q-p_{GL2P2}-t\,x),\{x,D_{GL12},D_{GL12}+D_{GL2P2}\}])\,/$$
$$(2\,q_o),\{Q,p_{GL12}+t\,D_{GL12},2\,q_o\}]+$$
$$\text{Integrate}[(\text{Integrate}[\delta\,(Q-p_{GL2M2}-t\,x),\{x,D_{GL12},D_{GL12}+D_{GL2M2}\}])\,/\,(2\,q_o),$$
$$\{Q,p_{GL12},p_{GL12}+t\,D_{GL12}\}];$$

(* iii. 0.541… <δ≤0.5984*)

In[∘]:= $$p_{GL13}=\text{Root}\left[8\,\delta^4\,\#1^4+144\,q_o^4+225\,\delta\,q_o^4-272\,\delta^2\,q_o^4+64\,\delta^3\,q_o^4+\right.$$
$$\#1^3\left(-16\,\delta^2\,q_o-94\,\delta^3\,q_o\right)+\#1^2\left(124\,\delta\,q_o^2+376\,\delta^2\,q_o^2-30\,\delta^3\,q_o^2-8\,\delta^4\,q_o^2\right)+$$
$$\left.\#1\left(-224\,q_o^3-480\,\delta\,q_o^3+104\,\delta^2\,q_o^3+64\,\delta^3\,q_o^3\right)\,\&,\,1\right];$$

$$D_{GL13}=\frac{2\,q_o\,(-2+\delta)+\sqrt{-8\,q_o^2\,(-2+\delta)-8\,p_{GL13}\,q_o\,\delta+(p_{GL13})^2\,\delta^2}}{t\,\delta};$$

$$p_{GL2P3}=\frac{2\,q_o+p_{GL13}-t\,D_{GL13}}{4};$$

$$D_{GL2P3}=\frac{2\,q_o+p_{GL13}-t\,D_{GL13}}{4\,t};$$

$$\Pi_{GL3}=p_{GL13}\,D_{GL13}+\frac{2\,q_o-p_{GL13}-t\,D_{GL13}}{2\,q_o}\,p_{GL2P3}\,D_{GL2P3};$$

$$CS_{GL3}=\text{Integrate}[(\text{Integrate}[Q-p_{GL13}-t\,x,\{x,0,D_{GL13}\}])\,/\,(2\,q_o),\{Q,0,2\,q_o\}]+$$
$$\text{Integrate}[(\text{Integrate}[\delta\,(Q-p_{GL2P3}-t\,x),\{x,D_{GL13},D_{GL13}+D_{GL2P3}\}])\,/\,(2\,q_o),$$
$$\{Q,p_{GL13}+t\,D_{GL13},2\,q_o\}];$$

(* iv. 0.5984<δ≤ 0.777… *)

In[∘]:=

$$p_{GL14} = \frac{1}{51\,\delta}\,2\,q_o\left(66+40\sqrt{2}-2\left(5+2\sqrt{2}\right)\delta-\sqrt{7556+5280\sqrt{2}-2\left(2153+1480\sqrt{2}\right)\delta+\left(1305+896\sqrt{2}\right)\delta^2}\right);$$

$$D_{GL14} = -\frac{2\,q_o\,(-2+\delta)+\sqrt{8\,p_{GL14}\,q_o\,\delta-(p_{GL14})^2\,\delta^2+8\,q_o^2\left(2-3\,\delta+\delta^2\right)}}{t\,\delta};$$

$$p_{GL2P4} = \frac{2\,q_o+p_{GL14}-t\,D_{GL14}}{4};$$

$$p_{GL2M4} = \frac{2\,p_{GL14}+t\,D_{GL14}}{4};$$

$$D_{GL2P4} = \frac{2\,q_o+p_{GL14}-t\,D_{GL14}}{4\,t};$$

$$D_{GL2M4} = \frac{2\,p_{GL14}-3\,t\,D_{GL14}}{4\,t};$$

$$\Pi_{GL4} = p_{GL14}\,D_{GL14}+\frac{2\,q_o-p_{GL14}-t\,D_{GL14}}{2\,q_o}\,p_{GL2P4}\,D_{GL2P4}+\frac{t\,D_{GL14}}{2\,q_o}\,(p_{GL2M4}\,D_{GL2M4}-D_{GL14}\,(p_{GL14}-p_{GL2M4}));$$

$$CS_{GL4} = \text{Integrate}\left[\left(\text{Integrate}\left[Q-p_{GL14}-t\,x+\frac{t\,D_{GL14}}{2\,q_o}\,\delta\,(p_{GL14}-p_{GL2M4}),\{x,0,D_{GL14}\}\right]\right)\Big/(2\,q_o),\{Q,0,2\,q_o\}\right]+\text{Integrate}[(\text{Integrate}[\delta\,(Q-p_{GL2P4}-t\,x),\{x,D_{GL14},D_{GL14}+D_{GL2P4}\}])/(2\,q_o),\{Q,p_{GL14}+t\,D_{GL14},2\,q_o\}]+\text{Integrate}[(\text{Integrate}[\delta\,(Q-p_{GL2M4}-t\,x),\{x,D_{GL14},D_{GL14}+D_{GL2M4}\}])/(2\,q_o),\{Q,p_{GL14},p_{GL14}+t\,D_{GL14}\}];$$

(* v. 0.777... <δ≤ 0.827... *)

In[ ]:=

$$p_{GL15} = \mathrm{Root}\Big[1125\,\#1^6 + 60928\,q_o^6 + \#1^5\left(2160\,q_o - \frac{23640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151312\,q_o^2}{\delta^2} - \frac{12096\,q_o^2}{\delta}\right) + \#1^3\left(-20160\,q_o^3 - \frac{205056\,q_o^3}{\delta^3} - \frac{193152\,q_o^3}{\delta^2} + \frac{73664\,q_o^3}{\delta}\right) + \#1^2\left(-40624\,q_o^4 - \frac{574464\,q_o^4}{\delta^4} + \frac{1165824\,q_o^4}{\delta^3} - \frac{582720\,q_o^4}{\delta^2} + \frac{266496\,q_o^4}{\delta}\right) + \#1\left(27648\,q_o^5 + \frac{67584\,q_o^5}{\delta^4} + \frac{170496\,q_o^5}{\delta^3} - \frac{376320\,q_o^5}{\delta^2} + \frac{58240\,q_o^5}{\delta}\right) + \frac{236544\,q_o^6}{\delta^4} - \frac{766464\,q_o^6}{\delta^3} + \frac{859904\,q_o^6}{\delta^2} - \frac{388608\,q_o^6}{\delta}\ \&,\ 3\Big];$$

$$D_{GL15} = -\frac{2\,q_o\,(-2+\delta) + \sqrt{8\,p_{GL15}\,q_o\,\delta - (p_{GL15})^2\,\delta^2 + 8\,q_o^2\,(2 - 3\,\delta + \delta^2)}}{t\,\delta};$$

$$p_{GL2P5} = \frac{2\,q_o + p_{GL15} - t\,D_{GL15}}{4};$$

$$p_{GL2M5} = \frac{2\,p_{GL15} + t\,D_{GL15}}{4};$$

$$D_{GL2P5} = \frac{2\,q_o + p_{GL15} - t\,D_{GL15}}{4\,t};$$

$$D_{GL2M5} = \frac{2\,p_{GL15} - 3\,t\,D_{GL15}}{4\,t};$$

$$\Pi_{GL5} = p_{GL15}\,D_{GL15} + \frac{2\,q_o - p_{GL15} - t\,D_{GL15}}{2\,q_o}\,p_{GL2P5}\,D_{GL2P5} + \frac{t\,D_{GL15}}{2\,q_o}\,(p_{GL2M5}\,D_{GL2M5} - D_{GL15}\,(p_{GL15} - p_{GL2M5}));$$

$$CS_{GL5} = \mathrm{Integrate}\left[\left(\mathrm{Integrate}\left[Q - p_{GL15} - t\,x + \frac{t\,D_{GL15}}{2\,q_o}\,\delta\,(p_{GL15} - p_{GL2M5}),\ \{x,\ 0,\ D_{GL15}\}\right]\right)\Big/\ (2\,q_o),\ \{Q,\ 0,\ 2\,q_o\}\right] + \mathrm{Integrate}[(\mathrm{Integrate}[\delta\,(Q - p_{GL2P5} - t\,x),\ \{x,\ D_{GL15},\ D_{GL15} + D_{GL2P5}\}])\,/\,(2\,q_o),\ \{Q,\ p_{GL15} + t\,D_{GL15},\ 2\,q_o\}] + \mathrm{Integrate}[(\mathrm{Integrate}[\delta\,(Q - p_{GL2M5} - t\,x),\ \{x,\ D_{GL15},\ D_{GL15} + D_{GL2M5}\}])\,/\,(2\,q_o),\ \{Q,\ p_{GL15},\ p_{GL15} + t\,D_{GL15}\}];$$

```
(* vi. 0.827… <δ<1*)
```

In[∘]:=

$$p_{GL16} = \frac{16\,q_o}{17};$$

$$D_{GL16} = \frac{-p_{GL16} + q_o}{t};$$

$$p_{GL2P6} = \frac{2\,q_o + p_{GL16} + t\,D_{GL16}}{4};$$

$$p_{GL2M6} = \frac{2\,p_{GL16} + t\,D_{GL16}}{4};$$

$$p_{GL2N6} = \frac{p_{GL16}}{4};$$

$$D_{GL2P6} = \frac{2\,q_o + p_{GL16} - 3\,t\,D_{GL16}}{4\,t};$$

$$D_{GL2M6} = \frac{2\,p_{GL16} - 3\,t\,D_{GL16}}{4\,t};$$

$$D_{GL2N6} = \frac{p_{GL16} - 4\,t\,D_{GL16}}{4\,t};$$

$$\Pi_{GL6} = p_{GL16}\,D_{GL16} + \frac{2\,q_o - p_{GL16} - t\,D_{GL16}}{2\,q_o}\,(p_{GL2P6}\,D_{GL2P6} - D_{GL16}\,(p_{GL16} - p_{GL2P6})) + \frac{t\,D_{GL16}}{2\,q_o}\,(p_{GL2M6}\,D_{GL2M6} - D_{GL16}\,(p_{GL16} - p_{GL2M6})) + \frac{p_{GL16}}{2\,q_o}\,(p_{GL2N6}\,D_{GL2N6} - D_{GL16}\,(p_{GL16} - p_{GL2N6}));$$

$$CS_{GL6} = \text{Integrate}\Big[\Big(\text{Integrate}\Big[Q - p_{GL16} - t\,x + \delta\,\Big(\frac{2\,q_o - p_{GL16} - t\,D_{GL16}}{2\,q_o}\,(p_{GL16} - p_{GL2P6}) + \frac{t\,D_{GL16}}{2\,q_o}\,(p_{GL16} - p_{GL2M6}) + \frac{p_{GL16}}{2\,q_o}\,(p_{GL16} - p_{GL2N6})\Big),\ \{x,\ 0,\ D_{GL16}\}\Big]\Big) \Big/ (2\,q_o),\ \{Q,\ 0,\ 2\,q_o\}\Big] +$$

$$\text{Integrate}[(\text{Integrate}[\delta\,(Q - p_{GL2P6} - t\,x),\ \{x,\ D_{GL16},\ D_{GL16} + D_{GL2P6}\}])\,/\,(2\,q_o),\ \{Q,\ p_{GL16} + t\,D_{GL16},\ 2\,q_o\}] +$$

$$\text{Integrate}[(\text{Integrate}[\delta\,(Q - p_{GL2M6} - t\,x),\ \{x,\ D_{GL16},\ D_{GL16} + D_{GL2M6}\}])\,/\,(2\,q_o),\ \{Q,\ p_{GL16},\ p_{GL16} + t\,D_{GL16}\}] + \text{Integrate}[(\text{Integrate}[\delta\,(Q - p_{GL2N6} - t\,x),\ \{x,\ D_{GL16},\ D_{GL16} + D_{GL2N6}\}])\,/\,(2\,q_o),\ \{Q,\ 0,\ p_{GL16}\}];$$

```
(***Proof of Proposition 9: Price comparison***)

(*Part (i)*) (*Compare p1GN* and p1GL**)

In[•]:= Reduce[pGN1 ≥ pGL11 && qo > 0 && 0 < δ ≤ 0.467…]
Reduce[pGN1 ≥ pGL12 && qo > 0 && 0.467… < δ ≤ 0.540855]
Reduce[pGN1 ≥ pGL13 && qo > 0 && 0.540855 < δ ≤ 0.5984]
Reduce[pGN1 ≥ pGL14 && qo > 0 && 0.5984 < δ ≤ 0.777…]
Reduce[pGN1 ≥ pGL15 && qo > 0 && 0.777… < δ ≤ 0.827…]
Reduce[pGN1 ≥ pGL16 && qo > 0 && 0.827… < δ < 1]

Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False
Out[•]= False

(*Hence, p1GN*<p1GL**)

(*Part (ii)*) (*We only have to compare p2GN* and p2RGL* when p2RGL*<p1GL**)

Reduce[pGN2 ≥ pGL2P6 && qo > 0 && 0.827… < δ < 1]
(*Compare p2GN* and p2PGL* where p2PGL*<p1GL**)
Reduce[pGN2 ≤ pGL2M2 && qo > 0 && 0.467… < δ ≤ 0.540855]
(*Compare p2GN* and p2MGL* where p2MGL*<p1GL* when 0.467…<δ≤0.540855*)
Reduce[pGN2 ≤ pGL2M4 && qo > 0 && 0.5984 < δ ≤ 0.777…]
(*Compare p2GN* and p2MGL* where p2MGL*<p1GL* when 0.5984<δ≤0.777…*)
Reduce[pGN2 ≤ pGL2M5 && qo > 0 && 0.777… < δ ≤ 0.827…]
(*Compare p2GN* and p2MGL* where p2MGL*<p1GL* when 0.777…<δ≤0.827…*)
Reduce[pGN2 ≤ pGL2M6 && qo > 0 && 0.827… < δ < 1]
(*Compare p2GN* and p2MGL* where p2MGL*<p1GL* when 0.827…<δ<1*)
Reduce[pGN2 ≤ pGL2N6 && qo > 0 && 0.827… < δ < 1]
(*Compare p2GN* and p2NGL* where p2NGL*<p1GL* when 0.827…<δ<1*)

Out[•]= False
Out[•]= False
```

Out[ ]=
```
False
```
Out[ ]=
```
False
```
Out[ ]=
```
False
```
Out[ ]=
```
False
```

(*Hence, p_2^{GN*}<p_{2P}^{GL*}, p_2^{GN*}>p_{2M}^{GL*} when p_{2M}^{GL*}<p_1^{GL*}, and p_2^{GN*}>p_{2N}^{GL*} whne p_{2N}^{GL*}<p_1^{GL*}*)

```
(***Proof of Proposition 10: Profit and consumer surplus comparisons***)

(*Part (i)*)(*Compare Π^GN and Π^GL*)

Reduce[Π_GN ≥ Π_GL1 && t > 2 q_o > 0 && 0 < δ ≤ 0.467…]
Reduce[Π_GN ≥ Π_GL2 && t > 2 q_o > 0 && 0.467… < δ ≤ 0.540855]
Reduce[Π_GN ≥ Π_GL3 && t > 2 q_o > 0 && 0.540855 < δ ≤ 0.5984]
Reduce[Π_GN ≥ Π_GL4 && t > 2 q_o > 0 && 0.5984 < δ ≤ 0.777…]
Reduce[Π_GN ≥ Π_GL5 && t > 2 q_o > 0 && 0.777… < δ ≤ 0.827…]
Reduce[Π_GN ≥ Π_GL6 && t > 2 q_o > 0 && 0.827… < δ < 1]
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
False

(*Hence, Π^GN<Π^GL always holds true*)

(*Part (ii)*)(*Compare CS^GN and CS^GL*)

In[•]:= Reduce[CS_GN ≤ CS_GL1 && t > 2 q_o > 0 && 0 < δ ≤ 0.467…]
Reduce[CS_GN ≤ CS_GL2 && t > 2 q_o > 0 && 0.467… < δ ≤ 0.540855]
Reduce[CS_GN ≥ CS_GL3 && t > 2 q_o > 0 && 0.540855 < δ ≤ 0.5984]
Reduce[CS_GN ≥ CS_GL4 && t > 2 q_o > 0 && 0.5984 < δ ≤ 0.777…]
Reduce[CS_GN ≥ CS_GL5 && t > 2 q_o > 0 && 0.777… < δ ≤ 0.827…]
Reduce[CS_GN ≥ CS_GL6 && t > 2 q_o > 0 && 0.827… < δ < 1]
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
False
Out[•]=
False
```

```
(*Hence, when 0<δ≤0.540855 CS^GN>CS^GL; when 0.540855<δ<1, CS^GN<CS^GL*)
```

Comparisons across all scenarios

(*Results of case CL*)

(* (i). 0<δ≤ 0.119… *)

In[∘]:=

$$p_{CL11} = \text{Root}\left[7424\, q_o^5 - 11136\, q_o^5\, \delta + 5568\, q_o^5\, \delta^2 + 9\, \#1^5\, \delta^3 - 928\, q_o^5\, \delta^3 + \#1^4\left(12\, q_o\, \delta^2 + 70\, q_o\, \delta^3\right) + \#1^3\left(-960\, q_o^2\, \delta + 864\, q_o^2\, \delta^2 - 520\, q_o^2\, \delta^3\right) + \#1^2\left(-6912\, q_o^3 + 1920\, q_o^3\, \delta - 672\, q_o^3\, \delta^2 + 720\, q_o^3\, \delta^3\right) + \#1\left(-6656\, q_o^4 + 11712\, q_o^4\, \delta - 4992\, q_o^4\, \delta^2 + 400\, q_o^4\, \delta^3\right)\ \&,\ 2\right];$$

$$D_{CL11} = \frac{8\, p_{CL11}\, q_o + 4\, q_o^2\, (-2+\delta) - p_{CL11}{}^2\, \delta}{4\, t\, q_o\, (-2+\delta) - 2\, t\, p_{CL11}\, \delta};$$

$$p_{CL2P1} = \frac{2\, q_o + p_{CL11} - t\, D_{CL11}}{4};$$

$$D_{CL2P1} = \frac{2\, q_o + p_{CL11} - t\, D_{CL11}}{4\, t};$$

$$p_{CL2M1} = \frac{2\, p_{CL11} - t\, D_{CL11}}{4};$$

$$D_{CL2M1} = \frac{2\, p_{CL11} - t\, D_{CL11}}{4\, t};$$

$$\Pi_{CL1} = \text{Simplify}\left[p_{CL11}\, D_{CL11} + \frac{2\, q_o - p_{CL11} - t\, D_{CL11}}{2\, q_o}\, p_{CL2P1}\, D_{CL2P1} + \frac{t\, D_{CL11}}{2\, q_o}\, p_{CL2M1}\, D_{CL2M1}\right];$$

$$\begin{aligned} CS_{CL1} = {} & \text{Integrate}[\text{Integrate}[Q - p_{CL11} - t\,x,\ \{x,\ 0,\ D_{CL11}\}] / (2\, q_o),\ \{Q,\ 0,\ 2\, q_o\}] + \\ & \text{Integrate}[\text{Integrate}[\delta\ (Q - p_{CL2P1} - t\,x),\ \{x,\ D_{CL11},\ D_{CL11} + D_{CL2P1}\}] / (2\, q_o), \\ & \{Q,\ p_{CL11} + t\, D_{CL11},\ 2\, q_o\}] + \text{Integrate}[\\ & \text{Integrate}[\delta\ (Q - p_{CL2M1} - t\,x),\ \{x,\ D_{CL11},\ D_{CL11} + D_{CL2M1}\}] / (2\, q_o),\ \{Q,\ p_{CL11},\ p_{CL11} + t\, D_{CL11}\}]; \end{aligned}$$

(* (ii). 0.119… <δ≤ 0.391… *)

In[∘]:=

$$p_{CL12} = \frac{2\, q_o\, (-2+\delta)}{-6+\delta};$$

$$D_{CL12} = \frac{8\, p_{CL12}\, q_o + 4\, q_o^2\, (-2+\delta) - p_{CL12}{}^2\, \delta}{4\, t\, q_o\, (-2+\delta) - 2\, t\, p_{CL12}\, \delta};$$

$$p_{CL2P2} = \frac{2\, q_o + p_{CL12} - t\, D_{CL12}}{4};$$

$$D_{CL2P2} = \frac{2\, q_o + p_{CL12} - t\, D_{CL12}}{4\, t};$$

$$p_{CL2M2} = \frac{2\, p_{CL12} - t\, D_{CL12}}{4};$$

$$D_{CL2M2} = \frac{2\, p_{CL12} - t\, D_{CL12}}{4\, t};$$

$$\Pi_{CL2} = p_{CL12}\, D_{CL12} + \frac{2\, q_o - p_{CL12} - t\, D_{CL12}}{2\, q_o}\, p_{CL2P2}\, D_{CL2P2} + \frac{t\, D_{CL12}}{2\, q_o}\, p_{CL2M2}\, D_{CL2M2};$$

$$\begin{aligned} CS_{CL2} = {} & \text{Integrate}[\text{Integrate}[Q - p_{CL12} - t\,x,\ \{x,\ 0,\ D_{CL12}\}] / (2\, q_o),\ \{Q,\ 0,\ 2\, q_o\}] + \\ & \text{Integrate}[\text{Integrate}[\delta\ (Q - p_{CL2P2} - t\,x),\ \{x,\ D_{CL12},\ D_{CL12} + D_{CL2P2}\}] / (2\, q_o), \\ & \{Q,\ p_{CL12} + t\, D_{CL12},\ 2\, q_o\}] + \text{Integrate}[\\ & \text{Integrate}[\delta\ (Q - p_{CL2M2} - t\,x),\ \{x,\ D_{CL12},\ D_{CL12} + D_{CL2M2}\}] / (2\, q_o),\ \{Q,\ p_{CL12},\ p_{CL12} + t\, D_{CL12}\}]; \end{aligned}$$

(*(iii) 0.391… <δ<1*)

In[∘]:=
$$p_{CL13} = \frac{q_o \left(100 - 64\,\delta + \delta^2 - \sqrt{10000 - 13592\,\delta + 5088\,\delta^2 - 326\,\delta^3 + \delta^4}\right)}{6\,\delta};$$

$$D_{CL13} = \frac{q_o\,(2-\delta) - 2\,p_{CL13}}{t\,(2-\delta)};$$

$$p_{CL2P3} = \frac{2\,q_o + p_{CL13} - t\,D_{CL13}}{4};$$

$$D_{CL2P3} = \frac{2\,q_o + p_{CL13} - t\,D_{CL13}}{4\,t};$$

$$p_{CL2M3} = \frac{2\,p_{CL13} - t\,D_{CL13}}{4};$$

$$D_{CL2M3} = \frac{2\,p_{CL13} - t\,D_{CL13}}{4\,t};$$

$$p_{CL2N3} = \frac{p_{CL13} - 2\,t\,D_{CL13}}{4};$$

$$D_{CL2N3} = \frac{p_{CL13} - 2\,t\,D_{CL13}}{4\,t};$$

$$\Pi_{CL3} = p_{CL13}\,D_{CL13} + \frac{2\,q_o - p_{CL13} - t\,D_{CL13}}{2\,q_o}\,p_{CL2P3}\,D_{CL2P3} + \frac{t\,D_{CL13}}{2\,q_o}\,p_{CL2M3}\,D_{CL2M3} + \frac{p_{CL13}}{2\,q_o}\,p_{CL2N3}\,D_{CL2N3};$$

```
CS_CL3 = Integrate[Integrate[Q - p_CL13 - t x, {x, 0, D_CL13}] / (2 q_o), {Q, 0, 2 q_o}] + Integrate[
    Integrate[δ (Q - p_CL2P3 - t x), {x, D_CL13, D_CL13 + D_CL2P3}] / (2 q_o), {Q, p_CL13 + t D_CL13, 2 q_o}] +
   Integrate[Integrate[δ (Q - p_CL2M3 - t x), {x, D_CL13, D_CL13 + D_CL2M3}] / (2 q_o),
    {Q, p_CL13, p_CL13 + t D_CL13}] +
   Integrate[Integrate[δ (Q - p_CL2N3 - t x), {x, D_CL13, D_CL13 + D_CL2N3}] / (2 q_o), {Q, 0, p_CL13}];
```

```
(*Results of case GL*)

(* i. 0<δ≤ 0.467… *)
```

In[∘]:=
$$p_{GL11} = \frac{q_o \left(2 - 2\sqrt{1-\delta} + \left(-3 + 2\sqrt{1-\delta}\right)\delta + 2\,\delta^2\right)}{2\,\delta^2};;$$

$$D_{GL11} = \frac{-p_{GL11}\,\delta + 2\,q_o\left(-1 + \sqrt{1-\delta} + \delta\right)}{t\,\delta};$$

$$p_{GL2P1} = p_{GL11};$$

$$D_{GL2P1} = \frac{2\,q_o - p_{GL11} - t\,D_{GL11}}{2\,t};$$

$$\Pi_{GL1} = p_{GL11}\,D_{GL11} + \frac{2\,q_o - p_{GL11} - t\,D_{GL11}}{2\,q_o}\,p_{GL2P1}\,D_{GL2P1};$$

```
CS_GL1 = Integrate[(Integrate[Q - p_GL11 - t x, {x, 0, D_GL11}]) / (2 q_o), {Q, 0, 2 q_o}] +
   Integrate[(Integrate[δ (Q - p_GL2P1 - t x), {x, D_GL11, D_GL11 + D_GL2P1}]) / (2 q_o),
    {Q, p_GL11 + t D_GL11, 2 q_o}];

(* ii.  0.467… <δ≤0.540855*)
```

In[ ]:=

$$p_{GL12} = \frac{2\,q_o\left(\sqrt{\left(33+20\sqrt{2}-4\,\delta\right)(1-\delta)}+\left(5+2\sqrt{2}\right)\delta-5-2\sqrt{2}\right)}{\left(7+2\sqrt{2}\right)\delta};$$

$$D_{GL12} = \frac{-4\,p_{GL12}\,q_o\,(-1+\delta)+4\,q_o^2\,(-1+\delta)+(p_{GL12})^2\,\delta}{4\,t\,q_o\,(-1+\delta)-2\,t\,p_{GL12}\,\delta};$$

$$p_{GL2P2} = p_{GL12};$$

$$p_{GL2M2} = \frac{2\,p_{GL12}+t\,D_{GL12}}{4};$$

$$D_{GL2P2} = \frac{2\,q_o-p_{GL12}-t\,D_{GL12}}{2\,t};$$

$$D_{GL2M2} = \frac{2\,p_{GL12}-3\,t\,D_{GL12}}{4\,t};$$

$$\Pi_{GL2} = p_{GL12}\,D_{GL12}+\frac{2\,q_o-p_{GL12}-t\,D_{GL12}}{2\,q_o}\,p_{GL2P2}\,D_{GL2P2}+\frac{t\,D_{GL12}}{2\,q_o}\,(p_{GL2M2}\,D_{GL2M2}-D_{GL12}\,(p_{GL12}-p_{GL2M2}));$$

$$\begin{aligned}CS_{GL2} = {} & \text{Integrate}\left[\left(\text{Integrate}\left[Q-p_{GL12}-t\,x+\frac{t\,D_{GL12}}{2\,q_o}\,\delta\,(p_{GL12}-p_{GL2M2}),\ \{x,0,D_{GL12}\}\right]\right)\Big/\ (2\,q_o),\right.\\ & \left.\{Q,0,2\,q_o\}\right]+\text{Integrate}[(\text{Integrate}[\delta\,(Q-p_{GL2P2}-t\,x),\ \{x,D_{GL12},D_{GL12}+D_{GL2P2}\}])\ /\\ & (2\,q_o),\ \{Q,p_{GL12}+t\,D_{GL12},2\,q_o\}]+\\ & \text{Integrate}[(\text{Integrate}[\delta\,(Q-p_{GL2M2}-t\,x),\ \{x,D_{GL12},D_{GL12}+D_{GL2M2}\}])\ /\ (2\,q_o),\\ & \{Q,p_{GL12},p_{GL12}+t\,D_{GL12}\}];\end{aligned}$$

```
(* iii. 0.540855<δ≤0.5984*)
```

In[ ]:=

$$\begin{aligned}p_{GL13} = {} & \text{Root}\left[8\,\delta^4\,\#1^4+144\,q_o^4+225\,\delta\,q_o^4-272\,\delta^2\,q_o^4+64\,\delta^3\,q_o^4+\right.\\ & \#1^3\left(-16\,\delta^2\,q_o-94\,\delta^3\,q_o\right)+\#1^2\left(124\,\delta\,q_o^2+376\,\delta^2\,q_o^2-30\,\delta^3\,q_o^2-8\,\delta^4\,q_o^2\right)+\\ & \left.\#1\left(-224\,q_o^3-480\,\delta\,q_o^3+104\,\delta^2\,q_o^3+64\,\delta^3\,q_o^3\right)\ \&,\ 1\right];\end{aligned}$$

$$D_{GL13} = \frac{2\,q_o\,(-2+\delta)+\sqrt{-8\,q_o^2\,(-2+\delta)-8\,p_{GL13}\,q_o\,\delta+(p_{GL13})^2\,\delta^2}}{t\,\delta};$$

$$p_{GL2P3} = \frac{2\,q_o+p_{GL13}-t\,D_{GL13}}{4};$$

$$D_{GL2P3} = \frac{2\,q_o+p_{GL13}-t\,D_{GL13}}{4\,t};$$

$$\Pi_{GL3} = p_{GL13}\,D_{GL13}+\frac{2\,q_o-p_{GL13}-t\,D_{GL13}}{2\,q_o}\,p_{GL2P3}\,D_{GL2P3};$$

$$\begin{aligned}CS_{GL3} = {} & \text{Integrate}[(\text{Integrate}[Q-p_{GL13}-t\,x,\ \{x,0,D_{GL13}\}])\ /\ (2\,q_o),\ \{Q,0,2\,q_o\}]+\\ & \text{Integrate}[(\text{Integrate}[\delta\,(Q-p_{GL2P3}-t\,x),\ \{x,D_{GL13},D_{GL13}+D_{GL2P3}\}])\ /\ (2\,q_o),\\ & \{Q,p_{GL13}+t\,D_{GL13},2\,q_o\}];\end{aligned}$$

```
(* iv. 0.5984<δ≤ 0.777… *)
```

In[ ]:=
$$p_{GL14} = \frac{1}{51\,\delta} 2\, q_o \left(66 + 40\sqrt{2} - 2\left(5 + 2\sqrt{2}\right)\delta - \sqrt{7556 + 5280\sqrt{2} - 2\left(2153 + 1480\sqrt{2}\right)\delta + \left(1305 + 896\sqrt{2}\right)\delta^2}\right);$$

$$D_{GL14} = -\frac{2\, q_o\,(-2+\delta) + \sqrt{8\, p_{GL14}\, q_o\, \delta - (p_{GL14})^2\,\delta^2 + 8\, q_o^2\left(2 - 3\,\delta + \delta^2\right)}}{t\,\delta};$$

$$p_{GL2P4} = \frac{2\, q_o + p_{GL14} - t\, D_{GL14}}{4};$$

$$p_{GL2M4} = \frac{2\, p_{GL14} + t\, D_{GL14}}{4};$$

$$D_{GL2P4} = \frac{2\, q_o + p_{GL14} - t\, D_{GL14}}{4\, t};$$

$$D_{GL2M4} = \frac{2\, p_{GL14} - 3\, t\, D_{GL14}}{4\, t};$$

$$\Pi_{GL4} = p_{GL14}\, D_{GL14} + \frac{2\, q_o - p_{GL14} - t\, D_{GL14}}{2\, q_o}\, p_{GL2P4}\, D_{GL2P4} + \frac{t\, D_{GL14}}{2\, q_o}\,(p_{GL2M4}\, D_{GL2M4} - D_{GL14}\,(p_{GL14} - p_{GL2M4}));$$

$$CS_{GL4} = \text{Integrate}\left[\left(\text{Integrate}\left[Q - p_{GL14} - t\,x + \frac{t\, D_{GL14}}{2\, q_o}\,\delta\,(p_{GL14} - p_{GL2M4}),\ \{x,\ 0,\ D_{GL14}\}\right]\right)\Big/\ (2\, q_o),\ \{Q,\ 0,\ 2\, q_o\}\right] + \text{Integrate}[(\text{Integrate}[\delta\,(Q - p_{GL2P4} - t\,x),\ \{x,\ D_{GL14},\ D_{GL14} + D_{GL2P4}\}])\,/\,(2\, q_o),\ \{Q,\ p_{GL14} + t\, D_{GL14},\ 2\, q_o\}] + \text{Integrate}[(\text{Integrate}[\delta\,(Q - p_{GL2M4} - t\,x),\ \{x,\ D_{GL14},\ D_{GL14} + D_{GL2M4}\}])\,/\,(2\, q_o),\ \{Q,\ p_{GL14},\ p_{GL14} + t\, D_{GL14}\}];$$

```
(* v. 0.777... <δ≤ 0.827... *)
```

In[•]:= p_{GL15} =

$$\text{Root}\Big[1125\,\#1^6 + 60928\,q_o^6 + \#1^5\left(2160\,q_o - \frac{23640\,q_o}{\delta}\right) + \#1^4\left(312\,q_o^2 + \frac{151312\,q_o^2}{\delta^2} - \frac{12096\,q_o^2}{\delta}\right) + \#1^3\left(-20160\,q_o^3 - \frac{205056\,q_o^3}{\delta^3} - \frac{193152\,q_o^3}{\delta^2} + \frac{73664\,q_o^3}{\delta}\right) + \#1^2\left(-40624\,q_o^4 - \frac{574464\,q_o^4}{\delta^4} + \frac{1165824\,q_o^4}{\delta^3} - \frac{582720\,q_o^4}{\delta^2} + \frac{266496\,q_o^4}{\delta}\right) + \#1\left(27648\,q_o^5 + \frac{67584\,q_o^5}{\delta^4} + \frac{170496\,q_o^5}{\delta^3} - \frac{376320\,q_o^5}{\delta^2} + \frac{58240\,q_o^5}{\delta}\right) + \frac{236544\,q_o^6}{\delta^4} - \frac{766464\,q_o^6}{\delta^3} + \frac{859904\,q_o^6}{\delta^2} - \frac{388608\,q_o^6}{\delta}\ \&,\ 3\Big];$$

$$D_{GL15} = -\frac{2\,q_o\,(-2+\delta) + \sqrt{8\,p_{GL15}\,q_o\,\delta - (p_{GL15})^2\,\delta^2 + 8\,q_o^2\,(2 - 3\,\delta + \delta^2)}}{t\,\delta};$$

$$p_{GL2P5} = \frac{2\,q_o + p_{GL15} - t\,D_{GL15}}{4};$$

$$p_{GL2M5} = \frac{2\,p_{GL15} + t\,D_{GL15}}{4};$$

$$D_{GL2P5} = \frac{2\,q_o + p_{GL15} - t\,D_{GL15}}{4\,t};$$

$$D_{GL2M5} = \frac{2\,p_{GL15} - 3\,t\,D_{GL15}}{4\,t};$$

$$\Pi_{GL5} = p_{GL15}\,D_{GL15} + \frac{2\,q_o - p_{GL15} - t\,D_{GL15}}{2\,q_o}\,p_{GL2P5}\,D_{GL2P5} + \frac{t\,D_{GL15}}{2\,q_o}\,(p_{GL2M5}\,D_{GL2M5} - D_{GL15}\,(p_{GL15} - p_{GL2M5}));$$

$$CS_{GL5} = \text{Integrate}\left[\left(\text{Integrate}\left[Q - p_{GL15} - t\,x + \frac{t\,D_{GL15}}{2\,q_o}\,\delta\,(p_{GL15} - p_{GL2M5}),\ \{x, 0, D_{GL15}\}\right]\right)\Big/\,(2\,q_o),\ \{Q, 0, 2\,q_o\}\right] + \text{Integrate}[(\text{Integrate}[\delta\,(Q - p_{GL2P5} - t\,x),\ \{x, D_{GL15}, D_{GL15} + D_{GL2P5}\}]) / (2\,q_o),\ \{Q, p_{GL15} + t\,D_{GL15}, 2\,q_o\}] + \text{Integrate}[(\text{Integrate}[\delta\,(Q - p_{GL2M5} - t\,x),\ \{x, D_{GL15}, D_{GL15} + D_{GL2M5}\}]) / (2\,q_o),\ \{Q, p_{GL15}, p_{GL15} + t\,D_{GL15}\}];$$

(* vi. 0.827… <δ<1*)

In[∘]:=

$$p_{GL16} = \frac{16\, q_o}{17};$$

$$D_{GL16} = \frac{-p_{GL16} + q_o}{t};$$

$$p_{GL2P6} = \frac{2\, q_o + p_{GL16} + t\, D_{GL16}}{4};$$

$$p_{GL2M6} = \frac{2\, p_{GL16} + t\, D_{GL16}}{4};$$

$$p_{GL2N6} = \frac{p_{GL16}}{4};$$

$$D_{GL2P6} = \frac{2\, q_o + p_{GL16} - 3\, t\, D_{GL16}}{4\, t};$$

$$D_{GL2M6} = \frac{2\, p_{GL16} - 3\, t\, D_{GL16}}{4\, t};$$

$$D_{GL2N6} = \frac{p_{GL16} - 4\, t\, D_{GL16}}{4\, t};$$

$$\Pi_{GL6} = p_{GL16}\, D_{GL16} + \frac{2\, q_o - p_{GL16} - t\, D_{GL16}}{2\, q_o} (p_{GL2P6}\, D_{GL2P6} - D_{GL16}\, (p_{GL16} - p_{GL2P6})) + \frac{t\, D_{GL16}}{2\, q_o} (p_{GL2M6}\, D_{GL2M6} - D_{GL16}\, (p_{GL16} - p_{GL2M6})) + \frac{p_{GL16}}{2\, q_o} (p_{GL2N6}\, D_{GL2N6} - D_{GL16}\, (p_{GL16} - p_{GL2N6}));$$

$$CS_{GL6} = \text{Integrate}\Big[\Big(\text{Integrate}\Big[Q - p_{GL16} - t\, x + \delta \Big(\frac{2\, q_o - p_{GL16} - t\, D_{GL16}}{2\, q_o} (p_{GL16} - p_{GL2P6}) + \frac{t\, D_{GL16}}{2\, q_o} (p_{GL16} - p_{GL2M6}) + \frac{p_{GL16}}{2\, q_o} (p_{GL16} - p_{GL2N6})\Big), \{x, 0, D_{GL16}\}\Big]\Big) \Big/ (2\, q_o), \{Q, 0, 2\, q_o\}\Big] +$$

$$\text{Integrate}[(\text{Integrate}[\delta\, (Q - p_{GL2P6} - t\, x), \{x, D_{GL16}, D_{GL16} + D_{GL2P6}\}]) / (2\, q_o), \{Q, p_{GL16} + t\, D_{GL16}, 2\, q_o\}] +$$

$$\text{Integrate}[(\text{Integrate}[\delta\, (Q - p_{GL2M6} - t\, x), \{x, D_{GL16}, D_{GL16} + D_{GL2M6}\}]) / (2\, q_o), \{Q, p_{GL16}, p_{GL16} + t\, D_{GL16}\}] + \text{Integrate}[(\text{Integrate}[\delta\, (Q - p_{GL2N6} - t\, x), \{x, D_{GL16}, D_{GL16} + D_{GL2N6}\}]) / (2\, q_o), \{Q, 0, p_{GL16}\}];$$

(*Proof of Proposition 12: Profit and consumer surplus comparisons*)

(*Part (i)*)(*Comparison between Π^{CL} and Π^{GL}*)(*There are 8 comparison scenarios*)

$$\text{Reduce}[\Pi_{CL1} \le \Pi_{GL1}\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0 < \delta \le 0.119\ldots]$$

$$\text{Reduce}[\Pi_{CL2} \le \Pi_{GL1}\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0.119\ldots < \delta \le 0.391\ldots]$$

$$\text{Reduce}[\Pi_{CL3} \le \Pi_{GL1}\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0.391\ldots < \delta \le 0.467\ldots]$$

$$\text{Reduce}[\Pi_{CL3} \le \Pi_{GL2}\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0.467\ldots < \delta \le 0.540855]$$

$$\text{Reduce}[\Pi_{CL3} \le \Pi_{GL3}\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0.540855 < \delta \le 0.5984]$$

$$\text{Reduce}[\Pi_{CL3} \le \Pi_{GL4}\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0.5984 < \delta \le 0.777\ldots]$$

$$\text{Reduce}[\Pi_{CL3} \ge \Pi_{GL5}\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0.777\ldots < \delta \le 0.827\ldots]$$

$$\text{Reduce}[\Pi_{CL3} \ge \Pi_{GL6}\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0.827\ldots < \delta < 1]$$

Out[∘]=

False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
$0.62602 \le \delta \le 0.777464$ && $q_o > 0$ && $t > 2.\ q_o$

Out[•]=
False

Out[•]=
False

```
(*Hence, when 0<δ<0.626, Π^CL>Π^GL; when 0.626<δ<1, Π^CL>Π^GL. We define δ'=0.626*)

(*Part (ii)*)(*Comparison between CS^CL and CS^GL*)(*There are 8 comparison scenarios*)
```

In[•]:=
```
Reduce[CS_CL1 ≥ CS_GL1 && t > 2 q_o > 0 && 0 < δ ≤ 0.119…]
Reduce[CS_CL2 ≤ CS_GL1 && t > 2 q_o > 0 && 0.119… < δ ≤ 0.391…]
Reduce[CS_CL3 ≤ CS_GL1 && t > 2 q_o > 0 && 0.391… < δ ≤ 0.467…]
Reduce[CS_CL3 ≤ CS_GL2 && t > 2 q_o > 0 && 0.467… < δ ≤ 0.540855]
Reduce[CS_CL3 ≤ CS_GL3 && t > 2 q_o > 0 && 0.540855 < δ ≤ 0.5984]
Reduce[CS_CL3 ≤ CS_GL4 && t > 2 q_o > 0 && 0.5984 < δ ≤ 0.777…]
Reduce[CS_CL3 ≤ CS_GL5 && t > 2 q_o > 0 && 0.777… < δ ≤ 0.827…]
Reduce[CS_CL3 ≤ CS_GL6 && t > 2 q_o > 0 && 0.827… < δ < 1]
```

Out[•]=
False

Out[•]=
$0.119\ldots < \delta \le 0.206\ldots$ && $q_o > 0$ && $t > 2\ q_o$

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

Out[•]=
False

```
(*Hence, when 0<δ<0.206…, CS^CL<CS^GL;
when 0.206…<δ<1, CS^CL>CS^GL. We define δ''=0.206…*)
```

```
(*Proof of Corollary 1: Win-win scenario*)
```

$CS_{GN} = \frac{q_o^2}{8\,t};$

In[∘]:= Reduce[$CS_{CL1} > CS_{GN}$ && $t > 2\,q_o > 0$ && $0 < \delta \le$ 0.119…]

Reduce[$CS_{CL2} > CS_{GN}$ && $t > 2\,q_o > 0$ && 0.119… $< \delta \le$ 0.391…]

Reduce[$CS_{CL3} < CS_{GN}$ && $t > 2\,q_o > 0$ && 0.391… $< \delta < 1$]

Out[∘]=

False

Out[∘]=

False

Out[∘]=

3 0.391… $< \delta <$ 8 0.435… && $q_o > 0$ && $t > 2\,q_o$

(*Hence, when $0<\delta<$ 8 0.435…, $CS^{CL}<CS^{GN}$; when 8 0.435… $<\delta<1$, $CS^{CL}>CS^{GN}$. We define $\delta'''=$ 8 0.435…. Based on the proof in Proposition 12, when $\delta'''<\delta<\delta'$, $\Pi^{CL}>\max\{\Pi^{CN},\Pi^{GN},\Pi^{GL}\}$ and $CS^{CL}>\max\{CS^{CN},CS^{GN},CS^{GL}\}$ hold true simultaneously. Therefore, contingent pricing with social learning can achieve a win-win situation for both the firm and consumers compared to price guarantee*)

Extension 7.1 : Fully Rational Consumers

Case CL. Contingent pricing with social learning

```
(*In the second period, consumer reviews will realize in three types,
including completely positive (P), mixed (M), and completely negative (N)*)
```

(*If $q_R^{CL}=q_P^{CL}=\frac{2q_o+p_1+t\ D_1}{2}$*)

$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4}$; (*The second-period price if $q_R^{CL}=q_P^{CL}$*)

$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t}$; (*The second-period demand if $q_R^{CL}=q_P^{CL}$*)

(*If $q_R^{CL}=q_M^{CL}=q$, $p_1<q<p_1+tD_1$*)

(*When $p_1<tD_1$ and $p_1<q<tD_1$*)

$p_{2M} = 0$;
$D_{2M} = 0$;

(*When $p_1<tD_1$ and $tD_1<q<p_1+tD_1$, or $p_1>tD_1$*)

$p_{2M} = \frac{Q - t\,D_1}{2}$; (*The second-period price if $q_R^{CL}=q$*)

(*We use Q to denote q when conducting mathematica operations*)

$D_{2M} = \frac{Q - t\,D_1}{2\,t}$; (*The second-period demand if $q_R^{CL}=q$*)

(*If $q_R^{CL}=q_N^{CL}=\frac{p_1}{2}$*)

$p_{2N} = \frac{p_1 - 2\,t\,D_1}{4}$; (*The second-period price if $q_R^{CL}=q_N^{CL}$*)

$D_{2N} = \frac{p_1 - 2\,t\,D_1}{4\,t}$; (*The second-period demand if $q_R^{CL}=q_N^{CL}$*)

(*Based on the above disscussion, there are three scenarios according to p_1,
i.e., $p_1 \le tD_1$, $tD_1<p_1\le 2tD_1$, and $p_1>2tD_1$*)

(*Scenario 1: $p_1 \le tD_1$,
consumers will not purchase upon observing completely negative reviews, i.e., $D_{2N}=0$*)

In[ ]:= $p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4}$;

$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t}$;

$p_{2M} = \frac{Q - t\,D_1}{2}$;

$D_{2M} = \frac{Q - t\,D_1}{2\,t}$;

In[•]:= $U_1 = q_o - p_1 - t\,D_1$;(*Consumers' expected utility when buying in the first period*)

$$U_2 = \delta_c \left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2\,P} - t\,D_1\right) + \text{Integrate}[(Q - p_{2\,M} - t\,D_1) / (2\,q_o),\ \{Q,\ t\,D_1,\ p_1 + t\,D_1\}]\right);$$

(*Consumers' expected utility when buying in the second period*)

In[•]:= Simplify[Solve[$U_1 == U_2$, D_1], $0 < \delta_c < 1$ && $t > 2\,q_o > 0$]

Out[•]=

$$\left\{\left\{D_1 \to \frac{2\,q_o\,(-2+\delta_c) - 2\,\sqrt{2}\,\sqrt{-q_o\,(q_o\,(-2+\delta_c) + p_1\,\delta_c)}}{t\,\delta_c}\right\},\ \left\{D_1 \to \frac{2\left(q_o\,(-2+\delta_c) + \sqrt{2}\,\sqrt{-q_o\,(q_o\,(-2+\delta_c) + p_1\,\delta_c)}\right)}{t\,\delta_c}\right\}\right\}$$

(*Check which solution is the feasible solution*)

In[•]:= $D_1 = \frac{2\,q_o\,(-2+\delta_c) - 2\,\sqrt{2}\,\sqrt{-q_o\,(q_o\,(-2+\delta_c) + p_1\,\delta_c)}}{t\,\delta_c}$;

Reduce[$0 < p_1 \le t\,D_1$ && $t > 2\,q_o > 0$ && $0 < \delta_c < 1$]

Out[•]=

False

In[•]:= $D_1 = \frac{2\left(q_o\,(-2+\delta_c) + \sqrt{2}\,\sqrt{-q_o\,(q_o\,(-2+\delta_c) + p_1\,\delta_c)}\right)}{t\,\delta_c}$;

Reduce[$0 < p_1 \le t\,D_1$ && $t > 2\,q_o > 0$ && $0 < \delta_c < 1$]

Out[•]=

$$p_1 > 0 \,\&\&\, \left(\left(2\,p_1 < q_o < \frac{3\,p_1}{2} + \sqrt{\frac{5}{2}}\,\sqrt{p_1^2} \,\&\&\, t > 2\,q_o \,\&\&\, 0 < \delta_c \le \frac{-16\,p_1\,q_o + 8\,q_o^2}{p_1^2 - 4\,p_1\,q_o + 4\,q_o^2}\right) \,||\, \left(q_o == \frac{3\,p_1}{2} + \sqrt{\frac{5}{2}}\,\sqrt{p_1^2} \,\&\&\, t > 2\,q_o \,\&\&\, 0 < \delta_c < \frac{-16\,p_1\,q_o + 8\,q_o^2}{p_1^2 - 4\,p_1\,q_o + 4\,q_o^2}\right) \,||\, \left(q_o > \frac{3\,p_1}{2} + \sqrt{\frac{5}{2}}\,\sqrt{p_1^2} \,\&\&\, t > 2\,q_o \,\&\&\, 0 < \delta_c < 1\right)\right)$$

(*The second solution is feasible,
hence the optimal response function of the first-period demand is as follows*)

In[•]:= $D_1 = \frac{2\left(q_o\,(-2+\delta_c) + \sqrt{2}\,\sqrt{-q_o\,(q_o\,(-2+\delta_c) + p_1\,\delta_c)}\right)}{t\,\delta_c}$;

In[•]:= Π = Simplify$\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2\,P}\,D_{2\,P} + \text{Integrate}[p_{2\,M}\,D_{2\,M} / (2\,q_o),\ \{Q,\ t\,D_1,\ p_1 + t\,D_1\}],\ 0 < \delta_c < 1 \,\&\&\, t > 2\,q_o > 0\right]$;(*The firm's total profit function*)

In[•]:= Reduce[D[D[Π, p_1], p_1] ≥ 0 && $0 < p_1 \le t\,D_1$ && $0 < \delta_c < 1$ && $t > 2\,q_o > 0$]
(*Determine the sign of $\frac{\partial^2\Pi}{\partial p_1^2}$*)

Out[•]=

False

```
(* ∂²Π/∂p1² <0, meaning Π is concave and it has a maximum value at point where ∂Π/∂p1 =0*)

(*Construct KKT conditions*)

In[•]:= g = t D1 - p1; (*The condition of p1≤tD1*)
L = Simplify[
   -(p1 D1 + (2 qo - p1 - t D1)/(2 qo) p2P D2P + Integrate[p2M D2M / (2 qo), {Q, t D1, p1 + t D1}]) - λ g,
   0 < δc < 1 && t > 2 qo > 0]; (*The KKT Lagrange function*)

In[•]:= Simplify[Solve[{D[L, p1] == 0, λ g == 0}, {p1, λ}], 0 < δc < 1 && t > 2 qo > 0]
```

Out[∘]=

$$\Big\{\Big\{p_1 \to \mathrm{Root}\Big[\#1^5 + \#1^4\, q_o + 4096\, q_o^5 + \#1^3 \left(128\, q_o^2 + \frac{384\, q_o^2}{\delta_c^2} + \frac{1680\, q_o^2}{\delta_c}\right) + \#1^2 \left(128\, q_o^3 + \frac{1024\, q_o^3}{\delta_c^3} + \frac{1920\, q_o^3}{\delta_c^2} + \frac{17\,680\, q_o^3}{\delta_c}\right) + \#1 \left(4096\, q_o^4 - \frac{14\,336\, q_o^4}{\delta_c^3} - \frac{27\,072\, q_o^4}{\delta_c^2} + \frac{5120\, q_o^4}{\delta_c}\right) + \frac{9216\, q_o^5}{\delta_c^3} + \frac{14\,400\, q_o^5}{\delta_c^2} - \frac{17\,408\, q_o^5}{\delta_c}\ \&,\ 1\Big],$$

$$\lambda \to 0\Big\},\ \Big\{p_1 \to \mathrm{Root}\Big[\#1^5 + \#1^4\, q_o + 4096\, q_o^5 + \#1^3 \left(128\, q_o^2 + \frac{384\, q_o^2}{\delta_c^2} + \frac{1680\, q_o^2}{\delta_c}\right) + \#1^2 \left(128\, q_o^3 + \frac{1024\, q_o^3}{\delta_c^3} + \frac{1920\, q_o^3}{\delta_c^2} + \frac{17\,680\, q_o^3}{\delta_c}\right) + \#1 \left(4096\, q_o^4 - \frac{14\,336\, q_o^4}{\delta_c^3} - \frac{27\,072\, q_o^4}{\delta_c^2} + \frac{5120\, q_o^4}{\delta_c}\right) + \frac{9216\, q_o^5}{\delta_c^3} + \frac{14\,400\, q_o^5}{\delta_c^2} - \frac{17\,408\, q_o^5}{\delta_c}\ \&,\ 2\Big],$$

$$\lambda \to 0\Big\},\ \Big\{p_1 \to \mathrm{Root}\Big[\#1^5 + \#1^4\, q_o + 4096\, q_o^5 + \#1^3 \left(128\, q_o^2 + \frac{384\, q_o^2}{\delta_c^2} + \frac{1680\, q_o^2}{\delta_c}\right) + \#1^2 \left(128\, q_o^3 + \frac{1024\, q_o^3}{\delta_c^3} + \frac{1920\, q_o^3}{\delta_c^2} + \frac{17\,680\, q_o^3}{\delta_c}\right) + \#1 \left(4096\, q_o^4 - \frac{14\,336\, q_o^4}{\delta_c^3} - \frac{27\,072\, q_o^4}{\delta_c^2} + \frac{5120\, q_o^4}{\delta_c}\right) + \frac{9216\, q_o^5}{\delta_c^3} + \frac{14\,400\, q_o^5}{\delta_c^2} - \frac{17\,408\, q_o^5}{\delta_c}\ \&,\ 3\Big],$$

$$\lambda \to 0\Big\},\ \Big\{p_1 \to \mathrm{Root}\Big[\#1^5 + \#1^4\, q_o + 4096\, q_o^5 + \#1^3 \left(128\, q_o^2 + \frac{384\, q_o^2}{\delta_c^2} + \frac{1680\, q_o^2}{\delta_c}\right) + \#1^2 \left(128\, q_o^3 + \frac{1024\, q_o^3}{\delta_c^3} + \frac{1920\, q_o^3}{\delta_c^2} + \frac{17\,680\, q_o^3}{\delta_c}\right) + \#1 \left(4096\, q_o^4 - \frac{14\,336\, q_o^4}{\delta_c^3} - \frac{27\,072\, q_o^4}{\delta_c^2} + \frac{5120\, q_o^4}{\delta_c}\right) + \frac{9216\, q_o^5}{\delta_c^3} + \frac{14\,400\, q_o^5}{\delta_c^2} - \frac{17\,408\, q_o^5}{\delta_c}\ \&,\ 4\Big],$$

$$\lambda \to 0\Big\},\ \Big\{p_1 \to \mathrm{Root}\Big[\#1^5 + \#1^4\, q_o + 4096\, q_o^5 + \#1^3 \left(128\, q_o^2 + \frac{384\, q_o^2}{\delta_c^2} + \frac{1680\, q_o^2}{\delta_c}\right) + \#1^2 \left(128\, q_o^3 + \frac{1024\, q_o^3}{\delta_c^3} + \frac{1920\, q_o^3}{\delta_c^2} + \frac{17\,680\, q_o^3}{\delta_c}\right) + \#1 \left(4096\, q_o^4 - \frac{14\,336\, q_o^4}{\delta_c^3} - \frac{27\,072\, q_o^4}{\delta_c^2} + \frac{5120\, q_o^4}{\delta_c}\right) + \frac{9216\, q_o^5}{\delta_c^3} + \frac{14\,400\, q_o^5}{\delta_c^2} - \frac{17\,408\, q_o^5}{\delta_c}\ \&,\ 5\Big],$$

$$\lambda \to 0\Big\},\ \Big\{p_1 \to \frac{2\, q_o \left(-4 + \sqrt{16 - 6\,\delta_c} + \delta_c\right)}{\delta_c},$$

$$\lambda \to \frac{q_o \left(384 \left(-4 + \sqrt{16 - 6\,\delta_c}\right) - 8 \left(-256 + 55 \sqrt{16 - 6\,\delta_c}\right) \delta_c + \left(-688 + 94 \sqrt{16 - 6\,\delta_c}\right) \delta_c^2 + 51\, \delta_c^3\right)}{8\, t\, \delta_c^2\, (-8 + 3\, \delta_c)}\Big\},$$

$$\Big\{p_1 \to -\frac{2\, q_o \left(4 + \sqrt{16 - 6\,\delta_c} - \delta_c\right)}{\delta_c},$$

$$\lambda \to \frac{q_o \left(-384 \left(4 + \sqrt{16 - 6\,\delta_c}\right) + 8 \left(256 + 55 \sqrt{16 - 6\,\delta_c}\right) \delta_c - 2 \left(344 + 47 \sqrt{16 - 6\,\delta_c}\right) \delta_c^2 + 51\, \delta_c^3\right)}{8\, t\, \delta_c^2\, (-8 + 3\, \delta_c)}\Big\}\Big\}$$

```
(*There are 5 solutions,we check each solution if it satisfies conditions*)

(*Solution 1, interior solution*)
```

In[•]:=

$$p_1 = \text{Root}\Big[\#1^5 + \#1^4 q_o + 4096 q_o^5 + \#1^3 \left(128 q_o^2 + \frac{384 q_o^2}{\delta_c^2} + \frac{1680 q_o^2}{\delta_c}\right) + \#1^2 \left(128 q_o^3 + \frac{1024 q_o^3}{\delta_c^3} + \frac{1920 q_o^3}{\delta_c^2} + \frac{17680 q_o^3}{\delta_c}\right) + \#1 \left(4096 q_o^4 - \frac{14336 q_o^4}{\delta_c^3} - \frac{27072 q_o^4}{\delta_c^2} + \frac{5120 q_o^4}{\delta_c}\right) + \frac{9216 q_o^5}{\delta_c^3} + \frac{14400 q_o^5}{\delta_c^2} - \frac{17408 q_o^5}{\delta_c} \,\&,\, 1\Big];$$

$$\text{Reduce}[0 < p_1 < t\, D_1 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta_c < 1]$$

Out[•]=

```
False

(*Solution 2, interior solution*)
```

In[•]:=

$$p_1 = \text{Root}\Big[\#1^5 + \#1^4 q_o + 4096 q_o^5 + \#1^3 \left(128 q_o^2 + \frac{384 q_o^2}{\delta_c^2} + \frac{1680 q_o^2}{\delta_c}\right) + \#1^2 \left(128 q_o^3 + \frac{1024 q_o^3}{\delta_c^3} + \frac{1920 q_o^3}{\delta_c^2} + \frac{17680 q_o^3}{\delta_c}\right) + \#1 \left(4096 q_o^4 - \frac{14336 q_o^4}{\delta_c^3} - \frac{27072 q_o^4}{\delta_c^2} + \frac{5120 q_o^4}{\delta_c}\right) + \frac{9216 q_o^5}{\delta_c^3} + \frac{14400 q_o^5}{\delta_c^2} - \frac{17408 q_o^5}{\delta_c} \,\&,\, 2\Big];$$

$$\text{Reduce}[0 < p_1 < t\, D_1 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta_c < 1]$$

Out[•]=

```
False

(*Solution 3, interior solution*)
```

In[•]:=

$$p_1 = \text{Root}\Big[\#1^5 + \#1^4 q_o + 4096 q_o^5 + \#1^3 \left(128 q_o^2 + \frac{384 q_o^2}{\delta_c^2} + \frac{1680 q_o^2}{\delta_c}\right) + \#1^2 \left(128 q_o^3 + \frac{1024 q_o^3}{\delta_c^3} + \frac{1920 q_o^3}{\delta_c^2} + \frac{17680 q_o^3}{\delta_c}\right) + \#1 \left(4096 q_o^4 - \frac{14336 q_o^4}{\delta_c^3} - \frac{27072 q_o^4}{\delta_c^2} + \frac{5120 q_o^4}{\delta_c}\right) + \frac{9216 q_o^5}{\delta_c^3} + \frac{14400 q_o^5}{\delta_c^2} - \frac{17408 q_o^5}{\delta_c} \,\&,\, 3\Big];$$

$$\text{Reduce}[0 < p_1 < t\, D_1 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta_c < 1]$$

Out[•]=

```
False

(*Solution 4, interior solution*)
```

In[•]:=

$$p_1 = \text{Root}\Big[\#1^5 + \#1^4 q_o + 4096 q_o^5 + \#1^3 \left(128 q_o^2 + \frac{384 q_o^2}{\delta_c^2} + \frac{1680 q_o^2}{\delta_c}\right) + \#1^2 \left(128 q_o^3 + \frac{1024 q_o^3}{\delta_c^3} + \frac{1920 q_o^3}{\delta_c^2} + \frac{17680 q_o^3}{\delta_c}\right) + \#1 \left(4096 q_o^4 - \frac{14336 q_o^4}{\delta_c^3} - \frac{27072 q_o^4}{\delta_c^2} + \frac{5120 q_o^4}{\delta_c}\right) + \frac{9216 q_o^5}{\delta_c^3} + \frac{14400 q_o^5}{\delta_c^2} - \frac{17408 q_o^5}{\delta_c} \,\&,\, 4\Big];$$

$$\text{Reduce}[0 < p_1 < t\, D_1 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta_c < 1]$$

Out[•]=

```
False

(*Solution 5, interior solution*)
```

In[∘]:=
$$p_1 = \text{Root}\Big[\#1^5 + \#1^4 q_o + 4096 q_o^5 + \#1^3\left(128 q_o^2 + \frac{384 q_o^2}{\delta_c^2} + \frac{1680 q_o^2}{\delta_c}\right) + \#1^2\left(128 q_o^3 + \frac{1024 q_o^3}{\delta_c^3} + \frac{1920 q_o^3}{\delta_c^2} + \frac{17\,680 q_o^3}{\delta_c}\right) + \#1\left(4096 q_o^4 - \frac{14\,336 q_o^4}{\delta_c^3} - \frac{27\,072 q_o^4}{\delta_c^2} + \frac{5120 q_o^4}{\delta_c}\right) + \frac{9216 q_o^5}{\delta_c^3} + \frac{14\,400 q_o^5}{\delta_c^2} - \frac{17\,408 q_o^5}{\delta_c} \;\&,\; 5\Big];$$

$\text{Reduce}[0 < p_1 < t D_1 \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1]$

Out[∘]=
False

(*Solution 6, boundary solution, which is the solution of p_1=tD_1*)

In[∘]:=
$$p_1 = \frac{2 q_o \left(-4 + \sqrt{16 - 6 \delta_c} + \delta_c\right)}{\delta_c};$$

$$\lambda = \frac{q_o \left(384 \left(-4 + \sqrt{16 - 6 \delta_c}\right) - 8 \left(-256 + 55 \sqrt{16 - 6 \delta_c}\right) \delta_c + \left(-688 + 94 \sqrt{16 - 6 \delta_c}\right) \delta_c^2 + 51 \delta_c^3\right)}{8 t \delta_c^2 (-8 + 3 \delta_c)};$$

$\text{Reduce}[\lambda > 0 \;\&\&\; D_1 > 0 \;\&\&\; 0 < p_1 \le t D_1 \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1]$

Out[∘]=
$q_o > 0 \;\&\&\; t > 2 q_o \;\&\&\; 0 < \delta_c < 1$

(*Solution 7, boundary solution, which is the solution of p_1=tD_1*)

In[∘]:=
$$p_1 = -\frac{2 q_o \left(4 + \sqrt{16 - 6 \delta_c} - \delta_c\right)}{\delta_c};$$

$$\lambda = \frac{q_o \left(-384 \left(4 + \sqrt{16 - 6 \delta_c}\right) + 8 \left(256 + 55 \sqrt{16 - 6 \delta_c}\right) \delta_c - 2 \left(344 + 47 \sqrt{16 - 6 \delta_c}\right) \delta_c^2 + 51 \delta_c^3\right)}{8 t \delta_c^2 (-8 + 3 \delta_c)};$$

$\text{Reduce}[\lambda > 0 \;\&\&\; D_1 > 0 \;\&\&\; 0 < p_1 \le t D_1 \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1]$

Out[∘]=
False

(*Therefore, $p_1 = \frac{2 q_o \left(-4 + \sqrt{16 - 6 \delta_c} + \delta_c\right)}{\delta_c}$*)

(*Scenario 2: tD_1<p_1≤2tD_1*)

In[∘]:=
$$p_{2P} = \frac{2 q_o + p_1 - t D_1}{4};$$

$$D_{2P} = \frac{2 q_o + p_1 - t D_1}{4 t};$$

$$p_{2M} = \frac{Q - t D_1}{2};$$

$$D_{2M} = \frac{Q - t D_1}{2 t};$$

In[∘]:= $U_1 = q_o - p_1 - t\,D_1;$
$U_2 = \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \right.$
$\left.\text{Integrate}[(Q - p_{2M} - t D_1) / (2 q_o), \{Q, p_1, p_1 + t D_1\}]\right);$

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1], $t > 2 q_o > 0$ && $0 < \delta_c < 1$]

Out[∘]=
$$\left\{\left\{D_1 \to \frac{8 p_1 q_o + 4 q_o^2 (-2 + \delta_c) - p_1^2 \delta_c}{4 t q_o (-2 + \delta_c) - 2 t p_1 \delta_c}\right\}\right\}$$

In[∘]:= $D_1 = \frac{8 p_1 q_o + 4 q_o^2 (-2 + \delta_c) - p_1^2 \delta_c}{4 t q_o (-2 + \delta_c) - 2 t p_1 \delta_c};$

In[∘]:= Simplify[Reduce[$D_1 > 0$ && $t D_1 < p_1 \le 2 t D_1$ && $t > 2 q_o > 0$ && $0 < \delta_c < 1$]]

Out[∘]=
$t > 2 q_o$ && $p_1 > 0$ &&
$\left(\left(\frac{8 q_o (-2 p_1 + q_o)}{(p_1 - 2 q_o)^2} < \delta_c \right.\right.$ && $\left(\left(5 p_1 == 2 q_o \right.\right.$ && $\left.\delta_c < \frac{6 p_1 - 4 q_o}{p_1 - 2 q_o}\right)$ || $\left(2 p_1 < q_o\right.$ && $2 q_o < 5 p_1$ &&
$\left.\delta_c \le \frac{6 p_1 - 4 q_o}{p_1 - 2 q_o}\right)$ || $\left(5 p_1 < 2 q_o\right.$ && $2 q_o < 3 p_1 + \sqrt{10}\sqrt{p_1^2}$ && $\left.\left.\left.\delta_c < 1\right)\right)\right)$ ||
$\left(\delta_c > 0\right.$ && $3 p_1 < 2 q_o$ && $q_o \le 2 p_1$ && $\left.\left.\delta_c \le \frac{6 p_1 - 4 q_o}{p_1 - 2 q_o}\right)\right)$

In[∘]:= Π = Simplify$\left[p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P} + \text{Integrate}[p_{2M} D_{2M} / (2 q_o), \{Q, p_1, p_1 + t D_1\}],\right.$
$\left. t > 2 q_o > 0 \text{ \&\& } 0 < \delta_c < 1\right]$;(*The firm's total profit function*)

In[∘]:= Reduce[D[D[Π, p_1], p_1] ≥ 0 && $D_1 > 0$ && $t D_1 < p_1 \le 2 t D_1$ && $t > 2 q_o > 0$ && $0 < \delta_c < 1$]
(*Determine the sign of $\frac{\partial^2 \Pi}{\partial p_1^2}$*)

Out[∘]=
False

(*$\frac{\partial^2 \Pi}{\partial p_1^2}$<0,
meaning that Π is concave and it has a maximum value at the point where $\frac{\partial \Pi}{\partial p_1}$=0*)

(*Construct KKT conditions*)

In[∘]:= $g_1 = p_1 - t D_1;$
$g_2 = 2 t D_1 - p_1;$

In[∘]:= $L = -\left(p_1 D_1 + \frac{2 q_o - p_1 - t D_1}{2 q_o} p_{2P} D_{2P} + \text{Integrate}[p_{2M} D_{2M} / (2 q_o), \{Q, p_1, p_1 + t D_1\}]\right) -$
$\lambda_1 g_1 - \lambda_2 g_2;$

In[∘]:= Simplify[Solve[{D[L, p_1] == 0, $\lambda_1 g_1 == 0$, $\lambda_2 g_2 == 0$}, {p_1, λ_1, λ_2}], $q_o > 0$ && $0 < \delta_c < 1$]

Out[∘]=
$\{\{p_1 \to \text{Root}[57\,344 q_o^6 - 116\,224 q_o^6 \delta_c + 88\,320 q_o^6 \delta_c^2 - 29\,824 q_o^6 \delta_c^3 + 19 \#1^6 \delta_c^4 + 3776 q_o^6 \delta_c^4 +$
$\#1^5 \left(88 q_o \delta_c^3 + 100 q_o \delta_c^4\right) + \#1^4 \left(-1920 q_o^2 \delta_c^2 + 2328 q_o^2 \delta_c^3 - 1324 q_o^2 \delta_c^4\right) +$
$\#1^3 \left(-20\,480 q_o^3 \delta_c + 14\,208 q_o^3 \delta_c^2 - 9088 q_o^3 \delta_c^3 + 3552 q_o^3 \delta_c^4\right) +$
$\#1^2 \left(-57\,344 q_o^4 + 28\,672 q_o^4 \delta_c + 12\,096 q_o^4 \delta_c^2 - 1856 q_o^4 \delta_c^3 - 2096 q_o^4 \delta_c^4\right) +$

$$\#1\ \left(-49\,152\ q_o^5+132\,608\ q_o^5\ \delta_c-109\,824\ q_o^5\ \delta_c^2+34\,944\ q_o^5\ \delta_c^3-3520\ q_o^5\ \delta_c^4\right)\ \&,\ 1\Big],\ \lambda_1\to 0,$$

$$\lambda_2\to 0\Big\},\ \Big\{p_1\to \text{Root}\Big[57\,344\ q_o^6-116\,224\ q_o^6\ \delta_c+88\,320\ q_o^6\ \delta_c^2-29\,824\ q_o^6\ \delta_c^3+19\ \#1^6\ \delta_c^4+$$

$$3776\ q_o^6\ \delta_c^4+\#1^5\ \left(88\ q_o\ \delta_c^3+100\ q_o\ \delta_c^4\right)+\#1^4\ \left(-1920\ q_o^2\ \delta_c^2+2328\ q_o^2\ \delta_c^3-1324\ q_o^2\ \delta_c^4\right)+$$

$$\#1^3\ \left(-20\,480\ q_o^3\ \delta_c+14\,208\ q_o^3\ \delta_c^2-9088\ q_o^3\ \delta_c^3+3552\ q_o^3\ \delta_c^4\right)+$$

$$\#1^2\ \left(-57\,344\ q_o^4+28\,672\ q_o^4\ \delta_c+12\,096\ q_o^4\ \delta_c^2-1856\ q_o^4\ \delta_c^3-2096\ q_o^4\ \delta_c^4\right)+$$

$$\#1\ \left(-49\,152\ q_o^5+132\,608\ q_o^5\ \delta_c-109\,824\ q_o^5\ \delta_c^2+34\,944\ q_o^5\ \delta_c^3-3520\ q_o^5\ \delta_c^4\right)\ \&,\ 2\Big],\ \lambda_1\to 0,$$

$$\lambda_2\to 0\Big\},\ \Big\{p_1\to \text{Root}\Big[57\,344\ q_o^6-116\,224\ q_o^6\ \delta_c+88\,320\ q_o^6\ \delta_c^2-29\,824\ q_o^6\ \delta_c^3+19\ \#1^6\ \delta_c^4+$$

$$3776\ q_o^6\ \delta_c^4+\#1^5\ \left(88\ q_o\ \delta_c^3+100\ q_o\ \delta_c^4\right)+\#1^4\ \left(-1920\ q_o^2\ \delta_c^2+2328\ q_o^2\ \delta_c^3-1324\ q_o^2\ \delta_c^4\right)+$$

$$\#1^3\ \left(-20\,480\ q_o^3\ \delta_c+14\,208\ q_o^3\ \delta_c^2-9088\ q_o^3\ \delta_c^3+3552\ q_o^3\ \delta_c^4\right)+$$

$$\#1^2\ \left(-57\,344\ q_o^4+28\,672\ q_o^4\ \delta_c+12\,096\ q_o^4\ \delta_c^2-1856\ q_o^4\ \delta_c^3-2096\ q_o^4\ \delta_c^4\right)+$$

$$\#1\ \left(-49\,152\ q_o^5+132\,608\ q_o^5\ \delta_c-109\,824\ q_o^5\ \delta_c^2+34\,944\ q_o^5\ \delta_c^3-3520\ q_o^5\ \delta_c^4\right)\ \&,\ 3\Big],\ \lambda_1\to 0,$$

$$\lambda_2\to 0\Big\},\ \Big\{p_1\to \text{Root}\Big[57\,344\ q_o^6-116\,224\ q_o^6\ \delta_c+88\,320\ q_o^6\ \delta_c^2-29\,824\ q_o^6\ \delta_c^3+19\ \#1^6\ \delta_c^4+$$

$$3776\ q_o^6\ \delta_c^4+\#1^5\ \left(88\ q_o\ \delta_c^3+100\ q_o\ \delta_c^4\right)+\#1^4\ \left(-1920\ q_o^2\ \delta_c^2+2328\ q_o^2\ \delta_c^3-1324\ q_o^2\ \delta_c^4\right)+$$

$$\#1^3\ \left(-20\,480\ q_o^3\ \delta_c+14\,208\ q_o^3\ \delta_c^2-9088\ q_o^3\ \delta_c^3+3552\ q_o^3\ \delta_c^4\right)+$$

$$\#1^2\ \left(-57\,344\ q_o^4+28\,672\ q_o^4\ \delta_c+12\,096\ q_o^4\ \delta_c^2-1856\ q_o^4\ \delta_c^3-2096\ q_o^4\ \delta_c^4\right)+$$

$$\#1\ \left(-49\,152\ q_o^5+132\,608\ q_o^5\ \delta_c-109\,824\ q_o^5\ \delta_c^2+34\,944\ q_o^5\ \delta_c^3-3520\ q_o^5\ \delta_c^4\right)\ \&,\ 4\Big],\ \lambda_1\to 0,$$

$$\lambda_2\to 0\Big\},\ \Big\{p_1\to \text{Root}\Big[57\,344\ q_o^6-116\,224\ q_o^6\ \delta_c+88\,320\ q_o^6\ \delta_c^2-29\,824\ q_o^6\ \delta_c^3+19\ \#1^6\ \delta_c^4+$$

$$3776\ q_o^6\ \delta_c^4+\#1^5\ \left(88\ q_o\ \delta_c^3+100\ q_o\ \delta_c^4\right)+\#1^4\ \left(-1920\ q_o^2\ \delta_c^2+2328\ q_o^2\ \delta_c^3-1324\ q_o^2\ \delta_c^4\right)+$$

$$\#1^3\ \left(-20\,480\ q_o^3\ \delta_c+14\,208\ q_o^3\ \delta_c^2-9088\ q_o^3\ \delta_c^3+3552\ q_o^3\ \delta_c^4\right)+$$

$$\#1^2\ \left(-57\,344\ q_o^4+28\,672\ q_o^4\ \delta_c+12\,096\ q_o^4\ \delta_c^2-1856\ q_o^4\ \delta_c^3-2096\ q_o^4\ \delta_c^4\right)+$$

$$\#1\ \left(-49\,152\ q_o^5+132\,608\ q_o^5\ \delta_c-109\,824\ q_o^5\ \delta_c^2+34\,944\ q_o^5\ \delta_c^3-3520\ q_o^5\ \delta_c^4\right)\ \&,\ 5\Big],\ \lambda_1\to 0,$$

$$\lambda_2\to 0\Big\},\ \Big\{p_1\to \text{Root}\Big[57\,344\ q_o^6-116\,224\ q_o^6\ \delta_c+88\,320\ q_o^6\ \delta_c^2-29\,824\ q_o^6\ \delta_c^3+19\ \#1^6\ \delta_c^4+$$

$$3776\ q_o^6\ \delta_c^4+\#1^5\ \left(88\ q_o\ \delta_c^3+100\ q_o\ \delta_c^4\right)+\#1^4\ \left(-1920\ q_o^2\ \delta_c^2+2328\ q_o^2\ \delta_c^3-1324\ q_o^2\ \delta_c^4\right)+$$

$$\#1^3\ \left(-20\,480\ q_o^3\ \delta_c+14\,208\ q_o^3\ \delta_c^2-9088\ q_o^3\ \delta_c^3+3552\ q_o^3\ \delta_c^4\right)+$$

$$\#1^2\ \left(-57\,344\ q_o^4+28\,672\ q_o^4\ \delta_c+12\,096\ q_o^4\ \delta_c^2-1856\ q_o^4\ \delta_c^3-2096\ q_o^4\ \delta_c^4\right)+$$

$$\#1\ \left(-49\,152\ q_o^5+132\,608\ q_o^5\ \delta_c-109\,824\ q_o^5\ \delta_c^2+34\,944\ q_o^5\ \delta_c^3-3520\ q_o^5\ \delta_c^4\right)\ \&,\ 6\Big],$$

$$\lambda_1\to 0,\ \lambda_2\to 0\Big\},\ \Big\{p_1\to \frac{2\ q_o\ (-2+\delta_c)}{-6+\delta_c},\ \lambda_1\to 0,$$

$$\lambda_2\to -\frac{q_o\ \left(384-2664\ \delta_c+1164\ \delta_c^2-478\ \delta_c^3+65\ \delta_c^4\right)}{64\ t\ (-6+\delta_c)^4}\Big\},$$

$$\Big\{p_1\to -\frac{2\ q_o\ \left(4+\sqrt{16-6\ \delta_c}-\delta_c\right)}{\delta_c},$$

$$\lambda_1\to \frac{q_o\ \left(384\ \left(4+\sqrt{16-6\ \delta_c}\right)-8\ \left(256+55\ \sqrt{16-6\ \delta_c}\right)\ \delta_c+\left(688+94\ \sqrt{16-6\ \delta_c}\right)\ \delta_c^2-51\ \delta_c^3\right)}{8\ t\ \delta_c^2\ (-8+3\ \delta_c)},$$

$$\lambda_2\to 0\Big\},$$

$$\Big\{p_1\to \frac{2\ q_o\ \left(-4+\sqrt{16-6\ \delta_c}+\delta_c\right)}{\delta_c},$$

$$\lambda_1\to$$

$$\frac{q_o\ \left(-384\ \left(-4+\sqrt{16-6\ \delta_c}\right)+8\ \left(-256+55\ \sqrt{16-6\ \delta_c}\right)\ \delta_c+\left(688-94\ \sqrt{16-6\ \delta_c}\right)\ \delta_c^2-51\ \delta_c^3\right)}{8\ t\ \delta_c^2\ (-8+3\ \delta_c)},$$

$$\lambda_2\to 0\Big\}\Big\}$$

(*There are 9 solutions, we check each solution if it satisfies conditions*)

(*Solution 1, interior solution*)

In[*]:=
$$\begin{aligned}&\mathbf{p_1 = Root\Big[57\,344\,q_o^6 - 116\,224\,q_o^6\,\delta_c + 88\,320\,q_o^6\,\delta_c^2 - 29\,824\,q_o^6\,\delta_c^3 + 19\,\#1^6\,\delta_c^4 +}\\&\quad\mathbf{3776\,q_o^6\,\delta_c^4 + \#1^5\,\left(88\,q_o\,\delta_c^3 + 100\,q_o\,\delta_c^4\right) + \#1^4\,\left(-1920\,q_o^2\,\delta_c^2 + 2328\,q_o^2\,\delta_c^3 - 1324\,q_o^2\,\delta_c^4\right) +}\\&\quad\mathbf{\#1^3\,\left(-20\,480\,q_o^3\,\delta_c + 14\,208\,q_o^3\,\delta_c^2 - 9088\,q_o^3\,\delta_c^3 + 3552\,q_o^3\,\delta_c^4\right) +}\\&\quad\mathbf{\#1^2\,\left(-57\,344\,q_o^4 + 28\,672\,q_o^4\,\delta_c + 12\,096\,q_o^4\,\delta_c^2 - 1856\,q_o^4\,\delta_c^3 - 2096\,q_o^4\,\delta_c^4\right) +}\\&\quad\mathbf{\#1\,\left(-49\,152\,q_o^5 + 132\,608\,q_o^5\,\delta_c - 109\,824\,q_o^5\,\delta_c^2 + 34\,944\,q_o^5\,\delta_c^3 - 3520\,q_o^5\,\delta_c^4\right)\,\&,\,1\Big];}\\&\mathbf{Reduce[t\,D_1 < p_1 < 2\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta_c < 1,\,Reals]}\end{aligned}$$

Out[*]=

False

(*Solution 2, interior solution*)

In[*]:=
$$\begin{aligned}&\mathbf{p_1 = Root\Big[57\,344\,q_o^6 - 116\,224\,q_o^6\,\delta_c + 88\,320\,q_o^6\,\delta_c^2 - 29\,824\,q_o^6\,\delta_c^3 + 19\,\#1^6\,\delta_c^4 +}\\&\quad\mathbf{3776\,q_o^6\,\delta_c^4 + \#1^5\,\left(88\,q_o\,\delta_c^3 + 100\,q_o\,\delta_c^4\right) + \#1^4\,\left(-1920\,q_o^2\,\delta_c^2 + 2328\,q_o^2\,\delta_c^3 - 1324\,q_o^2\,\delta_c^4\right) +}\\&\quad\mathbf{\#1^3\,\left(-20\,480\,q_o^3\,\delta_c + 14\,208\,q_o^3\,\delta_c^2 - 9088\,q_o^3\,\delta_c^3 + 3552\,q_o^3\,\delta_c^4\right) +}\\&\quad\mathbf{\#1^2\,\left(-57\,344\,q_o^4 + 28\,672\,q_o^4\,\delta_c + 12\,096\,q_o^4\,\delta_c^2 - 1856\,q_o^4\,\delta_c^3 - 2096\,q_o^4\,\delta_c^4\right) +}\\&\quad\mathbf{\#1\,\left(-49\,152\,q_o^5 + 132\,608\,q_o^5\,\delta_c - 109\,824\,q_o^5\,\delta_c^2 + 34\,944\,q_o^5\,\delta_c^3 - 3520\,q_o^5\,\delta_c^4\right)\,\&,\,2\Big];}\\&\mathbf{Reduce[t\,D_1 < p_1 < 2\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta_c < 1,\,Reals]}\end{aligned}$$

Out[*]=

False

(*Solution 3, interior solution*)

In[*]:=
$$\begin{aligned}&\mathbf{p_1 = Root\Big[57\,344\,q_o^6 - 116\,224\,q_o^6\,\delta_c + 88\,320\,q_o^6\,\delta_c^2 - 29\,824\,q_o^6\,\delta_c^3 + 19\,\#1^6\,\delta_c^4 +}\\&\quad\mathbf{3776\,q_o^6\,\delta_c^4 + \#1^5\,\left(88\,q_o\,\delta_c^3 + 100\,q_o\,\delta_c^4\right) + \#1^4\,\left(-1920\,q_o^2\,\delta_c^2 + 2328\,q_o^2\,\delta_c^3 - 1324\,q_o^2\,\delta_c^4\right) +}\\&\quad\mathbf{\#1^3\,\left(-20\,480\,q_o^3\,\delta_c + 14\,208\,q_o^3\,\delta_c^2 - 9088\,q_o^3\,\delta_c^3 + 3552\,q_o^3\,\delta_c^4\right) +}\\&\quad\mathbf{\#1^2\,\left(-57\,344\,q_o^4 + 28\,672\,q_o^4\,\delta_c + 12\,096\,q_o^4\,\delta_c^2 - 1856\,q_o^4\,\delta_c^3 - 2096\,q_o^4\,\delta_c^4\right) +}\\&\quad\mathbf{\#1\,\left(-49\,152\,q_o^5 + 132\,608\,q_o^5\,\delta_c - 109\,824\,q_o^5\,\delta_c^2 + 34\,944\,q_o^5\,\delta_c^3 - 3520\,q_o^5\,\delta_c^4\right)\,\&,\,3\Big];}\\&\mathbf{Reduce[t\,D_1 < p_1 < 2\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta_c < 1,\,Reals]}\end{aligned}$$

Out[*]=

$$q_o > 0\;\&\&\;t > 2\,q_o\;\&\&\;0 < \delta_c < \frac{2}{13}$$

(*Solution 4, interior solution*)

In[*]:=
$$\begin{aligned}&\mathbf{p_1 = Root\Big[57\,344\,q_o^6 - 116\,224\,q_o^6\,\delta_c + 88\,320\,q_o^6\,\delta_c^2 - 29\,824\,q_o^6\,\delta_c^3 + 19\,\#1^6\,\delta_c^4 +}\\&\quad\mathbf{3776\,q_o^6\,\delta_c^4 + \#1^5\,\left(88\,q_o\,\delta_c^3 + 100\,q_o\,\delta_c^4\right) + \#1^4\,\left(-1920\,q_o^2\,\delta_c^2 + 2328\,q_o^2\,\delta_c^3 - 1324\,q_o^2\,\delta_c^4\right) +}\\&\quad\mathbf{\#1^3\,\left(-20\,480\,q_o^3\,\delta_c + 14\,208\,q_o^3\,\delta_c^2 - 9088\,q_o^3\,\delta_c^3 + 3552\,q_o^3\,\delta_c^4\right) +}\\&\quad\mathbf{\#1^2\,\left(-57\,344\,q_o^4 + 28\,672\,q_o^4\,\delta_c + 12\,096\,q_o^4\,\delta_c^2 - 1856\,q_o^4\,\delta_c^3 - 2096\,q_o^4\,\delta_c^4\right) +}\\&\quad\mathbf{\#1\,\left(-49\,152\,q_o^5 + 132\,608\,q_o^5\,\delta_c - 109\,824\,q_o^5\,\delta_c^2 + 34\,944\,q_o^5\,\delta_c^3 - 3520\,q_o^5\,\delta_c^4\right)\,\&,\,4\Big];}\\&\mathbf{Reduce[t\,D_1 < p_1 < 2\,t\,D_1\;\&\&\;D_1 > 0\;\&\&\;t > 2\,q_o > 0\;\&\&\;0 < \delta_c < 1,\,Reals]}\end{aligned}$$

Out[*]=

False

(*Solution 5, interior solution*)

In[ ]:= $p_1 = \text{Root}\big[57344\, q_o^6 - 116224\, q_o^6\, \delta_c + 88320\, q_o^6\, \delta_c^2 - 29824\, q_o^6\, \delta_c^3 + 19\, \#1^6\, \delta_c^4 + 3776\, q_o^6\, \delta_c^4 + \#1^5\, (88\, q_o\, \delta_c^3 + 100\, q_o\, \delta_c^4) + \#1^4\, (-1920\, q_o^2\, \delta_c^2 + 2328\, q_o^2\, \delta_c^3 - 1324\, q_o^2\, \delta_c^4) + \#1^3\, (-20480\, q_o^3\, \delta_c + 14208\, q_o^3\, \delta_c^2 - 9088\, q_o^3\, \delta_c^3 + 3552\, q_o^3\, \delta_c^4) + \#1^2\, (-57344\, q_o^4 + 28672\, q_o^4\, \delta_c + 12096\, q_o^4\, \delta_c^2 - 1856\, q_o^4\, \delta_c^3 - 2096\, q_o^4\, \delta_c^4) + \#1\, (-49152\, q_o^5 + 132608\, q_o^5\, \delta_c - 109824\, q_o^5\, \delta_c^2 + 34944\, q_o^5\, \delta_c^3 - 3520\, q_o^5\, \delta_c^4)\ \&,\ 5\big];$

$\text{Reduce}[t\, D_1 < p_1 < 2\, t\, D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0 < \delta_c < 1,\ \text{Reals}]$

Out[ ]=

False

(*Solution 6, interior solution*)

In[ ]:= $p_1 = \text{Root}\big[57344\, q_o^6 - 116224\, q_o^6\, \delta_c + 88320\, q_o^6\, \delta_c^2 - 29824\, q_o^6\, \delta_c^3 + 19\, \#1^6\, \delta_c^4 + 3776\, q_o^6\, \delta_c^4 + \#1^5\, (88\, q_o\, \delta_c^3 + 100\, q_o\, \delta_c^4) + \#1^4\, (-1920\, q_o^2\, \delta_c^2 + 2328\, q_o^2\, \delta_c^3 - 1324\, q_o^2\, \delta_c^4) + \#1^3\, (-20480\, q_o^3\, \delta_c + 14208\, q_o^3\, \delta_c^2 - 9088\, q_o^3\, \delta_c^3 + 3552\, q_o^3\, \delta_c^4) + \#1^2\, (-57344\, q_o^4 + 28672\, q_o^4\, \delta_c + 12096\, q_o^4\, \delta_c^2 - 1856\, q_o^4\, \delta_c^3 - 2096\, q_o^4\, \delta_c^4) + \#1\, (-49152\, q_o^5 + 132608\, q_o^5\, \delta_c - 109824\, q_o^5\, \delta_c^2 + 34944\, q_o^5\, \delta_c^3 - 3520\, q_o^5\, \delta_c^4)\ \&,\ 6\big];$

$\text{Reduce}[t\, D_1 < p_1 < 2\, t\, D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0 < \delta_c < 1,\ \text{Reals}]$

Out[ ]=

False

(*Solution 7, boundary solution, which is the solution of p_1=2tD_1*)

In[ ]:= $p_1 = \frac{2\, q_o\, (-2 + \delta_c)}{-6 + \delta_c};$

$\lambda_1 = 0;$

$\lambda_2 = -\frac{q_o\, (384 - 2664\, \delta_c + 1164\, \delta_c^2 - 478\, \delta_c^3 + 65\, \delta_c^4)}{64\, t\, (-6 + \delta_c)^4};$

$\text{Reduce}[\lambda_2 > 0\ \&\&\ D_1 > 0\ \&\&\ t\, D_1 < p_1 \le 2\, t\, D_1\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0 < \delta_c < 1]$

Out[ ]=

$q_o > 0\ \&\&\ t > 2\, q_o\ \&\&\ \frac{2}{13} < \delta_c < 1$

(*Solution 8, boundary solution, which is the solution of p_1=tD_1*)

In[ ]:= $p_1 = -\frac{2\, q_o\, \left(4 + \sqrt{16 - 6\, \delta_c} - \delta_c\right)}{\delta_c};$

$\lambda_1 = \frac{q_o\, \left(384\, \left(4 + \sqrt{16 - 6\, \delta_c}\right) - 8\, \left(256 + 55\, \sqrt{16 - 6\, \delta_c}\right)\, \delta_c + \left(688 + 94\, \sqrt{16 - 6\, \delta_c}\right)\, \delta_c^2 - 51\, \delta_c^3\right)}{8\, t\, \delta_c^2\, (-8 + 3\, \delta_c)};$

$\lambda_2 = 0;$

$\text{Reduce}[\lambda_1 > 0\ \&\&\ D_1 > 0\ \&\&\ t\, D_1 \le p_1 < 2\, t\, D_1\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0 < \delta_c < 1]$

Out[ ]=

False

(*Solution 9, boundary solution, which is the solution of p_1=tD_1*)

In[•]:= $p_1 = \frac{2\,q_o\left(-4+\sqrt{16-6\,\delta_c}+\delta_c\right)}{\delta_c};$

$\lambda_1 = \frac{q_o\left(-384\left(-4+\sqrt{16-6\,\delta_c}\right)+8\left(-256+55\sqrt{16-6\,\delta_c}\right)\delta_c+\left(688-94\sqrt{16-6\,\delta_c}\right)\delta_c^2-51\,\delta_c^3\right)}{8\,t\,\delta_c^2\,(-8+3\,\delta_c)};$

$\lambda_2 = 0;$

Reduce[$\lambda_1 > 0$ && $D_1 > 0$ && $t\,D_1 \le p_1 < 2\,t\,D_1$ && $t > 2\,q_o > 0$ && $0 < \delta_c < 1$]

Out[•]=

False

(*Hence, the optimal solution is that when $0<\delta_c<\frac{2}{13}$, $p_1=P_2^{CL}(q_o,\delta_c)$, where $P_2^{CL}(q_o,\delta_c)$=Root$[57344\,q_o^6-116224\,q_o^6\,\delta_c+88320\,q_o^6\,\delta_c^2-29824\,q_o^6\,\delta_c^3+19\,\#1^6\,\delta_c^4+3776\,q_o^6\,\delta_c^4+\#1^5\left(88\,q_o\,\delta_c^3+100\,q_o\,\delta_c^4\right)+\#1^4\left(-1920\,q_o^2\,\delta_c^2+2328\,q_o^2\,\delta_c^3-1324\,q_o^2\,\delta_c^4\right)+\#1^3\left(-20480\,q_o^3\,\delta_c+14208\,q_o^3\,\delta_c^2-9088\,q_o^3\,\delta_c^3+3552\,q_o^3\,\delta_c^4\right)+\#1^2\left(-57344\,q_o^4+28672\,q_o^4\,\delta_c+12096\,q_o^4\,\delta_c^2-1856\,q_o^4\,\delta_c^3-2096\,q_o^4\,\delta_c^4\right)+\#1\left(-49152\,q_o^5+132608\,q_o^5\,\delta_c-109824\,q_o^5\,\delta_c^2+34944\,q_o^5\,\delta_c^3-3520\,q_o^5\,\delta_c^4\right)\&,3]$; when $\frac{2}{13}<\delta_c<1$, $p_1=\frac{2\,(-2+\delta_c)\,q_o}{-6+\delta_c}$*)

(*Scenario 3: $p_1>2tD_1$*)

In[•]:= $p_{2P} = \frac{2\,q_o+p_1-t\,D_1}{4};$

$D_{2P} = \frac{2\,q_o+p_1-t\,D_1}{4\,t};$

$p_{2M} = \frac{Q-t\,D_1}{2};$

$D_{2M} = \frac{Q-t\,D_1}{2\,t};$

$p_{2N} = \frac{p_1-2\,t\,D_1}{4};$

$D_{2N} = \frac{p_1-2\,t\,D_1}{4\,t};$

In[•]:= $U_1 = q_o - p_1 - t\,D_1;$

$U_2 = \delta_c\left(\frac{2\,q_o-p_1-t\,D_1}{2\,q_o}\left(\frac{2\,q_o+p_1+t\,D_1}{2}-p_{2P}-t\,D_1\right)+\right.$ Integrate[$(Q-p_{2M}-t\,D_1)/(2\,q_o)$, {Q, p_1, $p_1+t\,D_1$}] $\left.+\frac{p_1}{2\,q_o}\left(\frac{p_1}{2}-p_{2N}-t\,D_1\right)\right);$

In[•]:= Simplify[Solve[$U_1 == U_2$, D_1], $t > 2\,q_o > 0$ && $0 < \delta_c < 1$]

Out[•]=

$\left\{\left\{D_1 \to \frac{q_o+\frac{2\,p_1}{-2+\delta_c}}{t}\right\}\right\}$

In[•]:= $D_1 = \frac{q_o\,(2-\delta_c)-2\,p_1}{t\,(2-\delta_c)};$

In[•]:= Simplify[Reduce[$D_1 > 0$ && $p_1 > 2\,t\,D_1$ && $t > 2\,q_o > 0$ && $0 < \delta_c < 1$]]

Out[∘]=

$$t > 2\,q_o \,\&\&\, p_1 > 0 \,\&\&\, \left(\left(\frac{2\,p_1}{q_o} + \delta_c < 2 \,\&\&\, \left((\delta_c > 0 \,\&\&\, p_1 < q_o \,\&\&\, 2\,q_o \le 3\,p_1) \,||\, \left(3\,p_1 < 2\,q_o \,\&\&\, \frac{6\,p_1 - 4\,q_o}{p_1 - 2\,q_o} < \delta_c \,\&\&\, q_o \le 2\,p_1\right)\right)\right) \,||\, \left(2\,p_1 < q_o \,\&\&\, \frac{6\,p_1 - 4\,q_o}{p_1 - 2\,q_o} < \delta_c \,\&\&\, 2\,q_o < 5\,p_1 \,\&\&\, \delta_c < 1\right)\right)$$

In[∘]:=

$$\Pi = \text{Simplify}\left[p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P} + \text{Integrate}[\,p_{2M}\,D_{2M}\,/\,(2\,q_o),\,\{Q,\,p_1,\,p_1 + t\,D_1\}] + \frac{p_1}{2\,q_o}\,p_{2N}\,D_{2N}\right];\text{(*The firm's total profit function*)}$$

In[∘]:=

$$\text{Reduce}[D[D[\Pi,\,p_1],\,p_1] \ge 0 \,\&\&\, D_1 > 0 \,\&\&\, p_1 \ge 2\,t\,D_1 \,\&\&\, t > 2\,q_o > 0 \,\&\&\, 0 < \delta_c < 1]$$
(*Determine the sign of $\frac{\partial^2\Pi}{\partial p_1^2}$*)

Out[∘]=

False

(*$\frac{\partial^2\Pi}{\partial p_1^2}$<0,
meaning that Π is concave and it has a maximum value at the point where $\frac{\partial\Pi}{\partial p_1}$=0*)

(*Construct KKT conditions*)

In[∘]:=

$$g = p_1 - 2\,t\,D_1;$$
$$L = -\left(p_1\,D_1 + \frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P} + \text{Integrate}[\,p_{2M}\,D_{2M}\,/\,(2\,q_o),\,\{Q,\,p_1,\,p_1 + t\,D_1\}] + \frac{p_1}{2\,q_o}\,p_{2N}\,D_{2N}\right) - \lambda\,g;$$

In[∘]:=

$$\text{Simplify}[\text{Solve}[\{D[L,\,p_1] == 0,\,\lambda\,g == 0\},\,\{p_1,\,\lambda\}],\,q_o > 0 \,\&\&\, 0 < \delta_c < 1]$$

Out[∘]=

$$\left\{\left\{p_1 \to \frac{2\,q_o\,(-2+\delta_c)}{-6+\delta_c},\,\lambda \to -\frac{q_o\left(32 - 12\,\delta_c - 156\,\delta_c^2 + 29\,\delta_c^3\right)}{32\,t\,(-6+\delta_c)^3}\right\},\right.$$
$$\left\{p_1 \to \frac{q_o\,(-2+\delta_c)\left(96 - 64\,\delta_c + \delta_c^2 + \sqrt{9728 - 13\,320\,\delta_c + 5068\,\delta_c^2 - 326\,\delta_c^3 + \delta_c^4}\right)}{8 - 12\,\delta_c + 6\,\delta_c^2},\,\lambda \to 0\right\},$$
$$\left.\left\{p_1 \to \frac{q_o\,(-2+\delta_c)\left(96 - 64\,\delta_c + \delta_c^2 - \sqrt{9728 - 13\,320\,\delta_c + 5068\,\delta_c^2 - 326\,\delta_c^3 + \delta_c^4}\right)}{8 - 12\,\delta_c + 6\,\delta_c^2},\,\lambda \to 0\right\}\right\}$$

(*There are 3 solutions, we check each solution if it satisfies conditions*)

(*Solution 1, boundary solution, which is the solution of p_1=2tD_1*)

In[∘]:=

$$p_1 = \frac{2\,q_o\,(-2+\delta_c)}{-6+\delta_c};$$
$$\lambda = -\frac{q_o\left(32 - 12\,\delta_c - 156\,\delta_c^2 + 29\,\delta_c^3\right)}{32\,t\,(-6+\delta_c)^3};$$
$$\text{Reduce}[\lambda > 0 \,\&\&\, D_1 > 0 \,\&\&\, p_1 \ge 2\,t\,D_1 \,\&\&\, t > 2\,q_o > 0 \,\&\&\, 0 < \delta_c < 1]$$

Out[∘]=

$$q_o > 0 \,\&\&\, t > 2\,q_o \,\&\&\, 0 < \delta_c < \boxed{\sqrt[3]{\cdot}\ 0.432\ldots}$$

```
(*Solution 2, interior solution*)
```

In[∘]:= $p_1 = \frac{q_o\,(-2+\delta_c)\,\left(96-64\,\delta_c+\delta_c^2+\sqrt{9728-13\,320\,\delta_c+5068\,\delta_c^2-326\,\delta_c^3+\delta_c^4}\right)}{8-12\,\delta_c+6\,\delta_c^2}$;

Reduce[$D_1 > 0$ && $p_1 \geq 2\,t\,D_1$ && $t > 2\,q_o > 0$ && $0 < \delta_c < 1$]

Out[∘]=

False

```
(*Solution 3, interior solution*)
```

In[∘]:= $p_1 = \frac{q_o\,(-2+\delta_c)\,\left(96-64\,\delta_c+\delta_c^2-\sqrt{9728-13\,320\,\delta_c+5068\,\delta_c^2-326\,\delta_c^3+\delta_c^4}\right)}{8-12\,\delta_c+6\,\delta_c^2}$;

Reduce[$D_1 > 0$ && $p_1 \geq 2\,t\,D_1$ && $t > 2\,q_o > 0$ && $0 < \delta_c < 1$]

Out[∘]=

$q_o > 0$ && $t > 2\,q_o$ && $\sqrt[3]{\ }\,0.432\ldots \leq \delta_c < 1$

(*Therefore, when $0<\delta_c<$ 0.432…, $p_1=\frac{2\ q_o\ (-2+\delta_c)}{-6+\delta_c}$;

when 0.432… $\leq\delta_c<1$, $p_1=\frac{q_o\ (-2+\delta_c)\ \left(96-64\ \delta_c+\delta_c^2-\sqrt{9728-13320\ \delta_c+5068\ \delta_c^2-326\ \delta_c^3+\delta_c^4}\right)}{8-12\ \delta_c+6\ \delta_c^2}$ *)

```
(*Profit comparison*)

(*Based on the above 3 scenarios,
we then compare the firm's profits across (0,1) of δ*)
```

(*Scenario 1, $0<\delta_c<1$*)

In[∘]:= $p_1 = \frac{2\,q_o\,\left(-4+\sqrt{16-6\,\delta_c}+\delta_c\right)}{\delta_c}$;

$D_1 = \frac{2\,\left(q_o\,(-2+\delta_c)+\sqrt{2}\,\sqrt{-q_o\,(q_o\,(-2+\delta_c)+p_1\,\delta_c)}\right)}{t\,\delta_c}$;

$p_{2\,P} = \frac{2\,q_o+p_1-t\,D_1}{4}$;

$D_{2\,P} = \frac{2\,q_o+p_1-t\,D_1}{4\,t}$;

$p_{2\,M} = \frac{Q-t\,D_1}{2}$;

$D_{2\,M} = \frac{Q-t\,D_1}{2\,t}$;

Π_1 = Simplify$\Big[p_1\,D_1+\frac{2\,q_o-p_1-t\,D_1}{2\,q_o}\,p_{2\,P}\,D_{2\,P}+$

Integrate[$p_{2\,M}\,D_{2\,M}/(2\,q_o)$, {Q, $t\,D_1$, $p_1+t\,D_1$}], $0<\delta_c<1$ && $t > 2\,q_o > 0\Big]$;

(*Scenario 2(i), $0<\delta_c\leq\frac{2}{13}$*)

```
In[∘]:= p1 = Root[57 344 qo^6 - 116 224 qo^6 δc + 88 320 qo^6 δc^2 - 29 824 qo^6 δc^3 + 19 #1^6 δc^4 +
          3776 qo^6 δc^4 + #1^5 (88 qo δc^3 + 100 qo δc^4) + #1^4 (-1920 qo^2 δc^2 + 2328 qo^2 δc^3 - 1324 qo^2 δc^4) +
          #1^3 (-20 480 qo^3 δc + 14 208 qo^3 δc^2 - 9088 qo^3 δc^3 + 3552 qo^3 δc^4) +
          #1^2 (-57 344 qo^4 + 28 672 qo^4 δc + 12 096 qo^4 δc^2 - 1856 qo^4 δc^3 - 2096 qo^4 δc^4) +
          #1 (-49 152 qo^5 + 132 608 qo^5 δc - 109 824 qo^5 δc^2 + 34 944 qo^5 δc^3 - 3520 qo^5 δc^4) &, 3];
```

$$D_1 = \frac{8\, p_1\, q_o + 4\, q_o^2\, (-2 + \delta_c) - p_1^2\, \delta_c}{4\, t\, q_o\, (-2 + \delta_c) - 2\, t\, p_1\, \delta_c};$$

$$p_{2\,P} = \frac{2\, q_o + p_1 - t\, D_1}{4};$$

$$D_{2\,P} = \frac{2\, q_o + p_1 - t\, D_1}{4\, t};$$

$$p_{2\,M} = \frac{Q - t\, D_1}{2};$$

$$D_{2\,M} = \frac{Q - t\, D_1}{2\, t};$$

$$\Pi_{21} = \text{Simplify}\left[p_1\, D_1 + \frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2\,P}\, D_{2\,P} + \text{Integrate}[p_{2\,M}\, D_{2\,M} / (2\, q_o),\ \{Q,\ p_1,\ p_1 + t\, D_1\}],\ 0 < \delta_c < 1\, \&\&\, t > 2\, q_o > 0\right];$$

(*Scenario 2(ii), $\frac{2}{13}<\delta_c<1$*)

In[∘]:= $$p_1 = \frac{2\, q_o\, (-2 + \delta_c)}{-6 + \delta_c};$$

$$D_1 = \frac{8\, p_1\, q_o + 4\, q_o^2\, (-2 + \delta_c) - p_1^2\, \delta_c}{4\, t\, q_o\, (-2 + \delta_c) - 2\, t\, p_1\, \delta_c};$$

$$p_{2\,P} = \frac{2\, q_o + p_1 - t\, D_1}{4};$$

$$D_{2\,P} = \frac{2\, q_o + p_1 - t\, D_1}{4\, t};$$

$$p_{2\,M} = \frac{Q - t\, D_1}{2};$$

$$D_{2\,M} = \frac{Q - t\, D_1}{2\, t};$$

$$\Pi_{22} = \text{Simplify}\left[p_1\, D_1 + \frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2\,P}\, D_{2\,P} + \text{Integrate}[p_{2\,M}\, D_{2\,M} / (2\, q_o),\ \{Q,\ p_1,\ p_1 + t\, D_1\}],\ 0 < \delta_c < 1\, \&\&\, t > 2\, q_o > 0\right];$$

(*Scenario 3(i), $0<\delta_c<$ 0.432... *)

In[∘]:= $p_1 = \frac{2\, q_o\, (-2 + \delta_c)}{-6 + \delta_c};$

$D_1 = \frac{q_o\, (2 - \delta_c) - 2\, p_1}{t\, (2 - \delta_c)};$

$p_{2P} = \frac{2\, q_o + p_1 - t\, D_1}{4};$

$D_{2P} = \frac{2\, q_o + p_1 - t\, D_1}{4\, t};$

$p_{2M} = \frac{Q - t\, D_1}{2};$

$D_{2M} = \frac{Q - t\, D_1}{2\, t};$

$p_{2N} = \frac{p_1 - 2\, t\, D_1}{4};$

$D_{2N} = \frac{p_1 - 2\, t\, D_1}{4\, t};$

$\Pi_{31} = \text{Simplify}\Big[p_1\, D_1 + \frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2P}\, D_{2P} +$

$\quad \text{Integrate}[\, p_{2M}\, D_{2M} / (2\, q_o),\, \{Q,\, p_1,\, p_1 + t\, D_1\}] + \frac{p_1}{2\, q_o}\, p_{2N}\, D_{2N}\Big];$

(*Scenario 3(ii), 0.432... < δ_c < 1*)

In[∘]:= $p_1 = \frac{q_o\, (-2 + \delta_c)\, \left(96 - 64\, \delta_c + \delta_c^2 - \sqrt{9728 - 13\,320\, \delta_c + 5068\, \delta_c^2 - 326\, \delta_c^3 + \delta_c^4}\right)}{8 - 12\, \delta_c + 6\, \delta_c^2};$

$D_1 = \frac{q_o\, (2 - \delta_c) - 2\, p_1}{t\, (2 - \delta_c)};$

$p_{2P} = \frac{2\, q_o + p_1 - t\, D_1}{4};$

$D_{2P} = \frac{2\, q_o + p_1 - t\, D_1}{4\, t};$

$p_{2M} = \frac{Q - t\, D_1}{2};$

$D_{2M} = \frac{Q - t\, D_1}{2\, t};$

$p_{2N} = \frac{p_1 - 2\, t\, D_1}{4};$

$D_{2N} = \frac{p_1 - 2\, t\, D_1}{4\, t};$

$\Pi_{32} = \text{Simplify}\Big[p_1\, D_1 + \frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2P}\, D_{2P} +$

$\quad \text{Integrate}[\, p_{2M}\, D_{2M} / (2\, q_o),\, \{Q,\, p_1,\, p_1 + t\, D_1\}] + \frac{p_1}{2\, q_o}\, p_{2N}\, D_{2N}\Big];$

(*Comparison 1: when 0 < δ_c < $\frac{2}{13}$,
we have to compare profits under scenarios 1 2(i), and 3(i)*)

In[∘]:= $\text{Reduce}\left[\Pi_{31} < \Pi_1 \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0 < \delta_c < \frac{2}{13}\right]$

$\text{Reduce}\left[\Pi_{21} < \Pi_{31} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0 < \delta_c < \frac{2}{13}\right]$

Out[∘]=

False

Out[∘]=

False

(*Hence, when $0<\delta_c<\frac{2}{13}$,

$p_1=\text{Root}\big[57344\ q_o^6-116224\ q_o^6\ \delta_c+88320\ q_o^6\ \delta_c^2-29824\ q_o^6\ \delta_c^3+19\ \#1^6\ \delta_c^4+3776\ q_o^6\ \delta_c^4+ \#1^5\ (88\ q_o\ \delta_c^3+100\ q_o\ \delta_c^4)+\#1^4\ (-1920\ q_o^2\ \delta_c^2+2328\ q_o^2\ \delta_c^3-1324\ q_o^2\ \delta_c^4)+ \#1^3\ (-20480\ q_o^3\ \delta_c+14208\ q_o^3\ \delta_c^2-9088\ q_o^3\ \delta_c^3+3552\ q_o^3\ \delta_c^4)+ \#1^2\ (-57344\ q_o^4+28672\ q_o^4\ \delta_c+12096\ q_o^4\ \delta_c^2-1856\ q_o^4\ \delta_c^3-2096\ q_o^4\ \delta_c^4)+ \#1\ (-49152\ q_o^5+132608\ q_o^5\ \delta_c-109824\ q_o^5\ \delta_c^2+34944\ q_o^5\ \delta_c^3-3520\ q_o^5\ \delta_c^4)\&,3\big]$*)

(*Comparison 2: when $\frac{2}{13}<\delta_c<$ 0.432…,

we have to compare profits under scenarios 1, 2(ii), and 3(i)*)

In[∘]:= $\text{Reduce}\left[\Pi_{31} < \Pi_1 \,\&\&\, t > 2\, q_o > 0 \,\&\&\, \frac{2}{13} < \delta_c < 0.432\ldots\right]$

$\text{Reduce}\left[\Pi_{22} == \Pi_{31} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, \frac{2}{13} < \delta_c < 0.432\ldots\right]$

Out[∘]=

False

Out[∘]=

$t > 0 \,\&\&\, 0 < q_o < \frac{t}{2} \,\&\&\, \frac{2}{13} < \delta_c <$ 0.432…

(*Hence, when $\frac{2}{13}<\delta_c<$ 0.432…, $p_1=\frac{2\ q_o\ (-2+\delta_c)}{-6+\delta_c}$ *)

(*Comparison 3: when 0.432… $<\delta_c<1$,

we have to compare profits under scenarios 1, 2(ii), and 3(ii)*)

In[∘]:= $\text{Reduce}\left[\Pi_{22} < \Pi_1 \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0.432\ldots < \delta_c < 1\right]$

$\text{Reduce}\left[\Pi_{32} < \Pi_{22} \,\&\&\, t > 2\, q_o > 0 \,\&\&\, 0.432\ldots < \delta_c < 1\right]$

Out[∘]=

False

Out[∘]=

False

(*Hence, when 0.432… $<\delta_c<1$, $p_1=\frac{q_o\ (-2+\delta_c)\ \left(96-64\ \delta_c+\delta_c^2-\sqrt{9728-13320\ \delta_c+5068\ \delta_c^2-326\ \delta_c^3+\delta_c^4}\right)}{8-12\ \delta_c+6\ \delta_c^2}$ *)

(*The unique contingent pricing strategy,
firm profit, and consumer surplus in equilibrium*)

(*(i) $0<\delta_c<\frac{2}{13}$*)

```
pCL11 = Root[57 344 qo^6 - 116 224 qo^6 δc + 88 320 qo^6 δc^2 - 29 824 qo^6 δc^3 + 19 #1^6 δc^4 +
    3776 qo^6 δc^4 + #1^5 (88 qo δc^3 + 100 qo δc^4) + #1^4 (-1920 qo^2 δc^2 + 2328 qo^2 δc^3 - 1324 qo^2 δc^4) +
    #1^3 (-20 480 qo^3 δc + 14 208 qo^3 δc^2 - 9088 qo^3 δc^3 + 3552 qo^3 δc^4) +
    #1^2 (-57 344 qo^4 + 28 672 qo^4 δc + 12 096 qo^4 δc^2 - 1856 qo^4 δc^3 - 2096 qo^4 δc^4) +
    #1 (-49 152 qo^5 + 132 608 qo^5 δc - 109 824 qo^5 δc^2 + 34 944 qo^5 δc^3 - 3520 qo^5 δc^4) &, 3];
DCL11 = (8 pCL11 qo + 4 qo^2 (-2 + δc) - pCL11^2 δc) / (4 t qo (-2 + δc) - 2 t pCL11 δc);
pCL121 = (2 qo + pCL11 - t DCL11) / 4;
DCL121 = (2 qo + pCL11 - t DCL11) / (4 t);
pCL122 = (Q - t DCL11) / 2;
DCL122 = (Q - t DCL11) / (2 t);
ΠCL1 = Simplify[pCL11 DCL11 + (2 qo - pCL11 - t DCL11) / (2 qo) pCL121 DCL121 +
    Integrate[pCL122 DCL122 / (2 qo), {Q, pCL11, pCL11 + t DCL11}], 0 < δc < 1 && t > 2 qo > 0];
CSCL1 = Integrate[Integrate[Q - pCL11 - t x, {x, 0, DCL11}] / (2 qo), {Q, 0, 2 qo}] +
   Integrate[Integrate[δc (Q - pCL121 - t x), {x, DCL11, DCL11 + DCL121}] / (2 qo),
    {Q, pCL11 + t DCL11, 2 qo}] +
   Integrate[Integrate[δc (Q - pCL122 - t x), {x, DCL11, DCL11 + DCL122}] / (2 qo),
    {Q, pCL11, pCL11 + t DCL11}];

(*(ii) 2/13 < δc < 0.432... *)
```

$$p_{CL21} = \frac{2\,q_o\,(-2+\delta_c)}{-6+\delta_c};$$

$$D_{CL21} = \frac{q_o\,(2-\delta_c) - 2\,p_{CL21}}{t\,(2-\delta_c)};$$

$$p_{CL221} = \frac{2\,q_o + p_{CL21} - t\,D_{CL21}}{4};$$

$$D_{CL221} = \frac{2\,q_o + p_{CL21} - t\,D_{CL21}}{4\,t};$$

$$p_{CL222} = \frac{Q - t\,D_{CL21}}{2};$$

$$D_{CL222} = \frac{Q - t\,D_{CL21}}{2\,t};$$

$$p_{CL223} = \frac{p_{CL21} - 2\,t\,D_{CL21}}{4};$$

$$D_{CL223} = \frac{p_{CL21} - 2\,t\,D_{CL21}}{4\,t};$$

$$\Pi_{CL2} = \text{Simplify}\Big[p_{CL21}\,D_{CL21} + \frac{2\,q_o - p_{CL21} - t\,D_{CL21}}{2\,q_o}\,p_{CL221}\,D_{CL221} + \text{Integrate}[p_{CL222}\,D_{CL222}\,/\,(2\,q_o),\ \{Q,\ p_{CL21},\ p_{CL21} + t\,D_{CL21}\}] + \frac{p_{CL21}}{2\,q_o}\,p_{CL223}\,D_{CL223}\Big];$$

$$CS_{CL2} = \text{Integrate}[\text{Integrate}[Q - p_{CL21} - t\,x,\ \{x,\ 0,\ D_{CL21}\}]\,/\,(2\,q_o),\ \{Q,\ 0,\ 2\,q_o\}] + \text{Integrate}[\text{Integrate}[\delta_c\,(Q - p_{CL221} - t\,x),\ \{x,\ D_{CL21},\ D_{CL21} + D_{CL221}\}]\,/\,(2\,q_o),\ \{Q,\ p_{CL21} + t\,D_{CL21},\ 2\,q_o\}] + \text{Integrate}[\text{Integrate}[\delta_c\,(Q - p_{CL222} - t\,x),\ \{x,\ D_{CL21},\ D_{CL21} + D_{CL222}\}]\,/\,(2\,q_o),\ \{Q,\ p_{CL21},\ p_{CL21} + t\,D_{CL21}\}];$$

(*(iii) 0.432... < δ_c < 1*)

$$p_{CL31} = \frac{q_o\ (-2 + \delta_c)\ \left(96 - 64\ \delta_c + \delta_c^2 - \sqrt{9728 - 13\,320\ \delta_c + 5068\ \delta_c^2 - 326\ \delta_c^3 + \delta_c^4}\right)}{8 - 12\ \delta_c + 6\ \delta_c^2};$$

$$D_{CL31} = \frac{q_o\ (2 - \delta_c) - 2\ p_{CL31}}{t\ (2 - \delta_c)};$$

$$p_{CL321} = \frac{2\ q_o + p_{CL31} - t\ D_{CL31}}{4};$$

$$D_{CL321} = \frac{2\ q_o + p_{CL31} - t\ D_{CL31}}{4\ t};$$

$$p_{CL322} = \frac{Q - t\ D_{CL31}}{2};$$

$$D_{CL322} = \frac{Q - t\ D_{CL31}}{2\ t};$$

$$p_{CL323} = \frac{p_{CL31} - 2\ t\ D_{CL31}}{4};$$

$$D_{CL323} = \frac{p_{CL31} - 2\ t\ D_{CL31}}{4\ t};$$

$$\Pi_{CL3} = \text{Simplify}\Big[p_{CL31}\ D_{CL31} + \frac{2\ q_o - p_{CL31} - t\ D_{CL31}}{2\ q_o}\ p_{CL321}\ D_{CL321} +$$
$$\text{Integrate}[\ p_{CL322}\ D_{CL322} / (2\ q_o),\ \{Q,\ p_{CL31},\ p_{CL31} + t\ D_{CL31}\}] + \frac{p_{CL31}}{2\ q_o}\ p_{CL323}\ D_{CL323}\Big];$$

$$CS_{CL3} = \text{Integrate}[\text{Integrate}[Q - p_{CL31} - t\ x,\ \{x,\ 0,\ D_{CL31}\}] / (2\ q_o),\ \{Q,\ 0,\ 2\ q_o\}] +$$
$$\text{Integrate}[\text{Integrate}[\delta_c\ (Q - p_{CL321} - t\ x),\ \{x,\ D_{CL31},\ D_{CL31} + D_{CL321}\}] / (2\ q_o),$$
$$\{Q,\ p_{CL31} + t\ D_{CL31},\ 2\ q_o\}] +$$
$$\text{Integrate}[\text{Integrate}[\delta_c\ (Q - p_{CL322} - t\ x),\ \{x,\ D_{CL31},\ D_{CL31} + D_{CL322}\}] / (2\ q_o),$$
$$\{Q,\ p_{CL31},\ p_{CL31} + t\ D_{CL31}\}] +$$
$$\text{Integrate}[\text{Integrate}[\delta_c\ (Q - p_{CL323} - t\ x),\ \{x,\ D_{CL31},\ D_{CL31} + D_{CL323}\}] / (2\ q_o),\ \{Q,\ 0,\ p_{CL31}\}];$$

Case GL. Price guarantee with social learning

(* Combination 1. The conditions are $0<p_1<\frac{2q_o-tD_1}{3}$, $p_1 \le tD_1$, and $p_1 \le 16tD_1$*)

In[∘]:= $p_{2P} = \frac{2q_o + p_1 - tD_1}{4}$; (*The second-period price under completely positive reviews*)

$p_{2M} = p_1$; (*The second-period price under mixed reviews*)

$p_{2N} = p_1$; (*The second-period price under completely negative reviews*)

$D_{2P} = \frac{2q_o + p_1 - tD_1}{4t}$; (*The second-period demand under completely positive reviews*)

$D_{2M} = 0$; (*The second-period demand under mixed reviews*)

$D_{2N} = 0$; (*The second-period demand under completely negative reviews*)

In[∘]:= $U_1 = q_o - p_1 - tD_1$; (*Consumers' expected utility purchasing in the first period*)

$U_2 = \delta_c\left(\frac{2q_o - p_1 - tD_1}{2q_o}\left(\frac{2q_o + p_1 + tD_1}{2} - p_{2P} - tD_1\right)\right)$;

(*Consumers' expected utility purchasing in the second period*)

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[∘]=

$$\left\{\left\{D_1 \to \frac{2q_o(-2+\delta_c) - \sqrt{-8q_o^2(-2+\delta_c) - 8p_1q_o\delta_c + p_1^2\delta_c^2}}{t\delta_c}\right\}, \left\{D_1 \to \frac{2q_o(-2+\delta_c) + \sqrt{-8q_o^2(-2+\delta_c) - 8p_1q_o\delta_c + p_1^2\delta_c^2}}{t\delta_c}\right\}\right\}$$

(*There are two solutions of D_1, we then check if it satisfies conditions*)

In[∘]:= $D_1 = \frac{2q_o(-2+\delta_c) - \sqrt{-8q_o^2(-2+\delta_c) - 8p_1q_o\delta_c + p_1^2\delta_c^2}}{t\delta_c}$;

Reduce$\left[0 < p_1 < \frac{2q_o - tD_1}{3}\ \&\&\ p_1 \le tD_1\ \&\&\ p_1 \le 16tD_1\ \&\&\ 0 < D_1 < 1\ \&\&\ t > 2q_o > 0\ \&\&\ 0 < \delta_c < 1\right]$

Out[∘]=

False

(*The first solution does not satisfy conditions*)

In[∘]:= $D_1 = \frac{2q_o(-2+\delta_c) + \sqrt{-8q_o^2(-2+\delta_c) - 8p_1q_o\delta_c + p_1^2\delta_c^2}}{t\delta_c}$;

Simplify[

Reduce$\left[0 < p_1 < \frac{2q_o - tD_1}{3}\ \&\&\ p_1 \le tD_1\ \&\&\ p_1 \le 16tD_1\ \&\&\ 0 < D_1 < 1\ \&\&\ t > 2q_o > 0\ \&\&\ 0 < \delta_c < 1\right]$]

Out[∘]=

$$t > 2q_o\ \&\&\ p_1 > 0\ \&\&\ \delta_c > 0\ \&\&\ \left(\left(3p_1 == q_o\ \&\&\ \delta_c < \frac{4p_1 - 2q_o}{p_1 - q_o}\right) \,||\, \left(2p_1 < q_o\ \&\&\ q_o < 3p_1\ \&\&\ \delta_c \le \frac{4p_1 - 2q_o}{p_1 - q_o}\right) \,||\, (q_o > 3p_1\ \&\&\ \delta_c < 1)\right)$$

(*The second solution satisfies conditions, hence D_1 is given by*)

```
D₁ = (2 qₒ (-2 + δc) + Sqrt[-8 qₒ² (-2 + δc) - 8 p₁ qₒ δc + p₁² δc²]) / (t δc);
```

```
In[•]:= Π = Simplify[p₁ D₁ + (2 qₒ - p₁ - t D₁)/(2 qₒ) p2P D2P]; (*The firm's total profit*)
CS = Integrate[(Integrate[Q - p₁ - t x, {x, 0, D₁}]) / (2 qₒ), {Q, 0, 2 qₒ}] +
    Integrate[(Integrate[δ (Q - p2P - t x), {x, D₁, D₁ + D2P}]) / (2 qₒ),
     {Q, p₁ + t D₁, 2 qₒ}]; (*Consumer surplus*)
```

```
(*We then derive the optimal first-period price, profit,
and consumer surplus using numerical method by setting t=2.1 and qₒ=1,
which are the same as the base model. Then, we iterate systematically
 through all values of δc (0<δc<1) with a step size of 0.001*)
```

```
t = 2.1;
qₒ = 1;
results = {};
Results = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[p₁ = 0.001, p₁ < 1, p₁ += 0.001,
   If[Element[D₁, Reals] &&
     Element[Π, Reals] && (0 < p₁ < (2 qₒ - t D₁)/3 && p₁ ≤ t D₁ && p₁ ≤ 16 t D₁),
    AppendTo[results, {Π, p₁, CS}],
    AppendTo[results, {0, 0, 0}]]];
  {maxVal, maxP, maxCS} = Last@MaximalBy[results, First];
  AppendTo[Results, {δc, maxVal, maxP, maxCS}];
  results = {}
 ];
If[Results == {}, Print["No valid results found."], TableForm[Results,
  TableDirections → Row, TableHeadings → {None, {"δc", "Π", "p₁", "CS"}}]]
```

```
(* Combination 2. This scenario does not
 exist as no p₁ satisfies the stated conditions*)
```

```
(* Combination 3. The conditions are 0<p₁<(2qₒ-tD₁)/3, tD₁<p₁≤4tD₁, and p₁≤16tD₁*)
```

```
In[•]:= p2P = (2 qₒ + p₁ - t D₁)/4;
p2Mr = p₁; (*The second-period price under mixed reviews if p₁<q≤2 Sqrt[p₁tD₁] *)
p2Md = Q/2; (*The second-period price under mixed reviews if 2 Sqrt[p₁tD₁] <q≤p₁+tD₁*)
p2N = p₁;
D2P = (2 qₒ + p₁ - t D₁)/(4 t);
D2Mr = 0;
D2Md = (Q - 2 t D₁)/(2 t);
D2N = 0;
```

In[∘]:=
$$U_1 = \text{Simplify}\left[q_o - p_1 - t\,D_1 + \delta_c\,\text{Integrate}\left[(p_1 - p_{2\,Md}) / (2\,q_o),\ \left\{Q,\ 2\sqrt{p_1\,t\,D_1},\ p_1 + t\,D_1\right\}\right],\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\right];$$

$$U_2 = \text{Simplify}\left[\delta_c\left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2\,P} - t\,D_1\right) + \text{Integrate}\left[(Q - p_{2\,Md} - t\,D_1) / (2\,q_o),\ \left\{Q,\ 2\sqrt{p_1\,t\,D_1},\ p_1 + t\,D_1\right\}\right]\right),\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\right];$$

In[∘]:=
$$\text{Simplify}[\text{Solve}[U_1 == U_2,\ D_1,\ \text{Reals}],\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\ \&\&\ D_1 > 0]$$

Out[∘]=

$$\Big\{\Big\{D_1 \to \text{Root}\big[64\,p_1^2\,q_o^2 - 128\,p_1\,q_o^3 + 64\,q_o^4 - 48\,p_1^3\,q_o\,\delta_c + 48\,p_1^2\,q_o^2\,\delta_c + 64\,p_1\,q_o^3\,\delta_c - 64\,q_o^4\,\delta_c + t^4\,\#1^4\,\delta_c^2 + 9\,p_1^4\,\delta_c^2 - 24\,p_1^2\,q_o^2\,\delta_c^2 + 16\,q_o^4\,\delta_c^2 + \#1^3\left(-16\,t^3\,q_o\,\delta_c - 40\,t^3\,p_1\,\delta_c^2 + 8\,t^3\,q_o\,\delta_c^2\right) + \#1^2\left(64\,t^2\,q_o^2 - 208\,t^2\,p_1\,q_o\,\delta_c - 48\,t^2\,q_o^2\,\delta_c + 22\,t^2\,p_1^2\,\delta_c^2 + 96\,t^2\,p_1\,q_o\,\delta_c^2 + 8\,t^2\,q_o^2\,\delta_c^2\right) + \#1\left(128\,t\,p_1\,q_o^2 - 128\,t\,q_o^3 - 240\,t\,p_1^2\,q_o\,\delta_c + 128\,t\,p_1\,q_o^2\,\delta_c + 128\,t\,q_o^3\,\delta_c + 8\,t\,p_1^3\,\delta_c^2 + 24\,t\,p_1^2\,q_o\,\delta_c^2 - 96\,t\,p_1\,q_o^2\,\delta_c^2 - 32\,t\,q_o^3\,\delta_c^2\right)\ \&,\ 1\big]\ \text{if condition}\Big\},$$

$$\Big\{D_1 \to \text{Root}\big[64\,p_1^2\,q_o^2 - 128\,p_1\,q_o^3 + 64\,q_o^4 - 48\,p_1^3\,q_o\,\delta_c + 48\,p_1^2\,q_o^2\,\delta_c + 64\,p_1\,q_o^3\,\delta_c - 64\,q_o^4\,\delta_c + t^4\,\#1^4\,\delta_c^2 + 9\,p_1^4\,\delta_c^2 - 24\,p_1^2\,q_o^2\,\delta_c^2 + 16\,q_o^4\,\delta_c^2 + \#1^3\left(-16\,t^3\,q_o\,\delta_c - 40\,t^3\,p_1\,\delta_c^2 + 8\,t^3\,q_o\,\delta_c^2\right) + \#1^2\left(64\,t^2\,q_o^2 - 208\,t^2\,p_1\,q_o\,\delta_c - 48\,t^2\,q_o^2\,\delta_c + 22\,t^2\,p_1^2\,\delta_c^2 + 96\,t^2\,p_1\,q_o\,\delta_c^2 + 8\,t^2\,q_o^2\,\delta_c^2\right) + \#1\left(128\,t\,p_1\,q_o^2 - 128\,t\,q_o^3 - 240\,t\,p_1^2\,q_o\,\delta_c + 128\,t\,p_1\,q_o^2\,\delta_c + 128\,t\,q_o^3\,\delta_c + 8\,t\,p_1^3\,\delta_c^2 + 24\,t\,p_1^2\,q_o\,\delta_c^2 - 96\,t\,p_1\,q_o^2\,\delta_c^2 - 32\,t\,q_o^3\,\delta_c^2\right)\ \&,\ 2\big]\ \text{if condition}\Big\},$$

$$\Big\{D_1 \to \text{Root}\big[64\,p_1^2\,q_o^2 - 128\,p_1\,q_o^3 + 64\,q_o^4 - 48\,p_1^3\,q_o\,\delta_c + 48\,p_1^2\,q_o^2\,\delta_c + 64\,p_1\,q_o^3\,\delta_c - 64\,q_o^4\,\delta_c + t^4\,\#1^4\,\delta_c^2 + 9\,p_1^4\,\delta_c^2 - 24\,p_1^2\,q_o^2\,\delta_c^2 + 16\,q_o^4\,\delta_c^2 + \#1^3\left(-16\,t^3\,q_o\,\delta_c - 40\,t^3\,p_1\,\delta_c^2 + 8\,t^3\,q_o\,\delta_c^2\right) + \#1^2\left(64\,t^2\,q_o^2 - 208\,t^2\,p_1\,q_o\,\delta_c - 48\,t^2\,q_o^2\,\delta_c + 22\,t^2\,p_1^2\,\delta_c^2 + 96\,t^2\,p_1\,q_o\,\delta_c^2 + 8\,t^2\,q_o^2\,\delta_c^2\right) + \#1\left(128\,t\,p_1\,q_o^2 - 128\,t\,q_o^3 - 240\,t\,p_1^2\,q_o\,\delta_c + 128\,t\,p_1\,q_o^2\,\delta_c + 128\,t\,q_o^3\,\delta_c + 8\,t\,p_1^3\,\delta_c^2 + 24\,t\,p_1^2\,q_o\,\delta_c^2 - 96\,t\,p_1\,q_o^2\,\delta_c^2 - 32\,t\,q_o^3\,\delta_c^2\right)\ \&,\ 3\big]\ \text{if condition}\Big\},$$

$$\Big\{D_1 \to \text{Root}\big[64\,p_1^2\,q_o^2 - 128\,p_1\,q_o^3 + 64\,q_o^4 - 48\,p_1^3\,q_o\,\delta_c + 48\,p_1^2\,q_o^2\,\delta_c + 64\,p_1\,q_o^3\,\delta_c - 64\,q_o^4\,\delta_c + t^4\,\#1^4\,\delta_c^2 + 9\,p_1^4\,\delta_c^2 - 24\,p_1^2\,q_o^2\,\delta_c^2 + 16\,q_o^4\,\delta_c^2 + \#1^3\left(-16\,t^3\,q_o\,\delta_c - 40\,t^3\,p_1\,\delta_c^2 + 8\,t^3\,q_o\,\delta_c^2\right) + \#1^2\left(64\,t^2\,q_o^2 - 208\,t^2\,p_1\,q_o\,\delta_c - 48\,t^2\,q_o^2\,\delta_c + 22\,t^2\,p_1^2\,\delta_c^2 + 96\,t^2\,p_1\,q_o\,\delta_c^2 + 8\,t^2\,q_o^2\,\delta_c^2\right) + \#1\left(128\,t\,p_1\,q_o^2 - 128\,t\,q_o^3 - 240\,t\,p_1^2\,q_o\,\delta_c + 128\,t\,p_1\,q_o^2\,\delta_c + 128\,t\,q_o^3\,\delta_c + 8\,t\,p_1^3\,\delta_c^2 + 24\,t\,p_1^2\,q_o\,\delta_c^2 - 96\,t\,p_1\,q_o^2\,\delta_c^2 - 32\,t\,q_o^3\,\delta_c^2\right)\ \&,\ 4\big]\ \text{if condition}\Big\}\Big\}$$

```
(*There are 4 solutions, we then check each solution if it satisfies conditions*)

(*Solution 1*)
```

In[∘]:=
```
D1 = Root[64 p1^2 qo^2 - 128 p1 qo^3 + 64 qo^4 - 48 p1^3 qo δc + 48 p1^2 qo^2 δc + 64 p1 qo^3 δc - 64 qo^4 δc + t^4 #1^4 δc^2 +
    9 p1^4 δc^2 - 24 p1^2 qo^2 δc^2 + 16 qo^4 δc^2 + #1^3 (-16 t^3 qo δc - 40 t^3 p1 δc^2 + 8 t^3 qo δc^2) +
    #1^2 (64 t^2 qo^2 - 208 t^2 p1 qo δc - 48 t^2 qo^2 δc + 22 t^2 p1^2 δc^2 + 96 t^2 p1 qo δc^2 + 8 t^2 qo^2 δc^2) +
    #1 (128 t p1 qo^2 - 128 t qo^3 - 240 t p1^2 qo δc + 128 t p1 qo^2 δc + 128 t qo^3 δc +
       8 t p1^3 δc^2 + 24 t p1^2 qo δc^2 - 96 t p1 qo^2 δc^2 - 32 t qo^3 δc^2) &, 1];
Simplify[Reduce[
  0 < p1 < (2 qo - t D1)/3 && t D1 < p1 ≤ 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1 && t > 2 qo > 0 && 0 < δc < 1]]
```

Out[∘]=

$$p_1 > 0 \;\&\&\; t > 2 q_o \;\&\& \left(\left(\delta_c \le \frac{32\,(5 p_1 - 4 q_o)\, q_o}{17 p_1^2 + 16 p_1 q_o - 64 q_o^2} \;\&\& \left((2 p_1 == q_o \;\&\&\; \delta_c > 0) \;||\; \left(2 p_1 < q_o \;\&\&\; \frac{4 p_1 - 2 q_o}{p_1 - q_o} < \delta_c \;\&\&\; 8 q_o < 17 p_1\right) \;||\; \left(13 p_1 < 8 q_o \;\&\&\; q_o < 2 p_1 \;\&\&\; 2\sqrt{-\frac{p_1\,(3 p_1 - 2 q_o)\, q_o^2\,(2 p_1^2 - 3 p_1 q_o + q_o^2)^2}{(21 p_1^4 - 32 p_1^3 q_o + 22 p_1^2 q_o^2 - 8 p_1 q_o^3 + q_o^4)^2}} + \frac{q_o\,(6 p_1^3 - 3 p_1^2 q_o - 2 p_1 q_o^2 + q_o^3)}{21 p_1^4 - 32 p_1^3 q_o + 22 p_1^2 q_o^2 - 8 p_1 q_o^3 + q_o^4} < \delta_c\right)\right)\right) \;||\; \left(\frac{4 p_1 - 2 q_o}{p_1 - q_o} < \delta_c \;\&\& \left(\left(17 p_1 == 8 q_o \;\&\&\; \delta_c < \frac{32\,(5 p_1 - 4 q_o)\, q_o}{17 p_1^2 + 16 p_1 q_o - 64 q_o^2}\right) \;||\; (17 p_1 < 8 q_o \;\&\&\; q_o < 3 p_1 \;\&\&\; \delta_c < 1)\right)\right)\right)$$

```
(*Solution 2*)
```

In[∘]:=
```
D1 = Root[64 p1^2 qo^2 - 128 p1 qo^3 + 64 qo^4 - 48 p1^3 qo δc + 48 p1^2 qo^2 δc + 64 p1 qo^3 δc - 64 qo^4 δc + t^4 #1^4 δc^2 +
    9 p1^4 δc^2 - 24 p1^2 qo^2 δc^2 + 16 qo^4 δc^2 + #1^3 (-16 t^3 qo δc - 40 t^3 p1 δc^2 + 8 t^3 qo δc^2) +
    #1^2 (64 t^2 qo^2 - 208 t^2 p1 qo δc - 48 t^2 qo^2 δc + 22 t^2 p1^2 δc^2 + 96 t^2 p1 qo δc^2 + 8 t^2 qo^2 δc^2) +
    #1 (128 t p1 qo^2 - 128 t qo^3 - 240 t p1^2 qo δc + 128 t p1 qo^2 δc + 128 t qo^3 δc +
       8 t p1^3 δc^2 + 24 t p1^2 qo δc^2 - 96 t p1 qo^2 δc^2 - 32 t qo^3 δc^2) &, 2];
Simplify[Reduce[
  0 < p1 < (2 qo - t D1)/3 && t D1 < p1 ≤ 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1 && t > 2 qo > 0 && 0 < δc < 1]]
```

Out[∘]=
```
False
```

```
(*Solution 3*)
```

In[∘]:=
```
D1 = Root[64 p1^2 qo^2 - 128 p1 qo^3 + 64 qo^4 - 48 p1^3 qo δc + 48 p1^2 qo^2 δc + 64 p1 qo^3 δc - 64 qo^4 δc + t^4 #1^4 δc^2 +
    9 p1^4 δc^2 - 24 p1^2 qo^2 δc^2 + 16 qo^4 δc^2 + #1^3 (-16 t^3 qo δc - 40 t^3 p1 δc^2 + 8 t^3 qo δc^2) +
    #1^2 (64 t^2 qo^2 - 208 t^2 p1 qo δc - 48 t^2 qo^2 δc + 22 t^2 p1^2 δc^2 + 96 t^2 p1 qo δc^2 + 8 t^2 qo^2 δc^2) +
    #1 (128 t p1 qo^2 - 128 t qo^3 - 240 t p1^2 qo δc + 128 t p1 qo^2 δc + 128 t qo^3 δc +
       8 t p1^3 δc^2 + 24 t p1^2 qo δc^2 - 96 t p1 qo^2 δc^2 - 32 t qo^3 δc^2) &, 3];
Simplify[Reduce[
  0 < p1 < (2 qo - t D1)/3 && t D1 < p1 ≤ 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1 && t > 2 qo > 0 && 0 < δc < 1]]
```

Out[•]=

```
False
```

```
(*Solution 4*)
```

In[•]:=

```
D1 = Root[64 p1^2 qo^2 - 128 p1 qo^3 + 64 qo^4 - 48 p1^3 qo δc + 48 p1^2 qo^2 δc + 64 p1 qo^3 δc - 64 qo^4 δc + t^4 #1^4 δc^2 +
    9 p1^4 δc^2 - 24 p1^2 qo^2 δc^2 + 16 qo^4 δc^2 + #1^3 (-16 t^3 qo δc - 40 t^3 p1 δc^2 + 8 t^3 qo δc^2) +
    #1^2 (64 t^2 qo^2 - 208 t^2 p1 qo δc - 48 t^2 qo^2 δc + 22 t^2 p1^2 δc^2 + 96 t^2 p1 qo δc^2 + 8 t^2 qo^2 δc^2) +
    #1 (128 t p1 qo^2 - 128 t qo^3 - 240 t p1^2 qo δc + 128 t p1 qo^2 δc + 128 t qo^3 δc +
       8 t p1^3 δc^2 + 24 t p1^2 qo δc^2 - 96 t p1 qo^2 δc^2 - 32 t qo^3 δc^2) &, 4];
Simplify[Reduce[
  0 < p1 < (2 qo - t D1)/3 && t D1 < p1 ≤ 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1 && t > 2 qo > 0 && 0 < δc < 1]]
```

Out[•]=

```
False
```

```
(*Therefore, solution 1 is the feasible solution*)
```

```
D1 = Root[64 p1^2 qo^2 - 128 p1 qo^3 + 64 qo^4 - 48 p1^3 qo δc + 48 p1^2 qo^2 δc + 64 p1 qo^3 δc - 64 qo^4 δc + t^4 #1^4 δc^2 +
    9 p1^4 δc^2 - 24 p1^2 qo^2 δc^2 + 16 qo^4 δc^2 + #1^3 (-16 t^3 qo δc - 40 t^3 p1 δc^2 + 8 t^3 qo δc^2) +
    #1^2 (64 t^2 qo^2 - 208 t^2 p1 qo δc - 48 t^2 qo^2 δc + 22 t^2 p1^2 δc^2 + 96 t^2 p1 qo δc^2 + 8 t^2 qo^2 δc^2) +
    #1 (128 t p1 qo^2 - 128 t qo^3 - 240 t p1^2 qo δc + 128 t p1 qo^2 δc +
       128 t qo^3 δc + 8 t p1^3 δc^2 + 24 t p1^2 qo δc^2 - 96 t p1 qo^2 δc^2 - 32 t qo^3 δc^2) &, 1];
Π = p1 D1 + (2 qo - p1 - t D1)/(2 qo) p2P D2P +
   Integrate[(p2Md D2Md - D1 (p1 - p2Md)) / (2 qo), {Q, 2 Sqrt[p1 t D1], p1 + t D1}];
CS = Integrate[
    (Integrate[Q - p1 - t x + δc Integrate[(p1 - p2Md) / (2 qo), {Q, 2 Sqrt[p1 t D1], p1 + t D1}],
       {x, 0, D1}]) / (2 qo), {Q, 0, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2P - t x), {x, D1, D1 + D2P}]) / (2 qo), {Q, p1 + t D1, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2Md - t x), {x, D1, D1 + D2Md}]) / (2 qo),
    {Q, 2 Sqrt[p1 t D1], p1 + t D1}];
```

```
t = 2.1;
qo = 1;
results = {};
Results = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[p1 = 0.001, p1 < 1, p1 += 0.001,
   If[Element[D1, Reals] && Element[Π, Reals] &&
     (0 < p1 < (2 qo - t D1)/3 && t D1 < p1 ≤ 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1),
    AppendTo[results, {Π, p1, CS}],
    AppendTo[results, {0, 0, 0}]]];
  {maxVal, maxP, maxCS} = Last@MaximalBy[results, First];
  AppendTo[Results, {δc, maxVal, maxP, maxCS}];
  results = {}
 ];
If[Results == {}, Print["No valid results found."], TableForm[Results,
  TableDirections → Row, TableHeadings → {None, {"δc", "Π", "p1", "CS"}}]]
```

```
(*Combination 4. This scenario does not
 exist as no p1 satisfies the stated conditions*)
```

(*Combination 5. The conditions are $0<p_1<\frac{2q_o-tD_1}{3}$, $p_1>4tD_1$, and $p_1\leq 16tD_1$*)

In[*]:=
$$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4};$$
$$p_{2M} = \frac{2\,p_1 + t\,D_1}{4};$$
$$p_{2N} = p_1;$$
$$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t};$$
$$D_{2M} = \frac{2\,p_1 - 3\,t\,D_1}{4\,t};$$
$$D_{2N} = 0;$$

In[*]:=
$$U_1 = q_o - p_1 - t\,D_1 + \delta_c\,\frac{t\,D_1}{2\,q_o}\,(p_1 - p_{2M});$$

(*Consumers' expected utility purchasing in the first period*)

$$U_2 = \delta_c\left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2P} - t\,D_1\right) + \frac{t\,D_1}{2\,q_o}\left(\frac{2\,p_1 + t\,D_1}{2} - p_{2M} - t\,D_1\right)\right);$$

(*Consumers' expected utility purchasing in the second period*)

In[*]:=
```
Simplify[Solve[U1 == U2, D1], p1 > 0 && t > 2 qo > 0 && 0 < δc < 1]
```

Out[*]=
$$\left\{\left\{D_1 \to -\frac{2\,q_o\,(-2+\delta_c) + \sqrt{8\,p_1\,q_o\,\delta_c - p_1^2\,\delta_c^2 + 8\,q_o^2\,(2 - 3\,\delta_c + \delta_c^2)}}{t\,\delta_c}\right\},\right.$$
$$\left.\left\{D_1 \to \frac{-2\,q_o\,(-2+\delta_c) + \sqrt{8\,p_1\,q_o\,\delta_c - p_1^2\,\delta_c^2 + 8\,q_o^2\,(2 - 3\,\delta_c + \delta_c^2)}}{t\,\delta_c}\right\}\right\}$$

(*There are two solutions of D1,
we then check each solution if it satisfies conditions*)

In[*]:=
$$D_1 = -\frac{2\,q_o\,(-2+\delta_c) + \sqrt{8\,p_1\,q_o\,\delta_c - p_1^2\,\delta_c^2 + 8\,q_o^2\,(2 - 3\,\delta_c + \delta_c^2)}}{t\,\delta_c};$$

$$\text{Simplify}\left[\text{Reduce}\left[0 < p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > 4\,t\,D_1\ \&\&\ 0 < p_1 \leq 16\,t\,D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\right]\right]$$

Out[*]=
$$t > 2\,q_o\ \&\&\ p_1 > 0\ \&\&\ \left(\left(\frac{32\,(5\,p_1 - 4\,q_o)\,q_o}{17\,p_1^2 + 16\,p_1\,q_o - 64\,q_o^2} < \delta_c\ \&\&\right.\right.$$
$$\left(\left(33\,p_1 + 8\,\sqrt{13}\,\sqrt{p_1^2} == 32\,q_o\ \&\&\ \delta_c < \frac{128\,(17\,p_1 - 16\,q_o)\,q_o}{257\,p_1^2 + 64\,p_1\,q_o - 1024\,q_o^2}\right)\ ||\right.$$
$$\left(13\,p_1 < 8\,q_o\ \&\&\ 32\,q_o < 33\,p_1 + 8\,\sqrt{13}\,\sqrt{p_1^2}\ \&\&\ \delta_c \leq \frac{128\,(17\,p_1 - 16\,q_o)\,q_o}{257\,p_1^2 + 64\,p_1\,q_o - 1024\,q_o^2}\right)\ ||$$
$$\left.\left.\left(33\,p_1 + 8\,\sqrt{13}\,\sqrt{p_1^2} < 32\,q_o\ \&\&\ 8\,q_o < 17\,p_1\ \&\&\ \delta_c < 1\right)\right)\right)\ ||$$
$$\left.\left(49\,p_1 < 32\,q_o\ \&\&\ \frac{4\,q_o\,(-2\,p_1 + q_o)}{5\,p_1^2 - 12\,p_1\,q_o + 4\,q_o^2} < \delta_c\ \&\&\ 8\,q_o \leq 13\,p_1\ \&\&\ \delta_c \leq \frac{128\,(17\,p_1 - 16\,q_o)\,q_o}{257\,p_1^2 + 64\,p_1\,q_o - 1024\,q_o^2}\right)\right)$$

In[ ]:=
```
D₁ = (-2 q₀ (-2 + δc) + Sqrt[8 p₁ q₀ δc - p₁^2 δc^2 + 8 q₀^2 (2 - 3 δc + δc^2)]) / (t δc);
Simplify[
 Reduce[0 < p₁ < (2 q₀ - t D₁)/3 && p₁ > 4 t D₁ && 0 < p₁ ≤ 16 t D₁ && D₁ > 0 && t > 2 q₀ > 0 && 0 < δc < 1]]
```

$$D_1 = \frac{-2\, q_o\, (-2 + \delta_c) + \sqrt{8\, p_1\, q_o\, \delta_c - p_1^2\, \delta_c^2 + 8\, q_o^2 \left(2 - 3\, \delta_c + \delta_c^2\right)}}{t\, \delta_c};$$

$$\text{Simplify}\left[\text{Reduce}\left[0 < p_1 < \frac{2\, q_o - t\, D_1}{3} \;\&\&\; p_1 > 4\, t\, D_1 \;\&\&\; 0 < p_1 \le 16\, t\, D_1 \;\&\&\; D_1 > 0 \;\&\&\; t > 2\, q_o > 0 \;\&\&\; 0 < \delta_c < 1\right]\right]$$

Out[ ]=

```
False
```

```
(*Hence, the first solution satisfies conditions*)
```

$$D_1 = \frac{-2\ (-2+\delta)\ q_o + \sqrt{-\delta^2\ p_1^2 + 8\ \delta\ p_1\ q_o + 8\ \left(2 - 3\ \delta + \delta^2\right)\ q_o^2}}{t\ \delta};$$

$$\text{Reduce}\left[0 < p_1 < \frac{2\ q_o - t\ D_1}{3}\ \&\&\ p_1 > 4\ t\ D_1\ \&\&\ 0 < p_1 \le 16\ t\ D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\ q_o > 0\ \&\&\ 0 < \delta_c < 1\right]$$

Out[ ]=

```
False
```

```
(*Hence, the second solution does not satisfy conditions*)
```

$$D_1 = -\frac{2\ q_o\ (-2+\delta_c) + \sqrt{8\ p_1\ q_o\ \delta_c - p_1^2\ \delta_c^2 + 8\ q_o^2\ \left(2 - 3\ \delta_c + \delta_c^2\right)}}{t\ \delta_c};$$

```
(*The first-period demand function*)
```

$$\Pi = \text{Simplify}\left[p_1\ D_1 + \frac{2\ q_o - p_1 - t\ D_1}{2\ q_o}\ p_{2P}\ D_{2P} + \frac{t\ D_1}{2\ q_o}\ \left(p_{2M}\ D_{2M} - D_1\ (p_1 - p_{2M})\right)\right];$$

```
(*The firm's total profit*)
```

$$CS = \text{Integrate}\left[\left(\text{Integrate}\left[Q - p_1 - t\ x + \frac{t\ D_1}{2\ q_o}\ \delta_c\ (p_1 - p_{2M}),\ \{x, 0, D_1\}\right]\right) \Big/ (2\ q_o),\ \{Q, 0, 2\ q_o\}\right] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c\ (Q - p_{2P} - t\ x),\ \{x, D_1, D_1 + D_{2P}\}]) / (2\ q_o),\ \{Q, p_1 + t\ D_1, 2\ q_o\}] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c\ (Q - p_{2M} - t\ x),\ \{x, D_1, D_1 + D_{2M}\}]) / (2\ q_o),\ \{Q, p_1, p_1 + t\ D_1\}];\text{(*Consumer surplus*)}$$

```
t = 2.1;
qo = 1;
results = {};
Results = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[p1 = 0.001, p1 < 1, p1 += 0.001,
   If[Element[D1, Reals] && Element[Π, Reals] &&
     (0 < p1 < (2 qo - t D1)/3 && p1 > 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1),
    AppendTo[results, {Π, p1, CS}],
    AppendTo[results, {0, 0, 0}]]];
  {maxVal, maxP, maxCS} = Last@MaximalBy[results, First];
  AppendTo[Results, {δc, maxVal, maxP, maxCS}];
  results = {}
 ];
If[Results == {}, Print["No valid results found."], TableForm[Results,
  TableDirections → Row, TableHeadings → {None, {"δc", "Π", "p1", "CS"}}]]
```

(* Combiantion 6. The conditions are $0<p_1<\frac{2q_o-tD_1}{3}$, $p_1>4tD_1$, and $p_1>16tD_1$*)

In[◦]:= $p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4}$; (*The second-period price under completely positive reviews*)

$p_{2M} = \frac{2\,p_1 + t\,D_1}{4}$; (*The second-period price under mixed reviews*)

$p_{2N} = \frac{p_1}{4}$; (*The second-period price under completely negative reviews*)

$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t}$; (*The second-period demand under completely positive reviews*)

$D_{2M} = \frac{2\,p_1 - 3\,t\,D_1}{4\,t}$; (*The second-period demand under mixed reviews*)

$D_{2N} = \frac{p_1 - 4\,t\,D_1}{4\,t}$; (*The second-period demand under completely negative reviews*)

In[◦]:= $U_1 = q_o - p_1 - t\,D_1 + \delta_c \left(\frac{t\,D_1}{2\,q_o}\,(p_1 - p_{2M}) + \frac{p_1}{2\,q_o}\,(p_1 - p_{2N})\right)$;

(*Consumers' expected utility purchasing in the first period*)

$$U_2 = \delta_c \left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2P} - t\,D_1\right) + \frac{t\,D_1}{2\,q_o}\left(\frac{2\,p_1 + t\,D_1}{2} - p_{2M} - t\,D_1\right) + \frac{p_1}{2\,q_o}\left(\frac{p_1}{2} - p_{2N} - t\,D_1\right)\right);$$

(*Consumers' expected utility purchasing in the second period*)

In[◦]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[◦]=

$$\left\{\left\{D_1 \to -\frac{2\,q_o\,(-2+\delta_c) + 2\,p_1\,\delta_c + \sqrt{8\,p_1\,q_o\,(-1+\delta_c)\,\delta_c + p_1^2\,\delta_c^2 + 8\,q_o^2\,(2 - 3\,\delta_c + \delta_c^2)}}{t\,\delta_c}\right\}, \left\{D_1 \to \frac{-2\,q_o\,(-2+\delta_c) - 2\,p_1\,\delta_c + \sqrt{8\,p_1\,q_o\,(-1+\delta_c)\,\delta_c + p_1^2\,\delta_c^2 + 8\,q_o^2\,(2 - 3\,\delta_c + \delta_c^2)}}{t\,\delta_c}\right\}\right\}$$

(*Hence, there are two solutions of D_1. We then check each solution if it satisfies conditions*)

In[◦]:= $$D_1 = -\frac{2\,q_o\,(-2+\delta_c) + 2\,p_1\,\delta_c + \sqrt{8\,p_1\,q_o\,(-1+\delta_c)\,\delta_c + p_1^2\,\delta_c^2 + 8\,q_o^2\,(2 - 3\,\delta_c + \delta_c^2)}}{t\,\delta_c};$$

Reduce$\left[0 < p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > 4\,t\,D_1\ \&\&\ p_1 > 16\,tD_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\right]$

Out[◦]=

False

(*The first solution does not satisfy conditions*)

In[◦]:= $$D_1 = \frac{-2\,q_o\,(-2+\delta_c) - 2\,p_1\,\delta_c + \sqrt{8\,p_1\,q_o\,(-1+\delta_c)\,\delta_c + p_1^2\,\delta_c^2 + 8\,q_o^2\,(2 - 3\,\delta_c + \delta_c^2)}}{t\,\delta_c};$$

Reduce$\left[0 < p_1 < \frac{2\,q_o - t\,D_1}{3}\ \&\&\ p_1 > 4\,t\,D_1\ \&\&\ p_1 > 16\,tD_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\right]$

Out[◦]=

False

(*The second solution does not satisfy conditions*)

```
(*Therefore, there are no feasible solutions for combination 6*)
```

(* Combiantion 7. The conditions are $\frac{2q_o-tD_1}{3} \le p_1 \le \frac{2q_o+tD_1}{3}$, $p_1 \le tD_1$, and $p_1 \le 16tD_1$*)

In[∘]:=

$p_{2P} = p_1$; (*The second-period price under completely positive reviews*)
$p_{2M} = p_1$; (*The second-period price under mixed reviews*)
$p_{2N} = p_1$; (*The second-period price under completely negative reviews*)
$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2 t}$; (*The second-period demand under completely positive reviews*)
$D_{2M} = 0$; (*The second-period demand under mixed reviews*)
$D_{2N} = 0$; (*The second-period demand under completely negative reviews*)

In[∘]:=

$U_1 = q_o - p_1 - t D_1$; (*Consumers' expected utility purchasing in the first period*)
$U_2 = \delta_c \frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right)$;
(*Consumers' expected utility purchasing in the second period*)

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1], t > 0]

Out[∘]=

$$\left\{\left\{D_1 \to -\frac{2 q_o \left(1 + \sqrt{1 - \delta_c} - \delta_c\right) + p_1 \delta_c}{t \delta_c}\right\}, \left\{D_1 \to \frac{-p_1 \delta_c + 2 q_o \left(-1 + \sqrt{1 - \delta_c} + \delta_c\right)}{t \delta_c}\right\}\right\}$$

(*There are two solution of D_1. We then check each solution if it satisfies conditions*)

In[∘]:=

$$D_1 = -\frac{2 q_o \left(1 + \sqrt{1 - \delta_c} - \delta_c\right) + p_1 \delta_c}{t \delta_c};$$

$$\text{Reduce}\left[0 < \frac{2 q_o - t D_1}{3} \le p_1 \le \frac{2 q_o + t D_1}{3} \;\&\&\; 0 < p_1 \le t D_1 \;\&\&\; p_1 \le 16\, tD_1 \;\&\&\; D_1 > 0 \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1\right]$$

Out[∘]=

False

(*The first solution does not satisfy conditions*)

In[∘]:=

$$D_1 = \frac{-p_1 \delta_c + 2 q_o \left(-1 + \sqrt{1 - \delta_c} + \delta_c\right)}{t \delta_c};$$

$$\text{Reduce}\left[0 < \frac{2 q_o - t D_1}{3} \le p_1 \le \frac{2 q_o + t D_1}{3} \;\&\&\; 0 < p_1 \le t D_1 \;\&\&\; p_1 \le 16\, tD_1 \;\&\&\; D_1 > 0 \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1\right]$$

Out[∘]=

False

(*Hence, there is no feasible solution in combination 7*)

(* Combiantion 8. This scenario does not exist as no p_1 satisfies the stated conditions*)

(* Combiantion 9. The conditions are $\frac{2q_o-tD_1}{3} \le p_1 \le \frac{2q_o+tD_1}{3}$, $tD_1 < p_1 \le 4tD_1$, and $p_1 \le 16tD_1$*)

```
In[∘]:= p2P = p1;(*The second-period price under completely positive reviews*)
        p2Mr = p1;(*The second-period price under mixed reviews if p1<q≤2 √(p1tD1) *)
        p2Md = Q/2;(*The second-period price under mixed reviews if 2 √(p1tD1) <q≤p1+tD1*)
        p2N = p1;(*The second-period price under completely negative reviews*)
        D2P = (2 qo - p1 - t D1)/(2 t);(*The second-period demand under completely positive reviews*)
        D2Mr = 0;(*The second-
         period demand if keeping prices consistent under mixed reviews*)
        D2Md = (Q - 2 t D1)/(2 t);(*The second-period demand if marking down under mixed reviews*)
        D2N = 0;(*The second-period demand under completely negative reviews*)

In[∘]:= U1 = Simplify[qo - p1 - t D1 + δc Integrate[(p1 - p2Md) / (2 qo), {Q, 2 √(p1 t D1), p1 + t D1}]];
        (*Consumers' expected utility purchasing in the first period*)
        U2 = Simplify[δc ((2 qo - p1 - t D1)/(2 qo) ((2 qo + p1 + t D1)/2 - p2P - t D1) +
              Integrate[(Q - p2Md - t D1) / (2 qo), {Q, 2 √(p1 t D1), p1 + t D1}])];
         (*Consumers' expected utility purchasing in the second period*)

In[∘]:= Simplify[Solve[U1 == U2, D1, Reals], t > 2 qo > 0 && 0 < p1 < qo && 0 < D1 < 1 && 0 < δc < 1]
```

$$\Big\{\Big\{D_1 \to \text{Root}\Big[-p_1^2\, q_o^2 + 2\, p_1\, q_o^3 - q_o^4 + 2\, p_1^2\, q_o^2\, \delta_c - 4\, p_1\, q_o^3\, \delta_c + 2\, q_o^4\, \delta_c + t^3\, \#1^3\, p_1\, \delta_c^2 - p_1^2\, q_o^2\, \delta_c^2 + 2\, p_1\, q_o^3\, \delta_c^2 - q_o^4\, \delta_c^2 + \#1^2\, \big(-t^2\, q_o^2 + 2\, t^2\, p_1\, q_o\, \delta_c + 2\, t^2\, q_o^2\, \delta_c + t^2\, p_1^2\, \delta_c^2 - 2\, t^2\, p_1\, q_o\, \delta_c^2 - t^2\, q_o^2\, \delta_c^2\big) + \#1\, \big(-2\, t\, p_1\, q_o^2 + 2\, t\, q_o^3 + 2\, t\, p_1^2\, q_o\, \delta_c + 2\, t\, p_1\, q_o^2\, \delta_c - 4\, t\, q_o^3\, \delta_c + t\, p_1^3\, \delta_c^2 - 2\, t\, p_1^2\, q_o\, \delta_c^2 + 2\, t\, q_o^3\, \delta_c^2\big)\ \&,\ 1\Big]$$
$$\text{if Root}\Big[4\, p_1\, q_o^4 - 4\, q_o^5 + \#1^2\, \big(20\, p_1^3\, q_o^2 - 12\, p_1^2\, q_o^3 + 15\, p_1\, q_o^4 - 24\, q_o^5\big) + \#1^4\, \big(3\, p_1^5 - 16\, p_1^4\, q_o + 34\, p_1^3\, q_o^2 - 36\, p_1^2\, q_o^3 + 19\, p_1\, q_o^4 - 4\, q_o^5\big) + \#1^3\, \big(16\, p_1^4\, q_o - 54\, p_1^3\, q_o^2 + 60\, p_1^2\, q_o^3 - 34\, p_1\, q_o^4 + 16\, q_o^5\big) + \#1\, \big(-12\, p_1^2\, q_o^3 - 4\, p_1\, q_o^4 + 16\, q_o^5\big)\ \&,\ 1\Big] \neq \delta_c\Big\},$$

$$\Big\{D_1 \to \text{Root}\Big[-p_1^2\, q_o^2 + 2\, p_1\, q_o^3 - q_o^4 + 2\, p_1^2\, q_o^2\, \delta_c - 4\, p_1\, q_o^3\, \delta_c + 2\, q_o^4\, \delta_c + t^3\, \#1^3\, p_1\, \delta_c^2 - p_1^2\, q_o^2\, \delta_c^2 + 2\, p_1\, q_o^3\, \delta_c^2 - q_o^4\, \delta_c^2 + \#1^2\, \big(-t^2\, q_o^2 + 2\, t^2\, p_1\, q_o\, \delta_c + 2\, t^2\, q_o^2\, \delta_c + t^2\, p_1^2\, \delta_c^2 - 2\, t^2\, p_1\, q_o\, \delta_c^2 - t^2\, q_o^2\, \delta_c^2\big) + \#1\, \big(-2\, t\, p_1\, q_o^2 + 2\, t\, q_o^3 + 2\, t\, p_1^2\, q_o\, \delta_c + 2\, t\, p_1\, q_o^2\, \delta_c - 4\, t\, q_o^3\, \delta_c + t\, p_1^3\, \delta_c^2 - 2\, t\, p_1^2\, q_o\, \delta_c^2 + 2\, t\, q_o^3\, \delta_c^2\big)\ \&,\ 2\Big]\ \text{if}$$
$$\delta_c > \text{Root}\Big[4\, p_1\, q_o^4 - 4\, q_o^5 + \#1^2\, \big(20\, p_1^3\, q_o^2 - 12\, p_1^2\, q_o^3 + 15\, p_1\, q_o^4 - 24\, q_o^5\big) + \#1^4\, \big(3\, p_1^5 - 16\, p_1^4\, q_o + 34\, p_1^3\, q_o^2 - 36\, p_1^2\, q_o^3 + 19\, p_1\, q_o^4 - 4\, q_o^5\big) + \#1^3\, \big(16\, p_1^4\, q_o - 54\, p_1^3\, q_o^2 + 60\, p_1^2\, q_o^3 - 34\, p_1\, q_o^4 + 16\, q_o^5\big) + \#1\, \big(-12\, p_1^2\, q_o^3 - 4\, p_1\, q_o^4 + 16\, q_o^5\big)\ \&,\ 2\Big]\Big\},$$

$$\Big\{D_1 \to \text{Root}\Big[-p_1^2\, q_o^2 + 2\, p_1\, q_o^3 - q_o^4 + 2\, p_1^2\, q_o^2\, \delta_c - 4\, p_1\, q_o^3\, \delta_c + 2\, q_o^4\, \delta_c + t^3\, \#1^3\, p_1\, \delta_c^2 - p_1^2\, q_o^2\, \delta_c^2 + 2\, p_1\, q_o^3\, \delta_c^2 - q_o^4\, \delta_c^2 + \#1^2\, \big(-t^2\, q_o^2 + 2\, t^2\, p_1\, q_o\, \delta_c + 2\, t^2\, q_o^2\, \delta_c + t^2\, p_1^2\, \delta_c^2 - 2\, t^2\, p_1\, q_o\, \delta_c^2 - t^2\, q_o^2\, \delta_c^2\big) + \#1\, \big(-2\, t\, p_1\, q_o^2 + 2\, t\, q_o^3 + 2\, t\, p_1^2\, q_o\, \delta_c + 2\, t\, p_1\, q_o^2\, \delta_c - 4\, t\, q_o^3\, \delta_c + t\, p_1^3\, \delta_c^2 - 2\, t\, p_1^2\, q_o\, \delta_c^2 + 2\, t\, q_o^3\, \delta_c^2\big)\ \&,\ 3\Big]\ \text{if}$$
$$\delta_c > \text{Root}\Big[4\, p_1\, q_o^4 - 4\, q_o^5 + \#1^2\, \big(20\, p_1^3\, q_o^2 - 12\, p_1^2\, q_o^3 + 15\, p_1\, q_o^4 - 24\, q_o^5\big) + \#1^4\, \big(3\, p_1^5 - 16\, p_1^4\, q_o + 34\, p_1^3\, q_o^2 - 36\, p_1^2\, q_o^3 + 19\, p_1\, q_o^4 - 4\, q_o^5\big) + \#1^3\, \big(16\, p_1^4\, q_o - 54\, p_1^3\, q_o^2 + 60\, p_1^2\, q_o^3 - 34\, p_1\, q_o^4 + 16\, q_o^5\big) + \#1\, \big(-12\, p_1^2\, q_o^3 - 4\, p_1\, q_o^4 + 16\, q_o^5\big)\ \&,\ 2\Big]\Big\}\Big\}$$

(*There are 3 solutions of D_1,
we then check each solution if it satisfies conditions*)

(*Solution 1*)

```
D1 = Root[
    -p1^2 qo^2 + 2 p1 qo^3 - qo^4 + 2 p1^2 qo^2 δc - 4 p1 qo^3 δc + 2 qo^4 δc + t^3 #1^3 p1 δc^2 - p1^2 qo^2 δc^2 + 2 p1 qo^3 δc^2 - qo^4 δc^2 +
      #1^2 (-t^2 qo^2 + 2 t^2 p1 qo δc + 2 t^2 qo^2 δc + t^2 p1^2 δc^2 - 2 t^2 p1 qo δc^2 - t^2 qo^2 δc^2) + #1 (-2 t p1 qo^2 +
          2 t qo^3 + 2 t p1^2 qo δc + 2 t p1 qo^2 δc - 4 t qo^3 δc + t p1^3 δc^2 - 2 t p1^2 qo δc^2 + 2 t qo^3 δc^2) &, 1];
Simplify[Reduce[Root[4 p1 qo^4 - 4 qo^5 + #1^2 (20 p1^3 qo^2 - 12 p1^2 qo^3 + 15 p1 qo^4 - 24 qo^5) +
         #1^4 (3 p1^5 - 16 p1^4 qo + 34 p1^3 qo^2 - 36 p1^2 qo^3 + 19 p1 qo^4 - 4 qo^5) +
         #1^3 (16 p1^4 qo - 54 p1^3 qo^2 + 60 p1^2 qo^3 - 34 p1 qo^4 + 16 qo^5) + #1 (-12 p1^2 qo^3 - 4 p1 qo^4 + 16 qo^5) &, 1] ≠
      δc && (2 qo - t D1)/3 ≤ p1 ≤ (2 qo + t D1)/3 && t D1 < p1 ≤ 4 t D1 && 0 < p1 ≤ 16 t D1 &&
    p1 < qo && D1 > 0 && t > 2 qo > 0 && 0 < δc < 1, Reals]]
```

Out[•]=

$$t > 2 q_o \,\&\&\, p_1 > 0 \,\&\&\, \left(\left(0 < \delta_c \le \frac{2 q_o (-5 p_1 + 4 q_o)}{3 p_1^2 - 10 p_1 q_o + 8 q_o^2} \,\&\&\, \frac{11 p_1}{8} < q_o \le \frac{13 p_1}{8}\right) \,||\, \left(0 < \delta_c \le 2 \sqrt{-\frac{p_1 (3 p_1 - 2 q_o) q_o^2 \left(2 p_1^2 - 3 p_1 q_o + q_o^2\right)^2}{\left(21 p_1^4 - 32 p_1^3 q_o + 22 p_1^2 q_o^2 - 8 p_1 q_o^3 + q_o^4\right)^2}} + \frac{q_o \left(6 p_1^3 - 3 p_1^2 q_o - 2 p_1 q_o^2 + q_o^3\right)}{21 p_1^4 - 32 p_1^3 q_o + 22 p_1^2 q_o^2 - 8 p_1 q_o^3 + q_o^4} \,\&\&\, \frac{13 p_1}{8} < q_o < 2 p_1\right) \,||\, \left(0 < \delta_c \le 2 \sqrt{\frac{p_1 (3 p_1 - 2 q_o) q_o^2 \left(8 p_1^2 - 10 p_1 q_o + 3 q_o^2\right)^2}{(p_1 - q_o)^4 \left(-39 p_1^2 + 14 p_1 q_o + 9 q_o^2\right)^2}} + \frac{q_o \left(-12 p_1^3 + p_1^2 q_o + 18 p_1 q_o^2 - 9 q_o^3\right)}{(p_1 - q_o)^2 \left(39 p_1^2 - 14 p_1 q_o - 9 q_o^2\right)} \,\&\&\, \frac{4 p_1}{3} < q_o \le \frac{11 p_1}{8}\right)\right)$$

```
(*Solution 2*)
```

In[•]:=
```
D1 = Root[
    -p1^2 qo^2 + 2 p1 qo^3 - qo^4 + 2 p1^2 qo^2 δc - 4 p1 qo^3 δc + 2 qo^4 δc + t^3 #1^3 p1 δc^2 - p1^2 qo^2 δc^2 + 2 p1 qo^3 δc^2 - qo^4 δc^2 +
      #1^2 (-t^2 qo^2 + 2 t^2 p1 qo δc + 2 t^2 qo^2 δc + t^2 p1^2 δc^2 - 2 t^2 p1 qo δc^2 - t^2 qo^2 δc^2) + #1 (-2 t p1 qo^2 +
          2 t qo^3 + 2 t p1^2 qo δc + 2 t p1 qo^2 δc - 4 t qo^3 δc + t p1^3 δc^2 - 2 t p1^2 qo δc^2 + 2 t qo^3 δc^2) &, 2];
Simplify[Reduce[δc > Root[4 p1 qo^4 - 4 qo^5 + #1^2 (20 p1^3 qo^2 - 12 p1^2 qo^3 + 15 p1 qo^4 - 24 qo^5) +
          #1^4 (3 p1^5 - 16 p1^4 qo + 34 p1^3 qo^2 - 36 p1^2 qo^3 + 19 p1 qo^4 - 4 qo^5) + #1^3
           (16 p1^4 qo - 54 p1^3 qo^2 + 60 p1^2 qo^3 - 34 p1 qo^4 + 16 qo^5) + #1 (-12 p1^2 qo^3 - 4 p1 qo^4 + 16 qo^5) &, 2] &&
    (2 qo - t D1)/3 ≤ p1 ≤ (2 qo + t D1)/3 && t D1 < p1 ≤ 4 t D1 && 0 < p1 ≤ 16 t D1 && p1 < qo &&
    D1 > 0 && t > 2 qo > 0 && 0 < δc < 1, Reals]]
```

Out[•]=
```
False
```

```
(*Solution 3*)
```

In[∘]:=
```
D₁ = Root[
   -p₁² q₀² + 2 p₁ q₀³ - q₀⁴ + 2 p₁² q₀² δc - 4 p₁ q₀³ δc + 2 q₀⁴ δc + t³ #1³ p₁ δc² - p₁² q₀² δc² + 2 p₁ q₀³ δc² - q₀⁴ δc² +
    #1² (-t² q₀² + 2 t² p₁ q₀ δc + 2 t² q₀² δc + t² p₁² δc² - 2 t² p₁ q₀ δc² - t² q₀² δc²) + #1 (-2 t p₁ q₀² +
       2 t q₀³ + 2 t p₁² q₀ δc + 2 t p₁ q₀² δc - 4 t q₀³ δc + t p₁³ δc² - 2 t p₁² q₀ δc² + 2 t q₀³ δc²) &, 3];
Simplify[Reduce[δc > Root[4 p₁ q₀⁴ - 4 q₀⁵ + #1² (20 p₁³ q₀² - 12 p₁² q₀³ + 15 p₁ q₀⁴ - 24 q₀⁵) +
       #1⁴ (3 p₁⁵ - 16 p₁⁴ q₀ + 34 p₁³ q₀² - 36 p₁² q₀³ + 19 p₁ q₀⁴ - 4 q₀⁵) + #1³
        (16 p₁⁴ q₀ - 54 p₁³ q₀² + 60 p₁² q₀³ - 34 p₁ q₀⁴ + 16 q₀⁵) + #1 (-12 p₁² q₀³ - 4 p₁ q₀⁴ + 16 q₀⁵) &, 2] &&
   (2 q₀ - t D₁)/3 ≤ p₁ ≤ (2 q₀ + t D₁)/3 && t D₁ < p₁ ≤ 4 t D₁ && 0 < p₁ ≤ 16 t D₁ && p₁ < q₀ &&
   D₁ > 0 && t > 2 q₀ > 0 && 0 < δc < 1, Reals]]
```

Out[∘]=
```
False
```

```
(*Hence, solution 1 is the feasible solution*)

D₁ = Root[
   -p₁² q₀² + 2 p₁ q₀³ - q₀⁴ + 2 p₁² q₀² δc - 4 p₁ q₀³ δc + 2 q₀⁴ δc + t³ #1³ p₁ δc² - p₁² q₀² δc² + 2 p₁ q₀³ δc² - q₀⁴ δc² +
    #1² (-t² q₀² + 2 t² p₁ q₀ δc + 2 t² q₀² δc + t² p₁² δc² - 2 t² p₁ q₀ δc² - t² q₀² δc²) + #1 (-2 t p₁ q₀² +
       2 t q₀³ + 2 t p₁² q₀ δc + 2 t p₁ q₀² δc - 4 t q₀³ δc + t p₁³ δc² - 2 t p₁² q₀ δc² + 2 t q₀³ δc²) &, 1];
Π = p₁ D₁ + (2 q₀ - p₁ - t D₁)/(2 q₀) p2P D2P +
   Integrate[(p2Md D2Md - D₁ (p₁ - p2Md)) / (2 q₀), {Q, 2 Sqrt[p₁ t D₁], p₁ + t D₁}];
CS = Integrate[
     (Integrate[Q - p₁ - t x + δ Integrate[(p₁ - p2Md) / (2 q₀), {Q, 2 Sqrt[p₁ t D₁], p₁ + t D₁}],
        {x, 0, D₁}]) / (2 q₀), {Q, 0, 2 q₀}] +
   Integrate[(Integrate[δ (Q - p2P - t x), {x, D₁, D₁ + D2P}]) / (2 q₀), {Q, p₁ + t D₁, 2 q₀}] +
   Integrate[(Integrate[δ (Q - p2Md - t x), {x, D₁, D₁ + D2Md}]) / (2 q₀),
    {Q, 2 Sqrt[p₁ t D₁], p₁ + t D₁}];
```

In[•]:=
```
t = 2.1;
qo = 1;
results = {};
Results = {};
For[δ = 0.001, δ < 1, δ += 0.001,
  For[p1 = 0.001, p1 < 1, p1 += 0.001,
   If[Element[D1, Reals] && Element[Π, Reals] &&
     ((2 qo - t D1)/3 ≤ p1 ≤ (2 qo + t D1)/3 && t D1 < p1 ≤ 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1),
    AppendTo[results, {Π, p1, CS}],
    AppendTo[results, {0, 0, 0}]]];
  {maxVal, maxP, maxCS} = Last@MaximalBy[results, First];
  AppendTo[Results, {δ, maxVal, maxP, maxCS}];
  results = {}
 ];
If[Results == {}, Print["No valid results found."], TableForm[Results,
  TableDirections → Row, TableHeadings → {None, {"δc", "Π", "p1", "CS"}}]]
```

(* Combiantion 10. This scenario does not exist as no p_1 satisfies the stated conditions*)

(* Combiantion 11. The conditions are $\frac{2q_o-tD_1}{3} \le p_1 \le \frac{2q_o+tD_1}{3}$, $p_1>4tD_1$, and $p_1 \le 16tD_1$*)

In[•]:=
$p_{2P} = p_1$; (*The second-period price under completely positive reviews*)

$p_{2M} = \frac{2 p_1 + t D_1}{4}$; (*The second-period price under mixed reviews*)

$p_{2N} = p_1$; (*The second-period price under completely negative reviews*)

$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2 t}$; (*The second-period demand under completely positive reviews*)

$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t}$; (*The second-period demand under mixed reviews*)

$D_{2N} = 0$; (*The second-period demand under completely negative reviews*)

In[•]:=
$U_1 = q_o - p_1 - t D_1 + \delta_c \frac{t D_1}{2 q_o} (p_1 - p_{2M})$;

(*Consumers' expected utility purchasing in the first period*)

$U_2 = \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right)\right)$;

(*Consumers' expected utility purchasing in the second period*)

In[•]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[•]=
$$\left\{\left\{D_1 \to \frac{-4 p_1 q_o (-1 + \delta_c) + 4 q_o^2 (-1 + \delta_c) + p_1^2 \delta_c}{4 t q_o (-1 + \delta_c) - 2 t p_1 \delta_c}\right\}\right\}$$

```
In[*]:= D1 = (-4 p1 qo (-1 + δc) + 4 qo^2 (-1 + δc) + p1^2 δc)/(4 t qo (-1 + δc) - 2 t p1 δc);(*The response function of D1*)
Simplify[Reduce[
  (2 qo - t D1)/3 ≤ p1 ≤ (2 qo + t D1)/3 && p1 > 4 t D1 && p1 ≤ 16 t D1 && D1 > 0 && t > 2 qo > 0 && 0 < δc < 1]
  (*Check if D1 satisfies conditions*)]

Out[*]=
t > 2 qo && p1 > 0 && (2 qo (-5 p1 + 4 qo))/(3 p1^2 - 10 p1 qo + 8 qo^2) < δc &&
 ((11 p1 < 8 qo && 32 qo ≤ 47 p1 && δc ≤ (4 qo (-4 p1 + 3 qo))/(7 p1^2 - 20 p1 qo + 12 qo^2)) ||
  (47 p1 < 32 qo && 32 qo ≤ 49 p1 && δc ≤ (-34 p1 qo + 32 qo^2)/(9 p1^2 - 34 p1 qo + 32 qo^2)) ||
  (49 p1 < 32 qo && 8 qo < 13 p1 && δc ≤ (4 qo (-2 p1 + qo))/(5 p1^2 - 12 p1 qo + 4 qo^2)))

(*Hence, the response function of D1 satisfies conditionsis and is given by*)

In[*]:= D1 = (-4 p1 qo (-1 + δc) + 4 qo^2 (-1 + δc) + p1^2 δc)/(4 t qo (-1 + δc) - 2 t p1 δc);
Π = Simplify[p1 D1 + (2 qo - p1 - t D1)/(2 qo) p2P D2P + (t D1)/(2 qo) (p2M D2M - D1 (p1 - p2M))];
(*The firm's total profit function*)
CS =
  Integrate[(Integrate[Q - p1 - t x + (t D1)/(2 qo) δc (p1 - p2M), {x, 0, D1}])/(2 qo), {Q, 0, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2P - t x), {x, D1, D1 + D2P}]) / (2 qo), {Q, p1 + t D1, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2M - t x), {x, D1, D1 + D2M}]) / (2 qo),
    {Q, p1, p1 + t D1}];(*Consumer surplus*)
```

```
t = 2.1;
qo = 1;
results = {};
Results = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[p1 = 0.001, p1 < 1, p1 += 0.001,
   If[Element[D1, Reals] && Element[Π, Reals] &&
     ((2 qo - t D1)/3 ≤ p1 ≤ (2 qo + t D1)/3 && p1 > 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1),
    AppendTo[results, {Π, p1, CS}],
    AppendTo[results, {0, 0, 0}]]];
  {maxVal, maxP, maxCS} = Last@MaximalBy[results, First];
  AppendTo[Results, {δc, maxVal, maxP, maxCS}];
  results = {}
 ];
If[Results == {}, Print["No valid results found."], TableForm[Results,
  TableDirections → Row, TableHeadings → {None, {"δc", "Π", "p1", "CS"}}]]
```

(* Combiantion 12. The conditions are $\frac{2q_o-tD_1}{3} \le p_1 \le \frac{2q_o+tD_1}{3}$, $p_1>4tD_1$, and $p_1>16tD_1$*)

In[∘]:=
```
p2P = p1; (*The second-period price under completely positive reviews*)
```
$p_{2M} = \frac{2 p_1 + t D_1}{4}$; (*The second-period price under mixed reviews*)

$p_{2N} = \frac{p_1}{4}$; (*The second-period price under completely negative reviews*)

$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2 t}$; (*The second-period demand under completely positive reviews*)

$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t}$; (*The second-period demand under mixed reviews*)

$D_{2N} = \frac{p_1 - 4 t D_1}{4 t}$; (*The second-period demand under completely negative reviews*)

In[∘]:= $U_1 = q_o - p_1 - t D_1 + \delta_c \left(\frac{t D_1}{2 q_o} (p_1 - p_{2M}) + \frac{p_1}{2 q_o} (p_1 - p_{2N})\right)$;

(*Consumers' expected utility purchasing in the first period*)

$U_2 = \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right) + \frac{p_1}{2 q_o} \left(\frac{p_1}{2} - p_{2N} - t D_1\right)\right)$;

(*Consumers' expected utility purchasing in the second period*)

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1], $t > 0$ && $q_o > 0$ && $0 < \delta_c < 1$]

Out[∘]=

$\left\{\left\{D_1 \to \frac{-p_1 + q_o}{t}\right\}\right\}$

In[∘]:= $D_1 = \frac{-p_1 + q_o}{t}$; (*The response function of D_1*)

Reduce[$\frac{2 q_o - t D_1}{3} \le p_1 \le \frac{2 q_o + t D_1}{3}$ && $p_1 > t D_1$ && $p_1 > 16 t D_1$ && $D_1 > 0$ && $t > 2 q_o > 0$ && $0 < \delta_c < 1$] (*We check if D_1 satisfies conditions*)

Out[∘]=

False

(*Hence, there are no feasible solutions for combination 12*)

(* Combiantion 13. This scenario does not exist as no p_1 satisfies the stated conditions*)

(* Combiantion 14. This scenario does not exist as no p_1 satisfies the stated conditions*)

(* Combiantion 15. The conditions are $p_1 > \frac{2q_o+tD_1}{3}$, $tD_1<p_1\le 4tD_1$, and $p_1\le 16tD_1$*)

```
In[]:= p2P = (2 qo + p1 + t D1)/4; (*The second-period price under completely positive reviews*)
p2Mr = p1; (*The second-period price under mixed reviews if p1<q≤2 Sqrt[p1tD1] *)
p2Md = Q/2; (*The second-period price under mixed reviews if 2 Sqrt[p1tD1] <q≤p1+tD1*)
p2N = p1; (*The second-period price under completely negative reviews*)
D2P = (2 qo + p1 - 3 t D1)/(4 t); (*The second-period demand under completely positive reviews*)
D2Mr = 0; (*The second-period demand under mixed reviews*)
D2Md = (Q - 2 t D1)/(2 t); (*The second-period demand under mixed reviews*)
D2N = 0; (*The second-period demand under completely negative reviews*)

In[]:= U1 = Simplify[qo - p1 - t D1 + δc
      ((2 qo - p1 - t D1)/(2 qo) (p1 - p2P) + Integrate[(p1 - p2Md) / (2 qo), {Q, 2 Sqrt[p1 t D1], p1 + t D1}])];
  (*Consumers' expected utility purchasing in the first period*)
U2 = Simplify[δc ((2 qo - p1 - t D1)/(2 qo) ((2 qo + p1 + t D1)/2 - p2P - t D1) +
      Integrate[(Q - p2Md - t D1) / (2 qo), {Q, 2 Sqrt[p1 t D1], p1 + t D1}])];
  (*Consumers' expected utility purchasing in the second period*)

In[]:= Simplify[Solve[U1 == U2, D1], t > 2 qo > 0 && 0 < δc < 1]
```

Out[]=

$$\Big\{\Big\{D_1 \to \frac{1}{6\,t\,p_1\,\delta_c^2}\Big(2\,q_o^2\,(-1+\delta_c)^2 + 4\,p_1\,q_o\,(-1+\delta_c)\,\delta_c - 2\,p_1^2\,\delta_c^2 + \Big(2\times 2^{1/3}\,\big(q_o^4\,(-1+\delta_c)^4 + 2\,p_1^3\,q_o\,(-1+\delta_c)\,\delta_c^3 - 2\,p_1^4\,\delta_c^4 + 2\,p_1^2\,q_o^2\,\delta_c^2\,(4-5\,\delta_c+\delta_c^2) - 2\,p_1\,q_o^3\,\delta_c\,(2-3\,\delta_c+\delta_c^3)\big)\Big)\Big/$$
$$\Big(2\,q_o^6\,(-1+\delta_c)^6 - 24\,p_1^5\,q_o\,(-1+\delta_c)\,\delta_c^5 + 7\,p_1^6\,\delta_c^6 - 6\,p_1\,q_o^5\,(-1+\delta_c)^4\,\delta_c\,(2+\delta_c) - 2\,p_1^3\,q_o^3\,(-1+\delta_c)^2\,\delta_c^3\,(23+13\,\delta_c) + 9\,p_1^2\,q_o^4\,(-1+\delta_c)^2\,\delta_c^2\,(4-2\,\delta_c+\delta_c^2) + 18\,p_1^4\,q_o^2\,\delta_c^4\,(1-3\,\delta_c+2\,\delta_c^2) + 3\,\sqrt{3}\,\delta_c^3\,\sqrt{\big(p_1^3\,(q_o^2\,(-1+\delta_c)+p_1^2\,\delta_c)^2\,\big(-4\,q_o^5\,(-1+\delta_c)^4 - 16\,p_1^4\,q_o\,(-1+\delta_c)\,\delta_c^3 + 3\,p_1^5\,\delta_c^4 - 12\,p_1^2\,q_o^3\,(-1+\delta_c)^2\,\delta_c\,(1+3\,\delta_c) + 2\,p_1^3\,q_o^2\,\delta_c^2\,(10-27\,\delta_c+17\,\delta_c^2) + p_1\,q_o^4\,(-1+\delta_c)^2\,(4+4\,\delta_c+19\,\delta_c^2)\big)\big)}\Big)^{1/3} +$$
$$2^{2/3}\Big(2\,q_o^6\,(-1+\delta_c)^6 - 24\,p_1^5\,q_o\,(-1+\delta_c)\,\delta_c^5 + 7\,p_1^6\,\delta_c^6 - 6\,p_1\,q_o^5\,(-1+\delta_c)^4\,\delta_c\,(2+\delta_c) - 2\,p_1^3\,q_o^3\,(-1+\delta_c)^2\,\delta_c^3\,(23+13\,\delta_c) + 9\,p_1^2\,q_o^4\,(-1+\delta_c)^2\,\delta_c^2\,(4-2\,\delta_c+\delta_c^2) + 18\,p_1^4\,q_o^2\,\delta_c^4\,(1-3\,\delta_c+2\,\delta_c^2) + 3\,\sqrt{3}\,\delta_c^3\,\sqrt{\big(p_1^3\,(q_o^2\,(-1+\delta_c)+p_1^2\,\delta_c)^2\,\big(-4\,q_o^5\,(-1+\delta_c)^4 - 16\,p_1^4\,q_o\,(-1+\delta_c)\,\delta_c^3 + 3\,p_1^5\,\delta_c^4 - 12\,p_1^2\,q_o^3\,(-1+\delta_c)^2\,\delta_c\,(1+3\,\delta_c) + 2\,p_1^3\,q_o^2\,\delta_c^2\,(10-27\,\delta_c+17\,\delta_c^2) + p_1\,q_o^4\,(-1+\delta_c)^2\,(4+4\,\delta_c+19\,\delta_c^2)\big)\big)}\Big)^{1/3}\Big)\Big\},$$

$$\Big\{D_1 \to \frac{1}{12\,t\,p_1\,\delta_c^2}\Big(4\,q_o^2\,(-1+\delta_c)^2 + 8\,p_1\,q_o\,(-1+\delta_c)\,\delta_c - 4\,p_1^2\,\delta_c^2 - \Big(2\,i\,2^{1/3}\,(-i+\sqrt{3})\,\big(q_o^4\,(-1+\delta_c)^4 + 2\,p_1^3\,q_o\,(-1+\delta_c)\,\delta_c^3 - 2\,p_1^4\,\delta_c^4 + 2\,p_1^2\,q_o^2\,\delta_c^2\,(4-5\,\delta_c+\delta_c^2) - 2\,p_1\,q_o^3\,\delta_c\,(2-3\,\delta_c+\delta_c^3)\big)\Big)\Big/$$
$$\Big(2\,q_o^6\,(-1+\delta_c)^6 - 24\,p_1^5\,q_o\,(-1+\delta_c)\,\delta_c^5 + 7\,p_1^6\,\delta_c^6 - 6\,p_1\,q_o^5\,(-1+\delta_c)^4\,\delta_c\,(2+\delta_c) -$$

$$2 p_1^3 q_o^3 (-1+\delta_c)^2 \delta_c^3 (23+13\delta_c) + 9 p_1^2 q_o^4 (-1+\delta_c)^2 \delta_c^2 \left(4-2\delta_c+\delta_c^2\right) +$$
$$18 p_1^4 q_o^2 \delta_c^4 \left(1-3\delta_c+2\delta_c^2\right) + 3\sqrt{3}\,\delta_c^3 \sqrt{\left(p_1^3 \left(q_o^2 (-1+\delta_c) + p_1^2 \delta_c\right)^2 \left(-4 q_o^5 (-1+\delta_c)^4 - 16 p_1^4 q_o (-1+\delta_c)\delta_c^3 + 3 p_1^5 \delta_c^4 - 12 p_1^2 q_o^3 (-1+\delta_c)^2 \delta_c (1+3\delta_c) + 2 p_1^3 q_o^2 \delta_c^2 \left(10-27\delta_c+17\delta_c^2\right) + p_1 q_o^4 (-1+\delta_c)^2 \left(4+4\delta_c+19\delta_c^2\right)\right)\right)}\Bigg)^{1/3} +$$
$$\mathbb{i}\, 2^{2/3} \left(\mathbb{i}+\sqrt{3}\right) \Bigg(2 q_o^6 (-1+\delta_c)^6 - 24 p_1^5 q_o (-1+\delta_c)\delta_c^5 + 7 p_1^6 \delta_c^6 - 6 p_1 q_o^5 (-1+\delta_c)^4 \delta_c (2+\delta_c) - 2 p_1^3 q_o^3 (-1+\delta_c)^2 \delta_c^3 (23+13\delta_c) + 9 p_1^2 q_o^4 (-1+\delta_c)^2 \delta_c^2 \left(4-2\delta_c+\delta_c^2\right) +$$
$$18 p_1^4 q_o^2 \delta_c^4 \left(1-3\delta_c+2\delta_c^2\right) + 3\sqrt{3}\,\delta_c^3 \sqrt{\left(p_1^3 \left(q_o^2 (-1+\delta_c) + p_1^2 \delta_c\right)^2 \left(-4 q_o^5 (-1+\delta_c)^4 - 16 p_1^4 q_o (-1+\delta_c)\delta_c^3 + 3 p_1^5 \delta_c^4 - 12 p_1^2 q_o^3 (-1+\delta_c)^2 \delta_c (1+3\delta_c) + 2 p_1^3 q_o^2 \delta_c^2 \left(10-27\delta_c+17\delta_c^2\right) + p_1 q_o^4 (-1+\delta_c)^2 \left(4+4\delta_c+19\delta_c^2\right)\right)\right)}\Bigg)^{1/3}\Bigg)\Bigg\},$$

$$\Bigg\{D_1 \to \frac{1}{12\, t\, p_1 \delta_c^2} \Bigg(4 q_o^2 (-1+\delta_c)^2 + 8 p_1 q_o (-1+\delta_c)\delta_c - 4 p_1^2 \delta_c^2 +$$
$$\left(2\,\mathbb{i}\, 2^{1/3} \left(\mathbb{i}+\sqrt{3}\right) \left(q_o^4 (-1+\delta_c)^4 + 2 p_1^3 q_o (-1+\delta_c)\delta_c^3 - 2 p_1^4 \delta_c^4 + 2 p_1^2 q_o^2 \delta_c^2 \left(4-5\delta_c+\delta_c^2\right) - 2 p_1 q_o^3 \delta_c \left(2-3\delta_c+\delta_c^3\right)\right)\right) \Big/$$
$$\Bigg(2 q_o^6 (-1+\delta_c)^6 - 24 p_1^5 q_o (-1+\delta_c)\delta_c^5 + 7 p_1^6 \delta_c^6 - 6 p_1 q_o^5 (-1+\delta_c)^4 \delta_c (2+\delta_c) - 2 p_1^3 q_o^3 (-1+\delta_c)^2 \delta_c^3 (23+13\delta_c) + 9 p_1^2 q_o^4 (-1+\delta_c)^2 \delta_c^2 \left(4-2\delta_c+\delta_c^2\right) +$$
$$18 p_1^4 q_o^2 \delta_c^4 \left(1-3\delta_c+2\delta_c^2\right) + 3\sqrt{3}\,\delta_c^3 \sqrt{\left(p_1^3 \left(q_o^2 (-1+\delta_c) + p_1^2 \delta_c\right)^2 \left(-4 q_o^5 (-1+\delta_c)^4 - 16 p_1^4 q_o (-1+\delta_c)\delta_c^3 + 3 p_1^5 \delta_c^4 - 12 p_1^2 q_o^3 (-1+\delta_c)^2 \delta_c (1+3\delta_c) + 2 p_1^3 q_o^2 \delta_c^2 \left(10-27\delta_c+17\delta_c^2\right) + p_1 q_o^4 (-1+\delta_c)^2 \left(4+4\delta_c+19\delta_c^2\right)\right)\right)}\Bigg)^{1/3} -$$
$$2^{2/3} \left(1+\mathbb{i}\sqrt{3}\right) \Bigg(2 q_o^6 (-1+\delta_c)^6 - 24 p_1^5 q_o (-1+\delta_c)\delta_c^5 + 7 p_1^6 \delta_c^6 - 6 p_1 q_o^5 (-1+\delta_c)^4 \delta_c (2+\delta_c) - 2 p_1^3 q_o^3 (-1+\delta_c)^2 \delta_c^3 (23+13\delta_c) + 9 p_1^2 q_o^4 (-1+\delta_c)^2 \delta_c^2 \left(4-2\delta_c+\delta_c^2\right) +$$
$$18 p_1^4 q_o^2 \delta_c^4 \left(1-3\delta_c+2\delta_c^2\right) + 3\sqrt{3}\,\delta_c^3 \sqrt{\left(p_1^3 \left(q_o^2 (-1+\delta_c) + p_1^2 \delta_c\right)^2 \left(-4 q_o^5 (-1+\delta_c)^4 - 16 p_1^4 q_o (-1+\delta_c)\delta_c^3 + 3 p_1^5 \delta_c^4 - 12 p_1^2 q_o^3 (-1+\delta_c)^2 \delta_c (1+3\delta_c) + 2 p_1^3 q_o^2 \delta_c^2 \left(10-27\delta_c+17\delta_c^2\right) + p_1 q_o^4 (-1+\delta_c)^2 \left(4+4\delta_c+19\delta_c^2\right)\right)\right)}\Bigg)^{1/3}\Bigg)\Bigg\}\Bigg\}$$

(*There are 3 solutions of D_1, but only the first solution is feasible since both the second and thrid ones are not real numbers*)

$$
\begin{aligned}
D_1 = {} & \frac{1}{6\, t\, p_1\, \delta_c^2} \\
& \Big(2\, q_o^2\, (-1+\delta_c)^2 + 4\, p_1\, q_o\, (-1+\delta_c)\, \delta_c - 2\, p_1^2\, \delta_c^2 + \Big(2 \times 2^{1/3}\, \Big(q_o^4\, (-1+\delta_c)^4 + 2\, p_1^3\, q_o\, (-1+\delta_c)\, \delta_c^3 - \\
& \quad 2\, p_1^4\, \delta_c^4 + 2\, p_1^2\, q_o^2\, \delta_c^2\, \big(4 - 5\, \delta_c + \delta_c^2\big) - 2\, p_1\, q_o^3\, \delta_c\, \big(2 - 3\, \delta_c + \delta_c^3\big)\Big)\Big) \Big/ \\
& \quad \Big(2\, q_o^6\, (-1+\delta_c)^6 - 24\, p_1^5\, q_o\, (-1+\delta_c)\, \delta_c^5 + 7\, p_1^6\, \delta_c^6 - 6\, p_1\, q_o^5\, (-1+\delta_c)^4\, \delta_c\, (2+\delta_c) - \\
& \quad 2\, p_1^3\, q_o^3\, (-1+\delta_c)^2\, \delta_c^3\, (23 + 13\, \delta_c) + 9\, p_1^2\, q_o^4\, (-1+\delta_c)^2\, \delta_c^2\, \big(4 - 2\, \delta_c + \delta_c^2\big) + \\
& \quad 18\, p_1^4\, q_o^2\, \delta_c^4\, \big(1 - 3\, \delta_c + 2\, \delta_c^2\big) + 3\, \sqrt{3}\, \delta_c^3 \sqrt{\Big(p_1^3\, \big(q_o^2\, (-1+\delta_c) + p_1^2\, \delta_c\big)^2\, \Big(-4\, q_o^5\, (-1+\delta_c)^4 -} \\
& \qquad \overline{16\, p_1^4\, q_o\, (-1+\delta_c)\, \delta_c^3 + 3\, p_1^5\, \delta_c^4 - 12\, p_1^2\, q_o^3\, (-1+\delta_c)^2\, \delta_c\, (1 + 3\, \delta_c) +} \\
& \qquad \overline{2\, p_1^3\, q_o^2\, \delta_c^2\, \big(10 - 27\, \delta_c + 17\, \delta_c^2\big) + p_1\, q_o^4\, (-1+\delta_c)^2\, \big(4 + 4\, \delta_c + 19\, \delta_c^2\big)\Big)\Big)}\,\Big)^{1/3} + \\
& 2^{2/3}\, \Big(2\, q_o^6\, (-1+\delta_c)^6 - 24\, p_1^5\, q_o\, (-1+\delta_c)\, \delta_c^5 + 7\, p_1^6\, \delta_c^6 - 6\, p_1\, q_o^5\, (-1+\delta_c)^4\, \delta_c\, (2+\delta_c) - \\
& \quad 2\, p_1^3\, q_o^3\, (-1+\delta_c)^2\, \delta_c^3\, (23 + 13\, \delta_c) + 9\, p_1^2\, q_o^4\, (-1+\delta_c)^2\, \delta_c^2\, \big(4 - 2\, \delta_c + \delta_c^2\big) + \\
& \quad 18\, p_1^4\, q_o^2\, \delta_c^4\, \big(1 - 3\, \delta_c + 2\, \delta_c^2\big) + 3\, \sqrt{3}\, \delta_c^3 \sqrt{\Big(p_1^3\, \big(q_o^2\, (-1+\delta_c) + p_1^2\, \delta_c\big)^2\, \Big(-4\, q_o^5\, (-1+\delta_c)^4 -} \\
& \qquad \overline{16\, p_1^4\, q_o\, (-1+\delta_c)\, \delta_c^3 + 3\, p_1^5\, \delta_c^4 - 12\, p_1^2\, q_o^3\, (-1+\delta_c)^2\, \delta_c\, (1 + 3\, \delta_c) +} \\
& \qquad \overline{2\, p_1^3\, q_o^2\, \delta_c^2\, \big(10 - 27\, \delta_c + 17\, \delta_c^2\big) + p_1\, q_o^4\, (-1+\delta_c)^2\, \big(4 + 4\, \delta_c + 19\, \delta_c^2\big)\Big)\Big)}\,\Big)^{1/3}\Big);
\end{aligned}
$$

$$
\begin{aligned}
\Pi = {} & p_1\, D_1 + \frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, (p_{2\,P}\, D_{2\,P} - D_1\, (p_1 - p_{2\,P})) + \\
& \mathrm{Integrate}\big[(p_{2\,Md}\, D_{2\,Md} - D_1\, (p_1 - p_{2\,Md})) / (2\, q_o),\ \big\{Q,\ 2\, \sqrt{p_1\, t\, D_1},\ p_1 + t\, D_1\big\}\big];
\end{aligned}
$$

$$
\begin{aligned}
CS = {} & \mathrm{Integrate}\Big[\\
& \Big(\mathrm{Integrate}\Big[Q - p_1 - t\, x + \delta_c\, \Big(\frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, (p_1 - p_{2\,P}) + \mathrm{Integrate}\big[(p_1 - p_{2\,Md}) / (2\, q_o), \\
& \qquad \big\{Q,\ 2\, \sqrt{p_1\, t\, D_1},\ p_1 + t\, D_1\big\}\big]\Big),\ \{x,\ 0,\ D_1\}\Big]\Big) \Big/ (2\, q_o),\ \{Q,\ 0,\ 2\, q_o\}\Big] + \\
& \mathrm{Integrate}[(\mathrm{Integrate}[\delta_c\, (Q - p_{2\,P} - t\, x),\ \{x,\ D_1,\ D_1 + D_{2\,P}\}]) / (2\, q_o), \\
& \quad \{Q,\ p_1 + t\, D_1,\ 2\, q_o\}] + \\
& \mathrm{Integrate}\big[(\mathrm{Integrate}[\delta_c\, (Q - p_{2\,Md} - t\, x),\ \{x,\ D_1,\ D_1 + D_{2\,Md}\}]) / (2\, q_o), \\
& \quad \big\{Q,\ 2\, \sqrt{p_1\, t\, D_1},\ p_1 + t\, D_1\big\}\big];
\end{aligned}
$$

```
t = 2.1;
qo = 1;
results = {};
Results = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[p1 = 0.001, p1 < 1, p1 += 0.001,
   If[Element[D1, Reals] && Element[Π, Reals] &&
     (p1 > (2 qo + t D1)/3 && t D1 < p1 ≤ 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1),
    AppendTo[results, {Π, p1, CS}],
    AppendTo[results, {0, 0, 0}]]];
  {maxVal, maxP, maxCS} = Last@MaximalBy[results, First];
  AppendTo[Results, {δc, maxVal, maxP, maxCS}];
  results = {}
 ];
If[Results == {}, Print["No valid results found."], TableForm[Results,
  TableDirections → Row, TableHeadings → {None, {"δc", "Π", "p1", "CS"}}]]
```

(* Combiantion 16. This scenario does not exist as no p_1 satisfies the stated conditions*)

(* Combiantion 17. The conditions are $p_1>\frac{2q_o+tD_1}{3}$, $p_1>4tD_1$, and $p_1\leq16tD_1$*)

In[ ]:= $p_{2P} = \frac{2 q_o + p_1 + t D_1}{4}$; (*The second-period price under completely positive reviews*)

$p_{2M} = \frac{2 p_1 + t D_1}{4}$; (*The second-period price under mixed reviews*)

$p_{2N} = p_1$; (*The second-period price under completely negative reviews*)

$D_{2P} = \frac{2 q_o + p_1 - 3 t D_1}{4 t}$; (*The second-period demand under completely positive reviews*)

$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t}$; (*The second-period demand under mixed reviews*)

$D_{2N} = 0$; (*The second-period demand under completely negative reviews*)

In[ ]:= $U_1 = q_o - p_1 - t D_1 + \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} (p_1 - p_{2P}) + \frac{t D_1}{2 q_o} (p_1 - p_{2M})\right)$;

(*Consumers' expected utility purchasing in the first period*)

$U_2 = \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right)\right)$;

(*Consumers' expected utility purchasing in the second period*)

In[ ]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[ ]=

$$\left\{\left\{D_1 \to \frac{-4 p_1 q_o (-1 + \delta_c) + 4 q_o^2 (-1 + \delta_c) + p_1^2 \delta_c}{4 t q_o (-1 + \delta_c) - 2 t p_1 \delta_c}\right\}\right\}$$

In[∘]:=
$$D_1 = \frac{-4\, p_1\, q_o\, (-1+\delta_c) + 4\, q_o^2\, (-1+\delta_c) + p_1^2\, \delta_c}{4\, t\, q_o\, (-1+\delta_c) - 2\, t\, p_1\, \delta_c};\ \text{(*The response function of } D_1\text{*)}$$

$$\text{Simplify}\left[\text{Reduce}\left[p_1 > \frac{2\, q_o + t\, D_1}{3}\ \&\&\ p_1 > 4\, t\, D_1\ \&\&\ p_1 \le 16\, t\, D_1\ \&\&\ D_1 > 0\ \&\&\ t > 2\, q_o > 0\ \&\&\ 0 < \delta_c < 1\right]\ \text{(*We check if } D_1 \text{ satisfies conditions*)}\right]$$

Out[∘]=
$$t > 2\, q_o\ \&\&\ p_1 > 0\ \&\&\ \delta_c \le \frac{-34\, p_1\, q_o + 32\, q_o^2}{9\, p_1^2 - 34\, p_1\, q_o + 32\, q_o^2}\ \&\&\ \Bigg(\left(\delta_c > 0\ \&\&\ 17\, p_1 < 16\, q_o\ \&\&\ 4\, q_o \le 5\, p_1\right)\ ||$$
$$\left(5\, p_1 < 4\, q_o\ \&\&\ \frac{2\, q_o\, (-5\, p_1 + 4\, q_o)}{3\, p_1^2 - 10\, p_1\, q_o + 8\, q_o^2} < \delta_c\ \&\&\ 8\, q_o \le 11\, p_1\right)\ ||$$
$$\left(11\, p_1 < 8\, q_o\ \&\&\ 32\, q_o < 47\, p_1\ \&\&\ \frac{4\, q_o\, (-4\, p_1 + 3\, q_o)}{7\, p_1^2 - 20\, p_1\, q_o + 12\, q_o^2} < \delta_c\right)\Bigg)$$

(*Hence, the response function of D_1 satisfies conditionsis and is given by*)

$$D_1 = \frac{-4\, p_1\, q_o\, (-1+\delta_c) + 4\, q_o^2\, (-1+\delta_c) + p_1^2\, \delta_c}{4\, t\, q_o\, (-1+\delta_c) - 2\, t\, p_1\, \delta_c};$$

$$\Pi = \text{Simplify}\left[p_1\, D_1 + \frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, (p_{2P}\, D_{2P} - D_1\, (p_1 - p_{2P})) + \frac{t\, D_1}{2\, q_o}\, (p_{2M}\, D_{2M} - D_1\, (p_1 - p_{2M}))\right];$$

(*The firm's total profit function*)

$$CS = \text{Integrate}\Bigg[\left(\text{Integrate}\left[Q - p_1 - t\, x + \delta_c\left(\frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, (p_1 - p_{2P}) + \frac{t\, D_1}{2\, q_o}\, (p_1 - p_{2M})\right), \{x, 0, D_1\}\right]\right) \Big/ (2\, q_o), \{Q, 0, 2\, q_o\}\Bigg] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c\, (Q - p_{2P} - t\, x), \{x, D_1, D_1 + D_{2P}\}]) / (2\, q_o), \{Q, p_1 + t\, D_1, 2\, q_o\}] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c\, (Q - p_{2M} - t\, x), \{x, D_1, D_1 + D_{2M}\}]) / (2\, q_o), \{Q, p_1, p_1 + t\, D_1\}];$$

```
t = 2.1;
qo = 1;
results = {};
Results = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[p1 = 0.001, p1 < 1, p1 += 0.001,
   If[Element[D1, Reals] && Element[Π, Reals] &&
     (p1 > (2 qo + t D1)/3 && p1 > 4 t D1 && p1 ≤ 16 t D1 && 0 < D1 < 1),
    AppendTo[results, {Π, p1, CS}],
    AppendTo[results, {0, 0, 0}]]];
  {maxVal, maxP, maxCS} = Last@MaximalBy[results, First];
  AppendTo[Results, {δc, maxVal, maxP, maxCS}];
  results = {}
 ];
If[Results == {}, Print["No valid results found."], TableForm[Results,
  TableDirections → Row, TableHeadings → {None, {"δc", "Π", "p1", "CS"}}]]
```

```
(* Combiantion 18. The conditions are p1>(2qo+tD1)/3, p1>4tD1, and p1>16tD1*)
```

In[∘]:=
$$p_{2P} = \frac{2 q_o + p_1 + t D_1}{4};\text{(*The second-period price under compltely positive reviews*)}$$
$$p_{2M} = \frac{2 p_1 + t D_1}{4};\text{(*The second-period price under mixed reviews*)}$$
$$p_{2N} = \frac{p_1}{4};\text{(*The second-period price under compltely negative reviews*)}$$
$$D_{2P} = \frac{2 q_o + p_1 - 3 t D_1}{4 t};\text{(*The second-period demand under compltely positive reviews*)}$$
$$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t};\text{(*The second-period demand under mixed reviews*)}$$
$$D_{2N} = \frac{p_1 - 4 t D_1}{4 t};\text{(*The second-period demand under compltely negative reviews*)}$$

In[∘]:=
$$U_1 = q_o - p_1 - t D_1 + \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} (p_1 - p_{2P}) + \frac{t D_1}{2 q_o} (p_1 - p_{2M}) + \frac{p_1}{2 q_o} (p_1 - p_{2N})\right);$$
(*Consumers' expected utility purchasing in the first period*)
$$U_2 = \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right) + \frac{p_1}{2 q_o} \left(\frac{p_1}{2} - p_{2N} - t D_1\right)\right);$$
(*Consumers' expected utility when purchasing in the second period*)

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[∘]=
$$\left\{\left\{D_1 \to \frac{-p_1 + q_o}{t}\right\}\right\}$$

In[∘]:=
$$D_1 = \frac{-p_1 + q_o}{t};\text{(*The response function of } D_1\text{*)}$$
$$\text{Reduce}\left[p_1 > \frac{2 q_o + t D_1}{3} \&\& p_1 > 4 t D_1 \&\& p_1 > 16 t D_1 \&\& D_1 > 0 \&\& t > 2 q_o > 0 \&\& 0 < \delta_c < 1\right]$$
(*We check if D_1 satisfies conditions*)

Out[∘]=
$$p_1 > 0 \&\& p_1 < q_o < \frac{17 p_1}{16} \&\& t > 2 q_o \&\& 0 < \delta_c < 1$$

(*Hence, the response function of D_1 satisfies conditionsis and is given by*)

In[∘]:=
$$D_1 = \frac{-p_1 + q_o}{t};$$

```
Π = Simplify[p1 D1 + (2 qo - p1 - t D1)/(2 qo) (p2P D2P - D1 (p1 - p2P)) + (t D1)/(2 qo) (p2M D2M - D1 (p1 - p2M)) +
    p1/(2 qo) (p2N D2N - D1 (p1 - p2N))];(*The firm's total profit function*)
CS = Integrate[
    (Integrate[Q - p1 - t x + δc ((2 qo - p1 - t D1)/(2 qo) (p1 - p2P) + (t D1)/(2 qo) (p1 - p2M) + p1/(2 qo) (p1 - p2N)),
        {x, 0, D1}]) / (2 qo), {Q, 0, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2P - t x), {x, D1, D1 + D2P}]) / (2 qo), {Q, p1 + t D1, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2M - t x), {x, D1, D1 + D2M}]) / (2 qo), {Q, p1, p1 + t D1}] +
   Integrate[(Integrate[δc (Q - p2N - t x), {x, D1, D1 + D2N}]) / (2 qo), {Q, 0, p1}];
t = 2.1;
qo = 1;
results = {};
Results = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[p1 = 0.001, p1 < 1, p1 += 0.001,
   If[
    Element[D1, Reals] && Element[Π, Reals] && (p1 ≥ (2 qo + t D1)/3 && p1 ≥ 4 t D1 && p1 ≥ 16 t D1),
    AppendTo[results, {Π, p1, CS}],
    AppendTo[results, {0, 0, 0}]]];
  {maxVal, maxP, maxCS} = Last@MaximalBy[results, First];
  AppendTo[Results, {δc, maxVal, maxP, maxCS}];
  results = {}
 ];
If[Results == {}, Print["No valid results found."], TableForm[Results,
  TableDirections → Row, TableHeadings → {None, {"δc", "Π", "p1", "CS"}}]]
```

Extension 7.2 : Discounted Future Profits

Case CL. Contingent pricing with social learning

```
(*Scenario 1: p1≤t D1/2, D2M=D2N=0*)
```

In[∘]:=

$$p_{2P} = \frac{2 q_o + p_1 - t D_1}{4};$$

$$D_{2P} = \frac{2 q_o + p_1 - t D_1}{4 t};$$

In[∘]:=

$$U_1 = q_o - p_1 - t D_1;$$

$$U_2 = \delta_c \frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right);$$

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[∘]=

$$\left\{\left\{D_1 \to \frac{2 q_o (-2 + \delta_c) - \sqrt{-8 q_o^2 (-2 + \delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c}\right\}, \left\{D_1 \to \frac{2 q_o (-2 + \delta_c) + \sqrt{-8 q_o^2 (-2 + \delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c}\right\}\right\}$$

```
(*There are 2 solutions of D1, we check each solution if it satisfies conditions*)
```

In[∘]:=

$$D_1 = \frac{2 q_o (-2 + \delta_c) - \sqrt{-8 q_o^2 (-2 + \delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c};$$

$$\text{Reduce}\left[0 < p_1 \le \frac{t D_1}{2} \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1\right]$$

Out[∘]= False

In[∘]:=

$$D_1 = \frac{2 q_o (-2 + \delta_c) + \sqrt{-8 q_o^2 (-2 + \delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c};$$

$$\text{Reduce}\left[0 < p_1 \le \frac{t D_1}{2} \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1\right]$$

Out[∘]=

$$p_1 > 0 \;\&\&\; \left(\left(3 p_1 < q_o < 2 p_1 + \frac{1}{2}\sqrt{19}\sqrt{p_1^2} \;\&\&\; t > 2 q_o \;\&\&\; 0 < \delta_c \le \frac{-24 p_1 q_o + 8 q_o^2}{3 p_1^2 - 8 p_1 q_o + 4 q_o^2}\right) \;||\; \left(q_o == 2 p_1 + \frac{1}{2}\sqrt{19}\sqrt{p_1^2} \;\&\&\; t > 2 q_o \;\&\&\; 0 < \delta_c < \frac{-24 p_1 q_o + 8 q_o^2}{3 p_1^2 - 8 p_1 q_o + 4 q_o^2}\right) \;||\; \left(q_o > 2 p_1 + \frac{1}{2}\sqrt{19}\sqrt{p_1^2} \;\&\&\; t > 2 q_o \;\&\&\; 0 < \delta_c < 1\right)\right)$$

```
(*Hence, the second solution is feasible*)
```

$$D_1 = \frac{2 q_o (-2 + \delta_c) + \sqrt{-8 q_o^2 (-2 + \delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c};$$

```
Π = Simplify[p1 D1 + δf (2 qo - p1 - t D1)/(2 qo) p2P D2P];
CS = Integrate[Integrate[Q - p1 - t x, {x, 0, D1}] / (2 qo), {Q, 0, 2 qo}] +
   Integrate[Integrate[δc (Q - p2P - t x), {x, D1, D1 + D2P}] / (2 qo), {Q, p1 + t D1, 2 qo}];

(*We then derive the optimal first-period price, profit,
and consumer surplus using numerical method by setting t=2.1 and qo=1,
which are the same as the base model. Then, we iterate systematically through
 all values of δc and δf (0<δc<1, 0<δf<1) with a step size of 0.001*)

t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[Element[D1, Reals] && (0 < D1 && 0 < p1 ≤ t D1 / 2),
     AppendTo[results, {Π, p1, CS}],
     AppendTo[results, {0, 0, 0}]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]

(*Scenario 2: tD1/2<p1≤2tD1, D2N=0*)

p2P = (2 qo + p1 - t D1)/4;
D2P = (2 qo + p1 - t D1)/(4 t);
p2M = (2 p1 - t D1)/4;
D2M = (2 p1 - t D1)/(4 t);
U1 = qo - p1 - t D1;
U2 = δc ((2 qo - p1 - t D1)/(2 qo) ((2 qo + p1 + t D1)/2 - p2P - t D1) + (t D1)/(2 qo) ((2 p1 + t D1)/2 - p2M - t D1));
```

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[∘]=

$$\left\{\left\{D_1 \to \frac{8\, p_1\, q_o + 4\, q_o^2\, (-2 + \delta_c) - p_1^2\, \delta_c}{4\, t\, q_o\, (-2 + \delta_c) - 2\, t\, p_1\, \delta_c}\right\}\right\}$$

In[∘]:= $D_1 = \frac{4\, q_o^2\, (2 - \delta_c) - 8\, p_1\, q_o + p_1^2\, \delta_c}{4\, t\, q_o\, (2 - \delta_c) + 2\, t\, p_1\, \delta_c}$;

$\Pi = \text{Simplify}\left[p_1\, D_1 + \delta_f \left(\frac{2\, q_o - p_1 - t\, D_1}{2\, q_o}\, p_{2P}\, D_{2P} + \frac{t\, D_1}{2\, q_o}\, p_{2M}\, D_{2M}\right)\right]$;

```
CS = Integrate[Integrate[Q - p1 - t x, {x, 0, D1}] / (2 qo), {Q, 0, 2 qo}] +
    Integrate[Integrate[δc (Q - p2P - t x), {x, D1, D1 + D2P}] / (2 qo), {Q, p1 + t D1, 2 qo}] +
    Integrate[Integrate[δc (Q - p2M - t x), {x, D1, D1 + D2M}] / (2 qo), {Q, p1, p1 + t D1}];

t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[Element[D1, Reals] && (0 < D1 && t D1 / 2 ≤ p1 ≤ 2 t D1),
     AppendTo[results, {Π, p1, CS}],
     AppendTo[results, {0, 0, 0}]]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]

(*Scenario 3: p1>2tD1*)
```

$$p_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4};$$

$$D_{2P} = \frac{2\,q_o + p_1 - t\,D_1}{4\,t};$$

$$p_{2M} = \frac{2\,p_1 - t\,D_1}{4};$$

$$D_{2M} = \frac{2\,p_1 - t\,D_1}{4\,t};$$

$$p_{2N} = \frac{p_1 - 2\,t\,D_1}{4};$$

$$D_{2N} = \frac{p_1 - 2\,t\,D_1}{4\,t};$$

$$U_1 = q_o - p_1 - t\,D_1;$$

$$U_2 = \delta_c \left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o} \left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2P} - t\,D_1\right) + \frac{t\,D_1}{2\,q_o} \left(\frac{2\,p_1 + t\,D_1}{2} - p_{2M} - t\,D_1\right) + \frac{p_1}{2\,q_o} \left(\frac{p_1}{2} - p_{2N} - t\,D_1\right)\right);$$

In[∘]:= **Simplify[Solve[U_1 == U_2, D_1]]**

Out[∘]=

$$\left\{\left\{D_1 \to \frac{q_o + \frac{2\,p_1}{-2+\delta_c}}{t}\right\}\right\}$$

In[∘]:=
$$D_1 = \frac{q_o\,(2 - \delta_c) - 2\,p_1}{t\,(2 - \delta_c)};$$

$$\Pi = \text{Simplify}\left[p_1\,D_1 + \delta_f \left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P} + \frac{t\,D_1}{2\,q_o}\,p_{2M}\,D_{2M} + \frac{p_1}{2\,q_o}\,p_{2N}\,D_{2N}\right)\right];$$

CS = Integrate[Integrate[$Q - p_1 - t\,x$, {x, 0, D_1}] / ($2\,q_o$), {Q, 0, $2\,q_o$}] +
Integrate[Integrate[$\delta_c\,(Q - p_{2P} - t\,x)$, {x, D_1, $D_1 + D_{2P}$}] / ($2\,q_o$), {Q, $p_1 + t\,D_1$, $2\,q_o$}] +
Integrate[Integrate[$\delta_c\,(Q - p_{2M} - t\,x)$, {x, D_1, $D_1 + D_{2M}$}] / ($2\,q_o$), {Q, p_1, $p_1 + t\,D_1$}] +
Integrate[Integrate[$\delta_c\,(Q - p_{2N} - t\,x)$, {x, D_1, $D_1 + D_{2N}$}] / ($2\,q_o$), {Q, 0, p_1}];

```
t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[Element[D1, Reals] && (0 < D1 && p1 ≥ 2 t D1),
     AppendTo[results, {Π, p1, CS}],
     AppendTo[results, {0, 0, 0}]]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
```

Case GL. Price guarantee with social learning

(*Combination 1. The conditions are $0<p_1\leq\frac{2q_o-tD_1}{3}$, $p_1\leq\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1\leq 16tD_1$*)

In[∘]:=
$$p_{2P} = \frac{2 q_o + p_1 - t D_1}{4};$$
$$p_{2M} = p_1;$$
$$p_{2N} = p_1;$$
$$D_{2P} = \frac{2 q_o + p_1 - t D_1}{4 t};$$
$$D_{2M} = 0;$$
$$D_{2N} = 0;$$

In[∘]:=
$$U_1 = q_o - p_1 - t D_1;$$
$$U_2 = \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right)\right);$$

In[∘]:= Simplify[Solve[U_1 == U_2, D_1]]

Out[∘]=
$$\left\{\left\{D_1 \to \frac{2 q_o (-2+\delta_c) - \sqrt{-8 q_o^2 (-2+\delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c}\right\}, \left\{D_1 \to \frac{2 q_o (-2+\delta_c) + \sqrt{-8 q_o^2 (-2+\delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c}\right\}\right\}$$

(*There are two solutions of D_1,
we then check each solution if it satisfies conditions*)

(*Solution 1*)

In[∘]:=
$$D_1 = \frac{2 q_o (-2+\delta_c) - \sqrt{-8 q_o^2 (-2+\delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c};$$

$$\text{Reduce}\left[0 < p_1 \leq \frac{2 q_o - t D_1}{3} \;\&\&\; p_1 \leq \frac{(3 + 2\sqrt{2})\, t D_1}{2} \;\&\&\; p_1 \leq 16\, t D_1 \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1\right]$$

Out[∘]=
False

(*Solution 2*)

In[∘]:=
$$D_1 = \frac{2 q_o (-2+\delta_c) + \sqrt{-8 q_o^2 (-2+\delta_c) - 8 p_1 q_o \delta_c + p_1^2 \delta_c^2}}{t \delta_c};$$

$$\text{Simplify}\left[\text{Reduce}\left[0 < p_1 \leq \frac{2 q_o - t D_1}{3} \;\&\&\; p_1 \leq \frac{(3 + 2\sqrt{2})\, t D_1}{2} \;\&\&\; p_1 \leq 16\, t D_1 \;\&\&\; t > 2 q_o > 0 \;\&\&\; 0 < \delta_c < 1\right]\right]$$

Out[•]=

$$p_1 > 0 \,\&\&\, t > 2\,q_o \,\&\&$$

$$\Bigg(\Bigg(\text{Root}\Big[\left(68+48\sqrt{2}\right)\#1^3 - 158\,p_1^3 - 104\sqrt{2}\,p_1^3 + \#1^2\left(-160\,p_1 - 112\sqrt{2}\,p_1\right) + \#1\left(171\,p_1^2 + 116\sqrt{2}\,p_1^2\right)\&,\,1\Big] == q_o \,\&\&\, \frac{q_o^2}{p_1} + p_1\,\delta_c == 2\,q_o\Bigg) \,||$$

$$\Bigg(\delta_c \le \frac{8\,q_o\left(\left(23+16\sqrt{2}\right)p_1 - \left(17+12\sqrt{2}\right)q_o\right)}{\left(13+12\sqrt{2}\right)p_1^2 + 8\left(3+2\sqrt{2}\right)p_1\,q_o - 4\left(17+12\sqrt{2}\right)q_o^2} \,\&\&$$

$$\Bigg(\Bigg(2\,p_1 == q_o \,\&\&\, \frac{(2\,p_1 - q_o)\,q_o}{p_1^2} < \delta_c\Bigg) \,||$$

$$\Bigg(\text{Root}\Big[\left(68+48\sqrt{2}\right)\#1^3 - 158\,p_1^3 - 104\sqrt{2}\,p_1^3 + \#1^2\left(-160\,p_1 - 112\sqrt{2}\,p_1\right) + \#1\left(171\,p_1^2 + 116\sqrt{2}\,p_1^2\right)\&,\,1\Big] < q_o \,\&\&\, q_o < 2\,p_1 \,\&\&\, \frac{(2\,p_1 - q_o)\,q_o}{p_1^2} \le \delta_c\Bigg) \,||\, \Bigg(\delta_c > 0 \,\&\&$$

$$2\,p_1 < q_o \,\&\&\, 4\left(-2+\sqrt{2}\right)p_1 + \left(-17+12\sqrt{2}\right)\sqrt{2659+1880\sqrt{2}}\,\sqrt{p_1^2} + 2\,q_o < 0\Bigg)\Bigg)\Bigg) \,||$$

$$\Bigg(\delta_c > 0 \,\&\&\, \Bigg(\Bigg(4\left(-2+\sqrt{2}\right)p_1 + \left(-17+12\sqrt{2}\right)\sqrt{2659+1880\sqrt{2}}\,\sqrt{p_1^2} + 2\,q_o == 0 \,\&\&$$

$$\delta_c < \frac{8\,q_o\left(\left(23+16\sqrt{2}\right)p_1 - \left(17+12\sqrt{2}\right)q_o\right)}{\left(13+12\sqrt{2}\right)p_1^2 + 8\left(3+2\sqrt{2}\right)p_1\,q_o - 4\left(17+12\sqrt{2}\right)q_o^2}\Bigg) \,||$$

$$\Bigg(4\left(-2+\sqrt{2}\right)p_1 + \left(-17+12\sqrt{2}\right)\sqrt{2659+1880\sqrt{2}}\,\sqrt{p_1^2} + 2\,q_o > 0 \,\&\&\, \delta_c < 1\Bigg)\Bigg)\Bigg)\Bigg)$$

(*Hence, the second solution is feasible*)

$$\mathbf{D_1 = \frac{2\,q_o\,(-2+\delta_c) + \sqrt{-8\,q_o^2\,(-2+\delta_c) - 8\,p_1\,q_o\,\delta_c + p_1^2\,\delta_c^2}}{t\,\delta_c};}$$

$$\mathbf{\Pi = Simplify\Big[p_1\,D_1 + \delta_f\,\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2P}\,D_{2P}\Big];}$$

CS = Integrate[(Integrate[Q − p_1 − t x, {x, 0, D_1}]) / (2 q_o), {Q, 0, 2 q_o}] +
Integrate[(Integrate[δ_c (Q − p_{2P} − t x), {x, D_1, D_1 + D_{2P}}]) / (2 q_o), {Q, p_1 + t D_1, 2 q_o}];

```
t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[Element[D1, Reals] &&
      (0 < p1 ≤ (2 qo - t D1)/3 && p1 ≤ ((3 + 2 √2) t D1)/2 && p1 ≤ 16 t D1 && 0 < D1 < 1),
     AppendTo[results, {Π, p1, CS}],
     AppendTo[results, {0, 0, 0}]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
(*Combination 2: N.A.*)
(*Combination 3: The conditions are 0<p1≤(2qo-tD1)/3, p1>((3+2√2)tD1)/2, and p1≤16tD1*)
```

In[∘]:=
```
p2P = (2 qo + p1 - t D1)/4;
p2M = (2 p1 + t D1)/4;
p2N = p1;
D2P = (2 qo + p1 - t D1)/(4 t);
D2M = (2 p1 - 3 t D1)/(4 t);
D2N = 0;
```

In[∘]:=
$$U_1 = q_o - p_1 - t\,D_1 + \delta_c \frac{t\,D_1}{2\,q_o}\,(p_1 - p_{2\,M});$$

$$U_2 = \delta_c \left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2\,P} - t\,D_1\right) + \frac{t\,D_1}{2\,q_o}\left(\frac{2\,p_1 + t\,D_1}{2} - p_{2\,M} - t\,D_1\right)\right);$$

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1], $p_1 > 0$ && $t > 2\,q_o > 0$ && $0 < \delta_c < 1$]

Out[∘]=

$$\left\{\left\{D_1 \to -\frac{2\,q_o\,(-2+\delta_c) + \sqrt{8\,p_1\,q_o\,\delta_c - p_1^2\,\delta_c^2 + 8\,q_o^2\,\left(2 - 3\,\delta_c + \delta_c^2\right)}}{t\,\delta_c}\right\},\right.$$
$$\left.\left\{D_1 \to \frac{-2\,q_o\,(-2+\delta_c) + \sqrt{8\,p_1\,q_o\,\delta_c - p_1^2\,\delta_c^2 + 8\,q_o^2\,\left(2 - 3\,\delta_c + \delta_c^2\right)}}{t\,\delta_c}\right\}\right\}$$

(*There are 2 solutions of D_1, we check each solution if it satisfies conditions*)

In[∘]:=
$$D_1 = -\frac{2\,q_o\,(-2+\delta_c) + \sqrt{8\,p_1\,q_o\,\delta_c - p_1^2\,\delta_c^2 + 8\,q_o^2\,\left(2 - 3\,\delta_c + \delta_c^2\right)}}{t\,\delta_c};$$

Simplify[

$$\text{Reduce}\left[0 < p_1 \le \frac{2\,q_o - t\,D_1}{3} \;\&\&\; p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2} \;\&\&\; p_1 \le 16\,t\,D_1 \;\&\&\; t > 2\,q_o > 0 \;\&\&\; 0 < \delta_c < 1\right]\Big]$$

Out[ ]=

$$p_1 > 0 \,\&\&\, t > 2\, q_o \,\&\&\, \Bigg(\Bigg(\delta_c \le \frac{128\,(17\,p_1 - 16\,q_o)\,q_o}{257\,p_1^2 + 64\,p_1\,q_o - 1024\,q_o^2} \,\&\&$$

$$\Bigg(\Bigg(\text{Root}\Big[\left(68 + 48\sqrt{2}\right) \#1^3 - 272\,p_1^3 - 184\sqrt{2}\,p_1^3 + \#1^2\left(-432\,p_1 - 304\sqrt{2}\,p_1\right) +$$

$$\#1\left(695\,p_1^2 + 484\sqrt{2}\,p_1^2\right)\&,\,2\Big] == q_o \,\&\&\, \frac{4\,q_o\,(-2\,p_1 + q_o)}{5\,p_1^2 - 12\,p_1\,q_o + 4\,q_o^2} < \delta_c\Bigg) \,||$$

$$\Bigg(\text{Root}\Big[\left(68 + 48\sqrt{2}\right) \#1^3 - 272\,p_1^3 - 184\sqrt{2}\,p_1^3 + \#1^2\left(-432\,p_1 - 304\sqrt{2}\,p_1\right) +$$

$$\#1\left(695\,p_1^2 + 484\sqrt{2}\,p_1^2\right)\&,\,2\Big] < q_o \,\&\&\, 32\,q_o < 33\,p_1 + 8\sqrt{13}\,\sqrt{p_1^2} \,\&\&$$

$$\frac{8\,q_o\left(\left(23 + 16\sqrt{2}\right)p_1 - \left(17 + 12\sqrt{2}\right)q_o\right)}{3\left(7 + 4\sqrt{2}\right)p_1^2 + 8\left(3 + 2\sqrt{2}\right)p_1\,q_o - 4\left(17 + 12\sqrt{2}\right)q_o^2} < \delta_c\Bigg) \,||$$

$$\Bigg(49\,p_1 < 32\,q_o \,\&\&\, q_o < \text{Root}\Big[\left(68 + 48\sqrt{2}\right) \#1^3 - 272\,p_1^3 - 184\sqrt{2}\,p_1^3 +$$

$$\#1^2\left(-432\,p_1 - 304\sqrt{2}\,p_1\right) + \#1\left(695\,p_1^2 + 484\sqrt{2}\,p_1^2\right)\&,\,2\Big] \,\&\&$$

$$\frac{4\,q_o\,(-2\,p_1 + q_o)}{5\,p_1^2 - 12\,p_1\,q_o + 4\,q_o^2} \le \delta_c\Bigg)\Bigg)\Bigg) \,||\, \Bigg(49\,p_1 == 32\,q_o \,\&\&\, \delta_c == \frac{4\,q_o\,(-2\,p_1 + q_o)}{5\,p_1^2 - 12\,p_1\,q_o + 4\,q_o^2}\Bigg) \,||$$

$$\Bigg(\frac{8\,q_o\left(\left(23 + 16\sqrt{2}\right)p_1 - \left(17 + 12\sqrt{2}\right)q_o\right)}{3\left(7 + 4\sqrt{2}\right)p_1^2 + 8\left(3 + 2\sqrt{2}\right)p_1\,q_o - 4\left(17 + 12\sqrt{2}\right)q_o^2} < \delta_c \,\&\&$$

$$\Bigg(\Bigg(33\,p_1 + 8\sqrt{13}\,\sqrt{p_1^2} == 32\,q_o \,\&\&\, \delta_c < \frac{128\,(17\,p_1 - 16\,q_o)\,q_o}{257\,p_1^2 + 64\,p_1\,q_o - 1024\,q_o^2}\Bigg) \,||$$

$$\Bigg(33\,p_1 + 8\sqrt{13}\,\sqrt{p_1^2} < 32\,q_o \,\&\&$$

$$4\left(-2 + \sqrt{2}\right)p_1 + \left(-17 + 12\sqrt{2}\right)\sqrt{2523 + 1784\sqrt{2}}\,\sqrt{p_1^2} + 2\,q_o < 0 \,\&\&\, \delta_c < 1\Bigg)\Bigg)\Bigg)\Bigg)$$

In[ ]:=

$$\mathbf{D_1 = \frac{-2\,q_o\,(-2 + \delta_c) + \sqrt{8\,p_1\,q_o\,\delta_c - p_1^2\,\delta_c^2 + 8\,q_o^2\left(2 - 3\,\delta_c + \delta_c^2\right)}}{t\,\delta_c};}$$

$$\mathbf{Simplify\Big[Reduce\Big[0 < p_1 \le \frac{2\,q_o - t\,D_1}{3} \,\&\&\, p_1 > \frac{\left(3 + 2\sqrt{2}\right)t\,D_1}{2} \,\&\&\, p_1 \le 16\,t\,D_1 \,\&\&\, t > 2\,q_o > 0 \,\&\&\, 0 < \delta_c < 1\Big]\Big]}$$

Out[ ]=

```
False
```

(*Hence, the first solution is feasible*)

$$\mathbf{D_1 = -\frac{2\,q_o\,(-2 + \delta_c) + \sqrt{8\,p_1\,q_o\,\delta_c - p_1^2\,\delta_c^2 + 8\,q_o^2\left(2 - 3\,\delta_c + \delta_c^2\right)}}{t\,\delta_c};}$$

```
Π = Simplify[p1 D1 + δf ((2 qo - p1 - t D1)/(2 qo) p2P D2P + (t D1)/(2 qo) (p2M D2M - D1 (p1 - p2M)))];
CS =
  Integrate[(Integrate[Q - p1 - t x + (t D1)/(2 qo) δc (p1 - p2M), {x, 0, D1}]) / (2 qo), {Q, 0, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2P - t x), {x, D1, D1 + D2P}]) / (2 qo), {Q, p1 + t D1, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2M - t x), {x, D1, D1 + D2M}]) / (2 qo), {Q, p1, p1 + t D1}];
t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[Element[D1, Reals] && 0 < p1 ≤ (2 qo - t D1)/3 && p1 > ((3 + 2 √2) t D1)/2 && p1 ≤ 16 t D1,
     AppendTo[results, {Π, p1, CS}],
     AppendTo[results, {0, 0, 0}]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
```

(*Combination 4: The conditions are $0<p_1\leq\frac{2q_o-tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1>16tD_1$*)

In[∘]:=
$$p_{2P}=\frac{2q_o+p_1-tD_1}{4};$$
$$p_{2M}=\frac{2p_1+tD_1}{4};$$
$$p_{2N}=\frac{p_1}{4};$$
$$D_{2P}=\frac{2q_o+p_1-tD_1}{4t};$$
$$D_{2M}=\frac{2p_1-3tD_1}{4t};$$
$$D_{2N}=\frac{p_1-4tD_1}{4t};$$

In[∘]:=
$$U_1=q_o-p_1-tD_1+\delta_c\left(\frac{tD_1}{2q_o}(p_1-p_{2M})+\frac{p_1}{2q_o}(p_1-p_{2N})\right);$$
$$U_2=\delta_c\left(\frac{2q_o-p_1-tD_1}{2q_o}\left(\frac{2q_o+p_1+tD_1}{2}-p_{2P}-tD_1\right)+\frac{tD_1}{2q_o}\left(\frac{2p_1+tD_1}{2}-p_{2M}-tD_1\right)+\frac{p_1}{2q_o}\left(\frac{p_1}{2}-p_{2N}-tD_1\right)\right);$$

In[∘]:= Simplify[Solve[U_1 == U_2, D_1]]

Out[∘]=
$$\left\{\left\{D_1\to-\frac{2q_o(-2+\delta_c)+2p_1\delta_c+\sqrt{8p_1q_o(-1+\delta_c)\delta_c+p_1^2\delta_c^2+8q_o^2\left(2-3\delta_c+\delta_c^2\right)}}{t\delta_c}\right\},\left\{D_1\to\frac{-2q_o(-2+\delta_c)-2p_1\delta_c+\sqrt{8p_1q_o(-1+\delta_c)\delta_c+p_1^2\delta_c^2+8q_o^2\left(2-3\delta_c+\delta_c^2\right)}}{t\delta_c}\right\}\right\}$$

(*There are 2 solutions of D_1, we check each solution if it satisfies conditions*)

In[∘]:=
$$D_1=-\frac{2q_o(-2+\delta_c)+2p_1\delta_c+\sqrt{8p_1q_o(-1+\delta_c)\delta_c+p_1^2\delta_c^2+8q_o^2\left(2-3\delta_c+\delta_c^2\right)}}{t\delta_c};$$
$$\text{Reduce}\left[0<p_1\leq\frac{2q_o-tD_1}{3}\;\&\&\;p_1>\frac{\left(3+2\sqrt{2}\right)tD_1}{2}\;\&\&\;p_1>16tD_1\;\&\&\;t>2q_o>0\;\&\&\;0<\delta_c<1\right]$$

Out[∘]=
False

In[∘]:=
$$D_1=\frac{-2q_o(-2+\delta_c)-2p_1\delta_c+\sqrt{8p_1q_o(-1+\delta_c)\delta_c+p_1^2\delta_c^2+8q_o^2\left(2-3\delta_c+\delta_c^2\right)}}{t\delta_c};$$
$$\text{Reduce}\left[0<p_1\leq\frac{2q_o-tD_1}{3}\;\&\&\;p_1>\frac{\left(3+2\sqrt{2}\right)tD_1}{2}\;\&\&\;p_1>16tD_1\;\&\&\;t>2q_o>0\;\&\&\;0<\delta_c<1\right]$$

Out[∘]=
False

(*Therefore, there is no feasible solutions for combination 4*)

(*Combination 5: The conditions are $\frac{2q_o-tD_1}{3}\leq p_1\leq\frac{2q_o+tD_1}{3}$, $p_1\leq\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1\leq 16tD_1$*)

In[•]:=
$$p_{2P} = p_1;$$
$$p_{2M} = p_1;$$
$$p_{2N} = p_1;$$
$$D_{2P} = \frac{2 q_o - p_1 - t D_1}{2 t};$$
$$D_{2M} = 0;$$
$$D_{2N} = 0;$$

In[•]:=
$$U_1 = q_o - p_1 - t D_1;$$
$$U_2 = \delta_c \frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right);$$

In[•]:= Simplify[Solve[$U_1 == U_2$, D_1], $t > 0$]

Out[•]=
$$\left\{\left\{D_1 \to -\frac{2 q_o \left(1 + \sqrt{1-\delta_c} - \delta_c\right) + p_1 \delta_c}{t \delta_c}\right\}, \left\{D_1 \to \frac{-p_1 \delta_c + 2 q_o \left(-1 + \sqrt{1-\delta_c} + \delta_c\right)}{t \delta_c}\right\}\right\}$$

(*There are two solutions of D_1, we check each solution if it satisfies conditions*)

(*Solution 1*)

In[•]:=
$$D_1 = -\frac{2 q_o \left(1 + \sqrt{1-\delta_c} - \delta_c\right) + p_1 \delta_c}{t \delta_c};$$
Simplify[Reduce[
$$\frac{2 q_o - t D_1}{3} \le p_1 \le \frac{2 q_o + t D_1}{3} \ \&\&\ p_1 \le \frac{\left(3 + 2\sqrt{2}\right) t D_1}{2} \ \&\&\ p_1 \le 16\, t D_1 \ \&\&\ t > 2 q_o > 0 \ \&\&\ 0 < \delta_c < 1]]$$

Out[•]=
False

(*Solution 2*)

In[•]:=
$$D_1 = \frac{-p_1 \delta_c + 2 q_o \left(-1 + \sqrt{1-\delta_c} + \delta_c\right)}{t \delta_c};$$
Simplify[Reduce[
$$\frac{2 q_o - t D_1}{3} \le p_1 \le \frac{2 q_o + t D_1}{3} \ \&\&\ p_1 \le \frac{\left(3 + 2\sqrt{2}\right) t D_1}{2} \ \&\&\ p_1 \le 16\, t D_1 \ \&\&\ t > 2 q_o > 0 \ \&\&\ 0 < \delta_c < 1]]$$

Out[•]=
$$t > 2 q_o \ \&\&\ p_1 > 0 \ \&\&$$
$$\left(\left(0 < \delta_c \le \frac{(2 p_1 - q_o)\, q_o}{p_1^2} \ \&\&\ \mathrm{Root}\left[\left(68 + 48\sqrt{2}\right) \#1^3 - 158 p_1^3 - 104\sqrt{2}\, p_1^3 + \#1^2 \left(-228 p_1 - 160\sqrt{2}\, p_1\right) + \#1 \left(285 p_1^2 + 196\sqrt{2}\, p_1^2\right) \&, 1\right] < q_o < 2 p_1\right) \ ||$$
$$\left(0 < \delta_c \le \frac{4 q_o \left(-\left(\left(23 + 16\sqrt{2}\right) p_1\right) + \left(17 + 12\sqrt{2}\right) q_o\right)}{\left(33 + 20\sqrt{2}\right) p_1^2 - 4\left(23 + 16\sqrt{2}\right) p_1 q_o + 4\left(17 + 12\sqrt{2}\right) q_o^2} \ \&\&$$
$$\left(7 - 4\sqrt{2}\right) p_1 < q_o \le \mathrm{Root}\left[\left(68 + 48\sqrt{2}\right) \#1^3 - 158 p_1^3 - 104\sqrt{2}\, p_1^3 + \#1^2 \left(-228 p_1 - 160\sqrt{2}\, p_1\right) + \#1 \left(285 p_1^2 + 196\sqrt{2}\, p_1^2\right) \&, 1\right]\right)\right)$$

```
(*Therefore, solution 2 is feasible*)
```

$$D_1 = \frac{-p_1\,\delta_c + 2\,q_o\,\left(-1+\sqrt{1-\delta_c}+\delta_c\right)}{t\,\delta_c};$$

$$\Pi = \text{Simplify}\left[p_1\,D_1 + \delta_f\,\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,p_{2\,P}\,D_{2\,P}\right];$$

```
CS = Integrate[(Integrate[Q - p1 - t x, {x, 0, D1}]) / (2 qo), {Q, 0, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2P - t x), {x, D1, D1 + D2P}]) / (2 qo), {Q, p1 + t D1, 2 qo}];
t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[
     Element[D1, Reals] && (2 qo - t D1)/3 ≤ p1 ≤ (2 qo + t D1)/3 && p1 ≤ ((3 + 2 Sqrt[2]) t D1)/2 && p1 ≤ 16 t D1,
     AppendTo[results, {Π, p1, CS}],
     AppendTo[results, {0, 0, 0}]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
(*Combination 6: N.A.*)
```

(*Combination 7: The conditions are $\frac{2q_o-tD_1}{3}\leq p_1\leq\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1\leq 16tD_1$*)

```
In[]:= p2P = p1;
       p2M = (2 p1 + t D1)/4;
       p2N = p1;
       D2P = (2 qo - p1 - t D1)/(2 t);
       D2M = (2 p1 - 3 t D1)/(4 t);
       D2N = 0;

In[]:= U1 = qo - p1 - t D1 + δc (t D1)/(2 qo) (p1 - p2M);
       U2 = δc ((2 qo - p1 - t D1)/(2 qo) ((2 qo + p1 + t D1)/2 - p2P - t D1) + (t D1)/(2 qo) ((2 p1 + t D1)/2 - p2M - t D1));

In[]:= Simplify[Solve[U1 == U2, D1]]
```

Out[]=

$$\left\{\left\{D_1 \to \frac{-4\, p_1\, q_o\, (-1+\delta_c) + 4\, q_o^2\, (-1+\delta_c) + p_1^2\, \delta_c}{4\, t\, q_o\, (-1+\delta_c) - 2\, t\, p_1\, \delta_c}\right\}\right\}$$

```
In[]:= D1 = (-4 p1 qo (-1 + δc) + 4 qo^2 (-1 + δc) + p1^2 δc)/(4 t qo (-1 + δc) - 2 t p1 δc);

In[]:= Simplify[Reduce[
         (2 qo - t D1)/3 ≤ p1 ≤ (2 qo + t D1)/3 && p1 > ((3 + 2 √2) t D1)/2 && p1 ≤ 16 t D1 && t > 2 qo > 0 && 0 < δc < 1]]
```

Out[]=

$$t > 2\, q_o \;\&\&\; p_1 > 0 \;\&\&\; \left(\left(\delta_c \le \frac{4\, q_o\, (-4\, p_1 + 3\, q_o)}{7\, p_1^2 - 20\, p_1\, q_o + 12\, q_o^2} \;\&\&\; \left(\left(\delta_c > 0 \;\&\&\; 4\, p_1 < 3\, q_o \;\&\&\; q_o \le \left(7 - 4\sqrt{2}\right) p_1\right) \;||\; \left(\left(7 - 4\sqrt{2}\right) p_1 < q_o \;\&\&\; \frac{4\, q_o\, \left(-\left(\left(5 + 2\sqrt{2}\right) p_1\right) + \left(3 + 2\sqrt{2}\right) q_o\right)}{(p_1 - 2\, q_o)\, \left(\left(7 + 2\sqrt{2}\right) p_1 - 2\left(3 + 2\sqrt{2}\right) q_o\right)} < \delta_c \;\&\&\; 32\, q_o \le 47\, p_1\right)\right)\right) \;||\; \left(\frac{4\, q_o\, \left(-\left(\left(5 + 2\sqrt{2}\right) p_1\right) + \left(3 + 2\sqrt{2}\right) q_o\right)}{(p_1 - 2\, q_o)\, \left(\left(7 + 2\sqrt{2}\right) p_1 - 2\left(3 + 2\sqrt{2}\right) q_o\right)} < \delta_c \;\&\&\; \left(\left(47\, p_1 < 32\, q_o \;\&\&\; 32\, q_o \le 49\, p_1 \;\&\&\; \delta_c \le \frac{-34\, p_1\, q_o + 32\, q_o^2}{9\, p_1^2 - 34\, p_1\, q_o + 32\, q_o^2}\right) \;||\; \left(49\, p_1 < 32\, q_o \;\&\&\; 2\left(3 + 2\sqrt{2}\right) q_o < \left(11 + 6\sqrt{2}\right) p_1 \;\&\&\; \delta_c \le \frac{4\, q_o\, (-2\, p_1 + q_o)}{5\, p_1^2 - 12\, p_1\, q_o + 4\, q_o^2}\right)\right)\right)\right)$$

```
Π = Simplify[p1 D1 + δf ((2 qo - p1 - t D1)/(2 qo) p2P D2P + (t D1)/(2 qo) (p2M D2M - D1 (p1 - p2M)))];
CS =
  Integrate[(Integrate[Q - p1 - t x + (t D1)/(2 qo) δc (p1 - p2M), {x, 0, D1}]) / (2 qo), {Q, 0, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2P - t x), {x, D1, D1 + D2P}]) / (2 qo), {Q, p1 + t D1, 2 qo}] +
   Integrate[(Integrate[δc (Q - p2M - t x), {x, D1, D1 + D2M}]) / (2 qo), {Q, p1, p1 + t D1}];
```

```
t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[
      Element[D1, Reals] && (2 qo - t D1)/3 ≤ p1 ≤ (2 qo + t D1)/3 && p1 > ((3 + 2 Sqrt[2]) t D1)/2 && p1 ≤ 16 t D1,
      AppendTo[results, {Π, p1, CS}],
      AppendTo[results, {0, 0, 0}]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
```

(*Combination 8: The conditions are $\frac{2q_o - tD_1}{3} \le p_1 \le \frac{2q_o + tD_1}{3}$, $p_1 > \frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1 > 16tD_1$*)

In[∘]:=
$$p_{2P} = p_1;$$
$$p_{2M} = \frac{2\,p_1 + t\,D_1}{4};$$
$$p_{2N} = \frac{p_1}{4};$$
$$D_{2P} = \frac{2\,q_o - p_1 - t\,D_1}{2\,t};$$
$$D_{2M} = \frac{2\,p_1 - 3\,t\,D_1}{4\,t};$$
$$D_{2N} = \frac{p_1 - 4\,t\,D_1}{4\,t};$$

In[∘]:=
$$U_1 = q_o - p_1 - t\,D_1 + \delta_c \left(\frac{t\,D_1}{2\,q_o}\,(p_1 - p_{2M}) + \frac{p_1}{2\,q_o}\,(p_1 - p_{2N})\right);$$
$$U_2 = \delta_c \left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2P} - t\,D_1\right) + \frac{t\,D_1}{2\,q_o}\left(\frac{2\,p_1 + t\,D_1}{2} - p_{2M} - t\,D_1\right) + \frac{p_1}{2\,q_o}\left(\frac{p_1}{2} - p_{2N} - t\,D_1\right)\right);$$

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1], $t > 0$ && $q_o > 0$ && $0 < \delta_c < 1$]

Out[∘]=
$$\left\{\left\{D_1 \to \frac{-p_1 + q_o}{t}\right\}\right\}$$

In[∘]:=
$$D_1 = \frac{-p_1 + q_o}{t};$$
Reduce[
$$\frac{2\,q_o - t\,D_1}{3} \le p_1 \le \frac{2\,q_o + t\,D_1}{3} \;\&\&\; p_1 > \frac{\left(3 + 2\sqrt{2}\right)t\,D_1}{2} \;\&\&\; p_1 > 16\,t\,D_1 \;\&\&\; t > 2\,q_o > 0 \;\&\&\; 0 < \delta_c < 1$$
]

Out[∘]=
False

(*Hence, there is no feasible solution for combination 8*)

```
(*Combination 9: The conditions are p1>(2qo+tD1)/3, p1≤((3+2√2)tD1)/2, and p1≤16tD1*)
```

In[∘]:=
$$p_{2P} = \frac{2\,q_o + p_1 + t\,D_1}{4};$$
$$p_{2M} = p_1;$$
$$p_{2N} = p_1;$$
$$D_{2P} = \frac{2\,q_o + p_1 - 3\,t\,D_1}{4\,t};$$
$$D_{2M} = 0;$$
$$D_{2N} = 0;$$

In[∘]:=
$$U_1 = q_o - p_1 - t\,D_1 + \delta_c\,\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,(p_1 - p_{2P});$$
$$U_2 = \delta_c\,\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2P} - t\,D_1\right);$$

In[∘]:= Simplify[Solve[$U_1 == U_2$, D_1], t > 0]

Out[∘]=
$$\left\{\left\{D_1 \to -\frac{2\,q_o\left(1 + \sqrt{1-\delta_c} - \delta_c\right) + p_1\,\delta_c}{t\,\delta_c}\right\},\ \left\{D_1 \to \frac{-p_1\,\delta_c + 2\,q_o\left(-1 + \sqrt{1-\delta_c} + \delta_c\right)}{t\,\delta_c}\right\}\right\}$$

```
(*There are two solutions of D1, we check each solution if it satisfies conditions*)
```

In[∘]:=
$$D_1 = -\frac{2\,q_o\left(1 + \sqrt{1-\delta_c} - \delta_c\right) + p_1\,\delta_c}{t\,\delta_c};$$
$$\text{Reduce}\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 \le \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\ \&\&\ p_1 \le 16\,t\,D_1\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\right]$$

Out[∘]=
False

In[∘]:=
$$D_1 = \frac{-p_1\,\delta_c + 2\,q_o\left(-1 + \sqrt{1-\delta_c} + \delta_c\right)}{t\,\delta_c};$$
$$\text{Reduce}\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 \le \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\ \&\&\ p_1 \le 16\,t\,D_1\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\right]$$

Out[∘]=
False

```
(*Therefore, there is no feasible solution for combination 9*)

(*Combination 10. N.A.*)

(*Combination 11: The conditions are p1>(2qo+tD1)/3, p1>((3+2√2)tD1)/2, and p1≤16tD1*)
```

In[•]:=
$$p_{2P} = \frac{2 q_o + p_1 + t D_1}{4};$$
$$p_{2M} = \frac{2 p_1 + t D_1}{4};$$
$$p_{2N} = p_1;$$
$$D_{2P} = \frac{2 q_o + p_1 - 3 t D_1}{4 t};$$
$$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t};$$
$$D_{2N} = 0;$$

In[•]:=
$$U_1 = q_o - p_1 - t D_1 + \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} (p_1 - p_{2P}) + \frac{t D_1}{2 q_o} (p_1 - p_{2M})\right);$$
$$U_2 = \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} \left(\frac{2 q_o + p_1 + t D_1}{2} - p_{2P} - t D_1\right) + \frac{t D_1}{2 q_o} \left(\frac{2 p_1 + t D_1}{2} - p_{2M} - t D_1\right)\right);$$

In[•]:= Simplify[Solve[$U_1 == U_2$, D_1]]

Out[•]=
$$\left\{\left\{D_1 \to \frac{-4 p_1 q_o (-1 + \delta_c) + 4 q_o^2 (-1 + \delta_c) + p_1^2 \delta_c}{4 t q_o (-1 + \delta_c) - 2 t p_1 \delta_c}\right\}\right\}$$

In[•]:=
$$D_1 = \frac{4 q_o^2 (1 - \delta_c) - 4 p_1 q_o (1 - \delta_c) - p_1^2 \delta_c}{4 t q_o (1 - \delta_c) + 2 t p_1 \delta_c};$$

In[•]:= Simplify[
$$\text{Reduce}\left[p_1 > \frac{2 q_o + t D_1}{3} \,\&\&\, p_1 > \frac{\left(3 + 2\sqrt{2}\right) t D_1}{2} \,\&\&\, p_1 \le 16 t D_1 \,\&\&\, t > 2 q_o > 0 \,\&\&\, 0 < \delta_c < 1\right]\Big]$$

Out[•]=
$$t > 2 q_o \,\&\&\, p_1 > 0 \,\&\&\, \delta_c \le \frac{-34 p_1 q_o + 32 q_o^2}{9 p_1^2 - 34 p_1 q_o + 32 q_o^2} \,\&\&\, \Big((\delta_c > 0 \,\&\&\, 17 p_1 < 16 q_o \,\&\&\, 3 q_o \le 4 p_1) \,||$$
$$\left(4 p_1 < 3 q_o \,\&\&\, 32 q_o < 47 p_1 \,\&\&\, \frac{4 q_o (-4 p_1 + 3 q_o)}{7 p_1^2 - 20 p_1 q_o + 12 q_o^2} < \delta_c\right)\Big)$$

$$D_1 = \frac{4 q_o^2 (1 - \delta_c) - 4 p_1 q_o (1 - \delta_c) - p_1^2 \delta_c}{4 t q_o (1 - \delta_c) + 2 t p_1 \delta_c};$$

Π =
$$\text{Simplify}\left[p_1 D_1 + \delta_f \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} (p_{2P} D_{2P} - D_1 (p_1 - p_{2P})) + \frac{t D_1}{2 q_o} (p_{2M} D_{2M} - D_1 (p_1 - p_{2M}))\right)\right];$$

CS = Integrate[
$$\left(\text{Integrate}\left[Q - p_1 - t x + \delta_c \left(\frac{2 q_o - p_1 - t D_1}{2 q_o} (p_1 - p_{2P}) + \frac{t D_1}{2 q_o} (p_1 - p_{2M})\right), \{x, 0, D_1\}\right]\right) \Big/$$
$$(2 q_o), \{Q, 0, 2 q_o\}\Big] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c (Q - p_{2P} - t x), \{x, D_1, D_1 + D_{2P}\}]) / (2 q_o), \{Q, p_1 + t D_1, 2 q_o\}] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c (Q - p_{2M} - t x), \{x, D_1, D_1 + D_{2M}\}]) / (2 q_o), \{Q, p_1, p_1 + t D_1\}];$$

```
t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[Element[D1, Reals] && p1 > (2 qo + t D1)/3 && p1 > ((3 + 2 √2) t D1)/2 && p1 ≤ 16 t D1,
     AppendTo[results, {Π, p1, CS}],
     AppendTo[results, {0, 0, 0}]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
```

(*Combination 12: The conditions are $p_1>\frac{2q_o+tD_1}{3}$, $p_1>\frac{(3+2\sqrt{2})tD_1}{2}$, and $p_1>16tD_1$*)

In[∘]:=

$$p_{2P} = \frac{2 q_o + p_1 + t D_1}{4};$$

$$p_{2M} = \frac{2 p_1 + t D_1}{4};$$

$$p_{2N} = \frac{p_1}{4};$$

$$D_{2P} = \frac{2 q_o + p_1 - 3 t D_1}{4 t};$$

$$D_{2M} = \frac{2 p_1 - 3 t D_1}{4 t};$$

$$D_{2N} = \frac{p_1 - 4 t D_1}{4 t};$$

In[∘]:=
$$U_1 = q_o - p_1 - t\,D_1 + \delta_c \left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,(p_1 - p_{2\,P}) + \frac{t\,D_1}{2\,q_o}\,(p_1 - p_{2\,M}) + \frac{p_1}{2\,q_o}\,(p_1 - p_{2\,N})\right);$$
$$U_2 = \delta_c \left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\left(\frac{2\,q_o + p_1 + t\,D_1}{2} - p_{2\,P} - t\,D_1\right) + \frac{t\,D_1}{2\,q_o}\left(\frac{2\,p_1 + t\,D_1}{2} - p_{2\,M} - t\,D_1\right) + \frac{p_1}{2\,q_o}\left(\frac{p_1}{2} - p_{2\,N} - t\,D_1\right)\right);$$

In[∘]:= `Simplify[Solve[U1 == U2, D1]]`

Out[∘]=
$$\left\{\left\{D_1 \to \frac{-p_1 + q_o}{t}\right\}\right\}$$

In[∘]:=
$$D_1 = \frac{-p_1 + q_o}{t};$$

In[∘]:=
$$\text{Reduce}\left[p_1 > \frac{2\,q_o + t\,D_1}{3}\ \&\&\ p_1 > \frac{\left(3 + 2\,\sqrt{2}\right)\,t\,D_1}{2}\ \&\&\ p_1 > 16\,t\,D_1\ \&\&\ t > 2\,q_o > 0\ \&\&\ 0 < \delta_c < 1\right]$$

Out[∘]=
$$p_1 > 0\ \&\&\ 0 < q_o < \frac{17\,p_1}{16}\ \&\&\ t > 2\,q_o\ \&\&\ 0 < \delta_c < 1$$

$$\Pi = \text{Simplify}\left[p_1\,D_1 + \delta_f\left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,(p_{2\,P}\,D_{2\,P} - D_1\,(p_1 - p_{2\,P})) + \frac{t\,D_1}{2\,q_o}\,(p_{2\,M}\,D_{2\,M} - D_1\,(p_1 - p_{2\,M})) + \frac{p_1}{2\,q_o}\,(p_{2\,N}\,D_{2\,N} - D_1\,(p_1 - p_{2\,N}))\right)\right];$$

$$CS = \text{Integrate}\left[\left(\text{Integrate}\left[Q - p_1 - t\,x + \delta_c\left(\frac{2\,q_o - p_1 - t\,D_1}{2\,q_o}\,(p_1 - p_{2\,P}) + \frac{t\,D_1}{2\,q_o}\,(p_1 - p_{2\,M}) + \frac{p_1}{2\,q_o}\,(p_1 - p_{2\,N})\right), \{x, 0, D_1\}\right]\right) \Big/ (2\,q_o), \{Q, 0, 2\,q_o\}\right] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c\,(Q - p_{2\,P} - t\,x), \{x, D_1, D_1 + D_{2\,P}\}]) / (2\,q_o), \{Q, p_1 + t\,D_1, 2\,q_o\}] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c\,(Q - p_{2\,M} - t\,x), \{x, D_1, D_1 + D_{2\,M}\}]) / (2\,q_o), \{Q, p_1, p_1 + t\,D_1\}] +$$
$$\text{Integrate}[(\text{Integrate}[\delta_c\,(Q - p_{2\,N} - t\,x), \{x, D_1, D_1 + D_{2\,N}\}]) / (2\,q_o), \{Q, 0, p_1\}];$$

```
t = 2.1;
qo = 1;
results = {};
ResultsVal = {};
ResultsP = {};
ResultsCon = {};
For[δc = 0.001, δc < 1, δc += 0.001,
  For[δf = 0.001, δf < 1, δf += 0.001,
   For[p1 = 0.001, p1 < 1, p1 += 0.001,
    If[Element[D1, Reals] && p1 > (2 qo + t D1)/3 && p1 > ((3 + 2 √2) t D1)/2 && p1 > 16 t D1,
     AppendTo[results, {Π, p1, CS}],
     AppendTo[results, {0, 0, 0}]]];
   {maxVal, P1, Consurplus} = Last@MaximalBy[results, First];
   AppendTo[ResultsVal, {maxVal}];
   AppendTo[ResultsP, {P1}];
   AppendTo[ResultsCon, {Consurplus}];
   results = {};
  ]];
PartitionedDataVal = Partition[Flatten[ResultsVal], 999];
PartitionedDataP = Partition[Flatten[ResultsP], 999];
PartitionedDataCon = Partition[Flatten[ResultsCon], 999];
TableForm[PartitionedDataVal,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataP,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
TableForm[PartitionedDataCon,
 TableHeadings → {Range[0.001, 0.999, 0.001], Range[0.001, 0.999, 0.001]}]
```