# MAT 201: Numerical Methods

## Lecture 2: Series and Sequence

Shahadat Hussain Parvez

# Series and Sequence

# Series and Sequence

- A sequence is a possibly infinite collection of numbers lined up in some order:

$$a_1, a_2, a_3 \quad \cdots$$

- A series is a possibly infinite sum:

$$a_1 + \ a_2 + \ a_3 \ + \cdots$$

# Convergence vs Divergence

- **Def 1.** A sequence $(a_j)_{j=0}^{\infty}$ is said to be -close to a number b if there exists a number N ≥ 0 (it can be very large), such that for all n ≥ N,

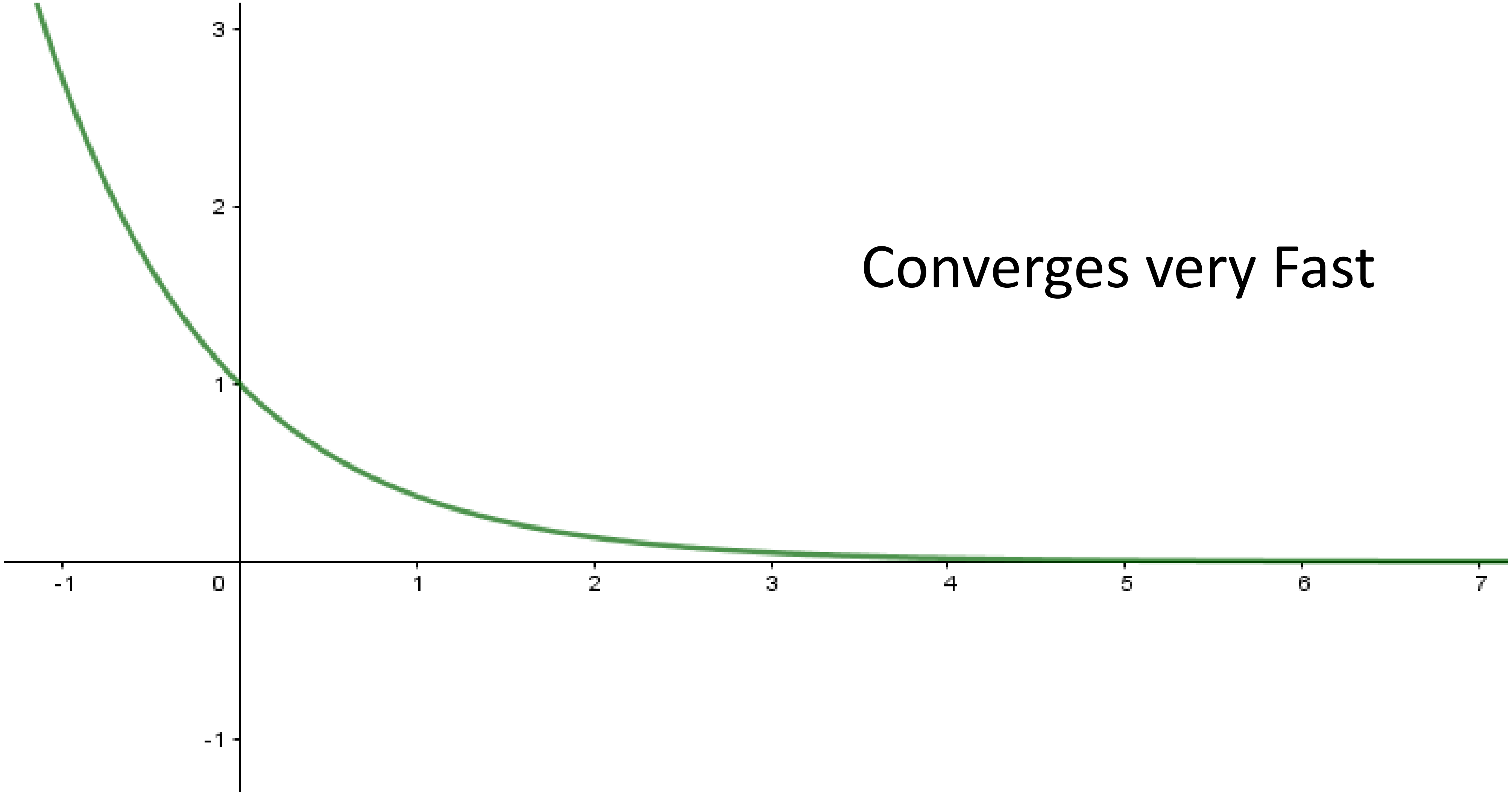
$$|a_j \ - b| \ \leq \varepsilon$$

- A sequence $(a_j)_{j=0}^{\infty}$ is said to converge to b if it is $\varepsilon$ -close to b for all f > 0 (however small). We then write $a_j \rightarrow b$, or $\lim_{j\to\infty} a_j = b$
- If a sequence does not converge we say that it diverges. Unbounded sequences, i.e., sequences that contain arbitrarily large numbers, always diverge. (So we never say “converges to infinity”, although it’s fine to say “diverges to infinity”.)

# Some examples

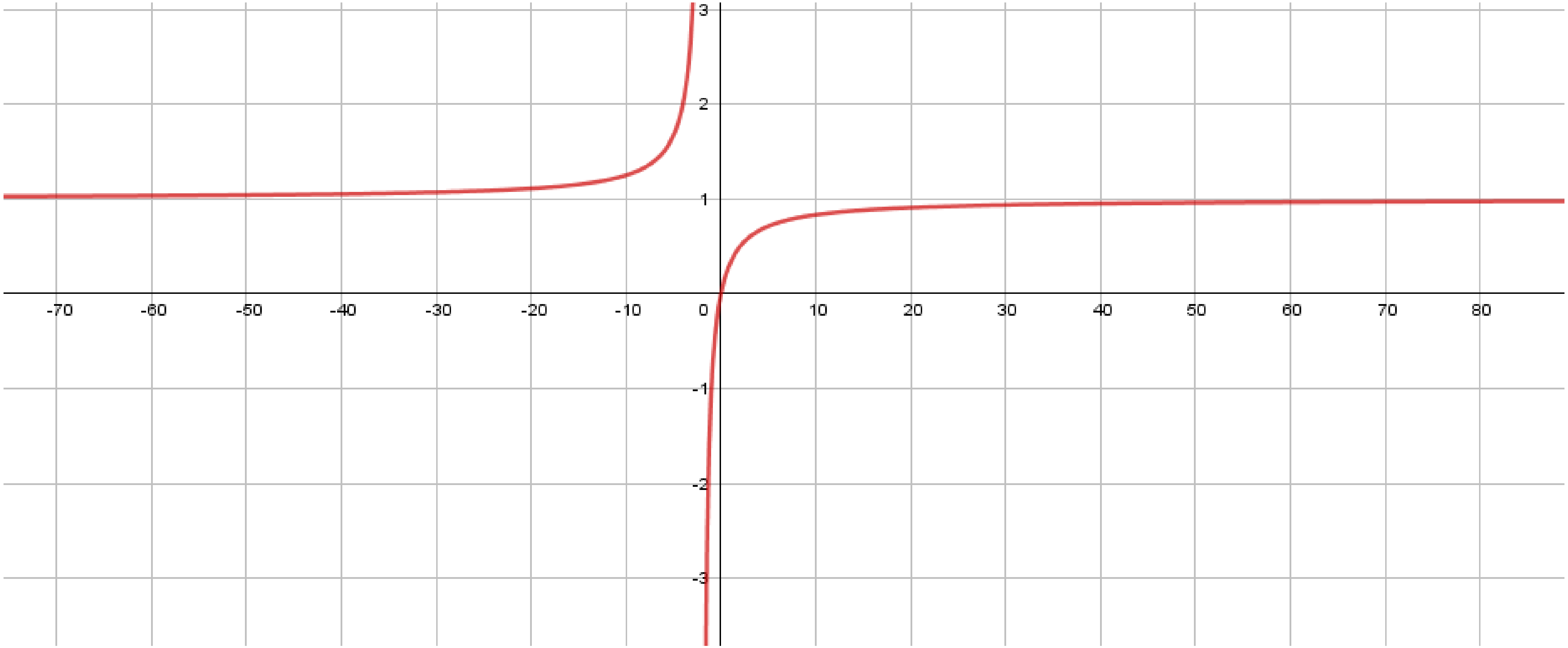
- $e^{-n} \to 0$ as $n \to \infty$, and convergence is very fast.
- $n/(n+2) \to 1$ as $n \to \infty$, and convergence is rather slow.
- $(-1)^n$ is bounded, but does not converge.
- $log(n) \to \infty$ as $n \to \infty$, so the sequence diverges. For a proof that $\log(n)$ takes on arbitrarily large values, fix any large integer $m$. Does there exist an $n$ such that $\log(n) \geq m$? Yes, it suffices to take $n \geq e^m$.

$$e^{-n}$$




Converges very Fast

$$n/(n+2) \to 1 \text{ as } n \to \infty$$




Converges Slowly

# Convergence vs Divergence

- **Def 2.** Consider a sequence $(a_j)_{j=0}^{\infty}$ . We define the N-th partial sum $S_N$ as

$$S_N = a_0 + a_1 + \cdots + a_N = \sum_{j=0}^{N} a_j$$

- We say that the series $\sum_j a_j$ converges if the sequence of partial sums $S_N$ converges to some number *b* as $N \rightarrow \infty$. Then we can write

$$\sum_{j=0}^{\infty} a_j = b$$

**Example 1.** *Consider* $\sum_{j=0}^{\infty} 2^{-j}$, *i.e.,*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots .$$

*This series converges to the limit 2. To prove this, consider the partial sum*

$$S_N = \sum_{j=0}^{N} 2^{-j}.$$

*Let us show by induction that* $S_N = 2 - 2^{-N}$. *The base case* $N = 0$ *is true since* $2^{-0} = 2 - 2^{-0}$. *For the induction case, assume* $S_N = 2 - 2^{-N}$. *We then write*

$$S_{N+1} = S_N + 2^{-(N+1)} = (2 - 2^{-N}) + 2^{-(N+1)} = 2 - 2^{-(N+1)},$$

*the desired conclusion.*

**Example 2.** *The previous example was the* $x = 1/2$ *special case of the so-called geometric series*

$$1 + x + x^2 + x^3 + \dots$$

*WIth a similar argument, we obtain the limit as*

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x},$$

*provided the condition* $|x| < 1$ *holds. This expression can also be seen as the Taylor expansion of* $1/(1-x)$*, centered at zero, and with radius of convergence 1.*

**Example 3.** *Consider the so-called harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots.$$

*This series diverges. To see this, let us show that the $N$ partial sum is comparable to* $\log(N)$. *We use the integral test*

$$S_N = \sum_{j=1}^{N} \frac{1}{j} \geq \int_1^{N+1} \frac{1}{x}\,dx.$$

*The latter integral is* $\log(N+1)$, *which diverges as a sequence. The partial sums, which are larger, must therefore also diverge.*

# Absolute Convergence

- **Def 3.** A series $\sum_{j=0}^{\infty} a_j$ is said to be absolutely convergent if $\sum_{j=0}^{\infty} |a_j|$ converges.
- If a series is not absolutely convergent, but nevertheless converges, we say that it is conditionally convergent.

# Taylor Series

# Taylor Series

- A Taylor series is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function ***f(x)*** about a point ***x=a*** is given by

$$f(x) =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2\,!}(x-a)^2 + \frac{f^{(3)}(a)}{3\,!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n\,!}(x-a)^n + R_n$$

- If ***a=0***, the expansion is known as Maclaurin Series.
- Taylor's theorem states that any function satisfying certain conditions can be expressed as a Taylor series.

# Taylor Series of common functions

$$\frac{1}{1-x} = \frac{1}{1-a} + \frac{x-a}{(1-a)^2} + \frac{(x-a)^2}{(1-a)^3} + \ldots$$

$$\cos x = \cos a - \sin a\,(x-a) - \frac{1}{2}\cos a\,(x-a)^2 + \frac{1}{6}\sin a\,(x-a)^3 + \ldots$$

$$e^x = e^a\left[1 + (x-a) + \frac{1}{2}(x-a)^2 + \frac{1}{6}(x-a)^3 + \ldots\right]$$

$$\ln x = \ln a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \ldots$$

$$\sin x = \sin a + \cos a\,(x-a) - \frac{1}{2}\sin a\,(x-a)^2 - \frac{1}{6}\cos a\,(x-a)^3 + \ldots$$

$$\tan x = \tan a + \sec^2 a\,(x-a) + \sec^2 a \tan a\,(x-a)^2 + \sec^2 a\left(\sec^2 a - \frac{2}{3}\right)(x-a)^3 + \ldots.$$

# Maclaurin Series

# Maclaurin Series

- A Maclaurin series is a Taylor series expansion of a function about 0,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

# Maclaurin Series of common functions

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$for\ -1 < x < 1$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$for\ all\ x$$

# Maclaurin Series of common functions

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

*for all x*

$$\sin x = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{50400}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

*for all x*

# Maclaurin Series of common functions

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty} \frac{-1}{n} x^n$$

$$for\ -1 < x \le 1$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots = \sum_{n=1}^{\infty} \frac{2}{(2n-1)} x^{2n-1}$$

$$for\ -1 < x < 1$$

***Check Lecture sheet for more functions***

# Computing Maclaurin Series of $e^4$

***See Similar example done in lecture 1***

# Estimating value of a function using Taylor Series

# Estimating value of a function using Taylor Series

- The accuracy of estimation increases with increase in the number of terms (Order of approximation)
- 1st term is known as zero order approximation
- As we increase more terms the order of approximation increases.
- Zero order approximation
  - Basically the function at new point is equal to that of old point
  - Accuracy will be very poor

$$f\left(x_{i+1}\right) \cong f\left(x_i\right)$$

# Estimating value of a function using Taylor Series

- First order approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

- Second order Approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)$$

# Estimating value of a function using Taylor Series

- The total approximation for n terms with the remainder can be written as

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{f^{(3)}(x_i)}{3!}(x_{i+1} - x_i)^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n$$

- $R_n$ term is added to account for the terms from n+1 to infinity.

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{(n+1)}$$

# Estimating value of a function using Taylor Series

- A more convenient form of Taylor series is to use

$$\boldsymbol{h = x_{i+1} - x_i}$$

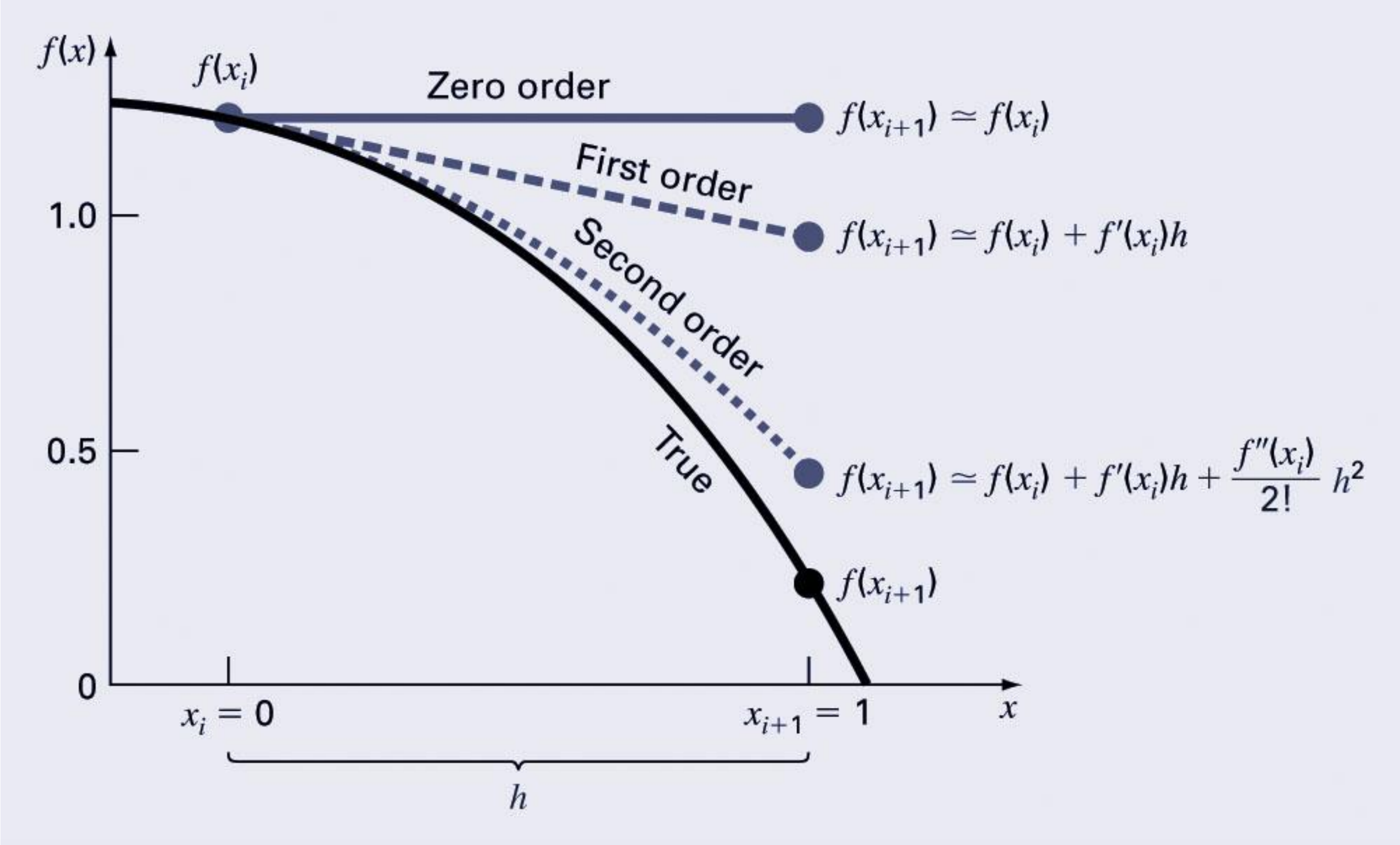
to get

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + R_n$$

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{(n+1)}$$

# Estimating value of a function using Taylor Series

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$



# Alternative Representation of Taylor Series

- As we have learned in previous section there is an alternative representation of Taylor series

$$f(x) =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2\,!}(x-a)^2 + \frac{f^{(3)}(a)}{3\,!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n\,!}(x-a)^n + R_n$$

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \quad \text{or} \quad R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{(n+1)}$$

## EXAMPLE 4.1 Taylor Series Approximation of a Polynomial

**Problem Statement.** Use zero- through fourth-order Taylor series expansions to approximate the function

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

from $x_i = 0$ with $h = 1$. That is, predict the function's value at $x_{i+1} = 1$.

**Solution.** Because we are dealing with a known function, we can compute values for $f(x)$ between 0 and 1. The results (Fig. 4.1) indicate that the function starts at $f(0) = 1.2$ and then curves downward to $f(1) = 0.2$. Thus, the true value that we are trying to predict is 0.2.

The Taylor series approximation with $n = 0$ is [Eq. (4.2)]

$$f(x_{i+1}) \simeq 1.2$$

Thus, as in Fig. 4.1, the zero-order approximation is a constant. Using this formulation results in a truncation error [recall Eq. (3.2)] of

$$E_t = 0.2 - 1.2 = -1.0$$

at $x = 1$.

For $n = 1$, the first derivative must be determined and evaluated at $x = 0$:

$$f'(0) = -0.4(0.0)^3 - 0.45(0.0)^2 - 1.0(0.0) - 0.25 = -0.25$$

Therefore, the first-order approximation is [Eq. (4.3)]

$$f(x_{i+1}) \simeq 1.2 - 0.25h$$

which can be used to compute $f(1) = 0.95$. Consequently, the approximation begins to capture the downward trajectory of the function in the form of a sloping straight line (Fig. 4.1). This results in a reduction of the truncation error to

$$E_t = 0.2 - 0.95 = -0.75$$

For $n = 2$, the second derivative is evaluated at $x = 0$:

$$f''(0) = -1.2(0.0)^2 - 0.9(0.0) - 1.0 = -1.0$$

Therefore, according to Eq. (4.4),

$$f(x_{i+1}) \simeq 1.2 - 0.25h - 0.5h^2$$

and substituting $h = 1$, $f(1) = 0.45$. The inclusion of the second derivative now adds some downward curvature resulting in an improved estimate, as seen in Fig. 4.1. The truncation error is reduced further to $0.2 - 0.45 = -0.25$.

Additional terms would improve the approximation even more. In fact, the inclusion of the third and the fourth derivatives results in exactly the same equation we started with:

$$f(x) = 1.2 - 0.25h - 0.5h^2 - 0.15h^3 - 0.1h^4$$

where the remainder term is

$$R_4 = \frac{f^{(5)}(\xi)}{5!}h^5 = 0$$

because the fifth derivative of a fourth-order polynomial is zero. Consequently, the Taylor series expansion to the fourth derivative yields an exact estimate at $x_{i+1} = 1$:

$$f(1) = 1.2 - 0.25(1) - 0.5(1)^2 - 0.15(1)^3 - 0.1(1)^4 = 0.2$$

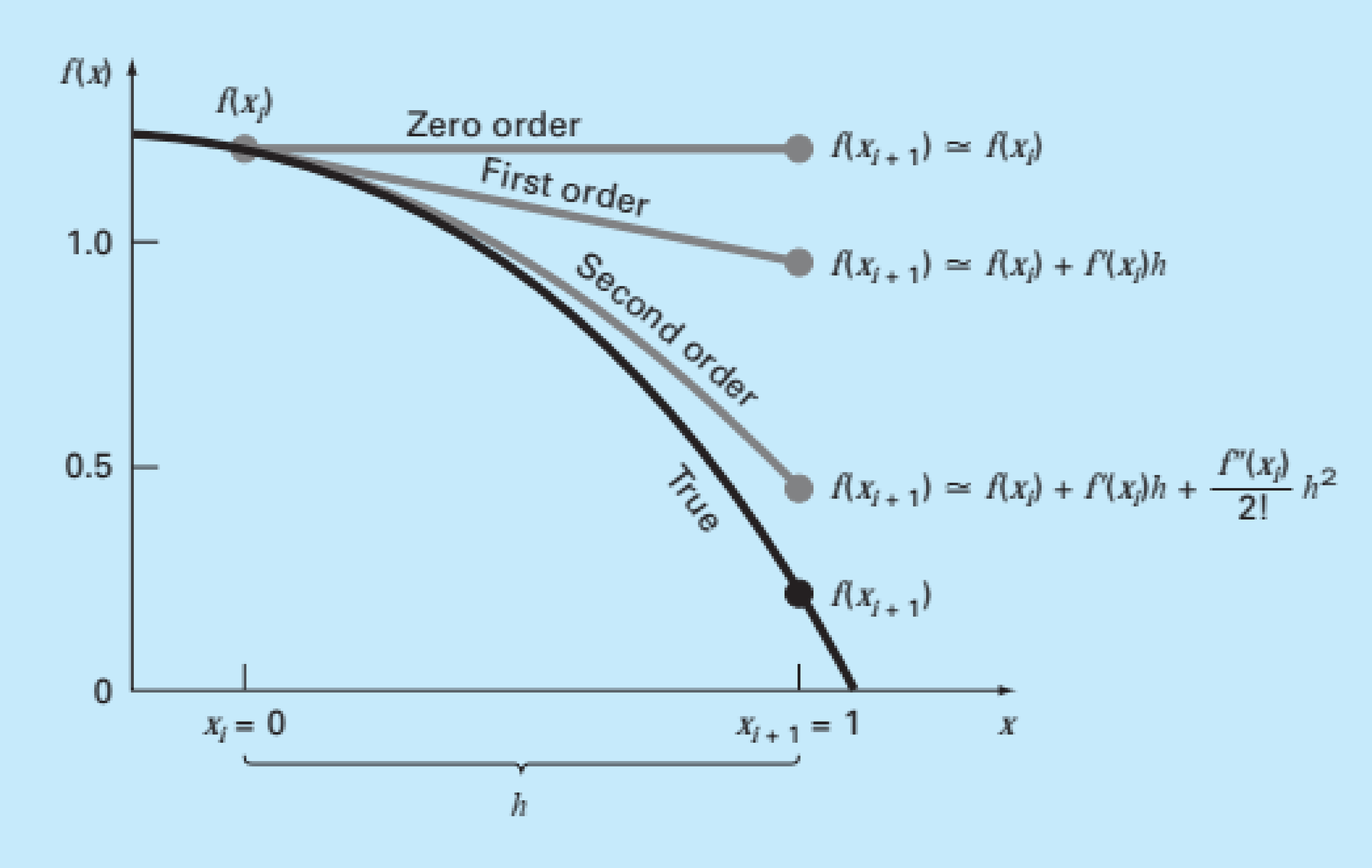



**FIGURE 4.1**
The approximation of $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ at $x = 1$ by zero-order, first-order, and second-order Taylor series expansions.

**EXAMPLE 4.2** Use of Taylor Series Expansion to Approximate a Function with an Infinite Number of Derivatives

Problem Statement. Use Taylor series expansions with $n = 0$ to 6 to approximate $f(x) = \cos x$ at $x_{i+1} = \pi/3$ on the basis of the value of $f(x)$ and its derivatives at $x_i = \pi/4$. Note that this means that $h = \pi/3 - \pi/4 = \pi/12$.

Solution. As with Example 4.1, our knowledge of the true function means that we can determine the correct value $f(\pi/3) = 0.5$.

The zero-order approximation is [Eq. (4.3)]

$$f\left(\frac{\pi}{3}\right) \cong \cos\left(\frac{\pi}{4}\right) = 0.707106781$$

which represents a percent relative error of

$$\varepsilon_t = \frac{0.5 - 0.707106781}{0.5} 100\% = -41.4\%$$

For the first-order approximation, we add the first derivative term where $f'(x) = -\sin x$:

$$f\left(\frac{\pi}{3}\right) \cong \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\left(\frac{\pi}{12}\right) = 0.521986659$$

which has $\varepsilon_t = -4.40$ percent.

For the second-order approximation, we add the second derivative term where $f''(x) = -\cos x$:

$$f\left(\frac{\pi}{3}\right) \cong \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\left(\frac{\pi}{12}\right) - \frac{\cos(\pi/4)}{2}\left(\frac{\pi}{12}\right)^2 = 0.497754491$$

with $\varepsilon_t = 0.449$ percent. Thus, the inclusion of additional terms results in an improved estimate.

The process can be continued and the results listed, as in Table 4.1. Notice that the derivatives never go to zero as was the case with the polynomial in Example 4.1. Therefore, each additional term results in some improvement in the estimate. However, also notice how most of the improvement comes with the initial terms. For this case, by the time we

**TABLE 4.1** Taylor series approximation of $f(x) = \cos x$ at $x_{i+1} = \pi/3$ using a base point of $\pi/4$. Values are shown for various orders ($n$) of approximation.

| Order $n$ | $f^{(n)}(x)$ | $f(\pi/3)$ | $\varepsilon_t$ |
|---|---|---|---|
| 0 | $\cos x$ | 0.707106781 | −41.4 |
| 1 | $-\sin x$ | 0.521986659 | −4.4 |
| 2 | $-\cos x$ | 0.497754491 | 0.449 |
| 3 | $\sin x$ | 0.499869147 | $2.62 \times 10^{-2}$ |
| 4 | $\cos x$ | 0.500007551 | $-1.51 \times 10^{-3}$ |
| 5 | $-\sin x$ | 0.500000304 | $-6.08 \times 10^{-5}$ |
| 6 | $-\cos x$ | 0.499999988 | $2.44 \times 10^{-6}$ |

have added the third-order term, the error is reduced to $2.62 \times 10^{-2}$ percent, which means that we have attained 99.9738 percent of the true value. Consequently, although the addition of more terms will reduce the error further, the improvement becomes negligible.

# Recommended Text for this lecture

- Numerical Methods for Engineers by Steven C. Chapra and Raymond P. Canale, McGraw Hills Education, $6^{th}$ Edition
  - Chapter 4

# Homework Based on Lecture 2

1. Use Maclaurin series to find approximation of the following functions until approximation error is below 0.1%. If the error is not below 6 terms you can stop
   a. $e^2$
   b. $e^{0.5}$
   c. $\sin 30^\circ$
   d. $\cos 45^\circ$
2. Use zero through third order Taylor series expansion to predict f(3) using a base point at x=1 $\boldsymbol{f(x)=25x^3-6x^2+7x-88}$. Compute the true percent relative error for each approximation

# Disclaimer

- This Presentation contains some edited version of slides provided by McGraw hills Education, organized by Dr. Michael R. Gustafson II, Duke University and Prof. Steve Chapra, Tufts University, to accompany the textbook.

- There is also some screenshots from different books.
- Some online images are also used.

*I have tried to cite any source. But if any citation is missed, kindly contact me to add your citation.

END