

Linear Algebra Self-Study

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Chapter 1

Vector and Matrices

1.1 My bad guys

So, the issue, right? I picked up LaTeX after I already finished chapter 1 , and nearly finished chapter 2. So to future me and any poor sods reading this, good luck lol.

Chapter 2

Solving Linear Equation $Ax = b$

2.1 Elimination and Back Substitution

I fucked up

2.2 Elimination Matrices and Inverse Matrices

This too.

2.3 Matrix Computation and $A = LU$

This one too.

2.4 Permutation and Transposes

Also this shit.

2.5 Derivatives and Finite Difference Matrices

Second difference matrices includes K, T, B . They all have the $-1, 2, -1$ pattern.

$$K_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Now we can approximate $-\frac{d^2u}{dx^2} = f(x)$. So, we want to compute $-\frac{d^2u}{dx^2}$ with a computer, but the computer can't understand derivative. So what we do is we turn $\frac{d^2u}{dx^2}$ into the matrix $\frac{K^2}{h}$, function $u(x)$ into vector u , and function $f(x)$ into F . We also need the boundary conditions, which are given where $u(0) = 0$ and $u(1) = 0$. We can't pick out the infinite space between 0 and 1, so we pick N equally spaced points at a regular interval. The space between each points (and the first and the last point) becomes meshwidth (h). If we have N internal points u_0, u_1, u_2, \dots plus two boundary points u_0 and u_{N+1} , we divide the total length into $N+1$ segments. Therefore the spacing is $h = \frac{1}{N+1}$. If we have 4 N , then the spacing is $h = \frac{1}{5}$. So instead of finding the continuous function $u(x)$, we will find the value at each internal points, and they becomes the unknown vector $U = [u_1, u_2, u_3, u_4]^T$.

$$-\frac{d^2u}{dx^2} = f(x) \text{ becomes } \frac{KU}{h^2} = F, \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \end{bmatrix}$$

The key point is that when divide the meshpoint into 4, therefore $N = 4$. Row 1 times U is $2u_1 - u_2$, and we already got the boundary where $u_0 = 0$ and $u_5 = 0$, making a typical row $\frac{(-u_1 + 2u_2 - u_3)}{h^2} = f(h)$. The division by h^2 makes $\frac{K}{h^2}$ a second difference matrix, replacing $-\frac{d^2u}{dx^2}$.

2.5.1 Properties of K

K has 4 properties. For the sake of example, we will use K for $N = 4$.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

1. K is symmetrical, as in $K_{ij} = K_{ji}$
2. K is banded. All the non-zeros $(-1, 2, -1)$ lie in a band around the main diagonal. The band has three diagonals, so K is a tridiagonal matrix.
3. K has a constant diagonals. A diagonal of -1, then 2, then -1 again. The matrix is called "shift-invariant", because the differential equation always have a constant coefficient of -1. The approximation to $-\frac{d^2u}{dx^2}$ is always $-1, 2, -1$ at every X .
4. X is invertible. It has an inverse matrix K^{-1} then $K^{-1}K = I$ and $KK^{-1} = I$.

$$K_4^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

K^{-1} is also symmetric but it is no diagonal. It is dense matrix, meaning no zeros.

5. Symmetric K_n matrices are positive definite.

Invertible, positive definite symmetric matrix, and semidefinite matrix are defined by their pivots.

1. Invertible matrices has nonzero pivots.
2. Positive definite symmetric matrices has positive nonzero pivots.
3. Positive semidefinite symmetric matrices has nonnegative pivots.

2.5.2 Free-fixed Matrice T_n

T_n and B_n are variations on K_n , where the variation comes from changing the boundary conditions. Think of it as an elastic band that are fixed at both ends, one end, or totally free at both ends. For example, T_n is very similar to K_n except that input $(1, 1)$ is switched from 2 to 1., representing a free boundary condition where $\frac{du}{dx} = 0$

$$\text{Free-fixed boundary conditions, still positive definite } T_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

T is no longer Toeplitz because its main constant, though it does have a simpler factorization than K; every pivot of T equals 1.

$$T = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ & 1 & -1 & 0 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} = LU$$

Note that $U = L^T$. Notice that U is a forward difference while L is a backward difference. Together, they add up to be a second difference, meaning that $x_{i+1} - 2x_i + x_{i-1}$ correspond to $[-1, 2, -1]$, meaning that T is a second difference.

2.5.3 The Free-Free Matrices B_n are Singular

In this context, "singular" means "not invertible". One test is simply seeing if determinant equals zero or not.

Theorem 2.5.1. *If B multiplies a nonzero vector x to produce $Bx = 0$, then B can't be invertible.*

For example, free-free matrix has 1 (and not 2) in its (1, 1) and (3, 3) input.

$$B_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ has } B_3 x = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

B_n is singular while T_n and K_n are invertible because T_n and K_n have a fixed end, which allows them to adjust, whereas B_n is free on both end so it can't adjust.

Chapter 3

The Four Fundamental Subspaces

How can we define "vector space"? Well, if we are talking about R^3 , the key operations are $v + w$ and cv . Notice that v and w could be matrices, so we could have matrix spaces and function spaces. Then inside R^n we could only allow x that satisfies $Ax = 0$, which will produce "nullspace of A ". All combinations of solutions to $Ax = 0$ are also solutions, meaning that the nullspace is a subspace. To point it simply, nullspace is just x in Ax that crushes it down to 0. Why is it called a space? Because it has structures and rules.

1. It must contain zero vector, $A \times 0 = 0$ always.
2. It must be closed under addition, meaning that if you take two vectors from the nullspace then add them together, they must still be in the nullspace.
3. It must be closed under scalar addition, meaning that if you take any vector from the nullspace then multiplies them by a constant, the result must still be in the nullspace.

Then, lastly, there are basis. A set of vectors that perfectly describes the space. Very important, some consider it the fundamental theorem. So, what it means is that basis are just sets of movement vectors for each dimension.

1. For a 3D space, you will need forward, right, and up.
2. For a 2D space, you will need right and up.
3. For a 1D space (line of sight, nullspace), you only need the direction of the line.

And so, $n - r$ special solutions to $Ax = 0$ are a basis for $N(A)$. It's called rank-nullity theorem.

Dimension of Input Space = Dimension of Column Space + Dimension of Nullspace, $n = r + (n - r)$

Note that $n - r$ is the dimension of the nullspace. One last note: THIS CHAPTER IS VERY IMPORTANT.

3.1 Vector Spaces and Subspaces

Here are a few fundamental points:

1. All linear combinations of $cv + dw$ must stay in the vector space (What is a vector space? Very simply, all the possible spaces that can be achieved by your given vector with vector addition or scalar multiplication), where c and d are scalar, and v and w are vectors.
2. The row space (meaning all possible linear combinations of the row vectors) is "spanned" (made up of) rows of A . While the column of A spans $C(A)$.
3. Matrices can be filled by more than just numbers. As long as it obeys the rules of a vector space, it can be treated as a vector. For example, we have two equations, $f(x) = x^2$ and $g(x) = 2x$. Can they be added together? Yes! $h(x) = x^2 + 2x$. Now if we fill it with the likes of \sin, \cos, x, x^2 , we can "span" and build a lot of other functions. For example, all quadratic polynomials are "spanned" by the functions $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$

R^n contains all column vector v to the length of n . For this case, the components from v_1 to v_n are all real numbers. However, if they allow for complex numbers (i), the R^n becomes C^n . To reiterate, all linear combination of $cv + dw$ must be in the vector space R^n . For example, all positive the set with all positive (meaning no vector consist of ANY nonpositive numbers) vectors (v_1, \dots, v_n) are NOT a vector space. Why? Simply take one simple vector, say $(1, 2)$ then multiply it by scalar of, say, $c = -1$. $(-1, -2)$ is NOT in our set, therefore it is not a vector space. Or for another example, a set of solution for $Ax = (1, \dots, 1)$ is not a vector space because a line in R^n is not a vector space unless it goes the central point $(0, \dots, 0)$.

3.1.1 Examples of Vector Spaces

Here are some examples of a neat vector space, the Z (zero vector) where $0 = (0, 0, \dots, 0)$. Combinations of $c0 + d0$ are all still 0, so still in the subspace. How about vector space of matrices? We can do that. $R^{3 \times 3}$ is a space that contains all 3×3 matrices. It does satisfy all eight rules, so why not? It's also a vector space. How about a vector space of functions? Sure can. The line of functions $y = ce^x$ (any c) is a line in a function space. This line contains all solutions to the differential equations of $\frac{dy}{dx} = y$. Yet another function space contains all quadratics $y = a + bx + cx^2$, where they are the solutions to $\frac{d^3y}{dx^3}$. And to reiterate, space in this context means all possible linear combination of the vectors or matrices or functions, and they all stay inside it.

3.1.2 Subspaces of Vector Spaces

What are subspaces? To put it simply, they are a flat plane inside the dimensional space, however, they are still the same dimension. Let's say, we got a R^3 space. We can make a plane any way we want as long as it passes $(0, 0, 0)$, what we get may look like a 2D plane, but it's still 3D. Therefore, the plane is a subspace of the full vector space R^3 .

Here is a list of possible subspaces of R^3 :

1. Any line through $(0, 0, 0)$
2. Any plane through $(0, 0, 0)$
3. The whole space R^3
4. The zero vector $(0, 0, 0)$

3.1.3 The Column Space of A

What we are trying to solve here is $Ax = b$. We want to know b , right? Well, b are a column space of A . Ax is just a combination of A , and to get every possible b , we need all possible x , which is just all linear combination of A , which is the column space of A , as written earlier. To build on that, vector space is made up of column vectors.

A crucial point to understand is that, **to solve $Ax = b$ is just to express b as a combination of the columns.** b got to be in the column space of A , otherwise, it doesn't exist!

Caution: columns of A do not form a subspace. Neither do invertible matrices, or singular matrices. Only all linear combinations.

3.1.4 The Row Space of A

The rows of A are the column of A^T , why do we do this? Because we like working with columns, so we use the column of A^T

The row space of A is just the column space of A^T

3.2 Computing the Nullspace by Elimination: $A = CR$

1. The **nullspace $N(A)$** in R^n contains all solutions x to $Ax = 0$, including $x = 0$.
2. CONTINUE THIS LATER

The goal of this section is to find all solutions for $Ax = 0$. If A is an invertible matrix, then the only solution is $x = 0$. In general, A has r independent columns, the other $n - r$ are a linear combination. Here is a matrix R with rank $r = 2$, with $n = 4$ columns. This means we have $n - r = 4 - 2 = 2$ independent solutions to $Rx = 0$. So the nullspace $N(R)$ will have 2 dimensions.

Example 1: $R = [IF]P = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix}$

Which means $Rx = 0$ is $x_1 + 3x_3 + 5x_4 = 0$ and $x_2 + 4x_3 + 6x_4 = 0$. We find the special solutions by just letting x_3 and x_4 equal 1 and 0 or 0 and 1. Set $x_3 = 1, x_4 = 0$ the equations will give $x_1 = -3, x_2 = -4$. Set $x_3 = 0, x_4 = 1$ the equations will give $x_1 = -5, x_2 = -6$. This gives us two special solutions: $s_1 = (-3, -4, 1, 0)$ and $s_2 = (-5, -6, 0, 1)$. They are both in the nullspace of R , as we can also see, $cs_1 + ds_2$ is still in the the nullspace. And so, s_1 and s_2 are the basis of nullspace.

Example 2: $R_0 = \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Which means $x_1 + 7x_2 + 0x_3 + 8x_4 = 0$, $x_3 + 9x_4 = 0$, and $0 = 0$ (wow on the last one). The matrix identity is inside column 1 and 3, and row 3 is all zero, which makes it a reduced row echelon form, even elimination can't make it simpler. We still have free variables for the special solution, namely, x_2 and x_4 . Set $x_2 = 1, x_4 = 0$, the equations give $x_1 = -7, x_3 = 0$. Set $x_2 = 0, x_4 = 1$, the equations give $x_1 = -8, x_3 = -9$. The special solutions are noe $s_1 = (-7, 1, 0, 0)$ and $s_2 = (-8, 0, -9, 1)$.

First, we start with any m by n matrix A , then apply elimination. That changes A into its reduced row echelon form, $R_0 = \text{rref}(A)$. Removing all zero rows of R_0 leaves R .

$r, m, n = 2, 2, 4$ Simplest Case $R = [IF]$ as in $\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix}$

$r, m, n = 2, 3, 4$ General Case $R_0 = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} P$ as in $\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hold up, what is rref, I, F here? For something to be rref, you need to satisfy 4 conditions.

1. The zero row is at the bottom
2. The first non-zero in row 1 is 1, the first non-zero entry in row 2 is 1.
3. Each pivot is to the right of the pivots in the rows above it.
4. Every pivot is only non-zero number in its entire column.

I in this context is still identity, but only if you take the pivot columns and align them chronologically as they were. F is free matrix, the other non-pivot columns, still in the same chronologically order.

In this case (R_0),

$$I \text{ is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, F \text{ is } \begin{bmatrix} 7 & 8 \\ 0 & 9 \\ 0 & 0 \end{bmatrix}$$

3.2.1 Elimination from A to $\text{rref}(A)$: Reduced Row Echelon Form

Refresher, how does elimination works?

1. Subtract a multiple of one row from another row
2. Multiple a row by nonzero number
3. Exchange any rows

For demonstration,

$$A = \begin{bmatrix} 1 & 2 & 11 & 17 \\ 3 & 7 & 37 & 57 \end{bmatrix} \text{ then } \begin{bmatrix} 1 & 2 & 11 & 17 \\ 0 & 1 & 4 & 6 \end{bmatrix} \text{ then } \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix}$$

So, what did elimination actually do? It inverted the leading 2 by 2 matrix, which we will call W .

$$W = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \text{ into } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We multiplied $W^{-1}A = W^{-1}[WH]$ to produce $R = [IW^{-1}H] = [IF]$. We always knew that free columns (H) is some combination of independent columns (W), but we now know that $H = WF$.

$$H = \begin{bmatrix} 11 & 17 \\ 37 & 57 \end{bmatrix} = WF = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

However you compute R from A , you will always get the same R . R is completely controlled by A .

For **example 2**, let us rref another A .

$$A = \begin{bmatrix} 1 & 7 & 3 & 35 \\ 2 & 14 & 6 & 70 \\ 2 & 14 & 9 & 97 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 3 & 35 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 27 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 27 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R_0$$

3.2.2 Elimination Column by Column: The Steps from A to R_0

We will now reduce what we learn to an easily applied algorithm. The big question is **does this new column $k+1$ join with I_k or F_k ?**

If l is all zero, the new column is **dependent** on the first k columns. Then u joins with F_k to become F_{k+1} .

If l is not all zero, then it is independent of the first k columns. Use the **largest** number, preferably, in l as the pivot. **Important** thing to remember here is that the column are talking about means the columns **UNDERNEATH** all pivots, not **ALL** columns. Then do elimination. Whatever is left becomes part of I as I_{k+1} .

From example 2, we can see that the combination of independent and dependent comes out to

$$C \times F = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 35 \\ 14 & 70 \\ 14 & 97 \end{bmatrix} = \text{depend columns of 2 and 4 of } A$$

Right back to where we came from, showing that C are almost like the ingredients and F are almost like the method.

3.2.3 The Matrix Factorization $A = CR$ and the Nullspace

In chapters prior, we know that $A = CR$ but we have no systemic way to find them, now we do. We apply elimination to reduce A to R_0 . Then I in R_0 locates the matrix C of independent columns in A . Removing zero row in R_0 produces R for $A = CR$

We have two special solution s_1 and s_2 for every column of F in R .

$$Rs_1 = 0, \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} -7 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Rs_1 = 0, \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} -8 \\ 0 \\ -9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

s_1 and s_2 are the easiest to see using the matrices $-F$ and I and P^T .

As a reminder, the two special solutions to $[IF]Px = 0$ are the columns of $P^T \begin{bmatrix} -F \\ I \end{bmatrix}$

It more correct than the other option because PP^T is the identity matrix of permutation matrix P:

$$\mathbf{R}\mathbf{x} = \mathbf{0}, [IF]P \times P^T \begin{bmatrix} -F \\ I \end{bmatrix} \text{ reduces to } [IF] \begin{bmatrix} -F \\ I \end{bmatrix} = [0]$$

Review Say, the m by n matrix A has rank r . We can find $n - r$ special solution to $Ax = 0$ by computing the rref R_0 of A . Remove the $m - r$ zero rows of R_0 to produce $R = [IF]P$ and $A = CR$. Then the special solutions to $Ax = 0$ are the $n - r$ columns of $P^T[-F, I]^T$

Example 3: Elimination of A gives R_0 and R . R reveals the nullspace of A

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R_0 \text{ with rank 2}$$

$$R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The independent columns are 1 and 3.

To solve $Ax = 0$ and $Rx = 0$, set $x_2 = 1$, which will get you $x_1 = -2, x_3 = 0$. Leave us special solution:

$$\mathbf{s} = (-2, 1, 0)$$

All solutions $x = (-2c, c, 0)$. And here it is, $A = CR$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} = CR = \begin{bmatrix} 1 & 1 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{columns basis in } C \times \text{row basis in } R.$$

For a lot matrices, the only solution to $Ax = 0$ is $x = 0$. Simply, all columns of A are independent. The nullspace $N(A)$ contains only the zero vector, no special solution. This case zero nullspace is **important** because it means that all columns of A is independent. But this can't happen if $n > m$ (column > row) because you can have n independent column in R^m .

Important Say A has more columns than rows ($n > m$), there will be at least one free variable. Meaning that $Ax = 0$ has at least one non-zero solution. Or to put it more specifically, there must be more than $n - m$ free columns. $Ax = 0$ must have nonzero solutions in $N(A)$.

Example 4: Find the nullspace of A, B, M and the two special solutions to $Mx = 0$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix}, M = [A \quad 2A] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

Solution The equation $Ax = 0$ has only the zero solution $x = 0$. The nullspace is only Z .

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R = I$$

No free variables, meaning A is invertible; therefore, no special solution.

The M matrix is different. It has extra columns instead of rows. That means that, with 4 columns and 2 rows, there will be 2 free columns leftover.

$$M = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 8 & 6 & 16 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = [I \quad F]$$

Again, to get special solutions out of it, we will let $x_3 = 1, x_4 = 0$ and $x_3 = 0, x_4 = 1$. What we will get is the special solution for the nullspace of M .

$$Mx = 0R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

3.2.4 Block Elimination in Three Steps: Final Thoughts

We will conclude nicely with three steps to block elimination.

Step 1 Exchange the columns and rows of P_C and P_R so that that r independent columns and rows come first in $P_R A P_C$

$$P_R A P_C = \begin{bmatrix} W & H \\ J & K \end{bmatrix}, C = \begin{bmatrix} W \\ J \end{bmatrix} \text{ and } B = \begin{bmatrix} W & H \end{bmatrix}$$

Step 2 Multiple the top rows by W^{-1} to produce $W^{-1}B = [I, W^{-1}H] = [I, F]$.

Step 3 Subtract $J[I, W^{-1}H]$ from $[J, K]$ to produce $[0, 0]$.

The results of the steps should be an rref form of R_0

$$P_R A P_C = \begin{bmatrix} W & H \\ J & K \end{bmatrix} \rightarrow \begin{bmatrix} I & W^{-1}H \\ J & K \end{bmatrix} \rightarrow \begin{bmatrix} I & W^{-1}H \\ 0 & 0 \end{bmatrix} = R_0$$

There are two things that need to be remembered.

1. W is invertible

2. The block satisfies $JW^{-1}H = K$

1. We must think back to $A = CR$. We can see that $B = WR$, and since B and R have the rank of r and W is also r by r , that means that W must have a rank of r and be invertible.

2. We know that the first row $[I, W^{-1}H]$ is linearly independent. Since A has the rank r , it means that the lower row $[J, K]$ must be a combination of the upper rows. This means for the combination to be valid, $JI = J$ and $JW^{-1}H = K$.

$$\text{The conclusion is that } P_R A P_C = \begin{bmatrix} W \\ J \end{bmatrix} W^{-1} \begin{bmatrix} W & H \end{bmatrix} = C W^{-1} B.$$

3.3 The Complete Solution to $Ax = b$

Our goal in this section will be about:

1. The complete solution to $Ax = b$: $x = x_p + x_n$, where p starts for any particular x and n nullspace.
2. Elimination from $Ax = b$ to $R_0x = d$: Solvable when zero rows of R_0 have zero in d .
3. When $R_0x = d$ is solvable, one x_p has all free variable equal to zero.
4. A has full column rank $r = n$ when its nullspace $N(A) = \text{zero vector}$: no free variables.
5. A has full row rank $r = m$ when its column space $C(A)$ is R^m : $Ax = b$ is always solvable.

The biggest thing that changed is that the b of $Ax = b$ is now not zero. Therefore, the row operation will also act on the right side, the b side. $Ax = b$ is reduced to a simpler $R_0x = d$ with the same, if any, solutions. One way to organize that is by augmenting (adding another column to the right side of the matrix). We can augment A with the right side of $(b_1, b_2, b_3) = (1, 6, 7)$ to produce augmented matrix $[Ab]$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \text{ has the augmented matrix } \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{bmatrix} = [Ab]$$

Now if we turn it into its rref form

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \text{ has the augmented matrix } \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [R_0d]$$

The last row is very important. Third equation became $0 = 0$, each means it can be solved. In the original matrix, the first row plus the second row equals the third row. Meaning to solve $Ax = b$ we need $b_1 + b_2 = b_3$, which led to $0 = 0$ in the third equation.

3.3.1 One Particular Solution $Ax_p = b$

To get an easy x_p , let the free variables be zeros: $x_2 = x_4 = 0$, and the two nonzero equations be the two pivot variables $x_1 = 1, x_3 = 6$. So our x_p is $x_p = (1, 0, 6, 0)$. To put it simply, **free variables = zero, pivots = variable from d**.

For a solution to exist, zero rows in R_0 must also be zero in d .

$$R_0 x_p = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

Notice how the complete solution includes all x_n :

$$x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Example 1 Find the condition on b_1, b_2, b_3 for $Ax = b$ for

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \text{ and } \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution Elimination using augmented matrix $[A, b]$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix} = [R_0 d]$$

And since there is no special solution ($n - r = 2 - 2 = 0$), the nullspace solution is $x_n = 0$. So the complete solution is

$$x = x_p + x_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Every matrix A with full column rank ($r = n$) has these properties :

1. All columns are independent, no free variables.
2. The nullspace $N(A)$ include only the zero vector $x = 0$
3. If $Ax = b$ has a solution, then it has only one solution.

With full column rank, $Ax = b$ will have **one or no solution only**.

3.3.2 Full Row Rank and the Complete Solution

Another extreme case is the full row rank. Now $Ax = b$ has one or infinitely many solutions. Full row rank requires that the matrix be a short and wide one ($m > n$, row $q > \text{col } q$), and every row is independent. For **example 2**, $Ax = b$ has 3 n but only two m .

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix} \text{ rank } r = m = 2$$

Imagine them as plane in xyz space. There are two planes, colliding into a line. The particular solution is a spot on that line, and the nullspace vector will move us along that line. $x = x_p + \text{all } x_n$ gives us the whole line solution.

Fast forward a bit, getting ex. 2 into the $[Ab]$ form gives us:

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} = [R, d]$$

This particular solution $(2, 1, 0)$ has free variable $x_3 = 0$. The special solution has $s_3 = 1$, and the $-x_1$ and $-x_2$ comes from the free column of R .

Check that x_p and s satisfies $Ax_p = b$ and $As = 0$

$$2 + 1 = 3, 2 + 2 = 4 - 3 + 2 + 1 = 0, -3 + 4 - 1 = 0$$

Remove that the nullspace solution x_n is just any multiple of s .

$$\text{Computer solution: } x = x_p + x_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Every matrix A with a full row rank ($r = m$) has all these properties:

1. All rows have pivot and R_0 has no zero rows: $R_0 = R$
2. $Ax = b$ has a solution for every right side b .
3. The column space of A is the whole space R^m .
4. If $m < n$, the equation has many solutions (called underdetermined in formal language).
5. The rows are linearly independent.

There are **four** possibilities for linear equation depend on rank r

1. $r = m$ and $r = n$ Square and invertible, $Ax = b$ has 1 solution.
2. $r = m$ and $r < n$ Short and wide, $Ax = b$ has infinite solutions.
3. $r < m$ and $r = n$ Tall and thin, $Ax = b$ has 0 or 1 solutions.
4. $r < m$ and $r < n$ Not full rank, $Ax = b$ has 0 or infinite solutions.

The reduced R_0 will fall in the same category as matrix A . For $R_0x = d$ and $Ax = b$ to be solvable, d must end in $m - r$ zeros.

Four types for R_0

$$\begin{bmatrix} I \end{bmatrix}, r = m = n$$

$$\begin{bmatrix} I & F \end{bmatrix} r = m < n$$

$$\begin{bmatrix} I \\ 0 \end{bmatrix} r = n < m$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} r < m, r, n$$

Case 1 and 2 have full row rank $r = m$. Case 1 and 3 have full column rank $r = n$.

3.4 Independence, Basis, Dimension

How big is the true size of a subspace? A matrix might be m by n but the column space is not necessarily n . The column space $C(A)$ is measured by independent columns. This will be clarified later.

Our goal here is to understand a **basis: independent vectors that "spans" a space**. Every vector in the space is a unique combination of the basis vectors. Some vague explanation of the terms:

1. Independent vectors, no extra vectors
2. Spanning a space, enough vectors to produce the rest.
3. Basis for a space, not too many and not too few
4. Dimension of a space, the number of vectors in every basis.

3.4.1 Linear Independence

Here's an odd definition for independence:

The columns of A are linearly independent when the only solution of $Ax = 0$ is $x = 0$. No other combination of Ax gives

To illustrate why: it is impossible for three vectors that are not in a plane to be in a plane **unless** x is 0, i.e.,

$$0v_1 + 0v_2 + 0v_3 + \dots$$

Or to put it in other words:

Linear independence only happens when $x_1v_1 + x_2v_2 + \dots = 0$ and all $x_i = 0$.

One point to drive is that vectors are either dependent or independent, no in between. For some examples:

1. $(1, 0)$ and $(1, 0.000001)$ are independent.
2. $(1, 1)$ and $(-1, -1)$ are dependent.
3. $(1, 1)$ and $(0, 0)$ are dependent because of the zero vector.
4. In \mathbb{R}^2 : Any three vectors $(a, b), (c, d), (e, f)$ are dependent.

Example 1: The columns of this A are dependent, $Ax = 0$ has a nonzero solution:

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{ is } -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Question, how do we fix this? Just do elimination.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \rightarrow R_0 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. F = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, x = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Remember that **columns of A are independent exactly when $r = n$** , there are n pivots and zero free variable. $x = 0$ is the only nullspace.

Also remember that any set of n vectors must be linearly dependent if $n > m$.

3.4.2 Vectors that Span a Subspace

To reiterate, column space consists of all combination of $x_1v_1 + \dots + x_nv_n$. The word "span" describes $C(A)$.

The columns of a matrix span its column space. They might be dependent.

Example 2: We will try to describe the column space and row space of A .

$$m = 3, n = 2 : A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}$$

We could say that the column space of A is a plane in R^3 spanned by two columns of A . The row space of A is spanned by three rows of A (which are columns in A^T) in R^2 . Oh, and the rows span R^n and columns span R^m . Yes, they are swapped.

3.4.3 A Basis for a Vector Space

To reiterate, the basis means just right. Two independent vectors can't span R^3 , four vectors can't be all independent even if they span R^3 . Three independent vectors for R^3 is just right.

Basis vectors are independent and they span the space

Note that there is **only one way to write v as a combination of the basis vector**

Example 3: The column of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the standard basis for R^2 .

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ are independent, and they span } R^2$$

Example 4: The column of every invertible n by n matrix give a basis for R^n :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

We can see that A is **invertible, independent**, with a rank = 3. B is singular matrix, which also have dependent columns, and it is not full column rank (column space $\neq R^3$). The only solution to $Ax = 0$ is $x = A^{-1}0 = 0$. However, for $Ax = b$ can always be solved by $x = A^{-1}b$. Everything comes together for invertible matrices. **The vector v_1, \dots, v_n are a basis for R^n exactly when they are the column of an n by n invertible matrix. R^n also has infinitely many bases** Or to put it more compactly, **every set of independent vectors can be extended to a basis; the spanning set of vectors can be reduced to a basis.**

Example 5: The matrix is not invertible so its column are not basis for anything.

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \rightarrow R_0 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Example 6: Find the bases for the column and row spaces of this rank two matrix.

$$R_0 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Like us see, so for column space of R_0 , we got two pivots in a R^3 , where $m = 3$; we got a subspace in R^3 . And for row space, we got two non-zero rows and a rank of R^4 where $n = 4$.

Question! Given five vectors in R^7 , how do we find their a basis that they span on? Two ways,

1. Make them into a row and eliminate to find their non-zero.
2. Make them into columns, and eliminate to find pivots. The pivot columns are the basis.

Another question, can another basis for the same vector space have more or less vectors? Answer, no! Number of vectors are tied to the space.

3.4.4 Dimension of a Vector Space

This chapter is proof heavy, but the gist of it is simply that there can be many vector bases to choose from but no matter what, to the same vector space, there will always be the same amount of vectors.

That is what dimension means, the number of basis vectors.

3.4.5 Bases for Matrix Spaces and Function Spaces

We should also know that **independence, basis, dimension** are not just limited to column vectors. We can also ask whether specific matrices A_1, A_2, A_3 are independent or not. Say, a 3×4 matrix space, what dimension is that? 12 dimensions since you need twelve matrices in every basis.

In differential equation, $\frac{d^2y}{dx^2} = y$ has a space of solutions. One of the basis is $y = e^x$ and $y = e^{-x}$. Oh, and the basis function gives dimension = 2 for this solution space because the linear equation starts with the second derivative.

Matrix space The vector space \mathbf{M} contains all 2 by 2 matrices, it's dimension is 4.

$$A_1, A_2, A_3, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A$$

These matrices are linearly independent, they can produce any matrix in \mathbf{M} .

We can see from $c_1A_1 + \dots + c_4A_4$ that the only way to get 0 is if all coefficients are 0. We can use this basis for many things. A_1, A_2, A_4 are a basis for the upper triangle matrices, A_1, A_4 for diagonal matrices. What about for symmetric matrices? $A_1, A_4, A_2 + A_3$.

If list out a few more dimensions:

1. The dimension of whole n by n matrix space is n^2
2. The dimension of the subspace of upper triangular matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$
3. The dimension of diagonal matrices is n .
4. The dimension of the subspace of symmetric matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$

Oh, and remember that Z is a vector that contains only zero vector. Its basis is a **empty set**, the one with no vector. NEVER allow zero vector into the basis because then nothing is independent.

For summary,

1. Columns of A are independent if $x = 0$ is the only solution to $Ax = 0$.
2. v_1, \dots, v_r **span** a space if their combinations fill that space.
3. Basis consists of linearly independent vectors that span the space. Ever vector in the space is a unique combination of the basis vector.
4. The number of basis vectors per basis is entirely dependent on the space. The number of vectors in a basis is the dimension of the space.
5. Pivot columns are one basis for the column space. The dimension of $C(A)$ is r

3.5 Dimensions of the Four Subspaces

1. The column space $C(A)$ and the row space $C(A^T)$ both have dimension r , the rank of A .
2. The nullspace $N(A)$ has dimension $n - r$. The left nullspace $N(A^T)$ has dimension $m - r$.
3. Elimination from A to R_0 changes $C(A)$ and $N(A^T)$ but their dimension doesn't change.

We must separate the difference between rank and dimension. The rank of a matrix counts independent columns, the dimension counts the number of vectors in a basis. We can count both pivots and basis vector. Rank reveals the dimension of the four fundamental subspaces.

1. Row space, $C(A^T)$, a subspace of R^n , its dimension is r
2. Column space, $C(A)$, a subspace of R^m , its dimension is r
3. Nullspace, $N(A)$, a subspace of R^n , its dimension is $n - r$
4. Left nullspace, $N(A^T)$, a subspae of R^m , its dimension is $m - r$

Keep in mind that row space is simply just the column space of A^T . Remember again that the row space and column space have the same dimension r .

3.5.1 The Four Subspaces for R_0

We will now try to identify the four subspaces in R_0 . Though the main point is that **the four dimensions are the same for A and R_0**

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can that see That

1. The pivot row is row 1 and 2.
2. The pivot column is column 1 and 3.
3. $r = 2$

Now, finding the nullspace, $n - r = 5 - 2 = 3$. We will need to find 3 special solutions.

1. Special solution 1 = $(-3, 1, 0, 0, 0)$
2. Special solution 2 = $(-5, 0, 1, 0, 0)$
3. Special solution 3 = $(-7, 0, 0, -2, 1)$

Now we will find the nullspace of $(R_0^T$ aka the left nullspace. Its dimension should be $m - r = 3 - 2 = 1$

$$(R_0^T y = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ is solved by } y = \begin{bmatrix} 0 \\ 0 \\ y_3 \end{bmatrix}$$

Wonder why it's called **left nullspace**? It's because we can transpose $(R_0^T y = 0$ to $y^T R_0 = 0^T$, the y^T is now a row vector to the left of R .

3.5.2 The Four Subspaces for A

A few things to remember,

1. A has the same row space as R_0 and R . Same dimension r and same basis. Elimination changes the row, but the row space remains untouched.
2. The column space of A has dimension r . The column rank **equals** the row ranks. The same combination (as in the same column number) are zero or otherwise (meaning both get zero or none get zero) for both A and R_0 .
3. A has the same nullspace as R_0 , same dimension $(n - r)$ and same basis. Elimination doesn't change the solutions to $Ax = 0$, including the special solutions.
4. The left nullspace of A (the nullspace of A^T) has dimension $m - r$. The counting rule for A was $r + (n - r) = n$ and the counting rule for A^T is $r + (m - r) = m$

Very IMPORTANT, This is a fundamental theorem of linear algebra, part 1: The column space and row space both have dimension r . The nullspaces have dimension $n - r$ and $m - r$

Imagine we have a 11 by 17 matrix with 187 nonzero entries, there are two key facts:

1. dimension of $C(A) = \text{dimension of } C(A^T) = \text{rank of } A$
2. dimension of $C(A) + \text{dimension of } N(A) = 17$

Example 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

It has $m = 2, n = 3, r = 1$. The row space is the line through $(1, 2, 3)$, the nullspace is the plane $x_1 + 2x_2 + 3x_3 = 0$. The dimension of the row space is 1 since it is a line, and the dimension of the

nullspace is 2 since it is a plane. We can do the equation $r + (n - r)$, where $n - r$ is the dimension of the nullspace and we get $1 + 2 = 3$ where $3 = n$, the number of column.

We can do the same for left nullspace and column space. The column space is $[1, 2]^T$ since we can see that everything is linearly dependent on it. And nullspace is perpendicular to column space, so try dotting it and see if it gets a 0 or not. In this case, the left nullspace is a line through $[2, 1]^T$

The y's in the left nullspace combine the rows of A to give the zero row,

Example 2: Time for a more practical question (finally!) Say, we have five equations with four unknowns, one for every nodes.

$$\begin{bmatrix} -1 & 1 & & & \\ -1 & & 1 & & \\ & -1 & 1 & & \\ & -1 & & 1 & \\ & & -1 & 1 & \end{bmatrix}$$

To find the nullspace $N(A)$, set $b = 0$. The first $x_1 = x_2$, the second $x_1 = x_3$, the third $x_2 = x_3$, the fourth $x_2 = x_4$, the fifth $x_3 = x_4$. So all of them are equal, $x = (c, c, c, c)$. Just one vector is enough to fill the nullspace of A , there it is a line in R^4 . We can say $(1, 1, 1, 1)$ is a basis for $N(A)$. $N(A)$ has a dim of 1, then r of A must be 3 since $4 - 3 = 1$. We now know the dimension of all four subspaces.

The column space $C(A)$ must be $r = 3$. We can again easily find out the independent column by rrefing it.

$$R_0 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The left nullspace $N(A^T)$, now we can solve $A^T y = 0$. Note that the rows give zero, because of that we can say row 3 = row 2 - row 1, so one of the solution is $y = (1, -1, 1, 0, 0)$. In case of the lower loop, row 3 = row 4 - row 5, $y = (0, 0, -1, 1, 1)$. The dimension of $m - r = 5 - 3 = 2$, giving us the basis for the left nullspace.

1. 1, 2, 3 forms a loop in the graph, (1, 2, 3) are dependent.
2. 1, 2, 4 forms a tree in the graph, (1, 2, 4) are independent

So, where the hell did loop and tree come from? Well, they didn't need to happen, we could've used elimination. But it is more elegant. Note that $Ax = b$ gives voltage at x_1, x_2, x_3, x_4 at the four nodes and $A^T y = 0$ gives currents y_1, y_2, y_3, y_4, y_5 on the five edges. These two equations are Kirchhoff's Voltage Law and Kirchhoff's Current Law. Now, I should've said this earlier, but we need a graph for this. I don't know how to make one **yet**. Check page **135** in Strang's Linear Algebra. Graphs are the most important model in discrete applied mathematics.

For summary, **Incidence matrix A** comes from a connected graph with n nodes and m edges. The row space and column space have dimensions of $r = n - 1$, the nullspaces of A and A^T have dimensions 1 and $m - n + 1$.

1. $N(A)$ the constant vectors (c, c, \dots, c) make up the nullspace of A : $\dim = 1$.
2. $C(A^T)$ the edge of any tree give r independent rows of A : $r = n - 1$.
3. $C(A)$ Voltage Law: The components of Ax add to zero around all loops: $\dim = n - 1$.
4. $N(A^T)$ (Current Law): $A^T y = \text{flow in} - \text{flow out} = 0$, which is solved by loop currents. There are $m - r = m - n + 1$ independent small loops in the graph.

3.5.3 Rank Two Matrices = Rank One plus Rank One

This one is a doozy. What it means is then any rank r matrix can be decomposed into r amount of rank 1 matrices.

For example, a rank 2 matrix:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 7 \\ 4 & 2 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

For reminder, C is literally just pivot columns and R is just nonzero-only rref.

If we put it in letters for columns and rows, we can see **rank 2 = rank 1 + rank 1**.

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \text{zero row} \end{bmatrix} = u_1 v_1^T + u_2 v_2^T$$

To put in Strang's word, **Columns of C times rows of R , every rank r is a sum of r rank one matrices.**

Chapter 4

Orthogonality

Two vectors are orthogonal (translated as right-angled from Greek) when their dot product is zero: $v \cdot w = v^T w = 0$. The vectors in the two subspaces, the vectors in a basis, the column vectors in Q . Orthogonal vectors also have a curious behavior where it is also like pythagoras's theorem.

$$v^T w = 0 \text{ and } \|v\|^2 + \|w\|^2 = \|v + w\|^2$$

Important part, **the fundamental subspaces are orthogonal**

1. $N(A)$ contains all vectors orthogonal to the row space $C(A^T)$.
2. $N(A^T)$ contains all vectors orthogonal to the column space $C(A)$.

$Ax = 0$ makes x orthogonal to each row, $A^T y = 0$ make y orthogonal to each column.

A key idea in this chapter is **projection**: If b is outside the column space of A , then we must find the closest point p that is still inside. The line from b to p shows the error e , and that line is perpendicular to the column space. The **least squares equation** $A^T Ax = A^T b$ produces the closest $p = Ax$ and smallest possible e , it also gives the best e when $Ax = b$ is unsolvable. The best x makes $\|Ax - b\|$ as small as possible, the least squares. $A^T Ax = A^T b$ is easy when $A^T A = I$. Then A has orthonormal columns perpendicular unit vector. Remember that Q has $Q^T Q = I$ and $QR = A$, where the R is upper triangle. Orthogonal matrices are perfect for computations, $A = QR$ is even better than $A + LU$

4.1 Orthogonality of Vectors and Subspaces

1. Orthogonal vectors have $v^T w = 0$, then $\|v\|^2 + \|w\|^2 = \|v + w\|^2$ as in $a^2 + b^2 = c^2$
2. Subspace V and W are orthogonal when $v^T w = 0$ for every v in V and every w in W .
3. Row space of A is orthogonal to nullspace, column space of A is orthogonal to left nullspace.
4. The dimensions add to $r + (n - r) = n$ and $r + (m - r) = m$: orthogonal complements.
5. If n vectors in R^n are independent, they span R^n . If n vectors span R^n , they are independent.

How can we proof that the entirety of the nullspace of A is orthogonal to the row space of A ? Look at $Ax = 0$.

$$Ax = \begin{bmatrix} \text{row 1 of } A \\ \vdots \\ \text{row } m \text{ of } A \end{bmatrix} [x] = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Notice how they are all zero? That means all of the row space is orthogonal to the nullspace.

But given that we like to work with columns, another way you can proof it is: $x^T(A^T y) = (Ax)^T y = 0^T y = 0$

Importantly, column space $C(A)$ and left nullspace $N(A^T)$ is also a perpendicular pair. The proof is more or less the same except we use A^T instead of A .

Example 1: The two rows of A are perpendicular to x in the nullspace of A :

$$Ax = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$A^T y = \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a fairly extreme case since $C(A)$ is all of R^2 and the nullspace of A^T is a zero vector. The total dimension is $2 + 1 = 3$. The subspaces accounted for all vectors in $R^3 = R^n$. The column space and left nullspace have a dimension of $2 + 0 = 2$, accounting for all vectors in $R^2 = R^m$.

Note that **if V and W are orthogonal subspaces in R^n then $\dim V + \dim W \leq n$** . Of course, imagine our house. We got a wall and a floor, they can't be orthogonal subspaces since they are both R^2 and our world is R^3 , $2 + 2 \geq 3$. So this is wrong. Some vector will lie in both the wall and the floor, which is the line where the wall meets the floor in both subspaces.

Two orthogonal subspaces that account for the whole space have a special name, **orthogonal complements**. Orthogonal complement of V^\perp of V contains all vectors orthogonal to V . So the two pairs of subspace in linear algebra are actually orthogonal complements.

1. Row space and null space, $r + (n - r) = n$
2. Column space and left nullspace $r + (m - r) = m$

Any vector x in R^n is the sum $x = x_{\text{row}} + x_{\text{null}}$ of its row space and component and its null space component. Same goes for y in R^m is the sum $y = y_{\text{col}} + y_{\text{null}}$, between its column space component and its component in $N(A^T)$.

Fundamental theorem of Linear Algebra, Part 2:

1. $N(A)$ is the orthogonal complement of the row space $C(A^T)$ in R^n
2. $N(A^T)$ is the orthogonal complement of the column space $C(A)$ in R^m

Check the figure on page 146 of Strang's LA. It will us that the complete solution to $Ax = b$ is $x = \text{one } x_r + \text{any } x_n$. Then the minimum norm solution to $Ax = b$ is $x = x_r$ from the row space plus $x_n = 0$ from the nullspace, from $\|x\|^2 = \|x_r\|^2 + \|x_n\|^2$

Every vector Ax is in the column space. Multiplying by A cannot do anything else. More importantly, every b in the column space comes from exactly one vector x_r in the row space.

Example 2: Every matrix of rank r has an r by r invertible submatrix. A has rank = 2:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 \end{bmatrix} \text{ contains } \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ in the pivot rows and pivot columns.}$$

So if the submatrix C has r independent columns, then C and C^T has r independent columns. This locates an r by r invertible submatrix of A .

4.1.1 Combing Bases from Subspaces

Basis have two properties:

1. They are linearly independent
2. They span the space

However, the two properties implies eachother. **If there are n columns of independent vector in A , then they span R^n , $Ax = b$ is solvable. If n vectors span R^n , they must be independent; $Ax = b$ has one solution. If $AB = I$ for square matrix, then $BA = I$ too.** We can also start from the opposite side, say $Ax = b$ can be solved for every b , forcing **existence of solution**. This means elimination produced no zero rows. There are n and not free variables. The nullspace contains only $x = 0$, ending in **uniqueness of solutions**.

Example 3:

$$\text{For } A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ split } x = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \text{ into } x_r + x_n = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The vector $(2, 4)$ is in the row space, while the orthogonal vector is from the nullspace $(2, -1)$.

Example 4: Suppose S a six dimensional subspace of nine-dimensional space R^9

1. What are the possible dimensions of subspaces orthogonal to S ? **0, 1, 2, 3** since S already took **6 of the total 9**.
2. What are the possible dimension of the orthogonal complement S^\perp of S ? **3, because they are asking for THE orthogonal complement, the one that contains everything else. So the biggest one, 3, is the answer.**
3. What is the smallest size of matrix A that has row space S ? **6 by 9, because A is a matrix of $m \times n$ and given $R^n = R^9$, $m \times 9$. For m , the dimension of the row space is the rank of m , so $m = 6$. Therefore, 6 times 9**
4. What is the smallest possible size of a matrix B that has nullspace S^\perp ? **6 by 9. We are told that R^9 have an n of 9. And that $S^\perp = 3$ from the second question, so we know that $r + \text{null} = n$, we got $r + 3 = 9$, $r = 6$. 6×9**

Notice how question 3 and 4 gives the same matrix? It is telling us that $C(A^T) = (N(A))^\perp$

4.1.2 Projection onto Lines and Subspaces

1. The projection of b onto the line through a is the closest point to b : $p = a(\frac{a^T b}{a^T a})$.
2. The error $e = b - p$ is perpendicular to a : the right triangle bpe has $\|p\|^2 + \|e\|^2 = \|b\|^2$.
3. The projection of b onto to a subspace S is the closest vector p in S ; $b - p$ is orthogonal to S .
4. $A^T A$ is invertible and symmetric when A has independent columns: $N(A^T A) = N(A)$.
5. The projection of b onto $C(A)$ is the vector $p = A(A^T A)^{-1} A^T b$.
6. The projection matrix onto $C(A)$ is $P = A(A^T A)^{-1} A^T$. It has $p = Pb$ and $P^2 = P = P^T$.

We can visual projection on a subspace S by having each vector b go the closest point p in S . The error vector $e = b - p$. If A has independent columns, the projection of b onto $C(A)$ is $p = A(A^T A)^{-1} A^T b$. The projection matrix is $P = A(A^T A)^{-1} A^T$. Its special property is $P^2 = P$, a second projection changes nothing, because it projects onto itself.

For some really nice subspaces, we can directly see their projection matrices.

1. What are the projections of $b = (2, 3, 4)$ onto the z axis and the xy plane.
2. What matrix P_1 and P_2 produce those projections onto a line and a plane?

When b is projected onto a line, its projection p is the part of the b along that line. If b is projected onto a plane, p is the part in that plane. The projection p is Pb .

We will call projection onto the z axis p_1 . Where projection p_2 drop straight down to the xy plane. To with $b = (2, 3, 4)$. $p_1 = (0, 0, 4)$, $p_2 = (2, 3, 0)$. We can see that

$$\text{Onto the } z \text{ axis: } P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ Onto the } xy \text{ plane: } P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p_1 = P_1 b = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, p_2 = P_2 b = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

In this case, the projections p_1 and p_2 are perpendicular. The xy plane and the z axis are orthogonal subspaces. More than that, they are orthogonal complements; their dimensions add to $1 + 2 = 3$. Every vector b in the whole space is the sum of its part in two subspaces.

1. The vectors give $p_1 + p_2 = b$.
2. The matrices give $P_1 + P_2$ and $P_1 P_2 = 0$.

For this example, our goals have been reached, and it is the same for any line and any plane and any n -dimensional subspaces in m dimension. The objective is to find the part p in each subspaces, and the projection matrix P that produces that part $p = Pb$. Every subspaces of R^m has its own m by m projection matrix P .

So, if the best description of a subspace is a basis, by putting the basis vectors into the columns A , we are now projecting onto the column space of A . We can say that the z axis is the column space of the $3 \times 1 A_1$. The xy plane is the column space of A_2 . That xy plane too is also the column space of A_3 . So $p_2 = p_3$ and $P_2 = P_3$.

$$A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 0 \end{bmatrix}$$

Our problem is to project any b onto the column space of any $m \times n$ matrix. Starting with a line, dimension $n = 1$, the matrix will have only one column; we will call it a .

4.2 Projections onto Lines and Subspaces

4.3 Least Squares Approximations

4.4 Orthogonal Matrices and Gram-Schmidt

4.5 The Pseudoinverse of a Matrix