Linear Algebra Self-Study

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Chapter 1

Vector and Matrices

1.1 My bad guys

So, the issue, right? I picked up LaTeX after I already finished chapter 1 , and nearly finished chapter 2. So to future me and any poor sods reading this, good luck lol.

Chapter 2

Solving Linear Equation $Ax = b$

2.1 Elimination and Back Substitution

I fucked up

2.2 Elimination Matrices and Inverse Matrices

This too.

2.3 Matrix Computation and $A = LU$

This one too.

2.4 Permutation and Tranposes

Also this shit.

2.5 Derivatives and Finite Difference Matrices

Second difference matrices includes K, T, B. They all have the $-1, 2, -1$ pattern.

$$K_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Now we can approximate $-\frac{d^2u}{dx^2} = f = (x)$ So, we want to compute $-\frac{d^2u}{dx^2}$ with a computer, but the computer can't understand derivative. So what we do is we turn $\frac{d^2u}{dx^2}$ into the matrix $\frac{K^2}{h}$, function $u(x)$ into vector u, and function $f(x)$ into F. We also need the boundary conditions, which are given where $u(0) = 0$ and $u(1) = 0$. We can't pick out the infinite space between 0 and 1, so we pick N equally spaced points at a regular interval. The space between each points (and the first and the last point) becomes meshwidth (h). If we have N interal points $u_0, u_1, u_2, \ldots$ plus two boundary points u_0 and u_{N+1}, we divide the total length into N+1 segments. Therefore the spacing is $h = \frac{1}{N+1}$. If we have 4 N, then the spacing is h $= \frac{1}{5}$. So instead of finding the continuous function $u(x)$, we will find the value at each internal points, and they becomes the unknown vector $U = [u_1, u_2, u_3, u_4]^T$.

$$-\frac{d^2u}{dx^2} = f(x) \text{ becomes } \frac{KU}{h^2} = F, \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \end{bmatrix}$$

The key point is that when divide the meshpoint into 4, therefore N = 4. Row 1 times U is $2u_1 - u_2$, and we already got the boundary where $u_0 = 0$ and $u_5 = 0$, making a typical row $\frac{(-u_1+2u_2-u_3)}{h^2} = f(h)$. The division by h^2 makes $\frac{K}{h^2}$ a second difference matrix, replacing $-\frac{d^2u}{dx^2}$.

2.5.1 Properties of K

K has 4 properties. For the sake of example, we will use K for $N = 4$.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

1. K is symmetrical, as in $K_{ij} = K_{ji}$

2. K is banded. All the non-zeros (-1, 2, -1) lie in a band around the main diagonal. The band has three diagonals, so K is a tridiagonal matrix.

3. K has a constant diagonals. A diagonal of -1, then 2, then -1 again. The matrix is called "shift-invariant", because the differential equation always have a constant coefficient of -1. The approximation to $-\frac{d^2u}{dx^2}$ is always $-1, 2, -1$ at every$X$.

4. X is invertible. It has an inverse matrix K^{-1} then $K^{-1}K = I$ and $KK^{-1} = I$.

$$K_4^{-1} = \frac{\mathbf{1}}{\mathbf{5}} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

   K^{-1} is aso symmetric but it is no diagonal. It is dense matrix, meaning no zeros.

5. Symmetric K_n matrices are positive definite.

Invertible, positive definite symmetric matrix, and semidefinite matrix are defined by their pivots.

1. Invertible matrices has nonzero pivots.

2. Positive definite symmetric matrices has positive nonzero pivots.

3. Positive semidefinite symmetric matrices has nonnegative pivots.

2.5.2 Free-fixed Matrice T_n

T_n and B_n are variations on K_n, where the variation comes from changing the boundary conditions. Think of it as an elastic band that are fixed at both ends, one end, or totally free at both ends. For example, T_n is very similar to K_n except that input $(1, 1)$ is switched from 2 to 1., representing a free boundary condition where $\frac{du}{dx} = 0$

$$\text{Free-fixed boundary conditions, still positive definite } T_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

T is no longer Toeplitz because its main constant, though it does have a simpler factorization than K; every pivot of T equals 1.

$$T = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ & 1 & -1 & 0 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} = LU$$

Note that $U = L^T$. Notice that U is a forward difference while L is a backward difference. Together, they add up to be a second difference, meaning that $x_{i+1} - 2x_i + x_{i-1}$ correspond to [-1, 2, -1], meaning that T is a second difference.

2.5.3 The Free-Free Matrices B_n are Singular

In this context, "singular" means "not invertible". One test is simply seeing if determinant equals zero or not.

Theorem 2.5.1. *If B multiplies a nonzero vector x to produce $Bx = 0$, then B can't be invertible.*

For example, free-free matrix has 1 (and not 2) in its (1, 1) and (3, 3) input.

$$B_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ has } B_3x = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

B_n is singular while T_n and K_n are invertible because T_n and K_n have a fixed end, which allows them to adjust, whereas B_n is free on both end so it can't adjust.

Chapter 3

The Four Fundamental Subspaces

How can we define "vector space"? Well, if we are talking about R^3, the key operation are $v + w$ and cv. Notice that v and w could be matrices, so we could have matrix spaces and function spaces. Then inside R^n we could only allow x that satisfies $Ax = 0$, which will produce "nullspace of A". All combination of solution to $Ax = 0$ are also solutions, meaning that the nullspace is a subspace. To point it simply, nullspace is just x in Ax that crushes it down to 0. Why is it called a space? Because it has structures and rules.

1. It must contain zero vector, $A \times 0 = 0$ always.
2. It must be closed under addition, meaning that if you take two vectors from the nullspace then add them together, they must still be in the nullspace.
3. It must be closed under scalar addition, meaning that if you take any vector from the nullspace then multiplies them by a constant, the result must stil be in the nullspace.

Then, lastly, there are basis. A set of vectors that perfectly describes the space. Very important, some considers it the fundamental theorem. So, what it means is that basis are just sets of movement vectors for each dimension.

1. For a 3D space, you will need forward, right, and up.
2. For a 2D space, you will need right and up.
3. For a 1D space (line of sight, nullspace), you only need the direction of the line.

And so, $n - r$ special solutions to $Ax = 0$ are a basis for $N(A)$. It's called rank-nullity theorem.

$$\text{Dimension of Input Space} = \text{Dimension of Column Space} + \text{Dimension of Nullspace}, \; n = r + (n - r)$$

Note that $n-r$ is the dimension of the nullspace. One last note: THIS CHAPTER IS VERY IMPORTANT.

3.1 Vector Spaces and Subspaces

3.2 The Nullspace of A: Solving $Ax = 0$

3.3 The Complete Solution to $Ax = b$

3.4 Independence, Basis, Dimension

3.5 Dimensions of the Four Subspaces