Linear Algebra Self-Study

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Contents

Chapter 1

Vector and Matrices

1.1 My bad guys

So, the issue, right? I picked up LaTeX after I already finished chapter 1 , and nearly finished chapter 2. So to future me and any poor sods reading this, good luck lol.

Chapter 2

Solving Linear Equation $Ax = b$

2.1 Elimination and Back Substitution

I fucked up

2.2 Elimination Matrices and Inverse Matrices

This too.

2.3 Matrix Computation and $A = LU$

This one too.

2.4 Permutation and Tranposes

Also this shit.

2.5 Derivatives and Finite Difference Matrices

Second difference matrices includes K, T, B. They all have the $-1, 2, -1$ pattern.

$$K_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Now we can approximate $-\frac{d^2u}{dx^2} = f = (x)$ So, we want to compute $-\frac{d^2u}{dx^2}$ with a computer, but the computer can't understand derivative. So what we do is we turn $\frac{d^2u}{dx^2}$ into the matrix $\frac{K^2}{h}$, function $u(x)$ into vector u, and function $f(x)$ into F. We also need the boundary conditions, which are given where $u(0) = 0$ and $u(1) = 0$. We can't pick out the infinite space between 0 and 1, so we pick N equally spaced points at a regular interval. The space between each points (and the first and the last point) becomes meshwidth (h). If we have N interal points $u_0, u_1, u_2, \ldots$ plus two boundary points u_0 and u_{N+1}, we divide the total length into N+1 segments. Therefore the spacing is $h = \frac{1}{N+1}$. If we have 4 N, then the spacing is h = $\frac{1}{5}$. So instead of finding the continuous function $u(x)$, we will find the value at each internal points, and they becomes the unknown vector $U = [u_1, u_2, u_3, u_4]^T$.

$$-\frac{d^2u}{dx^2} = f(x) \text{ becomes } \frac{KU}{h^2} = F, \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \end{bmatrix}$$

The key point is that when divide the meshpoint into 4, therefore N = 4. Row 1 times U is $2u_1 - u_2$, and we already got the boundary where $u_0 = 0$ and $u_5 = 0$, making a typical row $\frac{(-u_1+2u_2-u_3)}{h^2} = f(h)$. The division by h^2 makes $\frac{K}{h^2}$ a second difference matrix, replacing $-\frac{d^2u}{dx^2}$.

2.5.1 Properties of K

K has 4 properties. For the sake of example, we will use K for $N = 4$.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

1. K is symmetrical, as in $K_{ij} = K_{ji}$
2. K is banded. All the non-zeros (-1, 2, -1) lie in a band around the main diagonal. The band has three diagonals, so K is a tridiagonal matrix.
3. K has a constant diagonals. A diagonal of -1, then 2, then -1 again. The matrix is called "shift-invariant", because the differential equation always have a constant coefficient of -1. The approximation to $-\frac{d^2u}{dx^2}$ is always $-1, 2, -1$ at every$X$.
4. X is invertible. It has an inverse matrix K^{-1} then $K^{-1}K = I$ and $KK^{-1} = I$.

$$K_4^{-1} = \frac{\mathbf{1}}{\mathbf{5}} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

   K^{-1} is aso symmetric but it is no diagonal. It is dense matrix, meaning no zeros.

5. Symmetric K_n matrices are positive definite.

Invertible, positive definite symmetric matrix, and semidefinite matrix are defined by their pivots.

1. Invertible matrices has nonzero pivots.
2. Positive definite symmetric matrices has positive nonzero pivots.
3. Positive semidefinite symmetric matrices has nonnegative pivots.

2.5.2 Free-fixed Matrice T_n

T_n and B_n are variations on K_n, where the variation comes from changing the boundary conditions. Think of it as an elastic band that are fixed at both ends, one end, or totally free at both ends. For example, T_n is very similar to K_n except that input $(1, 1)$ is switched from 2 to 1., representing a free boundary condition where $\frac{du}{dx} = 0$

$$\text{Free-fixed boundary conditions, still positive definite } T_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

T is no longer Toeplitz because its main constant, though it does have a simpler factorization than K; every pivot of T equals 1.

$$T = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ & 1 & -1 & 0 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} = LU$$

Note that $U = L^T$. Notice that U is a forward difference while L is a backward difference. Together, they add up to be a second difference, meaning that $x_{i+1} - 2x_i + x_{i-1}$ correspond to [-1, 2, -1], meaning that T is a second difference.

2.5.3 The Free-Free Matrices B_n are Singular

In this context, "singular" means "not invertible". One test is simply seeing if determinant equals zero or not.

Theorem 2.5.1. *If B multiplies a nonzero vector x to produce $Bx = 0$, then B can't be invertible.*

For example, free-free matrix has 1 (and not 2) in its (1, 1) and (3, 3) input.

$$B_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ has } B_3x = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

B_n is singular while T_n and K_n are invertible because T_n and K_n have a fixed end, which allows them to adjust, whereas B_n is free on both end so it can't adjust.

Chapter 3

The Four Fundamental Subspaces

How can we define "vector space"? Well, if we are talking about R^3, the key operation are $v + w$ and cv. Notice that v and w could be matrices, so we could have matrix spaces and function spaces. Then inside R^n we could only allow x that satisfies $Ax = 0$, which will produce "nullspace of A". All combination of solution to $Ax = 0$ are also solutions, meaning that the nullspace is a subspace. To point it simply, nullspace is just x in Ax that crushes it down to 0. Why is it called a space? Because it has structures and rules.

1. It must contain zero vector, $A \times 0 = 0$ always.
2. It must be closed under addition, meaning that if you take two vectors from the nullspace then add them together, they must still be in the nullspace.
3. It must be closed under scalar addition, meaning that if you take any vector from the nullspace then multiplies them by a constant, the result must stil be in the nullspace.

Then, lastly, there are basis. A set of vectors that perfectly describes the space. Very important, some considers it the fundamental theorem. So, what it means is that basis are just sets of movement vectors for each dimension.

1. For a 3D space, you will need forward, right, and up.
2. For a 2D space, you will need right and up.
3. For a 1D space (line of sight, nullspace), you only need the direction of the line.

And so, $n - r$ special solutions to $Ax = 0$ are a basis for $N(A)$. It's called rank-nullity theorem.

$$\text{Dimension of Input Space} = \text{Dimension of Column Space} + \text{Dimension of Nullspace},\ n = r + (n - r)$$

Note that $n-r$ is the dimension of the nullspace. One last note: THIS CHAPTER IS VERY IMPORTANT.

3.1 Vector Spaces and Subspaces

Here are a few fundamental points:

1. All linear combination of $cv + dw$ must stay in the vector space (What is a vector space? Very simply, all the possible spaces tha can be achieved by your given vector with vector addition or scalar multiplication), where c and d are scalar, and v and w are vectors.
2. The row space (meaning all possible linear combination of the row vectors) is "spanned" (made up of) rows of A. While the column of A spans $C(A)$.
3. Matrices can be filled by more than just numbers. As long as it obeys the rules of a vector space, it can be treated as a vector. For example, we have two equations, $f(x) = x^2$ and $g(x) = 2x$. Can they be added together? Yes! $h(x) = x^2 + 2$. Now if we fill it with the likes of $\sin, \cos, x, x^2$, we can "span" and build a lot of other functions. For example, all quadratic polynomials are "spanned" by the functions $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$

R^n contains all column vector v to the length of n. For this case, the components from v_1 to v_n are all real numbers. However, if they allow for complex numbers (i), the R^n becomes C^n. To reiterate, all linear combination of $cv + dw$ must be in the vector space R^n. For example, all positive the set with all positive (meaning no vector consist of ANY nonpositive numbers) vectors $(v_1, \ldots, v_n)$ areNOT a vector space. Why? Simply take one simple vector, say $(1, 2)$ then multiply it by scalar of, say, c = -1. $(-1, -2)$ is NOT in our set, therefore it is not a vector space. Or for another example, a set of solution for $Ax = (1, \ldots, 1)$ is not a vector space because a line in R^n is not a vector space unless it goes the central point $(0, \ldots, 0)$.

3.1.1 Examples of Vector Spaces

Here are some examples of a neat vector space, the Z (zero vector) where $0 = (0, 0, \ldots, 0)$. Combinations of $c0 + d0$ are all still 0, so still in the subspace. How about vector space of matrices? We can do that. $R^{3\times3}$ is a space that contains all 3×3 matrices. It does satisfy all eight rules, so why not? It's also a vector space. How about a vector space of functions? Sure can. The line of functions $y = ce^x$ (any c) is a line in a function space. This line contains all solutions to the differential equations of $\frac{dy}{dx} = y$. Yet another function space contains all quadratics $y = a + bx + cx^2$, where they are the solutions to $\frac{d^3y}{dx^3}$ And to reiterate, space in this context means all possible linear combination of the vectors or matrices or functions, and they all stay inside it.

3.1.2 Subspaces of Vector Spaces

What ar subspaces? To put it simply, they are a flat plane inside the dimensional space, however, they are still the same dimension. Let's say, we got a R^3 space. We can make a plane any way we want as long as it passes $(0, 0, 0)$, what we get may look like a 2D plane, but it's still 3D. Therefore, the plane is a subspace of the full vector space R^3.
Here is a list of possible subspaces of R^3:

1. Any line through $(0, 0, 0)$
2. Any plane through $(0, 0, 0)$
3. The whole space R^3
4. The zero vector $(0, 0, 0)$

3.1.3 The Column Space of A

What we are trying to solve here is $Ax = b$. We want to know b, right? Well, b are a column space of A. Ax is just a combination of A, and to get every possible b, we need all possible x, which is just all linear combination of A, which is the column space of A, as written earlier. To build on that, vector space is made up of column vectors.

A crucial point to understand is that, **to solve $Ax = b$ is just to express b as a combination of the columns**. b got to be in the column space of A, otherwise, it doesn't exist!

Caution: columns of A do not form a subspace. Neither do invertible matrices, or singular matrices. Only all linear combinations.

3.1.4 The Row Space of A

The rows of A are the column of A^T, why do we do this? Because we like working with columns, so we use the column of A^T

The row space of A is just the column space of A^T

3.2 Computing the Nullspace by Elimination: $A = CR$

1. The **nullspace N(A)** in $\mathbf{R^n}$ contains all solutions x to $Ax = 0$, including $x = 0$.
2. CONTINUE THIS LATER

The goal of this section is to find all solutions for $Ax = 0$. If A is an invertible matrix, then the only solution is $x = 0$. In general, A has r independent columns, the other $n - r$ are a linear combination. Here is a matrix R with rank $r = 2$, with $n = 4$ columns. This means we have $n - r = 4 - 2 = 2$ independent solutions to $Rx = 0$. So the nullspace $N(R)$ will have 2 dimensions.

$$\textbf{Example 1: } R = [IF]P = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix}$$

Which means $Rx = 0$ is $x_1 + 3x_3 + 5x_4 = 0$ and $x_2 + 4x_3 + 6x_4 = 0$. We find the special solutions by just letting x_3 and x_4 equal 1 and 0 or 0 and 1. Set $x_3 = 1, x_4 = 0$ the equations will give $x_1 = -3, x_2 = -4$ Set $x_3 = 0, x_4 = 1$ the equations will give $x_1 = -5, x_2 = -6$ This gives us two special solutions: $s_1 = (-3, -4, 1, 0)$ and $s_2 = (-5, -6, 0, 1)$. They are both in the nullspace of R, as we can also see, $cs_1 + ds_2$ is still in the the nullspace. And so, s_1 and s_2 are the basis of nullspace.

$$\textbf{Example 2: } R_0 = \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which means $x_1 + 7x_2 + 0x_3 + 8x_4 = 0$, $x_3 + 9x_4 = 0$, and $0 = 0$ (wow on the last one). The matrix identity is inside column 1 and 3, and row 3 is all zero, which makes it a reduced row echelon form, even elimination can't make it simplier. We still have free variables for the special solution, namely, x_2 and x_4. Set $x_2 = 1, x_4 = 0$, the equations give $x_1 = -7, x_3 = 0$. Set $x_2 = 0, x_4 = 1$, the equations give $x_1 = -8, x_3 = -9$. The special solutions are noe $s_1 = (-7, 1, 0, 0)$ and $s_2 = (-8, 0, -9, 1)$.

First, we start with any m by n matrix A, then apply elimination. That changes A into its reduced row echelon form, $R_0 = \text{rref}(A)$. Removing all zero rows of R_0 leaves R.

$$r, m, n = 2, 2, 4 \text{ Simplest Case } R = [IF] \text{ as in } \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix}$$

$$r, m, n = 2, 3, 4 \text{ General Case } R_0 = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} P \text{ as in } \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hold up, what is rref, I, F here? For something to be rref, you need to satisfy 4 conditions.

1. The zero row is at the botto
2. The first non-zero in row 1 is 1, the first non-zero entry in row 2 is 1.
3. Each pivot is to the right of the pivots in the rows above it.
4. Every pivot is only non-zero number in its entire column.

I in this context is still identity, but only if you take the pivot columns and align them chronologically as they were. F is free matrix, the other non-pivot columns, still in the same chronologically order.

In this case (R_0),

$$\text{I is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ F is } \begin{bmatrix} 7 & 8 \\ 0 & 9 \\ 0 & 0 \end{bmatrix}$$

3.2.1 Elimination from A to rref (A): Reduced Row Echelon Form

Refresher, how does elimination works?

1. Subtruct a multiple of one row from another row
2. Multiple a row by nonzero number
3. Exchange any rows

For demonstration,

$$A = \begin{bmatrix} 1 & 2 & 11 & 17 \\ 3 & 7 & 37 & 57 \end{bmatrix} \text{ then } \begin{bmatrix} 1 & 2 & 11 & 17 \\ 0 & 1 & 4 & 6 \end{bmatrix} \text{ then } \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix}$$

So, what did elimination actually do? It inverted the leading 2 by 2 matrix, which we will call W.

$$W = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \text{ into } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We multiplied $W^{-1}A = W^{-1}[WH]$ to produce $R = [IW^{-1}H] = [IF]$. We always knew that free columns (H) is some combination of independent columns (W), but we now know that $H = WF$.

$$H = \begin{bmatrix} 11 & 17 \\ 37 & 57 \end{bmatrix} = WF = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

However you compute R from A, you will always get the same R. R is completely controlled by A.

For **example 2**, let us rref another A.

$$A = \begin{bmatrix} 1 & 7 & 3 & 35 \\ 2 & 14 & 6 & 70 \\ 2 & 14 & 9 & 97 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 3 & 35 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 27 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 27 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R_0$$

3.2.2 Elimination Column by Column: The Steps from A to R_0

We will now reduce what we learn to an easily applied algorithm. The big question is **does this new column $k+1$ join with I_k or F_k?**

If l is all zero , the new column is **dependent** on the first k columns. Then **u** joins with F_k to become F_{k+1}.

If l is not all zero , then it is independent of the first k columns. Use the **largest** number, preferably, in l as the pivot. **Important** thing to remember here is that the column are talking about means the columns **UNDERNEATH** all pivots, not **ALL** columns. Then do elimination. Whatever is left becomes part of **I** as $\mathbf{I_{k+1}}$.

From example 2, we can see that the combination of independent and dependent comes out to

$$C \times F = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 35 \\ 14 & 70 \\ 14 & 97 \end{bmatrix} = \text{depend columns of 2 and 4 of } A$$

Right back to where we came from, showing that C are almost like the ingredients and F are almost like the method.

3.2.3 The Matrix Factorization $A = CR$ and the Nullspace

In chapters prior, we know that $A = CR$ but we have no systemic way to find them, now we do. We apply elimination to reduce A to R_0. Then I in R_0 locates the matrix C of independent columns in A. Removing zero row in R_0 produces R for $A = CR$

We have two special solution s_1 and s_2 for every column of F in R.

$$Rs_1 = 0, \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} -7 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Rs_1 = 0, \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} -8 \\ 0 \\ -9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

s_1 and s_2 are the easiest to see using the matrices $-F$ and I and P^T.

As a reminder, the two special solutions to $[IF]Px = 0$ are the columns of $P^T \begin{bmatrix} -F \\ I \end{bmatrix}$

It more correct than the other option because PP^T is the identity matrix of permutation matrix P:

$$\mathbf{Rx} = \mathbf{0}, [IF]P \times P^T \begin{bmatrix} -F \\ I \end{bmatrix} \text{ reduces to } [IF] \begin{bmatrix} -F \\ I \end{bmatrix} = [0]$$

Review Say, the m by n matrix A has rank r. We can find $n - r$ special solution to $Ax = 0$ by comuting the rref R_0 of A. Remove the $m - r$ zero rows of R_0 to produce $R = [IF]P$ and $A = CR$. Then the special solutions to $Ax = 0$ are the $n - r$ columns of $P^T[-F, I]^T$

Example 3: Elimination of A gives R_0 and R. R reveals the nullspace of A

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R_0 \text{ with rank 2}$$

$$R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The independent columns are 1 and 3.

To solve $Ax = 0$ and $Rx = 0$, set $x_2 = 1$, which will get you $x_1 = -2, x_3 = 0$. Leave us special solution:

$$\mathbf{s} = (\mathbf{-2}, \mathbf{1}, \mathbf{0})$$

All solutions $x = (-2c, c, 0)$. And here it is, $A = CR$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} = CR = \begin{bmatrix} 1 & 1 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{columns basis in } C \ \times \text{row basis in } R.$$

For a lot matrices, the only solution to $Ax = 0$ is $x = 0$. Simply, all columns of A are independent. The nullspace $N(A)$ contains only the zero vector, no special solution. This case zero nullspace is **important** because it means that all columns of A is independent. But this can't happen if $n > m$ (column > row) because you can have n independent column in R^m.

Important Say A has more columns than rows ($n > m$), there will be at least one free variable. Meaning that $Ax = 0$ has at least one non-zero solution. Or to put it more specifically, there must be more than $n - m$ free columns. $Ax = 0$ must have nonzero solutions in $N(A)$.

Example 4: Find the nullspace of A, B, M and the two special solutions to $Mx = 0$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix}, M = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

Solution The equation $Ax = 0$ has only the zero solution $x = 0$. The nullspace is only Z.

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R = I$$

No free variables, meaning A is invertible; therefore, no special solution.

The M matrix is different. It has extra columns instead of rows. That means that, with 4 columns and 2 rows, there will be 2 free columns leftover.

$$M = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 8 & 6 & 16 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} I & F \end{bmatrix}$$

Again, to get special solutions out of it, we will let $x_3 = 1, x_4 = 0$ and $x_3 = 0, x_4 = 1$. What we will get is the special solution for the nullspace of M.

$$Mx = 0R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

3.2.4 Block Elimination in Three Steps: Final Thoughts

We will conclude nicely with three steps to block elimination.

Step 1 Exchange the columns and rows of P_C and P_R so that that r independent columns and rows come first in P_RAP_C

$$P_RAP_C = \begin{bmatrix} W & H \\ J & K \end{bmatrix}, C = \begin{bmatrix} W \\ J \end{bmatrix} \text{ and } , B = \begin{bmatrix} W & H \end{bmatrix}$$

Step 2 Multiple the top rows by W^{-1} to produce $W^{-1}B = [I, W^{-1}H] = [I, F]$.
Step 3 Subtract $J[I, W^{-1}H]$ from $[J, K]$ to produce $[0, 0]$.
The results of the steps should be an rref form of R_0

$$P_RAP_C = \begin{bmatrix} W & H \\ J & K \end{bmatrix} \to \begin{bmatrix} I & W^{-1}H \\ J & K \end{bmatrix} \to \begin{bmatrix} I & W^{-1}H \\ 0 & 0 \end{bmatrix} = R_0$$

There are two things that need to be remembered.

1. W is invertible

2. The block satisfies $JW^{-1}H = K$

1. We must think back to $A = CR$. We can see that $B = WR$, and since B and R have the rank of r and W is also r by r, that means that W must have a rank of r and be invertible.

2. We know that the first row $[I, W^{-1}H]$ is linearly independent. Since A has the rank r, it means that the lower row $[J, K]$ must be a combination of the upper rows. This means for the combination to be valid, $JI = J$ and $JW^{-1}H = K$.

$$\text{The conclusion is that } P_RAP_C = \begin{bmatrix} W \\ J \end{bmatrix} W^{-1} \begin{bmatrix} W & H \end{bmatrix} = CW^{-1}B.$$

3.3 The Complete Solution to $Ax = b$

Our goal in this section will be about:

1. The complete solution to $Ax = b$: $x = x_p + x_n$, where p starts for any particular x and n nullspace.

2. Elimination from $Ax = b$ to $R_0x = d$: Solvable when zero rows of R_0 have zero in d.

3. When $R_0x = d$ is solvable, one x_p has all free variable equal to zero.

4. A has full column rank $r = n$ when its nullspace $N(A) =$ zero vector: no free variables.

5. A has full row rank $r = m$ when its column space $C(A)$ is R^m: $Ax = b$ is always solvable.

The biggest thing that changed is that the b of $Ax = b$ is now not zero. Therefore, the row operation will also act on the right side, the b side. $Ax = b$ is reduced to a simpler $R_0x = d$ with the same, if any, solutions. One way to organize that is by augmenting (adding another column to the right side of the matrix). We can augment A with the right side of $(b_1, b_2, b_3) = (1, 6, 7)$ to produce augmented matrix $[Ab]$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \text{ has the augmented matrix } \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{bmatrix} = [Ab]$$

Now if we turn it into its rref form

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \text{ has the augmented matrix } \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [R_0d]$$

The last row is very important. Third equation became $0 = 0$, each means it can be solved. In the original matrix, the first row plus the second row equals the third row. Meaning to solve $Ax = b$ we need $b_1 + b_2 = b_3$, which led to $0 = 0$ in the third equation.

3.3.1 One Particular Solution $Ax_p = b$

To get an easy x_p, let the free variables be zeros: $x_2 = x_4 = 0$, and the two nonzero equations be the two pivot variables $x_1 = 1, x_3 = 6$. So our x_p is $x_p = (1, 0, 6, 0)$. To put it simply, **free variables = zero, pivots = variable from d**.

For a solution to exist, zero rows in R_0 must also be zero in d.

$$R_0 x_p = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

Notice how the complete solution includes all x_n:

$$x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Example 1 Find the condition on b_1, b_2, b_3 for $Ax = b$ for

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \text{ and } \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution Elimination using augmented matrix $[A, b]$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix} = [R_0 d]$$

And since there is no special solution ($n - r = 2 - 2 = 0$), the nullspace solution is $x_n = 0$. So the complete solution is

$$x = x_p + x_n = \begin{bmatrix} 2b_1 - b2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Every matrix A with full column rank $(r = n) hastheseproperties:$

1. All columns are independent, no free variables.
2. The nullspace $N(A)$ include only the zero vector $x = 0$
3. If $Ax = b$ has a solution, then it has only one solution.

With full column rank, $Ax = b$ will have **one or no solution only.**

3.3.2 Full Row Rank and the Complete Solution

Another extreme case is the full row rank. Now $Ax = b$ has one or infinitely many solutions. Full row rank requires that the matrix be a short and wide one (m > n, row q > col q), and every row is independent. For **example 2**, $Ax = b$ has 3 n but only two m.

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix} \text{ rank r = m = 2}$$

Imagine them as plane in xyz space. There are two planes, colliding into a line. The particular solution is a spot on that line, and the nullspace vector will move us along that line. $x = x_p +$ all x_n gives us the whole line solution.

Fast forward a bit, getting ex. 2 into the $[Ab]$ form gives us:

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} = [R, d]$$

This particular solution $(2, 1, 0)$ has free variable $x_3 = 0$. The special solution has $s_3 = 1$, and the $-x_1$ and $-x_2$ comes from the free column of R.

Check that x_p and s satisfies $Ax_p = b$ and $As = 0$

$$2+1=3, 2+2=4 - 3+2+1=0, -3+4-1=0$$

Remove that the nullspace solution x_n is just any multiple of s.

$$\text{Computer solution: } x = x_p + x_n = \begin{bmatrix}2\\1\\0\end{bmatrix} + x_3 \begin{bmatrix}-3\\2\\1\end{bmatrix}$$

Every matrix A with a full row rank ($r = m$) has all these properties:

1. All rows have pivot and R_0 has no zero rows: $R_0 = R$
2. $Ax = b$ has a solution for every right side b.
3. The column space of A is the whole space R^m.
4. If $m < n$, the equation has many solutions (called underdetermined in formal language).
5. The rows are linearly independent.

There are **four** possibilities for linear equation depend on rank r

1. $r = m$ and $r = n$ Square ad invertible, $Ax = b$ has 1 solution.
2. $r = m$ and $r < n$ Short and wide, $Ax = b$ has infinite solutions.
3. $r < m$ and $r = n$ Tall and thin, $Ax = b$ has 0 or 1 solutions.
4. $r < m$ and $r < n$ Not full rank, $Ax = b$ has 0 or infinite solutions.

The reduced R_0 will fall in the same category as matrix A. For $R_0x = d$ and $Ax = b$ to be solvable, d must end in $m - r$ zeros.

Four types for R_0

$$\begin{bmatrix}I\end{bmatrix}, r = m = n$$

$$\begin{bmatrix}I & F\end{bmatrix} r = m < n$$

$$\begin{bmatrix}I\\0\end{bmatrix} r = n < m$$

$$\begin{bmatrix}I & F\\0 & 0\end{bmatrix} r < m, r, n$$

Case 1 and 2 have full row rank $r = m$. Case 1 and 3 have full column rank $r = n$.

3.4 Independence, Basis, Dimension

How big is the true size of a subspace? A matrix might be m by n but the column space is not necessarily n. The column space $C(A)$ is measured by independent columns. This will be clarified later.

Our goal here is to understand a **basis: independent vectors that "spans" a space.** Every vector in the space is an unique combination of the basis vectors. Some vague explanation of the terms:

1. Independent vectors, no extra vectors
2. Spanning a space, enough vectors to produce the rest.
3. Basis for a space, not too many and not too few
4. Dimension of a space, the number of vectors in every basis.

3.4.1 Linear Independence

Here's an odd definition for independence:

The columns of A are linearly independent when the only solution of $Ax = 0$ is $x = 0$. No other combination of Ax gives

To illustrate why: it is impossible for three vectors that are not in a plane to be in a plane **unless** x is 0, i.e.,

$$0v_1 + 0v_2 + 0v_3 + \dots$$

Or to put it in other words:

Linear independence only happens when $x_1v_1 + x_2v_2 + \dots = 0$ and all $x_i = 0$.

One point to drive is that vectors are either dependent or independent, no in between. For some examples:

1. $(1, 0)$ and $(1, 0.000001)$ are independent.
2. $(1, 1)$ and $(-1, -1)$ are dependent.
3. $(1, 1)$ and $(0, 0)$ are dependent because of the zero vector.
4. In $\mathbb{R}^2$: Any three vectors $(a, b), (c, d), (e, f)$ are dependent.

Example 1: The columns of this A are dependent, $Ax = 0$ has a nonzero solution:

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{ is } -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Question, how do we fix this? Just do elimination.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \to R_0 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} .F = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, x = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Remember that **columns of A are independent exactly when r = n**, there are n pivots and zero free variable. $x = 0$ is the only nullspace.

Also remember that any set of n vectors must be linearly dependent if $n > m$.

3.4.2 Vectors that Span a Subspace

To reiterate, column space consists of all combination of $x_1v_1 + \dots + x_nv_n$. The word "span" describes $C(A)$.

The columns of a matrix span its column space. They might be dependent.

Example 2: We will try to describe the column space and row space of A.

$$m = 3, n = 2 : A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}$$

We could say that the column space of A is a plane in R^3 spanned by two columns of A The row space of A is spanned by three rows of A (which are columns in A^T) in R^2. Oh, and the rows span R^n and columns span R^m. Yes, they are swapped.

3.4.3 A Basis for a Vector Space

To reiterate, the basis means just right. Two independent vectors can't span R^3, four vectors can't be all independent even if they span R^3. Three independent vectors for R^3 is just right.

Basis vectors are independent and they span the space

Note that there is **only one way to write** v **as a combinaion of the basis vector**

Example 3: The column of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the standard basis for $\mathbf{R^2}$.

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ are independent, and they span } R^2$$

Example 4: The column of every invertible n by n matrix give a basis for R^n:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

We can see that A is **invertible**, **independent**, with a rank = 3. B is singular matrix, which also have dependent columns, and it is not full column rank (column space $\neq R^3$). The only solution to $Ax = 0$ is $x = A^{-1}0 = 0$. However, for $Ax = b$ can always be solved by $x = A^{-1}b$ Everything comes together for invertible matrices. **The vector** $v_1, \ldots, v_n$ **are a basis for** R^n **exactly when they are the column of an** n **by** n **invertible matrix.** R^n **also has infinitely many bases** Or to put it more compactly, **every set of independent vectors can be extended to a basis; the spanning set of vectors can be reduced to a basis.**

Example 5: The matrix is not invertible so its column are not basis for anything.

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \to R_0 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Example 6: Find the bases for the column and row spaces of this rank two matrix.

$$R_0 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Like us see, so for column space of R_0, we got two pivots in a R^3, where $m = 3$; we got a subspace in R^3 And for row space, we got two non-zero rows and a rank of R^4 where $n = 4$.

Question! Given five vectors in R^7, how do we find their a basis that they span on? Two ways,

1. Make them into a row and eliminate to find their non-zero.
2. Make them into columns, and eliminate to find pivots. The pivot columns are the basis.

Another question, can another basis for the same vector space have more or less vectors? Answer, no! Number of vectors are tied to the space.

3.4.4 Dimension of a Vector Space

3.5 Dimensions of the Four Subspaces