Discrete Structures
CSC160

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Relations and Grapahs

Let $A=\{a_1, a_2, ..a_k\}$ and $B=\{b_1,b_2,..b_m\}$.

The Cartesian product A x B is defined by a set of pairs

$\{(a_1\ b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k,b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Relations and Grapahs

- Relation or Binary relation R from set A to B is a subset of AxB which can be defined as:
- aRb $\leftrightarrow$ (a,b) € R $\leftrightarrow$ R(a,b).
- A Binary relation R on a single set A is defined as a subset of AxA.
- For two distinct set, A and B with cardinalities m and n, the maximum cardinality of the relation R from A to B is mn.

Relations and Grapahs

- The given sets A and B are as follows:
  - $A = \{0, 1, 2\}$
  - $B = \{u, v\}$
- The relation R is defined as follows:
  - $R = \{ (0,u), (0,v), (1,v), (2,u) \}$
- The Cartesian product of two sets, A and B, denoted as $A \times B$, is a set of ordered pairs where the first element of each ordered pair is from set A and the second element is from set B. In other words:
  - $A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$
- In your case, the Cartesian product $A \times B$ would be:
  - $A \times B = \{ (0, u), (0, v), (1, u), (1, v), (2, u), (2, v) \}$
- Now, let's compare this with the given relation R:
- $R = \{ (0,u), (0,v), (1,v), (2,u) \}$
- As you can see, every ordered pair in R is also in $A \times B$. Therefore, R is a valid subset of the Cartesian product $A \times B$.

Binary relation

<u>Definition:</u> Let A and B be two sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.
- We use the notation **a R b to denote (a,b) $\in$ R** and **a $\not R$ b to denote (a,b) $\notin$ R**. If **a R b**, we say a is related to b by R.

Example: Let A={a,b,c} and B={1,2,3}.

- Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**
- Is Q={(1,a),(2,b)} a relation from A to B? **No.**
- Is P={(a,a),(b,c),(b,a)} a relation from A to A? **Yes**

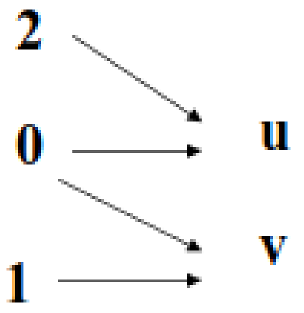
Relations and Graphs

- We can graphically represent a binary relation R as follows:
  - if **a R b** then draw an arrow from a to b.

$$\mathbf{a} \rightarrow \mathbf{b}$$

Example:

- Let $A = \{0, 1, 2\}$, $B = \{u,v\}$ and $R = \{ (0,u), (0,v), (1,v), (2,u) \}$
- Note: $R \subseteq A \times B$.
- **Graph:**



Relations and Graphs

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R.

Example:

- Let $A = \{0, 1, 2\}$, $B = \{u,v\}$ and $R = \{ (0,u), (0,v), (1,v), (2,u) \}$
- **Table:**

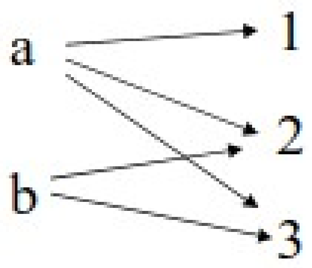
R	u	v
0	x	x
1		x
2	x	

or

R	u	v
0	1	1
1	0	1
2	1	0

Relations and functions

- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



- What is the difference between a **relation and a function from A to B**?

Relations and functions

- Relations represent **one to many relationships** between elements in A and B.
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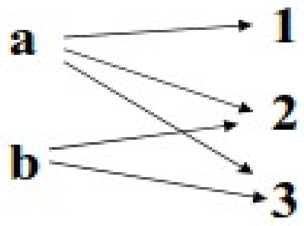
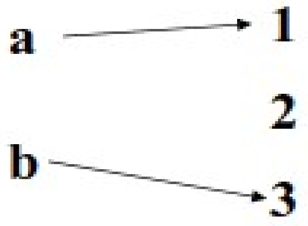


- What is the difference between a **relation and a function from A to B**? A function defined on sets A,B A → B assigns to each element in the domain set A exactly one element from B. So it is **a special relation.**



Relation on the set

Definition: **A relation on the set A** is a relation from A to itself.

Example 1:

- Let A = {1,2,3,4} and R_{div} = {(a,b)| a divides b}
- What does R_{div} consist of?
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
-

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relation on the set

Example:

- Let $A = \{1,2,3,4\}$.
- Define $a\ R_{\neq}\ b$ if and only if $a \neq b$.

Binary relations

- **Theorem:** The number of binary relations on a set A, where $|A| = n$ is:

$$2^{n^2}$$

- **Proof:**
- If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$.
- R is a binary relation on A if $R \subseteq A \times A$ (that is, R is a subset of A x A).
- The number of subsets of a set with k elements : 2^k
- The number of subsets of A x A is : $2^{|AxA|} = 2^{n^2}$

Relations and Graphs

- Discuss each Properties of relations with example each. (At your own)
  - Reflexive relations
  - Irreflexive relation
  - Symmetric relation
  - Antisymmetric relations
  - Transitive relations
  -

Relations and Graphs

- Combining Relations:
- Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.
- Consider the examples: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain
  - $R1 \cup R2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
  - $R1 \cap R2 = \{(1, 1)\}$,
  - $R1 - R2 = \{(2, 2), (3, 3)\}$,
  - $R2 - R1 = \{(1, 2), (1, 3), (1, 4)\}$.

Relations and Graphs

- N-ary Realtions
  - Relationships among elements of more than two sets often arise.
  - For instance, there is a relationship involving the name of a student, the student’s major, and the student’s grade point average.
  - Similarly, there is a relationship involving the airline, flight number, starting point, destination, departure time, and arrival time of a flight.
  - An example of such a relationship in mathematics involves three integers, where the first integer is larger than the second integer, which is larger than the third.

Relations and Graphs

- N-ary Relations
  - In mathematics and database theory, n-ary relations are relations or sets that involve more than two attributes or components.
  - In other words, instead of relating just two elements as in binary relations, n-ary relations relate n elements.
  - These relations are commonly used in databases to model complex relationships between entities. Here's an example of an n-ary relation:

Relations and Graphs

- N-ary Relations
  - Let's consider a university database where we want to represent the courses offered by the university.
  - In this database, we might have a ternary (3-ary) relation called "Teaches" that relates professors, courses, and semesters.
  - Each tuple in this relation represents a specific instance of a professor teaching a course in a particular semester.

Relations and Graphs

- N-ary Relations
  - The Teaches relation might look something like this:
  - Teaches(Professor, Course, Semester)
  - Here's an example of some tuples in this relation:
    - Teaches(Prof_A, Math101, Fall2023)
    - Teaches(Prof_B, Chem201, Spring2023)
    - Teaches(Prof_C, CS101, Fall2023)

- N-ary Relations
  - In this example:
  - Professor, Course, and Semester are the attributes (components) of the ternary relation.
  - Each tuple (e.g., the first tuple) represents a professor (Prof_A) teaching a course (Math101) in a specific semester (Fall2023).
  - The relation captures the relationships between professors, courses, and the semesters in which they are taught.
  - This is an example of an n-ary relation because it involves three components (attributes) instead of just two as in binary relations. N-ary relations can involve more than three components as well, depending on the complexity of the data being modeled.

Relations and Graphs

- N-ary Relations
  - **Definition**
  - Let A1 , A2 , . . . , An be sets.
  - An n-ary relation on these sets is a subset of A1 $\times$ A2 $\times \cdot \cdot \cdot \times$ An .
  - The sets A1 , A2 , . . . , An are called the domains of the relation, and n is called its degree.

Relations and Graphs

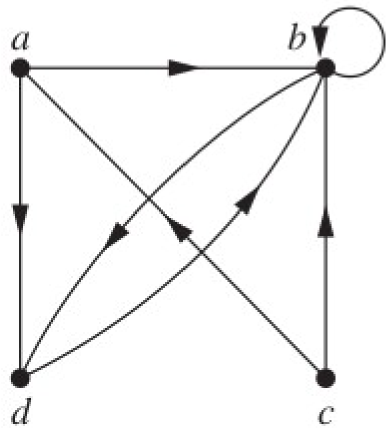
- Representing Relations using matrix
  - Let A = {a1 , a2 , a3 } and B = {b1 , b2 , b3 , b4 , b5 }. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

  - R = {(a1 , b2), (a2 , b1), (a2 , b3), (a2 , b4), (a3 , b1), (a3 , b3), (a3 , b5)}.
  - Because R consists of those ordered pairs (a_i , b_j) with m_{ij} = 1, it follows that

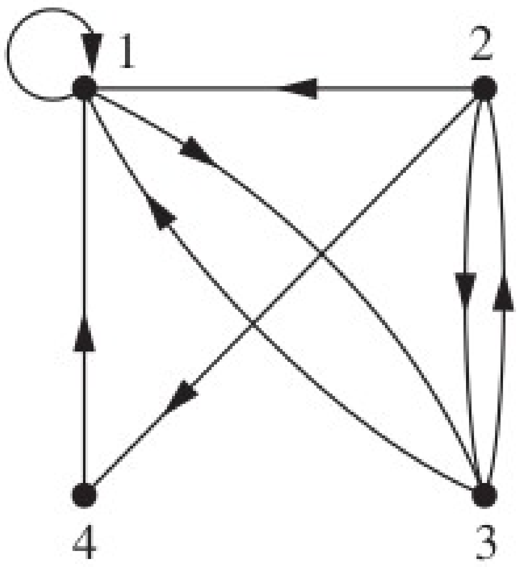
Relations and Graphs

- Representing Relations using Digraphs
  - A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).
  - The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.
  - An edge of the form (a, a) is represented using an arc from the vertex a back to itself.
  - Such an edge is called a loop.



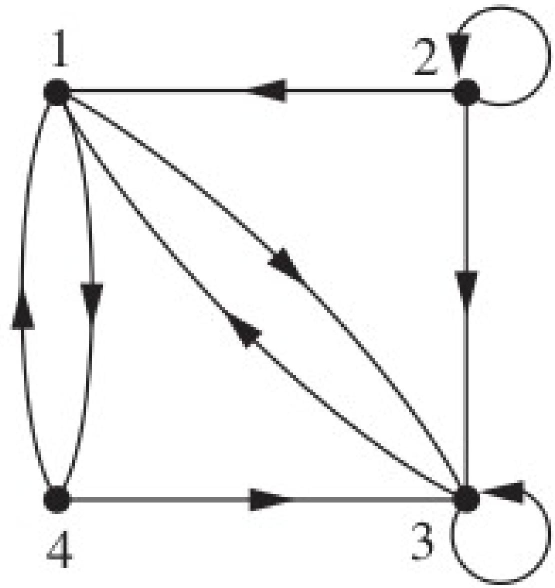
Relations and Graphs

- Representing Relations using Digraphs
  - The directed graph of the relation R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)} on the set {1, 2, 3, 4}



Relations and Graphs

- Representing Relations using Digraphs
  - What are the ordered pairs in the relation R represented by the given directed graph?



Relations and Graphs

- Closures of Relations
  - The closure of a relation is a fundamental concept in mathematics, particularly in the context of set theory and relational algebra.
  - It refers to the smallest or most specific relation that includes the original relation and satisfies certain properties or conditions.
  - There are different types of closures, such as
    - reflexive closure,
    - symmetric closure, and
    - transitive closure.

Relations and Graphs

- Closures of Relations
  - Reflexive Closure:
  - Given a relation R on a set A, its reflexive closure, denoted as R^+, is the smallest relation that contains all the original pairs from R and also includes all pairs (a, a) for every element a in A.
  - Example:
  - Let R be the relation on the set A = {1, 2, 3} defined as R = {(1, 2), (2, 3)}.
  - To find the reflexive closure R^+, we need to add (1, 1), (2, 2), and (3, 3) to R:
  - R^+ = {(1, 2), (2, 3), (1, 1), (2, 2), (3, 3)}

- Closure of Relations
  - Symmetric Closure:
  - Given a relation R on a set A, its symmetric closure, denoted as S(R), is the smallest relation that contains all the original pairs from R and also includes their reverse pairs.
  - Example:
  - Let R be the relation on the set A = {a, b, c} defined as R = {(a, b), (b, c)}. To find the symmetric closure S(R), we need to add the reverse pairs as well:
  - S(R) = {(a, b), (b, c), (b, a), (c, b)}

Relations and Graphs

- Closure of Relations
  - Transitive Closure:
  - Given a relation R on a set A, its transitive closure, denoted as T(R), is the smallest relation that contains all the original pairs from R and also includes any pairs that can be inferred transitively.
  - Example:
  - Let R be the relation on the set A = {1, 2, 3} defined as R = {(1, 2), (2, 3)}.
  - To find the transitive closure T(R), we need to add the transitive pairs.
  - In this case, we can infer that (1, 3) is transitive based on the existing pairs:
  - T(R) = {(1, 2), (2, 3), (1, 3)}

Relations and Graphs

- Equivalence relation
  - A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

- Equivalence relation: Example
  - Let R be the relation on the set of real numbers such that aRb if and only if a − b is an integer. Is R an equivalence relation?
  - Because a − a = 0 is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive.
  - Now suppose that aRb. Then a − b is an integer, so b − a is also an integer. Hence, bRa. It follows that R is symmetric.
  - If aRb and bRc, then a − b and b − c are integers.

    Therefore, a − c = (a − b) + (b − c) is also an integer. Hence, aRc. Thus, R is transitive.
  - Consequently, R is an equivalence relation.

- Partial Ordering:
  - We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary.
  - We schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins.
  - We order the set of integers using the relation containing the pairs (x, y), where x is less than y.
  - When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive.
  - These are properties that characterize relations used to order the elements of sets.

- Partial Ordering:
  - Definition:
  - A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive.
  - A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.

- Partial Ordering:
  - Show that the “greater than or equal” relation (≥) is a partial ordering on the set of integers.
  - Solution: Because a ≥ a for every integer a, ≥ is reflexive. If a ≥ b and b ≥ a, then a = b.
  - Hence, ≥ is antisymmetric. Finally, ≥ is transitive because a ≥ b and b ≥ c imply that a ≥ c.
  - It follows that ≥ is a partial ordering on the set of integers and (Z, ≥) is a poset.

Relations and Graphs

- Develop your understanding on following:
  - Lexicographic order
  - Lattices
  - Hasse Diagram

Relations and Graphs

- References:
  - *Kenneth H. Rosen, Discrete mathematics and its applications, Seventh Edition McGraw Hill Publication, 2012.*