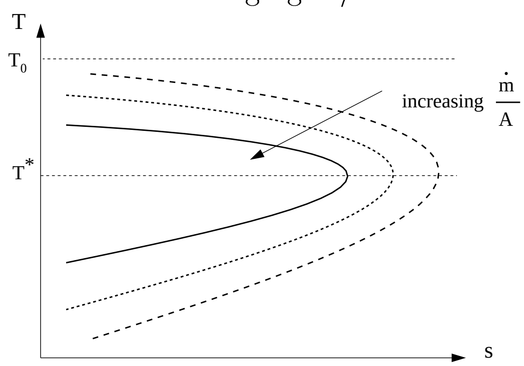
ME 131B Fluid Mechanics

Solutions to Week Six Problem Session: Fanno Flow (2/17/98)

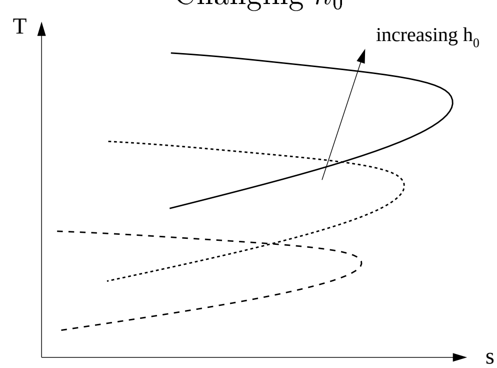
1. From which conservation principle(s) do we derive the Fanno curve?

   - The Fanno curve is derived from the conservation of mass and the conservation of energy principles.
   - All states on the *same* Fanno curve have the *same*
     - mass flux, $\dot{m}/A$
     - stagnation enthalpy, h_0
   - Effects of changing the mass flux and stagnation enthalpy on the Fanno curve are displayed in the following figures:

Changing $\dot{m}/A$



Changing h_0

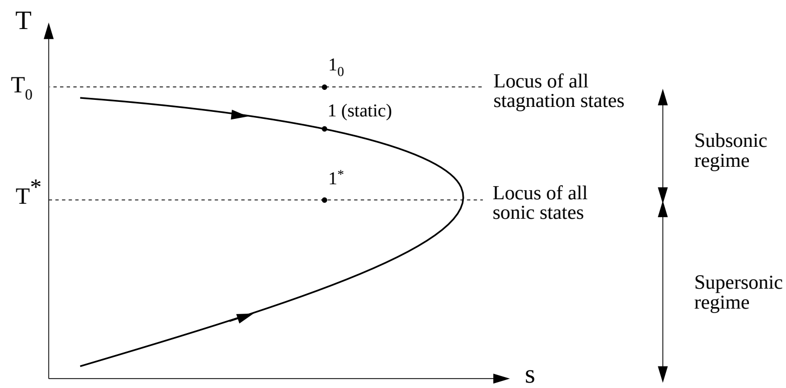


2. Trace out a Fanno curve on a $T-s$ plane and locate the corresponding stagnation and sonic states on the same diagram.

   - Since the stagnation temperature is constant for a Fanno flow (adiabatic), all the stagnation states lies on the horizontal line ($T = T_0$).
   - The ratio of sonic to stagnation temperature is a constant for any local static state

$$\frac{T_0}{T^*} = 1 + \frac{k-1}{2}$$

     Hence, the sonic temperature is also a constant. All the sonic states lies on the horizontal line ($T = T^*$).



3. Show that the Mach number corresponds to the maximum entropy point on a Fanno curve is unity.

- The road map to solve this problem is as follows:
  (a) Derive the equation of a Fanno curve on the $T - s$ plane.
  (b) Differentiate the equation with respect to temperature.
  (c) Solve for the maximum entropy point:

$$\frac{ds}{dT} = 0$$

- We first start with the Gibbs equation:

$$\begin{aligned} T\,ds &= du - \frac{P}{\rho^2}\,d\rho \\ ds &= C_v\,\frac{dT}{T} - R\,\frac{d\rho}{\rho} \qquad \text{(ideal gas)} \\ &= C_v\,\frac{dT}{T} + R\,\frac{dV}{V} \qquad (\rho V = \text{constant}) \end{aligned}$$

- Integrate the above equation (assume constant C_v), we obtain

$$s = C_v \log T + R \log V + \text{constant} \tag{1}$$

- From energy conservation, we have

$$\frac{V^2}{2} = h_0 - h = C_p\,(T_0 - T) \qquad \text{(perfect gas)}$$

$$\Rightarrow \qquad V = \sqrt{2\,C_p\,(T_0 - T)} \tag{2}$$

- Combine Equation (1) and Equation (2), we obtain the equation of a Fanno curve on the $T - s$ plane:

$$s = C_v \log T + \frac{R}{2} \log[\, 2\, C_p\, (T_0 - T)\,] + \text{constant}$$

- At the maximum entropy point,

$$\begin{aligned}
\frac{ds}{dT} &= 0 \\
\frac{C_v}{T} - \frac{R}{2}\,\frac{1}{T_0 - T} &= 0 \\
\frac{C_v}{T} - \frac{R}{2}\,\frac{2\,C_p}{V^2} &= 0 \qquad\qquad \text{From Equation (2)} \\
\Rightarrow \qquad V^2 &= \frac{C_p}{C_v}\, R\,T \\
&= k\, R\, T \\
&= c^2
\end{aligned}$$

We conclude that the flow speed at the maximum entropy point equals the speed of sound. Hence, it corresponds to a Mach 1.0 point.

4. Complete the following table with *increases, decreases, remains constant* for a flow through a constant-area, frictional duct:

	$M < 1$	$M > 1$
P	decreases	increases
ρ	decreases	increases
T	decreases	increases
V	increases	decreases
c	decreases	increases
M	increases	decreases
P_0	decreases	decreases
ρ_0	decreases	decreases
T_0	remains constant	remains constant
s	increases	increases
A^*	increases	increases
P^*	decreases	decreases
ρ^*	decreases	decreases
T^*	remains constant	remains constant
$P + \rho V^2$	decreases	decreases

5. A flow is supplied by a converging nozzle (unchoked).

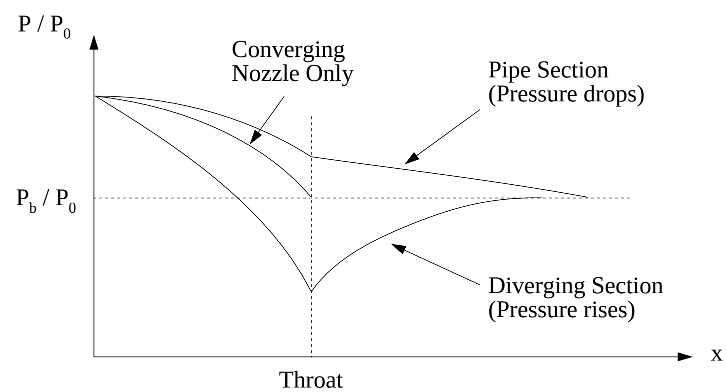
   (a) Will the addition of a diverging section *increase* or *decrease* the mass flow rate?

   - The addition of a diverging section will <u>increase</u> the mass flow rate. The reasons are outlined as follows:
     - Pressure *rises* in the diverging section for a subsonic flow which demands the pressure at the nozzle throat to be *lower* than the back pressure.
     - This leads to a *higher* Mach number at the nozzle throat, hence, a *higher* mass flow rate in the system.

   (b) What about adding a constant-area pipe? Will it *increase* or *decrease* the mass flow rate?

   - The addition of a constant-area pipe will <u>decrease</u> the mass flow rate. The reasons are outlined as follows:
     - Pressure *drops* along the pipe for a subsonic flow which demands the pressure at the nozzle exit to be *higher* than the back pressure.
     - This leads to a *lower* Mach number at the nozzle exit, hence, a *lower* mass flow rate in the system.

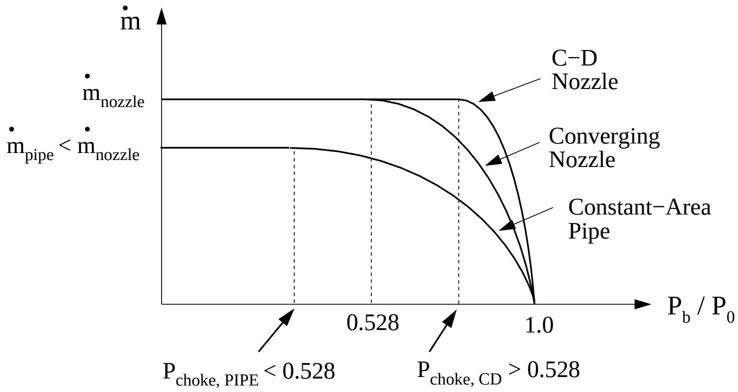
   Pressure distribution of the above three cases can be compared in the following figure:



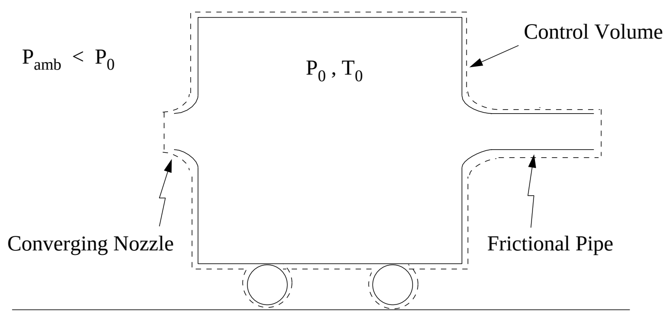
   (c) Sketch the variation of the mass flow rate, $\dot{m}$, with back-to-stagnation pressure ratio, P_b/P_0, for the above two cases on the same plot and highlight the differences.

   - The C-D nozzle is choked at a much higher pressure ratio (determined by the exit-to-throat area ratio) than the frictional pipe (determined by fL_{max}/D).
   - The mass flow rate out of a C-D nozzle is higher than that out of a frictional pipe.

   Mass flow rate of the above three cases can be compared in the following figure:



6. Consider the following system:



In what direction will the cart move? Explain your answer.

- As a first step of our analysis, let us choose a control volume as indicated above by the dotted line.
- We need to turn to the momentum equation

$$\underbrace{\vec{F}_S + \vec{F}_B}_{\text{external forces}} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \vec{V}\,(\rho\, dV)}_{\text{storage (acceleration)}} + \underbrace{\int_{CS} \vec{V}\left(\rho \vec{V} \cdot d\vec{A}\right)}_{\text{net momentum outflow}}$$

  and examine the differences in

  – momentum flux across the control surfaces

  – pressure forces on the control volume

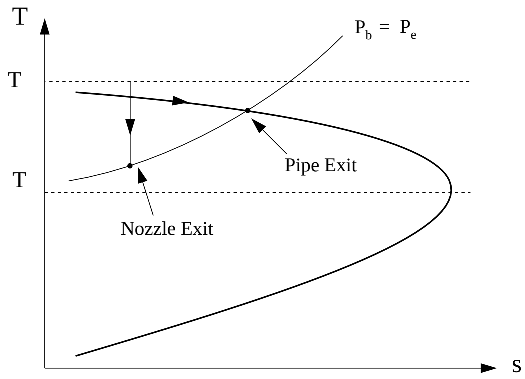
  before we conclude the motion of the cart.

- As we start lowering the back-to-stagnation pressure ratio, the exit plane pressure at the converging nozzle and the pipe are equal to the back pressure (subsonic flow). Hence, the net pressure force on the cart is zero.

- Let us examine the momentum flux

$$\rho_e V_e^2 A_e = \frac{P_e}{R T_e} V_e^2 A_e$$

for both cases. The two exit states can be represented on the $T - s$ diagram as follows:



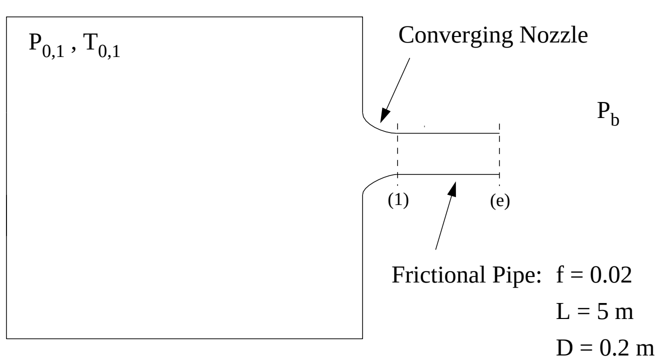
From the above diagram, we observe that

  - $P_{e,nozzle} = P_{e,pipe}$
  - $V_{e,nozzle} > V_{e,pipe}$
  - $T_{e,nozzle} < T_{e,pipe}$

Hence, we can conclude that the momentum flux out of the converging nozzle is *greater* than that out of the frictional pipe.

- As a conclusion, The cart will move to the right.

7. Fanno Flow in Subsonic Regime



Consider the above setup.

(a) Where can Mach 1.0 be realized?

- Mach 1.0 can *only* be realized at the pipe exit.

(b) Unchoked Case

In this part, we operate the above setup at a pressure ratio of $P_b/P_{0,1} = 0.60$. What is the flow conditions at the exit (P, T, M)?

- We have learned from Question 5 that the pipe is choked at a *lower* pressure ratio than the converging nozzle.
- In our case here, $P_b/P_{0,1} = 0.60 > 0.528$ (choking condition for converging nozzle). Hence, we can conclude
  - the pipe is not choked
  - flow is subsonic at the exit
  - exit plane pressure, P_e, equals the back pressure, P_b
- We can write the pressure ratio as follows:

$$\frac{P_b}{P_{0,1}} = \frac{P_b}{P_e} \frac{P_e}{P^*_{e,f}} \frac{P^*_{e,f}}{P^*_{1,f}} \frac{P^*_{1,f}}{P_1} \frac{P_1}{P_{0,1}}$$

- Let us examine the individual terms in the above equation:
  - $P_b/P_{0,1} = 0.60$ as given.
  - $P_b/P_e = 1$ because exit pressure equals back pressure for subsonic flow.
  - $P_e/P^*_{e,f}$ depends on the exit Mach number, M_e (from Fanno flow table).
  - $P^*_{e,f}/P^*_{1,f} = 1$ because Station (1) and Station (e) are on the same Fanno curve and are driven to the same reference $^*_f$ state.
  - $P^*_{1,f}/P_1$ depends on the inlet Mach number, M_1 (from Fanno flow table).
  - $P_1/P_{0,1}$ also depends on the inlet Mach number, M_1 (from isentropic flow table).

  We then simplify to the following relationship:

$$0.60 = \frac{P_e}{P^*_{e,f}} \frac{P^*_{1,f}}{P_1} \frac{P_1}{P_{0,1}} \tag{3}$$

- The pressure ratios in Equation (3) depend on two unknown Mach numbers (M_1, M_e). We can relate them by using the pipe geometry:

$$\frac{f\,L}{D} = \left(\frac{f\,L_{max}}{D}\right)_1 - \left(\frac{f\,L_{max}}{D}\right)_e$$

  where
  - $f L/D = 0.5$ as given by system specification.
  - $(f L_{max}/D)_1$ depends on the inlet Mach number, M_1 (from Fanno flow table).

- $(fL_{max}/D)_e$ depends on the exit Mach number, M_e (from Fanno flow table).

We have

$$0.5 = \left(\frac{f\,L_{max}}{D}\right)_1 - \left(\frac{f\,L_{max}}{D}\right)_e \tag{4}$$

- The solution procedure is an iterative one. It is outlined as follows:
  i. Guess M_1.
  ii. Look up the Fanno flow table to find out $(fL_{max}/D)_1$.
  iii. Calculate $(fL_{max}/D)_e$ from Equation (4).
  iv. Look up the Fanno flow table to find the corresponding M_e.
  v. With these two Mach numbers (M_1, M_e),
     - look up the Fanno flow table to find $P_e/P^*_{e,f}$
     - look up the Fanno flow table to find $P^*_{1,f}/P_1$
     - look up the isentropic flow table to find $P_1/P_{0,1}$
  vi. If the product of the above pressure ratios equals 0.60 (Equation (3)), the guess is correct in Step (i). Otherwise, keep guessing different values for M_1 and repeat the above procedure until the product of pressure ratios converges to 0.60.

(c) Choked Case

How much do we need to lower the back-to-stagnation pressure ratio to reach the choking condition? What does this ratio depend on? (Recall that it depends on the exit-to-throat area ratio for a C-D nozzle.)

- The solution procedure for the choked case is simpler than the unchoked case because we know
  - $M_e = 1$
  - $P_b = P^*_{1,f}$
- This also implies that

$$\frac{f\,L}{D} = \left(\frac{f\,L_{max}}{D}\right)_1 = 0.5$$

- From the Fanno flow table, we obtain the inlet Mach number to be

$$M_1 = 0.5978$$

- With $M_1 = 0.5978$,
  - the Fanno flow table gives

$$\frac{P_1}{P^*_{1,f}} = 1.7705$$

– the isentropic flow table gives

$$\frac{P_1}{P_{0,1}} = 0.78538$$

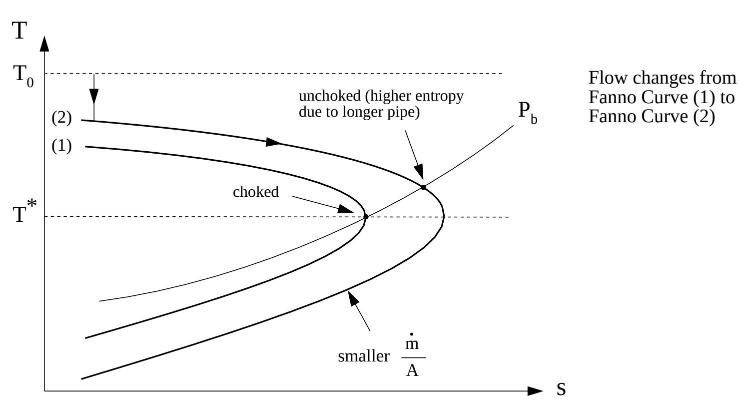
- The choking pressure ratio can then be computed as

$$\underline{\frac{P_b}{P_{0,1}} = \frac{P^*_{1,f}}{P_1}\frac{P_1}{P_{0,1}} = \left(\frac{1}{1.77051}\right)(0.78538) = 0.4436}$$

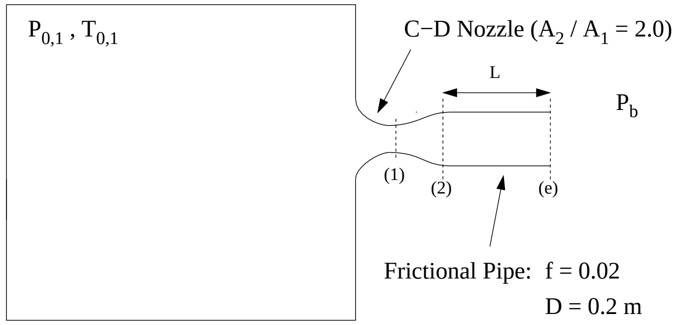
- This choking pressure ratio *depends on the value of fL/D of the pipe.*

(d) $\underline{L > L_{max}}$
After we lower the pressure ratio to the value we computed in Part (c), the flow is choked. What do you expect to happen if an extra two meters of pipe section is added to the existing system? Do you expect the flow is still choked at the exit? Explain your answer by showing the corresponding states on a $T - s$ diagram.

- After the flow is choked, addition of extra pipe section will *reduce* the mass flow rate inside the pipe. This corresponds to switching to another Fanno curve with a smaller mass flux value ($\dot{m}/A$) on the $T - s$ diagram.
- Since the exit pressure cannot be greater than the back pressure in subsonic flow (no shock mechanism), the flow leaves the exit *subsonically* with $P_e = P_b$. In other words, the addition of extra pipe section *unchokes* the system.
- This adjustment of flow conditions within the system is possible because *subsonic flow can communicate.* The addition of extra pipe section downstream can affect the pipe inlet condition upstream. In this case, it reduces the local Mach number at the pipe inlet. As we shall see in the next question, supersonic flow does not have this communication means. It can only adjust to extra pipe section by shock/expansion mechanism.
- The above conclusion can be summarized graphically in the following figure:



8. Fanno Flow in Supersonic Regime



(a) Slightly different from the last problem, there are two possible locations at which Mach 1.0 is attainable in the above setup. Where are they?

- Mach 1.0 can be realized at
  - the throat of the C-D nozzle
  - exit of the frictional pipe.

(b) $L = L_{max}$

In the supersonic operation mode, determine the pipe length L_{max} which gives a sonic flow right at the pipe exit.

- Under supersonic operation mode, the inlet Mach number, M_2, is governed by the area ratio of the C-D nozzle.
- For an area ratio of $A_2/A_1 = 2.0$, we obtain from the isentropic flow table

$$M_2 = 2.197$$

- From the Fanno flow table, this corresponds to

$$\left(\frac{f\,L_{max}}{D}\right)_2 = 0.36017$$

which gives a critical pipe length of

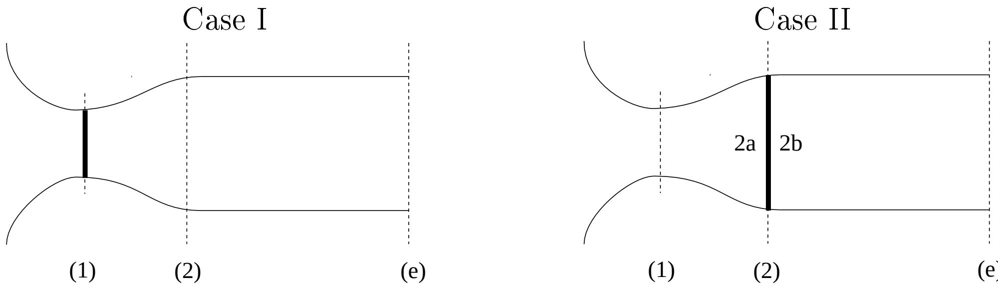
$$\underline{L_{max} = 3.602 \text{ m}}$$

(c) $L < L_{max}$

i. For $L = 2.0$ m, determine the range of the back-to-stagnation pressure ratio, $P_b/P_{0,1}$, over which

A. a normal shock appears in the diverging section of the C-D nozzle

- There are two limiting cases to consider here:



- Case I: upper limit
  - With the normal shock right at the nozzle throat, the flow goes *subsonic in the diverging section of C-D nozzle.*
  - Pipe inlet Mach number, M_2, is governed by the area ratio of the C-D nozzle (subsonic solution from isentropic flow table):

$$M_2 = 0.3060, \qquad \frac{P_2}{P_{0,2}} = 0.93712$$

  - With $M_2 = 0.3060$, the Fanno flow table gives

$$\frac{P_2}{P^*_{2,f}} = 3.5479 \qquad \left(\frac{f\,L_{max}}{D}\right)_2 = 5.031$$

  - This further gives the value of $f L_{max}/D$ at Station (e)

$$\left(\frac{f\,L_{max}}{D}\right)_e = \left(\frac{f\,L_{max}}{D}\right)_2 - \frac{f\,L}{D} = 4.831$$

  - From the Fanno flow table, this corresponds to an exit Mach number of 0.3105 and a pressure ratio

$$\frac{P_e}{P^*_{e,f}} = 3.4947$$

  - This gives the back-to-stagnation pressure ratio for Case I to be

$$\begin{aligned}\frac{P_b}{P_{0,1}} &= \frac{P_b}{P_e}\,\frac{P_e}{P^*_{e,f}}\,\frac{P^*_{e,f}}{P^*_{2,f}}\,\frac{P^*_{2,f}}{P_2}\,\frac{P_2}{P_{0,2}}\,\frac{P_{0,2}}{P_{0,1}}\\ &= (1)\,(3.4947)\,(1)\left(\frac{1}{3.5479}\right)(0.93712)\,(1)\\ &= 0.9231\end{aligned}$$

- Case II: lower limit
  - With the normal shock right at the pipe inlet, the flow goes *supersonic in the diverging section of C-D nozzle until it hits a shock at the nozzle exit, then goes subsonic right before entering the pipe section.*
  - The Mach number upstream of the shock is governed by the area ratio of C-D nozzle (supersonic solution from isentropic flow table):

$$M_{2a} = 2.197 \qquad \frac{P_{2a}}{P_{0,2a}} = 0.093936$$

  - With $M_{2a} = 2.197$, the normal shock table gives

$$M_{2b} = 0.54744 \qquad \frac{P_{2b}}{P_{2a}} = 5.4656$$

  - With $M_{2b} = 0.54744$, the Fanno flow table gives

$$\frac{P_{2b}}{P^*_{2b,f}} = 1.9438 \qquad \left(\frac{f\,L_{max}}{D}\right)_{2b} = 0.74305$$

  - This gives the value of $f L_{max}/D$ at Station (e)

$$\left(\frac{f\,L_{max}}{D}\right)_e = \left(\frac{f\,L_{max}}{D}\right)_{2b} - \frac{f\,L}{D} = 0.54305$$

  - From the Fanno flow table, this corresponds to an exit Mach number of 0.5874 and a pressure ratio

$$\frac{P_e}{P^*_{e,f}} = 1.8037$$

  - This gives the back-to-stagnation pressure ratio for Case II to be

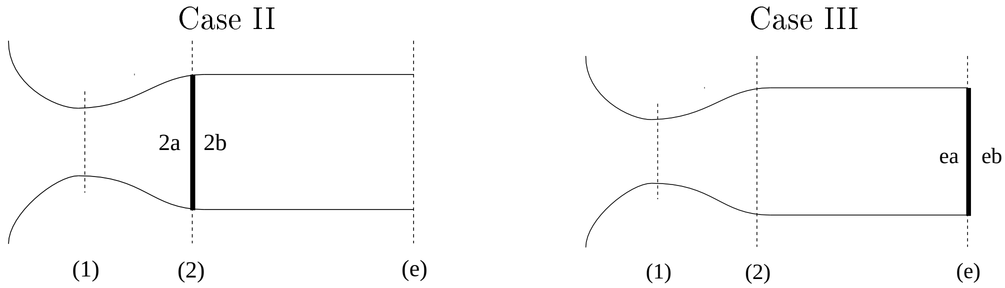
$$\begin{aligned}
\frac{P_b}{P_{0,1}} &= \frac{P_b}{P_e}\,\frac{P_e}{P^*_{e,f}}\,\frac{P^*_{e,f}}{P^*_{2b,f}}\,\frac{P^*_{2b,f}}{P_{2b}}\,\frac{P_{2b}}{P_{2a}}\,\frac{P_{2a}}{P_{0,2a}}\,\frac{P_{0,2a}}{P_{0,1}} \\
&= (1)\,(1.8037)\,(1)\left(\frac{1}{1.9438}\right)(5.4656)\,(0.093936)\,(1) \\
&= 0.4764
\end{aligned}$$

- Hence, a normal shock appears in the diverging section of the C-D nozzle when

$$\underline{0.4764 < \frac{P_b}{P_{0,1}} < 0.9231}$$

B. a normal shock appears in the pipe

- There are two limiting cases to consider here:



- Case II: upper limit It has already been studied in the previous part.
- Case III: lower limit
  - With the normal shock right at the pipe exit, the flow remains *supersonic from the nozzle throat all the way up to the pipe exit just before the shock, then exits subsonically after the shock.*
  - The pipe inlet Mach number, M_2, is again governed by the area ratio of C-D nozzle (supersonic solution from isentropic flow table). It has been found from the previous part:

$$M_2 = 2.197 \qquad \frac{P_2}{P_{0,2}} = 0.093936$$

  - With $M_2 = 2.197$, the Fanno flow table gives

$$\frac{P_2}{P^*_{2,f}} = 0.35567 \qquad \left(\frac{f\,L_{max}}{D}\right)_2 = 0.36012$$

  - This gives the value of fL_{max}/D at Station (ea)

$$\left(\frac{f\,L_{max}}{D}\right)_{ea} = \left(\frac{f\,L_{max}}{D}\right)_2 - \frac{f\,L}{D} = 0.16012$$

  - From the Fanno flow table, this corresponds to a Mach number of $M_{ea} = 1.566$ and a pressure ratio

$$\frac{P_{ea}}{P^*_{ea,f}} = 0.57292$$

  - With $M_{ea} = 1.566$, the normal shock table gives

$$M_{eb} = 0.6790 \qquad \frac{P_{eb}}{P_{ea}} = 2.6944$$

– This gives the back-to-stagnation pressure ratio for Case III to be

$$\begin{aligned}\frac{P_b}{P_{0,1}} &= \frac{P_b}{P_{eb}}\frac{P_{eb}}{P_{ea}}\frac{P_{ea}}{P^*_{ea,f}}\frac{P^*_{ea,f}}{P^*_{2,f}}\frac{P^*_{2,f}}{P_2}\frac{P_2}{P_{0,2}}\frac{P_{0,2}}{P_{0,1}} \\ &= (1)\,(2.6944)\,(0.57292)\,(1)\left(\frac{1}{0.35567}\right)(0.093936)\,(1) \\ &= 0.4077\end{aligned}$$

- Hence, a normal shock appears in the pipe section when

$$\underline{0.4077 < \frac{P_b}{P_{0,1}} < 0.4764}$$

For any back-to-stagnation pressure ratio which is lower than the critical value corresponds to Case III, the flow within the C-D nozzle and pipe section will be unaffected. All the pressure adjustment will take place outside the pipe. We will expect

- oblique shocks if the back pressure is higher than the design condition
- oblique expansion waves if the back pressure is lower than the design condition

The back-to-stagnation pressure ratio corresponds to the design condition (free of shock/expansion) is

$$\begin{aligned}\frac{P_b}{P_{0,1}} &= \frac{P_b}{P_e}\frac{P_e}{P^*_{e,f}}\frac{P^*_{e,f}}{P^*_{2,f}}\frac{P^*_{2,f}}{P_2}\frac{P_2}{P_{0,2}}\frac{P_{0,2}}{P_{0,1}} \\ &= (1)\,(0.57292)\,(1)\left(\frac{1}{0.35567}\right)(0.093936)\,(1) \\ &= 0.1513\end{aligned}$$

C. oblique shocks appear outside the pipe

- Oblique shocks appear outside the pipe *when the back-to-stagnation pressure ratio is between the design condition and the critical value corresponds to Case III:*

$$\underline{0.1513 < \frac{P_b}{P_{0,1}} < 0.4077}$$

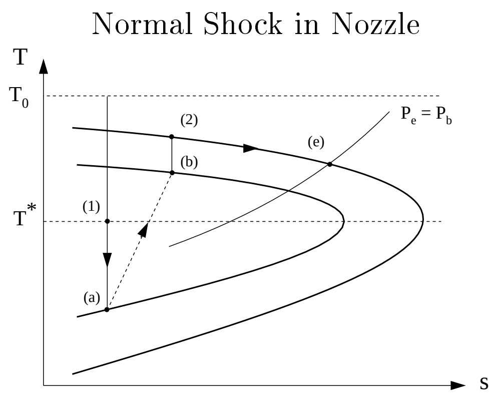
D. oblique expansion waves appear outside the pipe

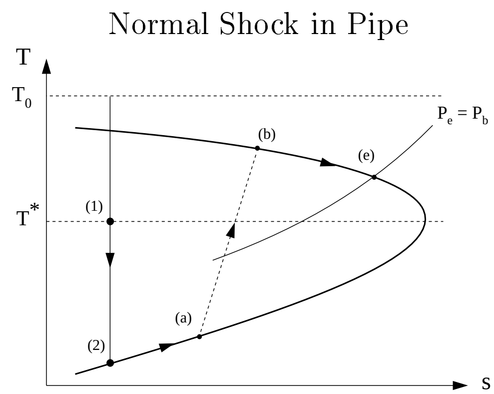
- Oblique expansion waves appear outside the pipe *when the back-to-stagnation pressure ratio is below the design condition:*

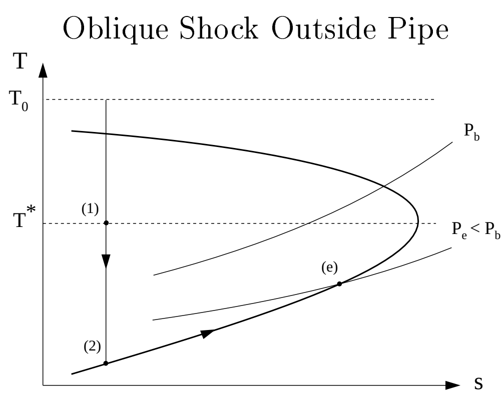
$$\underline{\frac{P_b}{P_{0,1}} < 0.1513}$$

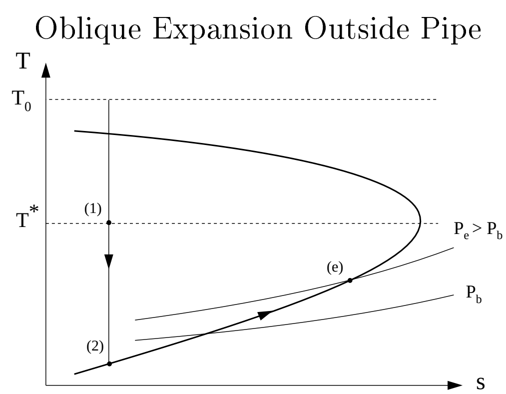
ii. For each of the above cases,

A. sketch the process path from the nozzle inlet to the pipe exit on a $T-s$ diagram.





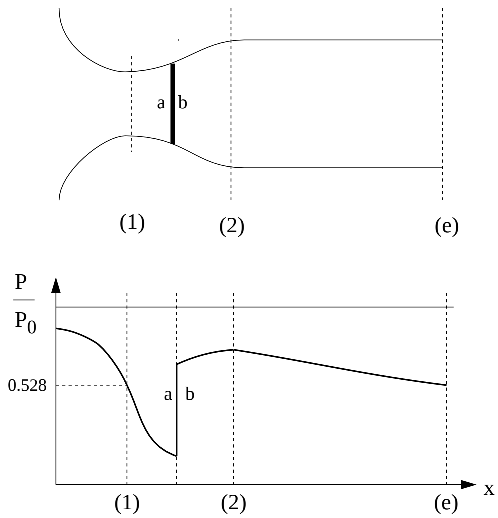




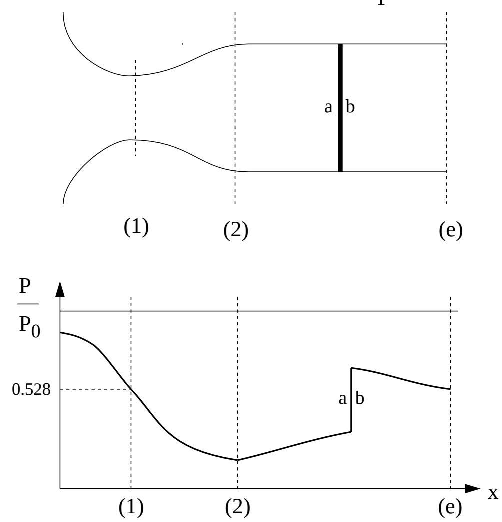
<u>Remarks:</u>
State (a) and State (b) are the upstream and downstream states of the normal shock respectively.

B. sketch the pressure distribution along the streamwise location from the nozzle inlet to the pipe exit.

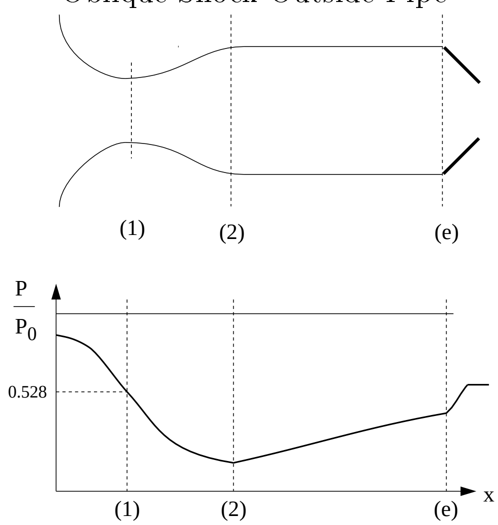
Normal Shock in Nozzle



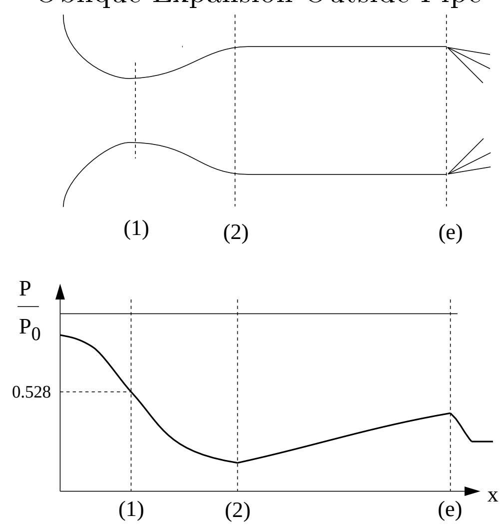
Normal Shock in Pipe



Oblique Shock Outside Pipe

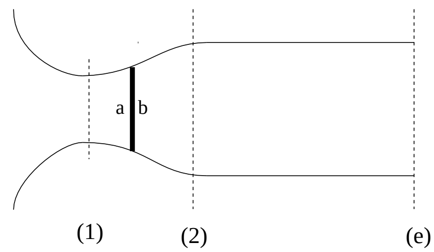


Oblique Expansion Outside Pipe



C. outline the solution procedure to locate the shock position in cases where normal shock appears.

- Similar to our previous procedure in locating a normal shock in the diverging section of a C-D nozzle, we need to solve this problem iteratively.
- The solution procedure is outlined below for the case with a normal shock standing in the diverging section of C-D nozzle (more difficult case):



A. Guess the Mach number upstream of the shock, M_a.

B. Obtain the pressure ratio, $P_a/P_{0,a}$, from the isentropic flow table (function of M_a).

C. Obtain the Mach number downstream of the shock, M_b, and the pressure ratios (P_b/P_a, $P_{0,b}/P_{0,a}$) from the normal shock table (function of M_a).

D. Relate M_b to the Mach number at the nozzle exit, M_2, as follows:

$$\frac{A_2}{A_1} = \frac{A_2}{A_2^*}\frac{A_2^*}{A_b^*}\frac{A_b^*}{A_a^*}\frac{A_a^*}{A_1}$$

- $A_2/A_1 = 2.0$ as given by system geometry.
- $A_2^*/A_b^* = 1$ because the flow is isentropic from Station (b) to Station (2).
- $A_b^*/A_a^* = P_{0,a}/P_{0,b}$ because the shock process is adiabatic. This ratio depends on M_a.
- $A_a^*/A_1 = 1$ because the isentropic flow from Station (1) to Station (a) is choked at Station (1).

E. We then obtain the simplified equation:

$$2.0 = \frac{A_2}{A_2^*}\frac{P_{0,a}}{P_{0,b}}$$

F. With the pressure ratio, $P_{0,a}/P_{0,b}$, determined in Step (C), the above equation gives the value of A_2/A_2^* which defines the Mach number at the nozzle exit, M_2 (from the isentropic flow table).

G. Obtain $P_2/P_{0,2}$ from the isentropic flow table (function of M_2).

H. Obtain $P_2/P_{2,f}^*$, $(fL_{max}/D)_2$ from the Fanno flow table (function of M_2).

I. Relate the Mach number at pipe exit, M_e, with the Mach number at pipe inlet, M_2, as follows:

$$\left(\frac{f\,L_{max}}{D}\right)_e = \left(\frac{f\,L_{max}}{D}\right)_2 - \left(\frac{f\,L}{D}\right)$$

J. With $fL/D = 0.2$ (given by system specification), the above equation gives the value of $(fL_{max}/D)_e$ which defines the Mach number at the pipe exit, M_e (subsonic solution from the Fanno flow table).

K. Obtain $P_e/P^*_{e,f}$ from the Fanno flow table (function of M_e).

L. The overall back-to-stagnation pressure ratio can then be computed as follows:

$$\frac{P_b}{P_{0,1}} = \frac{P_b}{P_e}\frac{P_e}{P^*_{e,f}}\frac{P^*_{e,f}}{P^*_{2,f}}\frac{P^*_{2,f}}{P_2}\frac{P_2}{P_{0,2}}\frac{P_{0,2}}{P_{0,b}}\frac{P_{0,b}}{P_{0,a}}\frac{P_{0,a}}{P_{0,1}}$$

– $P_b/P_e = 1$ because exit pressure equals back pressure for subsonic exit.

– $P^*_{e,f}/P^*_{2,f} = 1$ because Station (2) and Station (e) are on the same Fanno curve and are driven to the same reference $^*_f$ state.

– $P_{0,2}/P_{0,b} = 1$ because the flow is isentropic from Station (b) to Station (2).

– $P_{0,a}/P_{0,1} = 1$ because the flow is isentropic from Station (1) to Station (a).

M. We then simplify to the following relationship:

$$\frac{P_b}{P_{0,1}} = \underbrace{\frac{P_e}{P^*_{e,f}}}_{\text{Step K}}\;\underbrace{\frac{P^*_{2,f}}{P_2}}_{\text{Step H}}\;\underbrace{\frac{P_2}{P_{0,2}}}_{\text{Step G}}\;\underbrace{\frac{P_{0,b}}{P_{0,a}}}_{\text{Step C}}$$

All the pressure ratios have been determined in the previous steps. We just need to multiply all these pressure ratios together and check if their product equals the given back-to-stagnation pressure ratio. If it is, the guess is correct in Step (A). Otherwise, keep guessing different values of M_a and repeat the above procedure until the product of pressure ratios converges to the required value.

- The case with a normal shock standing in the pipe section can be analyzed in a similar manner and is easier!

(d) $\underline{L > L_{max}}$ (Common case)

- Since the fL_{max}/D values for supersonic flow is much smaller than those of subsonic flow, for most applications, $L > L_{max}$.
- Since $L > L_{max}$ and supersonic flow *cannot* communicate with downstream, a shock is *unavoidable within the pipe section.*
- The flow goes subsonic after the shock. It can *communicate* with the downstream condition. It either exits the pipe *subsonically, matching the back pressure* or *sonically with an exit pressure higher than the back pressure.*
- To decide between these two possible situations, we need to compare the back pressure, P_b, with the sonic pressure on the Fanno curve, P^*_f.

- If $P_b > P_f^*$, the flow exits *subsonically* and $P_e = P_b$.
- If $P_b < P_f^*$, the flow exits *sonically* and $P_e > P_b$. Expansion waves are expected to occur outside the pipe to adjust to the lower back pressure.

- In our present case, the sonic pressure on the Fanno curve, P_f^*, can be found as follows:

  i. From Part (b), we know that $M_2 = 2.197$ in supersonic operation mode.

  ii. From the isentropic flow table, we obtain

$$\frac{P_2}{P_{0,2}} = 0.093936$$

  iii. From the Fanno flow table, we obtain

$$\frac{P_2}{P_{2,f}^*} = 0.35567$$

  iv. Combine these two pressure ratios, we obtain

$$\begin{aligned}
\frac{P_{2,f}^*}{P_{0,1}} &= \frac{P_{2,f}^*}{P_2}\frac{P_2}{P_{0,2}}\frac{P_{0,2}}{P_{0,1}} \\
&= \left(\frac{1}{0.35567}\right)(0.093936)\ (1) \\
&= 0.2641
\end{aligned}$$

i. For $L = 5.0$ m, qualitatively describe the flow in the system for the following pressure ratios:

- In both cases, there is a normal shock within the system because $L > L_{max}$.

A. $P_b/P_{0,1} = 0.50$

- Since $P_b > P_f^*$, the flow exits <u>subsonically with exit pressure equals the back pressure.</u>
- No pressure adjustment is necessary outside the pipe.

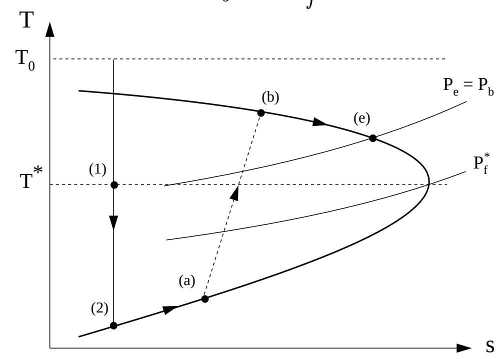
B. $P_b/P_{0,1} = 0.10$

- Since $P_b < P_f^*$, the flow exits <u>sonically with exit pressure higher than the back pressure.</u>
- Pressure adjustment in the form of *oblique expansion is expected to occur outside the pipe.*

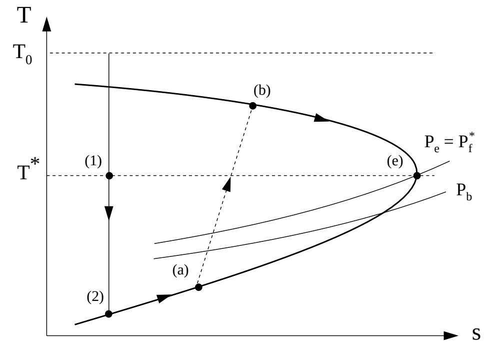
ii. For each of the above cases,

A. sketch the process path from the nozzle inlet to the pipe exit on a $T - s$ diagram.

$P_b > P_f^*$

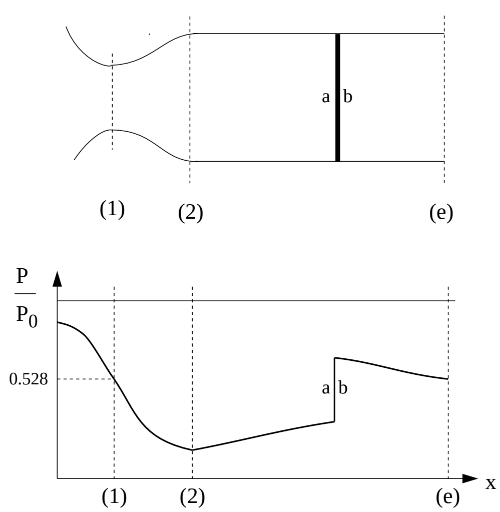


$P_b < P_f^*$



B. sketch the pressure distribution along the streamwise location from the nozzle inlet to the pipe exit.

$P_b > P_f^*$



$P_b < P_f^*$