Chapter 1

The Riemann Integral

1.1 Partitions

Definition 1.1.1. *Let X be a subset of . We say that X is* connected *iff X is nonempty and the following property is true: whenever x, y are elements in X such that $x < y$, the bounded interval $[x, y]$ is a subset of X (i.e., every number between x and y is also in X).*

Lemma 1.1.2. *Let X be a subset of the real line. Then the following two statements are logically equivalent:*

1. *X is bounded and either connected or empty.*
2. *X is a bounded interval.*

Proof. Both statements are logically equivalent when $X = \emptyset$ (which is vacuously true). So suppose that $X \neq \emptyset$.

We first show that X is bounded and connected implies X is a bounded interval. Since X is bounded, by i:5.5.9 we know that $\inf(X), \sup(X) \in$. Thus, $X \subseteq [\inf(X), \sup(X)]$. Now we split into four cases:

- If $\sup(X) \in X$ and $\inf(X) \in X$, then by i:11.1.1 X is connected implies $[\inf(X), \sup(X)] \subseteq X$. Thus, by i:3.1.18 we have $X = [\inf(X), \sup(X)]$.
- If $\sup(X) \in X$ and $\inf(X) \notin X$, then we claim that $\big(\inf(X), \sup(X)\big] \subseteq X$. This is true since X is connected and by i:11.1.1 we have $\big(a, \sup(X)\big] \subseteq X$ for every $a \in X$.
- If $\sup(X) \notin X$ and $\inf(X) \in X$, then we claim that $\big[\inf(X), \sup(X)\big) \subseteq X$. This is true since X is connected and by i:11.1.1 we have $\big[\inf(X), b\big) \subseteq X$ for every $b \in X$.
- If $\sup(X) \notin X$ and $\inf(X) \notin X$, then we claim that $\big(\inf(X), \sup(X)\big) \subseteq X$. This is true since X is connected and by i:11.1.1 we have $(a, b) \subseteq X$ for every $a, b \in X$ and $a < b$.

From all cases above, we conclude that X is a bounded interval.

Now we show that X is a bounded interval implies X is bounded and connected. Obviously X is bounded. Let $a, b \in$. Then X can be one of $(a, b), [a, b], (a, b], [a, b)$, and by i:11.1.1 all of which are connected. $\square$

Remark 1.1.3. *Recall that intervals are allowed to be singleton points, or even the empty set.*

Corollary 1.1.4. *If I and J are bounded intervals, then the intersection $I \cap J$ is also a bounded interval.*

Proof. If $I \cap J = \emptyset$, then $I \cap J$ is bounded interval. So suppose that $I \cap J \neq \emptyset$. Since I, J are bounded intervals, by i:11.1.4 we know that I, J are bounded and connected. Since I, J are bounded, $\exists M_1, M_2 \in$ such that $I \subseteq [-M_1, M_1]$ and $J \subseteq [-M_2, M_2]$. Let $M = \min(M_1, M_2)$. Then we have $I \cap J \subseteq [-M, M]$ and thus $I \cap J$ is bounded. Let $x, y \in I \cap J$ and $x < y$. Since I is connected and $I \cap J \subseteq I$, we have $[x, y] \subseteq I$. Similarly, since J is connected and $I \cap J \subseteq J$, we have $[x, y] \subseteq J$. Thus, $[x, y] \subseteq I \cap J$ and by i:11.1.1 $I \cap J$ is connected. Since $I \cap J$ is bounded and connected, by i:11.1.4 $I \cap J$ is bounded interval. □

Definition 1.1.5 (Length of intervals)**.** *If I is a bounded interval, we define the* length *of I, denoted I as follows. If I is one of the intervals $[a, b]$, (a, b), $[a, b)$, or $(a, b]$ for some real numbers $a < b$, then we define $Ib - a$. Otherwise, if I is a point or the empty set, we define $I = 0$.*

Definition 1.1.6 (Partitions)**.** *Let I be a bounded interval. A* partition *of I is a finite set $\mathbf{P}$ of bounded intervals contained in I, such that every x in I lies in exactly one of the bounded intervals J in $\mathbf{P}$.*

Remark 1.1.7. *Note that a partition is a set of intervals, while each interval is itself a set of real numbers. Thus, a partition is a set consisting of other sets.*

Theorem 1.1.8 (Length is finitely additive)**.** *Let I be a bounded interval, n be a natural number, and let $\mathbf{P}$ be a partition of I of cardinality n. Then*

$$I = \sum_{J \in \mathbf{P}} J.$$

Proof. We prove this by induction on n. More precisely, we let $P(n)$ be the property that whenever I is a bounded interval, and whenever $\mathbf{P}$ is a partition of I with cardinality n, that $I = \sum_{J \in \mathbf{P}} J$.

The base case $P(0)$ is trivial; the only way that I can be partitioned into an empty partition is if I is itself empty, at which point the claim is easy. The case $P(1)$ is also very easy; the only way that I can be partitioned into a singleton set J is if $J = I$, at which point the claim is again very easy.

Now suppose inductively that $P(n)$ is true for some $n \geq 1$, and now we prove $P(n+1)$. Let I be a bounded interval, and let $\mathbf{P}$ be a partition of I of cardinality $n + 1$.

If I is the empty set or a point, then all the intervals in $\mathbf{P}$ must also be either the empty set or a point, and so every interval has length zero and the claim is trivial. Thus, we will assume that I is an interval of the form (a, b), $(a, b]$, $[a, b)$, or $[a, b]$.

Let us first suppose that $b \in I$, i.e., I is either $(a, b]$ or $[a, b]$. Since $b \in I$, we know that one of the intervals K in $\mathbf{P}$ contains b. Since K is contained in I, it must therefore be of the form $(c, b]$, $[c, b]$, or b for some real number c, with $a \leq c \leq b$ (in the latter case of $K = b$, we set cb). In particular, this means that the set $I \setminus K$ is also an interval of the form $[a, c]$, (a, c), $(a, c]$, $[a, c)$ when $c > a$, or a point or empty set when $a = c$. Either way, we easily see that

$$I = K + I \setminus K.$$

On the other hand, since $\mathbf{P}$ forms a partition of I, we see that $\mathbf{P} \setminus K$ forms a partition of $I \setminus K$. By the induction hypothesis, we thus have

$$I \setminus K = \sum_{J \in \mathbf{P} \setminus K} J.$$

Combining these two identities (and using the laws of addition for finite sets, see i:7.1.11(e)) we obtain

$$I = \sum_{J \in \mathbf{P}} J$$

as desired.

Now suppose that $b \notin I$, i.e., I is either (a, b) or $[a, b)$. Then one of the intervals K also is of the form (c, b) or $[c, b)$ (see i:ex:11.1.3). In particular, this means that the set $I \setminus K$ is also an interval of the form $[a, c]$, (a, c), $(a, c]$, $[a, c)$ when $c > a$, or a point or empty set when $a = c$. The rest of the argument then proceeds as above. $\square$

Definition 1.1.9 (Finer and coarser partitions)**.** *Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. We say that $\mathbf{P}'$ is* finer *than $\mathbf{P}$ (or equivalently, that $\mathbf{P}$ is* coarser *than $\mathbf{P}'$) if for every J in $\mathbf{P}'$, there exists a K in $\mathbf{P}$ such that $J \subseteq K$.*

Remark 1.1.10. *There is no such thing as a "finest" partition of some interval I. (recall all partitions are assumed to be finite.) We do not compare partitions of different intervals.*

Definition 1.1.11 (Common refinement)**.** *Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. We define the* common refinement *$\mathbf{P}\#\mathbf{P}'$ of $\mathbf{P}$ and $\mathbf{P}'$ to be the set*

$$\mathbf{P}\#\mathbf{P}' K \cap J : K \in \mathbf{P} \text{ and } J \in \mathbf{P}'.$$

Corollary 1.1.12. *Let I be a bounded interval, and let $\mathbf{P}, \mathbf{P}'$ be two partitions of I. Then we have $I = \bigcup(\mathbf{P}\#\mathbf{P}')$.*

Proof. Let $x \in I$. By i:11.1.10 we know that $\exists!\ K \in \mathbf{P}$ such that $x \in K$. Similarly, $\exists!\ K' \in \mathbf{P}'$ such that $x \in K'$, thus $x \in K \cap K'$. By i:11.1.16 we know that $K \cap K' \in \mathbf{P}\#\mathbf{P}'$, thus $x \in \bigcup(\mathbf{P}\#\mathbf{P}')$. Since x was arbitrary, we have

$$I \subseteq \bigcup (\mathbf{P}\#\mathbf{P}').$$

Let $S \in \mathbf{P}\#\mathbf{P}'$. By i:11.1.16 we know that $\exists J \in \mathbf{P}$ and $\exists J' \in \mathbf{P}'$ such that $S = J \cap J'$. Since $S = J \cap J'$, we have $S \subseteq I$. Since S was arbitrary, we have

$$\bigcup (\mathbf{P}\#\mathbf{P}') \subseteq I.$$

Thus, by i:3.1.18 we have

$$I = \bigcup (\mathbf{P}\#\mathbf{P}').$$

$\square$

Corollary 1.1.13. *Let I be a bounded interval, and let $\mathbf{P}, \mathbf{P}'$ be two partitions of I. Then every element $x \in I$ contains in exactly one of the element $\mathbf{P}\#\mathbf{P}'$. In other words, $\exists!\ S \in \mathbf{P}\#\mathbf{P}$ such that $x \in S$.*

Proof. By i:ac:11.1.1 we know that at least one element in $\mathbf{P}\#\mathbf{P}'$ contains x. Suppose for the sake of contradiction that $\exists S_1, S_2 \in \mathbf{P}\#\mathbf{P}'$ such that $x \in S_1$ and $x \in S_2$ but $S_1 \neq S_2$. By i:11.1.16 we know that $S_1 = K \cap K'$ for some $K \in \mathbf{P}$ and $K' \in \mathbf{P}'$. Similarly, $S_2 = J \cap J'$ for some $J \in \mathbf{P}$ and $J' \in \mathbf{P}'$. We know that $x \in S_1$ implies $x \in K$. Similarly, $x \in S_2$ implies $x \in J$. But by i:11.1.10 we know that $K = J$, and a similar argument holds for $K' = J'$. Thus, we must have $S_1 = S_2$, a contradiction. $\square$

Corollary 1.1.14. *Let I be a bounded interval, and let $\mathbf{P}, \mathbf{P}'$ be two partitions of I. Then $\mathbf{P}\#\mathbf{P}'$ is finite and every element in $\mathbf{P}\#\mathbf{P}'$ is a bounded interval.*

Proof. Let $f : \mathbf{P} \times \mathbf{P}' \to \mathbf{P}\#\mathbf{P}'$ be a function where

$$f(K, K') = K \cap K' \text{ for every } (K, K') \in \mathbf{P} \times \mathbf{P}'.$$

By i:11.1.16 we see that f is surjective. By i:11.1.10 we know that both $\#(\mathbf{P}), \#(\mathbf{P}')$ are finite. Thus, by i:3.6.14(e) and i:ex:8.4.3 we have

$$\#(\mathbf{P} \times \mathbf{P}') = \#(\mathbf{P}) \times \#(\mathbf{P}') \geq \#(\mathbf{P}\#\mathbf{P}').$$

This means $\mathbf{P}\#\mathbf{P}'$ is finite.

By i:11.1.16 we know that for every $S \in \mathbf{P}\#\mathbf{P}'$, $S = K \cap K'$ for some $K \in \mathbf{P}$ and $K' \in \mathbf{P}'$. By i:11.1.10 we know that both K, K' are bounded interval, thus by i:11.1.6 we know that S is also a bounded interval. Since S was arbitrary, we conclude that every element in $\mathbf{P}\#\mathbf{P}'$ is a bounded interval. $\square$

Lemma 1.1.15. *Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. Then $\mathbf{P}\#\mathbf{P}'$ is also a partition of I, and is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$.*

Proof. By i:ac:11.1.1 we know that $I = \bigcup(\mathbf{P}\#\mathbf{P}')$. By i:ac:11.1.2 we know that every element in I contains in exactly one of the element $\mathbf{P}\#\mathbf{P}'$. By i:ac:11.1.3 we know that $\mathbf{P}\#\mathbf{P}'$ is finite and every element in $\mathbf{P}\#\mathbf{P}'$ is a bounded interval. Thus, by i:11.1.10 $\mathbf{P}\#\mathbf{P}'$ is a partition of I.

By i:11.1.16 we know that for every $S \in \mathbf{P}\#\mathbf{P}'$, $S = K \cap K'$ for some $K \in \mathbf{P}$ and $K' \in \mathbf{P}'$. This means $S \subseteq K$ and $S \subseteq K'$, thus by i:11.1.14 $\mathbf{P}\#\mathbf{P}'$ is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$ $\square$

Corollary 1.1.16. *Let I be a bounded interval, and let $\mathbf{P}, \mathbf{P}'$ be two partitions of I such that $\mathbf{P}'$ is finer than $\mathbf{P}$. For each $K \in \mathbf{P}$, we define $\mathbf{P}_K$ as follow:*

$$\mathbf{P}_K = K' \in \mathbf{P}' : K' \subseteq K.$$

Then $\mathbf{P}_K$ is a partition of K for every $K \in \mathbf{P}$, and $\bigcup_{K \in \mathbf{P}} \mathbf{P}_K = \mathbf{P}'$.

Proof. Since $\mathbf{P}_K \subseteq \mathbf{P}'$ and $\mathbf{P}'$ is a partition of I, by i:11.1.10 we know the following facts:

- $\mathbf{P}_K$ is finite.
- All distinct elements in $\mathbf{P}_K$ are disjoint.
- All elements in $\mathbf{P}_K$ are bounded interval.

To show that $\mathbf{P}_K$ is a partition of K, by i:11.1.10 it suffices to show that $K = \bigcup \mathbf{P}_K$.

Let $x \in K$. By i:11.1.10 we know that $x \in I$, thus $\exists!\ K' \in \mathbf{P}'$ such that $x \in K'$. Since $\mathbf{P}'$ is finer than $\mathbf{P}$, we must have $K' \subseteq K$. If not, then we have some $J \in \mathbf{P}$ such that $K' \subseteq J$, but $x \in J$ implies $J = K$, a contradiction. Since $K' \in \mathbf{P}'$ and $K' \subseteq K$, we have $K' \in \mathbf{P}_K$. Since x was arbitrary, we have $K \subseteq \bigcup \mathbf{P}_K$. By the definition of $\mathbf{P}_K$ we know that $\bigcup \mathbf{P}_K \subseteq K$, thus by i:3.1.18 we have $K = \bigcup \mathbf{P}_K$.

Now we show that $\bigcup_{K \in \mathbf{P}} \mathbf{P}_K = \mathbf{P}'$. We know that $\bigcup_{K \in \mathbf{P}} \mathbf{P}_K \subseteq \mathbf{P}'$. Let $K' \in \mathbf{P}'$. By i:11.1.18 we know that $\mathbf{P}'$ is finer than $\mathbf{P}$. By i:11.1.14 we know that $K' \subseteq K$ for some $K \in \mathbf{P}$. Thus, we have $K' \in \mathbf{P}_K$. Since K' was arbitrary, we have $\mathbf{P}' \subseteq \bigcup_{K \in \mathbf{P}} \mathbf{P}_K$. Thus, by i:3.1.18 we have $\bigcup_{K \in \mathbf{P}} \mathbf{P}_K = \mathbf{P}'$. □

Corollary 1.1.17. *Let I, J be bounded intervals such that $I \neq \emptyset$ and $I \subseteq J$, and let $\mathbf{P}$ be a partition of I. Let I_1, I_2 be the sets*

$$I_1 = x \in J : \big(x \leq \inf(I)\big) \wedge (x \notin I)$$

and

$$I_2 = x \in J : \big(x \geq \sup(I)\big) \wedge (x \notin I).$$

Then $\mathbf{P} \cup I_1, I_2$ is a partion of J.

Proof. First, we claim that I_1 is a bounded interval. If $I_1 = \emptyset$, then I_1 is a bounded interval. So suppose that $I_1 \neq \emptyset$. We know that $\inf(I) \in J$ since if $\inf(I) \notin J$, then by definition we would have $I_1 = \emptyset$, a contradiction. We must have $\inf(I_1) = \inf(J)$. If not, then we have $\inf(J) < \inf(I_1) \leq \inf(I)$. Since J is a bounded interval, we have $\inf(J) < x < \inf(I_1) \leq \inf(I)$ for some $x \in J$. But $x \in J$ and $x < \inf(I)$ implies $x \in I_1$, which contradict to $\inf(I_1) \leq x$. So we have $\inf(I_1) = \inf(J)$. Now we split into four cases:

- If $\inf(J) \in J$ and $\inf(I) \in I$, then $I_1 = \big[\inf(J), \inf(I)\big)$.
- If $\inf(J) \in J$ and $\inf(I) \notin I$, then $I_1 = [\inf(J), \inf(I)]$.
- If $\inf(J) \notin J$ and $\inf(I) \in I$, then $I_1 = \big(\inf(J), \inf(I)\big)$.
- If $\inf(J) \notin J$ and $\inf(I) \notin I$, then $I_1 = \big(\inf(J), \inf(I)]$.

From all cases above, we conclude that I_1 is a bounded interval.

Next we claim that I_2 is a bounded interval. If $I_2 = \emptyset$, then I_2 is a bounded interval. So suppose that $I_2 \neq \emptyset$. We know that $\sup(I) \in J$ since if $\sup(I) \notin J$, then by definition we would have $I_2 = \emptyset$, a contradiction. We must have $\sup(I_2) = \sup(J)$. If not, then we have $\sup(J) > \sup(I_2) \geq \sup(I)$. Since J is a bounded interval, we have $\sup(J) > x > \sup(I_2) \geq \sup(I)$ for some $x \in J$. But $x \in J$ and $x > \sup(I)$ implies $x \in I_2$, which contradict to $\sup(I_2) \geq x$. So we have $\sup(I_2) = \sup(J)$. Now we split into four cases:

- If $\sup(J) \in J$ and $\sup(I) \in I$, then $I_2 = \big(\sup(I), \sup(J)]$.
- If $\sup(J) \in J$ and $\sup(I) \notin I$, then $I_2 = [\sup(I), \sup(J)]$.
- If $\sup(J) \notin J$ and $\sup(I) \in I$, then $I_2 = \big(\sup(I), \sup(J)\big)$.
- If $\sup(J) \notin J$ and $\sup(I) \notin I$, then $I_2 = [\sup(I), \sup(J)\big)$.

From all cases above, we conclude that I_2 is a bounded interval.

Next we show that $I \cap I_1 = I \cap I_2 = I_1 \cap I_2 = \emptyset$. By definition we know that $I \cap I_1 = I \cap I_2 = \emptyset$. So we only need to show that $I_1 \cap I_2 = \emptyset$. If $(I_1 = \emptyset) \vee (I_2 = \emptyset)$, then we have $I_1 \cap I_2 = \emptyset$. So suppose that $(I_1 \neq \emptyset) \wedge (I_2 \neq \emptyset)$. Suppose for the sake of contradiction that $I_1 \cap I_2 \neq \emptyset$. Let $x \in I_1 \cap I_2$. Then we have $x \leq \inf(I) \leq \sup(I) \leq x$. Now we split into two cases:

- If $\inf(I) = \sup(I)$, then $I = a$ for some $a \in$. But $x \leq a \leq x$ implies $x = a$ and $x \in I$, which contradict to $x \notin I$.
- If $\inf(I) < \sup(I)$, then we have $x < x$, a contradiction.

From all cases above, we conclude that $I_1 \cap I_2 = \emptyset$.

Let $\mathbf{P}_J = \mathbf{P} \cup I_1, I_2$. By definition we know that $\bigcup \mathbf{P}_J \subseteq J$. Let $x \in J$. Now we split into two cases:

- If $x \in I$, then we have $x \in \bigcup \mathbf{P}$.
- If $x \notin I$, then we have $\big(x \leq \inf(I)\big) \vee \big(x \geq \sup(I)\big)$. Thus, $(x \in I_1) \vee (x \in I_2)$ and $x \in \bigcup \mathbf{P}$.

From all cases above, we conclude that $x \in \bigcup \mathbf{P}_J$. Since x was arbitrary, we have $J \subseteq \bigcup \mathbf{P}_J$. By i:3.1.18 we have $J = \bigcup \mathbf{P}_J$.

From the proofs above we have showed that $J = \bigcup \mathbf{P}_J$, all distinct element in $\mathbf{P}_J$ are disjoint, and all elements in $\mathbf{P}_J$ are bounded interval. Since $\mathbf{P}_J$ is finite ($\#(\mathbf{P}_J) = 3$), by i:11.1.10 $\mathbf{P}_J$ is a partition of J. □

Exercises

11.1.1 Prove i:11.1.4.

i:ex:11.1.1. See i:11.1.4. □

11.1.2 Prove i:11.1.6.

i:ex:11.1.2. Prove i:11.1.6. □

11.1.3 Let I be a bounded interval of the form $I = (a, b)$ or $I = [a, b)$ for some real numbers $a < b$. Let $I_1, \dots, I_n$ be a partition of I. Prove that one of the intervals I_j in this partition is of the form $I_j = (c, b)$ or $I_j = [c, b)$ for some $a \leq c \leq b$.

i:ex:11.1.3. Let $\mathbf{P} = I_1, \ldots, I_n$. If $c = b$, then $(c, b) = \emptyset$, and thus by i:11.1.10 $\mathbf{P} \cup \emptyset$ is a partition of I. So we only need to proof the cases where $a \leq c < b$. Suppose for the sake of contradiction that every interval I_j in the partition $\mathbf{P}$ is not of the form (c, b) or $[c, b)$. By i:11.1.10 this means for every $j \in 1, \ldots, n$, $x \in I_j$ implies $x \geq b$ or $x < c$. Since $I = (a, b)$ or $I = [a, b)$, we cannot have $x \geq b$, thus we must have $x < c$. This means $\sup(I_j) \leq c < b$ for every $j \in 1, \ldots, n$. But then we have $\sup(I) = b > \max \sup(I_j) : j \in 1, \ldots, n$, a contradiction. Thus, we must have one interval $I_j \in \mathbf{P}$ such that $I_j = (c, b)$ for some $a \leq c < b$. □

11.1.4 Prove i:11.1.18.

i:ex:11.1.4. Prove i:11.1.18. □