1 Convergence and limit laws

Definition 1.1 (Distance between two real numbers)**.** *Given two real numbers x and y, we define their distance $d(x, y)$ to be $d(x, y) := |x - y|$.*

Clearly, definition 1.1 is consistent with **??**. Further, **??** works just as well for real numbers as it does for rationals, because the real numbers obey all the rules of algebra that the rationals do.

Definition 1.2 (ε-close real numbers)**.** *Let $\varepsilon \in \mathbb{R}_{\geq 0}$. We say that two real numbers x, y are* ε-close *iff we have $d(y, x) \leq \varepsilon$.*

Again, it is clear that definition 1.2 is consistent with **??**.

Now let $(a_n)_{n=m}^{\infty}$ be a sequence of *real* numbers; i.e., we assign a real number a_n for every integer $n \geq m$. The starting index m is some integer; usually this will be 1, but in some cases we will start from some index other than 1. (The choice of label used to index this sequence is unimportant; we could use for instance $(a_k)_{k=m}^{\infty}$ and this would represent exactly the same sequence as $(a_n)_{n=m}^{\infty}$.) We can define the notion of a Cauchy sequence in the same manner as before.

Definition 1.3 (Cauchy sequences of reals)**.** *Let $\varepsilon \in \mathbb{R}^+$. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers starting at some integer index N is said to be* ε-steady *iff a_j and a_k are ε-close for every $j, k \in \mathbb{Z}_{\geq N}$. A sequence $(a_n)_{n=m}^{\infty}$ starting at some integer index m is said to be* eventually ε-steady *iff there exists an $N \in \mathbb{Z}_{\geq m}$ such that $(a_n)_{n=N}^{\infty}$ is ε-steady. We say that $(a_n)_{n=m}^{\infty}$ is a* Cauchy sequence *iff it is eventually ε-steady for every $\varepsilon \in \mathbb{R}^+$.*

To put it another way, a sequence $(a_n)_{n=m}^{\infty}$ of real numbers is a Cauchy sequence if, for every $\varepsilon \in \mathbb{R}^+$, there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - a_{n'}| \leq \varepsilon$ for all $n, n' \in \mathbb{Z}_{\geq N}$. These definitions are consistent with the corresponding definitions for rational numbers (**??????**), although verifying consistency for Cauchy sequences takes a little bit of care.

Proposition 1.4. *Let $(a_n)_{n=m}^{\infty}$ be a sequence of rational numbers starting at some integer index m. Then $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of* **??** *iff it is a Cauchy sequence in the sense of definition 1.3.*

i:6.1.4. Suppose first that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of definition 1.3; then it is eventually ε-steady for every $\varepsilon \in \mathbb{R}^+$. In particular, it is eventually ε-steady for every $\varepsilon \in \mathbb{Q}^+$, which makes it a Cauchy sequence in the sense of **??**.

Now suppose that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of **??**; then it is eventually ε'-steady for every $\varepsilon' \in \mathbb{Q}^+$. If $\varepsilon \in \mathbb{R}^+$, then there exists an $\varepsilon' \in \mathbb{Q}^+$ which is smaller than ε, by **??**. Since ε' is rational, we know that $(a_n)_{n=m}^{\infty}$ is eventually ε'-steady; since $\varepsilon' < \varepsilon$, this implies that $(a_n)_{n=m}^{\infty}$ is eventually ε-steady. Since ε is an arbitrary positive real number, we thus see that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of definition 1.3. □

Because of definition 1.4, we will no longer care about the distinction between **??** and definition 1.3, and view the concept of a Cauchy sequence as a single unified concept.

Definition 1.5 (Convergence of sequences)**.** *Let $\varepsilon \in \mathbb{R}^+$, and let $L \in \mathbb{R}$. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers is said to be* ε-close to L *iff a_n is ε-close to L for every $n \in \mathbb{Z}_{\geq N}$, i.e., we have $|a_n - L| \leq \varepsilon$ for every $n \in \mathbb{Z}_{\geq N}$. We say that a sequence $(a_n)_{n=m}^{\infty}$ is* eventually ε-close to L *iff there exists an $N \in \mathbb{Z}_{\geq m}$ such that $(a_n)_{n=N}^{\infty}$ is ε-close to L. We say that a sequence $(a_n)_{n=m}^{\infty}$* converges to L *iff it is eventually ε-close to L for every $\varepsilon \in \mathbb{R}^+$.*

Proposition 1.6 (Uniqueness of limits)**.** *Let $(a_n)_{n=m}^{\infty}$ be a real sequence starting at some integer index m, and let $L \neq L'$ be two distinct real numbers. Then it is not possible for $(a_n)_{n=m}^{\infty}$ to converge to L while also converging to L'.*

i:6.1.7. Suppose for the sake of contradiction that $(a_n)_{n=m}^{\infty}$ was converging to both L and L'. Let $\varepsilon = |L - L'|/3$. Note that ε is positive since $L \neq L'$. Since $(a_n)_{n=m}^{\infty}$ converges to L, we know that $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L; thus there is an $N \in \mathbb{Z}_{\geq m}$ such that $d(a_n, L) \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Similarly, there is an $M \in \mathbb{Z}_{\geq m}$ such that $d(a_n, L') \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq M}$. In particular, if we set $n := \max(N, M)$, then we have $d(a_n, L) \le \varepsilon$ and $d(a_n, L') \le \varepsilon$, hence by the triangle inequality $d(L, L') \le 2\varepsilon = 2|L - L'|/3$. But then we have $|L - L'| \le 2|L - L'|/3$, which contradicts the fact that $|L - L'| > 0$. Thus, it is not possible to converge to both L and L'. □

Definition 1.7 (Limits of sequences)**.** *If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L, we say that $(a_n)_{n=m}^{\infty}$ is* convergent *and that its* limit *is L; we write*

$$L = \lim_{n\to\infty} a_n$$

to denote this fact. If a sequence $(a_n)_{n=m}^{\infty}$ is not converging to any real number L, we say that the sequence $(a_n)_{n=m}^{\infty}$ is divergent *and we leave $\lim_{n\to\infty} a_n$ undefined.*

definition 1.6 ensures that a sequence can have at most one limit. Thus, if the limit exists, it is a single real number, otherwise it is undefined.

Remark 1.8. *The notation $\lim_{n\to\infty} a_n$ does not give any indication about the starting index m of the sequence, but the starting index is irrelevant (section 1). Thus, in the rest of this discussion we shall not be too careful as to where these sequences start, as we shall be mostly focused on their limits.*

We sometimes use the phrase "$a_n \to x$ as $n \to \infty$" as an alternate way of writing the statement "$(a_n)_{n=m}^{\infty}$ converges to x." Bear in mind, though, that the individual statements $a_n \to x$ and $n \to \infty$ do not have any rigorous meaning; this phrase is just a convention, though of course a very suggestive one.

Remark 1.9. *The exact choice of letter used to denote the index (in this case n) is irrelevant: the phrase $\lim_{n\to\infty} a_n$ has exactly the same meaning as $\lim_{k\to\infty} a_k$, for instance. Sometimes it will be convenient to change the label of the index to avoid conflicts of notation; for instance, we might want to change n to k because n is simultaneously being used for some other purpose, and we want to reduce confusion. See section 1.*

Proposition 1.10. *We have $\lim_{n\to\infty} 1/n = 0$.*

i:6.1.11. We have to show that the sequence $(a_n)_{n=1}^{\infty}$ converges to 0, where $a_n := 1/n$. In other words, for every $\varepsilon \in \mathbb{R}^+$, we need to show that the sequence $(a_n)_{n=1}^{\infty}$ is eventually ε-close to 0. So, let $\varepsilon \in \mathbb{R}^+$ be an arbitrary real number. We have to find an $N \in \mathbb{Z}^+$ such that $|a_n - 0| \le \varepsilon$ for every $n \in \mathbb{Z}_{\ge N}$. But if $n \in \mathbb{Z}_{\ge N}$, then

$$|a_n - 0| = |1/n - 0| = 1/n \le 1/N.$$

Thus, if we pick $N > 1/\varepsilon$ (which we can do by the Archimedean principle, **??**), then $1/N < \varepsilon$, and so $(a_n)_{n=N}^{\infty}$ is ε-close to 0. Thus, $(a_n)_{n=1}^{\infty}$ is eventually ε-close to 0. Since ε was arbitrary, $(a_n)_{n=1}^{\infty}$ converges to 0. □

Proposition 1.11 (Convergent sequences are Cauchy)**.** *Suppose that $(a_n)_{n=m}^{\infty}$ is a convergent sequence of real numbers. Then $(a_n)_{n=m}^{\infty}$ is also a Cauchy sequence.*

i:6.1.12. Suppose that $(a_n)_{n=m}^{\infty}$ converges to $L \in \mathbb{R}$. Let $\varepsilon \in \mathbb{R}^+$. Since $\lim_{n\to\infty} a_n = L$, by definition 1.5 there exists an $N \in \mathbb{Z}_{\ge m}$ such that $(a_n)_{n=N}^{\infty}$ is $\varepsilon/2$-close to L. This means $|a_n - L| \le \varepsilon/2$ for all $n \in \mathbb{Z}_{\ge N}$. Then we have

$$\begin{aligned} \forall j, k \in \mathbb{Z}_{\ge N}, |a_j - a_k| &\le |a_j - L| + |a_k - L| && i: ac: 5.4.1[f, g] \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. && i: 5.4.7[c, d] \end{aligned}$$

Thus, $(a_n)_{n=N}^{\infty}$ is ε-steady and $(a_n)_{n=m}^{\infty}$ is eventually ε-steady. Since ε was arbitrary, by definition 1.3 we see that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence of reals. □

Remark 1.12. *For a converse to definition 1.11, see* **??** *below.*

Proposition 1.13 (Formal limits are genuine limits)**.** *Suppose that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence of rational numbers. Then $(a_n)_{n=m}^{\infty}$ converges to $LIM_{n\to\infty} a_n$, i.e.*

$$LIM_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

i:6.1.15. Let $L = \text{LIM}_{n\to\infty} a_n$. By **??** we know that $L \in \mathbb{R}$. Thus, by **??** 1.4**??** 1.5 we can ask whether $(a_n)_{n=m}^{\infty}$ converges to L. Suppose for the sake of contradiction that $(a_n)_{n=m}^{\infty}$ does not converge to L. Then there must exist an $\varepsilon \in \mathbb{R}^+$ such that $(a_n)_{n=m}^{\infty}$ is not eventually ε-close to L. Fix such ε.

Since $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence of reals, we know that there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_j - a_k| \leq \frac{\varepsilon}{4}$ for all $j, k \in \mathbb{Z}_{\geq N}$. Fix such N. Since $(a_n)_{n=m}^{\infty}$ is not eventually ε-close to L, we know that $(a_n)_{n=m}^{\infty}$ is not eventually ε'-close to L for every $\varepsilon' \in \mathbb{R}_{0<\varepsilon}$. So $(a_n)_{n=m}^{\infty}$ is not eventually $\frac{2\varepsilon}{3}$-close to L, and we can find a $j \in \mathbb{Z}_{\geq N}$ such that $|a_j - L| > \frac{2\varepsilon}{3}$. Similarly, $(a_n)_{n=m}^{\infty}$ is not eventually $\frac{\varepsilon}{3}$-close to L, and we can find a $k \in \mathbb{Z}_{\geq N}$ such that $|a_k - L| > \frac{\varepsilon}{3}$. Fix such j and k. But then we have

$$\begin{aligned}
\frac{\varepsilon}{4} &\geq |a_j - a_k| && i:6.1.3 \\
&\geq |a_j - L| - |a_k - L| && i:4.3.3[f,g] \\
&\geq \frac{2\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3},
\end{aligned}$$

a contradiction. Thus, such ε does not exist. Therefore we must have $\lim_{n\to\infty} a_n = L = \text{LIM}_{n\to\infty} a_n$. □

Definition 1.14 (Bounded sequences)**.** *A sequence $(a_n)_{n=m}^{\infty}$ of reals is* bounded by *a real number $M \in \mathbb{R}_{\geq 0}$ iff we have $|a_n| \leq M$ for all $n \in \mathbb{Z}_{\geq m}$. We say that $(a_n)_{n=m}^{\infty}$ is bounded iff it is* bounded *by M for some real number $M \in \mathbb{R}_{\geq 0}$.*

definition 1.14 is consistent with **??**. See section 1.

Corollary 1.15. *Every convergent sequence of real numbers is bounded.*

i:6.1.17. Recall from **??** that every Cauchy sequence of rationals is bounded. An inspection of the proof of **??** shows that the same argument works for reals; every Cauchy sequence of reals is bounded. From definition 1.11 we see that every convergent sequence of reals is a Cauchy sequence. Thus, every convergent sequence of reals is bounded. □

Axiomatic Claim 1.16. *Let $x, y \in \mathbb{R}$. Then we have* $\min(x, y) = -\max(-x, -y)$. *Similarly, we have* $\max(x, y) = -\min(-x, -y)$.

i:ac:6.1.1. First, we show that $\min(x, y) = -\max(-x, -y)$. We split into two cases:

- If $x \leq y$, then we have $-x \geq -y$ by **??**. Thus,

$$\begin{aligned}
\min(x, y) &= x \\
&= -(-x) && i:ac:5.3.3 \\
&= -\max(-x, -y). && (-x \geq -y)
\end{aligned}$$

- If $x > y$, then we have $-x < -y$ by **??**. Thus,

$$\begin{aligned}
\min(x, y) &= y \\
&= -(-y) && i:ac:5.3.3 \\
&= -\max(-x, -y). && (-x < -y)
\end{aligned}$$

From all cases above, we see that $\min(x, y) = -\max(-x, -y)$. Thus, we conclude that $\min(x, y) = -\max(-x, -y)$.

Now we show that $\max(x, y) = -\min(-x, -y)$. This is true since

$$\begin{aligned}
\max(x, y) &= -(-\max(-(-x), -(-y))) && i:ac:5.3.3 \\
&= -\min(-x, -y). && \text{(from the proof above)}
\end{aligned}$$

□

Theorem 1.17 (Limit Laws)**.** *Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$.*

1. *The sequence $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$; in other words,*
$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$
2. *The sequence $(a_n b_n)_{n=m}^{\infty}$ converges to xy; in other words,*
$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n.$$
3. *For any real number c, the sequence $(ca_n)_{n=m}^{\infty}$ converges to cx; in other words,*
$$\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n.$$
4. *The sequence $(a_n - b_n)_{n=m}^{\infty}$ converges to $x - y$; in other words,*
$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$
5. *Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^{\infty}$ converges to y^{-1}; in other words,*
$$\lim_{n\to\infty} b_n^{-1} = \lim_{n\to\infty} b_n{}^{-1}.$$
6. *Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \in \mathbb{Z}_{\geq m}$. Then the sequence $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y; in other words,*
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$
7. *The sequence $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$; in other words,*
$$\lim_{n\to\infty} \max(a_n, b_n) = \max \lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n.$$
8. *The sequence $(\min(a_n, b_n))_{n=m}^{\infty}$ converges to $\min(x, y)$; in other words,*
$$\lim_{n\to\infty} \min(a_n, b_n) = \min \lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n.$$

i:6.1.19(a). Let $\varepsilon \in \mathbb{R}^+$. By definition 1.7, there exists an $N_a \in \mathbb{Z}_{\geq m}$ such that $|a_n - x| \leq \varepsilon/2$ for every $n \in \mathbb{Z}_{\geq N_a}$. Similarly, there exists an $N_b \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \varepsilon/2$ for every $n \in \mathbb{Z}_{\geq N_b}$. Now we fix both N_a and N_b. Let $N = \max(N_a, N_b)$. Then we have

$$\begin{aligned}
\forall n \in \mathbb{Z}_{\geq N}, |(a_n + b_n) - (x + y)| &= |(a_n - x) + (b_n - y)| && i : ac : 5.3.3\\
&\leq |a_n - x| + |b_n - y| && i : ac : 5.4.1\\
&\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. && i : 5.4.7[c, d]
\end{aligned}$$

Thus, by definition 1.5, $(a_n + b_n)_{n=N}^{\infty}$ is ε-close to $x + y$, and $(a_n + b_n)_{n=m}^{\infty}$ is eventually ε-close to $x + y$. Since ε was arbitrary, by definition 1.5 again we know that $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$. □

i:6.1.19(b). Since $x = \lim_{n\to\infty} a_n$ and $y = \lim_{n\to\infty} b_n$, by definition 1.15, there exist some $A, B \in \mathbb{R}_{\geq 0}$ such that $|a_n| \leq A$ and $|b_n| \leq B$ for all $n \in \mathbb{Z}_{\geq m}$. Fix both A and B. Clearly, we have $A < |x| + A + 1$ and $B < B + 1$, so we must have $|a_n| \leq |x| + A + 1$ and $|b_n| \leq B + 1$ for all $n \in \mathbb{Z}_{\geq m}$.

Let $\varepsilon \in \mathbb{R}^+$. Observe that $\dfrac{\varepsilon}{2(|x| + A + 1)} \in \mathbb{R}^+$ and $\dfrac{\varepsilon}{2(B + 1)} \in \mathbb{R}^+$. Since $x = \lim_{n\to\infty} a_n$, by definition 1.7, there exists an $N_a \in \mathbb{Z}_{\geq m}$ such that $|a_n - x| \leq \dfrac{\varepsilon}{2(B + 1)}$ for all $n \in \mathbb{Z}_{\geq N_a}$. Similarly, since

$y = \lim_{n\to\infty} b_n$, there exists an $N_b \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \dfrac{\varepsilon}{2(|x| + A + 1)}$ for all $n \in \mathbb{Z}_{\geq N_b}$. Now we fix both N_a and N_b. Let $N = \max(N_a, N_b)$. Then we have

$$\begin{aligned}
\forall n \in \mathbb{Z}_{\geq N}, |a_n b_n - xy| &= |a_n b_n - xy + xb_n - xb_n| && i : ac : 5.3.3 \\
&= |b_n(a_n - x) + x(b_n - y)| && i : ac : 5.3.3 \\
&\leq |b_n(a_n - x)| + |x(b_n - y)| && i : ac : 5.4.1 \\
&= |b_n|\,|a_n - x| + |x|\,|b_n - y| && i : ac : 5.4.1 \\
&\leq (B+1) \times \frac{\varepsilon}{2(B+1)} + |x|\,|b_n - y| && i : 5.4.7[c, d, e] \\
&\leq \frac{\varepsilon}{2} + (|x| + A + 1) \times \frac{\varepsilon}{2(|x| + A + 1)} && i : 5.4.7[c, d, e] \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, by definition 1.5, $(a_n b_n)_{n=N}^{\infty}$ is ε-close to xy, and $(a_n b_n)_{n=m}^{\infty}$ is eventually ε-close to xy. Since ε was arbitrary, by definition 1.5 again we know that $(a_n b_n)_{n=m}^{\infty}$ converges to xy. $\square$

i:6.1.19(c). Let $(c_n)_{n=m}^{\infty}$ be a sequence of reals where $c_n = c$ for all $n \in \mathbb{Z}_{\geq m}$. Clearly, we have $\lim_{n\to\infty} c_n = \lim_{n\to\infty} c = c$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} (c a_n) &= \lim_{n\to\infty} (c_n a_n) \\
&= \lim_{n\to\infty} c_n \lim_{n\to\infty} a_n && i : 6.1.19[b] \\
&= c \lim_{n\to\infty} a_n.
\end{aligned}$$

$\square$

i:6.1.19(d). We have

$$\begin{aligned}
\lim_{n\to\infty} (a_n - b_n) &= \lim_{n\to\infty} (a_n + (-1)(b_n)) && i : ac : 5.3.2 \\
&= \lim_{n\to\infty} a_n + \lim_{n\to\infty} ((-1)(b_n)) && i : 6.1.19[a] \\
&= \lim_{n\to\infty} a_n + (-1) \lim_{n\to\infty} b_n && i : 6.1.19[c] \\
&= \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n. && i : ac : 5.3.2
\end{aligned}$$

$\square$

i:6.1.19(e). First, we show that $(b_n)_{n=m}^{\infty}$ is bounded away from zero. Since $y \neq 0$, we know that $|y| > 0$. Since $y = \lim_{n\to\infty} b_n$, we know that there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \dfrac{|y|}{2}$ for all $n \in \mathbb{Z}_{\geq N}$. Then we have

$$\begin{aligned}
&\forall n \in \mathbb{Z}_{\geq N}, \frac{-|y|}{2} \leq b_n - y \leq \frac{|y|}{2} && i : ac : 5.4.1 \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} \dfrac{y}{2} \leq b_n \leq \dfrac{3y}{2} & \text{if } y \in \mathbb{R}^+ \\ \dfrac{3y}{2} \leq b_n \leq \dfrac{y}{2} & \text{if } y \in \mathbb{R}^- \end{cases} && i : 5.4.7[c, d] \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} \dfrac{y}{2} \leq b_n & \text{if } y \in \mathbb{R}^+ \\ \dfrac{-y}{2} \leq -b_n & \text{if } y \in \mathbb{R}^- \end{cases} && i : ex : 4.2.6 \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, \left|\frac{y}{2}\right| \leq |b_n|. && i : ac : 5.4.1
\end{aligned}$$

Since $y \neq 0$, we see that $\left|\dfrac{y}{2}\right| \in \mathbb{R}^+$. Thus, $(b_n)_{n=N}^{\infty}$ is bounded away from zero. Since $b_n \neq 0$ for all $n \in \mathbb{Z}_{\geq m}$, we see that $(b_n)_{n=m}^{N-1}$ is also bounded away from zero. Combining the results we see that $(b_n)_{n=m}^{\infty}$ is bounded away from zero.

Now we show that $\lim_{n\to\infty} b_n^{-1} = y^{-1}$. Let $\varepsilon \in \mathbb{R}^+$. Since $(b_n)_{n=m}^{\infty}$ is bounded away from zero, there exists an $M \in \mathbb{R}^+$ such that $|b_n| \geq M$ for all $n \in \mathbb{Z}_{\geq m}$. Fix such M. Clearly, we have $\varepsilon M\,|y| \in \mathbb{R}^+$ and $\dfrac{1}{|b_n|} \leq \dfrac{1}{M}$ for all $n \in \mathbb{Z}_{\geq m}$. Since $y = \lim_{n\to\infty} b_n \neq 0$, by definition 1.7, there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \varepsilon M\,|y|$ for all $n \in \mathbb{Z}_{\geq N}$. Fix such N. Then we have

$$\begin{aligned}
\forall n \in \mathbb{Z}_{\geq N}, \left|b_n^{-1} - y^{-1}\right| &= \left|\frac{1}{b_n} - \frac{1}{y}\right| && i:5.6.2\\
&= \left|\frac{y - b_n}{b_n y}\right| && i:ac:5.3.3\\
&= |y - b_n|\,\frac{1}{|b_n|\,|y|} && i:ac:5.4.1\\
&\leq |y - b_n|\,\frac{1}{M\,|y|} && i:5.4.7[e]\\
&\leq \varepsilon M\,|y|\,\frac{1}{M\,|y|} = \varepsilon. && i:5.4.7[e]
\end{aligned}$$

Thus, by definition 1.5, ${b_n^{-1}}_{n=N}^{\infty}$ is ε-close to y^{-1}, and ${b_n^{-1}}_{n=m}^{\infty}$ is eventually ε-close to y^{-1}. Since ε was arbitrary, by definition 1.5 again we know that ${b_n^{-1}}_{n=m}^{\infty}$ converges to y^{-1}. □

i:6.1.19(f). We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} a_n b_n^{-1} && i:ac:5.3.5\\
&= \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n^{-1} && i:6.1.19[b]\\
&= \lim_{n\to\infty} a_n \lim_{n\to\infty} {b_n}^{-1} && i:6.1.19[e]\\
&= \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}. && i:ac:5.3.5
\end{aligned}$$

□

i:6.1.19(g). First, suppose that $x = y$. Let $\varepsilon \in \mathbb{R}^+$. By definition 1.7, there exists an $N_a \in \mathbb{Z}_{\geq m}$ such that $|a_n - x| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N_a}$. Similarly, there exists an $N_b \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N_b}$. Fix both N_a and N_b. Let $N = \max(N_a, N_b)$. Then we have $|a_n - x| \leq \varepsilon$ and $|b_n - y| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Since $x = y$, we have $\max(x, y) = x$. Thus,

$$\begin{aligned}
&\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} |a_n - \max(x,y)| = |a_n - x| \leq \varepsilon \\ |b_n - \max(x,y)| = |b_n - x| = |b_n - y| \leq \varepsilon \end{cases}\\
\implies &\forall n \in \mathbb{Z}_{\geq N}, |\max(a_n, b_n) - \max(x,y)| = \begin{cases} |a_n - x| \leq \varepsilon & \text{if } a_n \geq b_n \\ |b_n - x| \leq \varepsilon & \text{if } a_n < b_n \end{cases}\\
\implies &\forall n \in \mathbb{Z}_{\geq N}, |\max(a_n, b_n) - \max(x,y)| \leq \varepsilon.
\end{aligned}$$

By definition 1.5, $\max(a_n, b_n)_{n=N}^{\infty}$ is ε-close to $\max(x, y)$, and $\max(a_n, b_n)_{n=m}^{\infty}$ is eventually ε-close to $\max(x, y)$. Since ε was arbitrary, by definition 1.5 again we know that $\max(a_n, b_n)_{n=m}^{\infty}$ converges to $\max(x, y)$.

Now suppose that $x \neq y$. We have either $x < y$ or $x > y$, so without the loss of generality, suppose that $x < y$. Then we have $\dfrac{y-x}{2} \in \mathbb{R}^+$. Let $\varepsilon \in \mathbb{R}^+$. Clearly, we have $\min \varepsilon, \dfrac{y-x}{2} \in \mathbb{R}^+$. By definition 1.7, there exists an $N_a \in \mathbb{Z}_{\geq m}$ such that $|a_n - x| \leq \min \varepsilon, \dfrac{y-x}{2}$ for all $n \in \mathbb{Z}_{\geq N_a}$. Similarly, there exists an $N_b \in \mathbb{Z}_{\geq m}$

such that $|b_n - y| \leq \min \varepsilon, \dfrac{y-x}{2}$ for all $n \in \mathbb{Z}_{\geq N_b}$. Fix both N_a and N_b. Let $N = \max(N_a, N_b)$. Then we have

$$\begin{aligned}
&\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} |a_n - x| \leq \min \varepsilon, \dfrac{y-x}{2} \leq \dfrac{y-x}{2} \\ |b_n - y| \leq \min \varepsilon, \dfrac{y-x}{2} \leq \dfrac{y-x}{2} \end{cases} && i : 5.4.7[c] \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} -\dfrac{y-x}{2} \leq a_n - x \leq \dfrac{y-x}{2} \\ -\dfrac{y-x}{2} \leq b_n - y \leq \dfrac{y-x}{2} \end{cases} && i : ac : 5.4.1 \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} a_n \leq \dfrac{y-x}{2} + x \\ y - \dfrac{y-x}{2} \leq b_n \end{cases} && i : 5.4.7[d] \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} a_n \leq \dfrac{x+y}{2} \\ \dfrac{x+y}{2} \leq b_n \end{cases} && i : ac : 5.3.3 \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, a_n \leq \dfrac{x+y}{2} \leq b_n. && i : 5.4.7[c]
\end{aligned}$$

This means $\max(a_n, b_n) = b_n$ for all $n \in \mathbb{Z}_{\geq N}$. Thus,

$$\begin{aligned}
\forall n \in \mathbb{Z}_{\geq N}, |\max(a_n, b_n) - \max(x, y)| &= |b_n - y| \\
&\leq \min \varepsilon, \frac{y-x}{2} \\
&\leq \varepsilon. && i : 5.4.7[c]
\end{aligned}$$

By definition 1.5, $\max(a_n, b_n)_{n=N}^{\infty}$ is ε-close to $\max(x, y)$, and $\max(a_n, b_n)_{n=m}^{\infty}$ is eventually ε-close to $\max(x, y)$. Since ε was arbitrary, by definition 1.5 again we know that $\max(a_n, b_n)_{n=m}^{\infty}$ converges to $\max(x, y)$. □

i:6.1.19(h). We have

$$\begin{aligned}
\lim_{n\to\infty} \min(a_n, b_n) &= \lim_{n\to\infty} -\max(-a_n, -b_n) && i : ac : 6.1.1 \\
&= -\lim_{n\to\infty} \max(-a_n, -b_n) && i : 6.1.19[c] \\
&= -\max \lim_{n\to\infty} -a_n, \lim_{n\to\infty} -b_n && i : 6.1.19[g] \\
&= -\max -\lim_{n\to\infty} a_n, -\lim_{n\to\infty} b_n && i : 6.1.19[c] \\
&= \min \lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n. && i : ac : 6.1.1
\end{aligned}$$

□

Let $(a_n)_{n=m}^{\infty}$ be a sequence of reals, such that $a_{n+1} > a_n$ for each $n \in \mathbb{Z}_{\geq m}$. Prove that whenever $j, k \in \mathbb{Z}_{\geq m}$ such that $j > k$, then we have $a_j > a_k$. (We refer to these sequences as *increasing* sequences.)

i:ex:6.1.1. Let $E = \{z \in \mathbb{Z}_{\geq m} : j \leq z \leq k\}$. Then E is finite (since $\#(E) = k - j + 1$) and non-empty (since $j, k \in E$). So $(a_n)_{n=j}^{k}$ is a finite sequence, and the elements in $(a_n)_{n=j}^{k}$ are $\{a_j, a_{j+1}, \ldots, a_{k-1}, a_k\}$. By hypothesis, we have $a_{n+1} > a_n$ for each $n \in \mathbb{Z}_{\geq m}$. Thus, we have $a_j < a_{j+1} < \cdots < a_{k-1} < a_k$, and by **??**(c) we have $a_j < a_k$. □

Let $(a_n)_{n=m}^{\infty}$ be a sequence of reals, and let $L \in \mathbb{R}$. Show that $(a_n)_{n=m}^{\infty}$ converges to L iff, given any $\varepsilon \in \mathbb{R}^+$, one can find an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - L| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$.

i:ex:6.1.2. We have

$$\begin{aligned}
&(a_n)_{n=m}^\infty \text{ converges to } L \\
\iff &\forall \varepsilon \in \mathbb{R}^+, (a_n)_{n=m}^\infty \text{ is eventually } \varepsilon\text{-close to } L && i:6.1.5 \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}_{\geq m} : (a_n)_{n=N}^\infty \text{ is } \varepsilon\text{-close to } L && i:6.1.5 \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}_{\geq m} : \forall n \in \mathbb{Z}_{\geq N}, |a_n - L| \leq \varepsilon. && i:6.1.5
\end{aligned}$$

□

Let $(a_n)_{n=m}^\infty$ be a sequence of reals, let $c \in \mathbb{R}$, and let $m' \in \mathbb{Z}_{\geq m}$. Show that $(a_n)_{n=m}^\infty$ converges to c iff $(a_n)_{n=m'}^\infty$ converges to c.

i:ex:6.1.3. First, suppose that $(a_n)_{n=m}^\infty$ converges to c. Let $\varepsilon \in \mathbb{R}^+$. By definition 1.5, there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Fix such N. Now we split into two cases:

- If $N \geq m'$, then we have found an $N \in \mathbb{Z}_{\geq m'}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$.
- If $N < m'$, then by setting $N' = m'$ we see that there exists an $N' \in \mathbb{Z}_{\geq m'}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N'}$.

From all cases above, we see that we can find an $M \in \mathbb{Z}_{\geq m'}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq M}$. Thus, by definition 1.5, we see that $(a_n)_{n=m'}^\infty$ is eventually ε-close to c. Since ε was arbitrary, by definition 1.5 again we see that $(a_n)_{n=m'}^\infty$ converges to c.

Now suppose that $(a_n)_{n=m'}^\infty$ converges to c. Let $\varepsilon \in \mathbb{R}^+$. By definition 1.5, there exists an $N \in \mathbb{Z}_{\geq m'}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Fix such N. Since $m \leq m'$, we have found an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Thus, by definition 1.5, we see that $(a_n)_{n=m}^\infty$ is eventually ε-close to c. Since ε was arbitrary, by definition 1.5 again we see that $(a_n)_{n=m}^\infty$ converges to c. □

Let $(a_n)_{n=m}^\infty$ be a sequence of reals, let $c \in \mathbb{R}$, and let $k \in \mathbb{Z}_{\geq 0}$. Show that $(a_n)_{n=m}^\infty$ converges to c iff $(a_{n+k})_{n=m}^\infty$ converges to c.

i:ex:6.1.4. First, observe that $(a_{n+k})_{n=m}^\infty = (a_n)_{n=m+k}^\infty$. Since $m + k \geq m$, by section 1 we see that $(a_n)_{n=m}^\infty$ converges to c iff $(a_n)_{n=m+k}^\infty$ converges to c. Thus, $(a_n)_{n=m}^\infty$ converges to c iff $(a_{n+k})_{n=m}^\infty$ converges to c. □

Prove definition 1.11.

i:ex:6.1.5. See definition 1.11. □

Prove definition 1.13.

i:ex:6.1.6. See definition 1.13. □

Show that definition 1.14 is consistent with **??** (i.e., prove an analogue of definition 1.4 for bounded sequences instead of Cauchy sequences).

i:ex:6.1.7. First, suppose that $(a_n)_{n=m}^\infty$ is a sequence of reals which is bounded in the sense of definition 1.14. Then there exists an $M \in \mathbb{R}_{\geq 0}$ such that $|a_n| \leq M$ for all $n \in \mathbb{Z}_{\geq m}$. By **??**, there exists an $M' \in \mathbb{Z}^+$ such that $M \leq M'$. Clearly, $M' \in \mathbb{Z}^+$ implies $M' \in \mathbb{Q}_{\geq 0}$. Thus, by **??**(c), we have $|a_n| \leq M'$ for all $n \in \mathbb{Z}_{\geq m}$. This means that $(a_n)_{n=m}^\infty$ is a bounded sequence in the sense of **??**.

Now suppose that $(a_n)_{n=m}^\infty$ is a sequence of reals which is bounded in the sense of **??**. Then there exists an $M \in \mathbb{Q}_{\geq 0}$ such that $|a_n| \leq M$ for all $n \in \mathbb{Z}_{\geq m}$. Since M is also a real number, we see that $(a_n)_{n=m}^\infty$ is a bounded sequence in the sense of definition 1.14. □

Proof definition 1.17.

i:ex:6.1.8. See definition 1.17. □

Explain why definition 1.17(f) fails when the limit of the denominator is 0. (To repair that problem requires *L'Hôpital's rule*, see **??**.)

i:ex:6.1.9. Suppose for the sake of contradiction that definition 1.17(f) works when denominator is 0. Let $(a_n)_{n=1}^{\infty} = (1/n)_{n=1}^{\infty}$. Then we have

$$\lim_{n\to\infty} \frac{a_n}{a_n} = \lim_{n\to\infty} \frac{1/n}{1/n} = \lim_{n\to\infty} 1 = 1.$$

But by definition 1.10 we also have

$$\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} a_n} = \frac{0}{0}$$

which is undefined. Thus, definition 1.17(f) fails when denominator is 0. □

Show that the concept of equivalent Cauchy sequence, as defined in **??**, does not change if ε is required to be positive real instead of positive rational. More precisely, if $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are sequences of reals, show that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close for every $\varepsilon \in \mathbb{Q}^+$ iff they are eventually ε-close for every $\varepsilon \in \mathbb{R}^+$.

i:ex:6.1.10. Suppose first that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close for all $\varepsilon \in \mathbb{Q}^+$. Let $\varepsilon' \in \mathbb{R}^+$. By **??**, there exists an $\varepsilon \in \mathbb{Q}^+$ such that $\varepsilon \leq \varepsilon'$. Fix such ε. Since $\varepsilon \in \mathbb{Q}^+$, by hypothesis we know that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close. This implies that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε'-close. Since ε' was arbitrary, we see that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε'-close for all $\varepsilon' \in \mathbb{R}^+$.

Now suppose that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε'-close for all $\varepsilon' \in \mathbb{R}^+$. This implies that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close for all $\varepsilon \in \mathbb{Q}^+$. Thus, we conclude that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close for all $\varepsilon \in \mathbb{Q}^+$ iff they are eventually ε'-close for all $\varepsilon' \in \mathbb{R}^+$. □