Chapter 1

The Natural Numbers

1.1 Addition

Definition 1.1.1 (Addition of natural numbers)**.** *Let m be a natural number. To add zero to m, we define $0 + mm$. Now suppose inductively that we have defined how to add n to m. Then we can add n++ to m by defining $(n{+}{+}) + m(n + m){+}{+}$.*

Corollary 1.1.2. *The sum $n + m$ of two natural numbers n, m is again a natural number.*

i:ac:2.2.1. Let m, n be a natural number. We induct on n. For $n = 0$, by i:2.2.1, we have $0 + m = m$, which is a natural number by the definition of m. So the base case holds. Suppose inductively that for some natural number n, we know that $n + m$ is a natural number. We want to show that $(n{+}{+}) + m$ is also a natural number. By i:2.2.1, we have $(n{+}{+}) + m = (n + m){+}{+}$. By the induction hypothesis, we know that $n + m$ is a natural number. Thus, by i:2.2, we know that $(n + m){+}{+}$ is again a natural number. This closes the induction. □

Lemma 1.1.3. *For any natural number n, we have $n + 0 = n$.*

i:2.2.2. We induct on n. The base case $0 + 0 = 0$ follows since we know that $0 + m = m$ for every natural number m (i:2.2.1), and 0 is a natural number (i:2.1). Now suppose inductively that $n + 0 = n$. We wish to show that $(n{+}{+}) + 0 = n{+}{+}$. But by i:2.2.1, $(n{+}{+}) + 0$ is equal to $(n + 0){+}{+}$, which is equal to $n{+}{+}$ since $n + 0 = n$. This closes the induction. □

Lemma 1.1.4. *For any natural numbers n and m, we have $n + (m{+}{+}) = (n + m){+}{+}$.*

i:2.2.3. We induct on n (keeping m fixed). For $n = 0$, we must prove $0 + (m{+}{+}) = (0 + m){+}{+}$. But by i:2.2.1, $0 + (m{+}{+}) = m{+}{+}$ and $0 + m = m$, so both sides are equal to $m{+}{+}$ and are thus equal to each other. So the base case holds. Suppose inductively that $n + (m{+}{+}) = (n + m){+}{+}$. We need to show that $(n{+}{+}) + (m{+}{+}) = ((n{+}{+}) + m){+}{+}$. The left-hand side is $(n + (m{+}{+})){+}{+}$ by i:2.2.1, which is equal to $((n + m){+}{+}){+}{+}$ by the inductive hypothesis. Similarly, we have $(n{+}{+}) + m = (n + m){+}{+}$ by i:2.2.1, and so the right-hand side is also equal to $((n + m){+}{+}){+}{+}$. Thus, both sides are equal, and we have closed the induction. □

Corollary 1.1.5. *For any natural number n, we have $n{+}{+} = n + 1$.*

i:ac:2.2.2. Since n is a natural number, by i:2.2 we know that $n{+}{+}$ is also a natural number. Thus, we can apply i:2.2.2 to derive the following fact:

$$\begin{aligned} n{+}{+} &= (n{+}{+}) + 0 && i : 2.2.2 \\ &= (n + 0){+}{+} && i : 2.2.1 \\ &= n + (0{+}{+}) && i : 2.2.3 \\ &= n + 1. \end{aligned}$$

□

Proposition 1.1.6 (Addition is commutative)**.** *For any natural numbers n and m, we have $n + m = m + n$.*

i:2.2.4. We induct on n. For $n = 0$, we need to show $0+m = m+0$. But by i:2.2.1, $0+m = m$, while by i:2.2.2, $m + 0 = m$. Thus, both sides are equal, and the base case holds. Suppose inductively that $n + m = m + n$. We must prove that $(n{+}{+}) + m = m + (n{+}{+})$ to close the induction. By i:2.2.1, $(n{+}{+}) + m = (n + m){+}{+}$. By i:2.2.3, $m + (n{+}{+}) = (m + n){+}{+}$, but this is equal to $(n + m){+}{+}$ by the induction hypothesis. Thus, $(n{+}{+}) + m = m + (n{+}{+})$, and we have closed the induction. □

Proposition 1.1.7 (Addition is associative)**.** *For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.*

i:2.2.5. We induct on c and keep both a and b fixed. For $c = 0$, we have

$$\begin{aligned} (a + b) + 0 &= a + b && i : 2.2.2 \\ &= a + (b + 0). && i : 2.2.2 \end{aligned}$$

Thus, the base case holds. Suppose inductively that $(a+b)+c = a+(b+c)$ for some natural number c. We want to show that $(a + b) + (c{+}{+}) = a + (b + (c{+}{+}))$. But this is true since

$$\begin{aligned} (a + b) + (c{+}{+}) &= ((a + b) + c){+}{+} && i : 2.2.3 \\ &= (a + (b + c)){+}{+} \\ &= a + (b + c){+}{+} && i : 2.2.3 \\ &= a + (b + (c{+}{+})). && i : 2.2.3 \end{aligned}$$

This closes the induction. □

Remark 1.1.8. *Because of associativity showed in i:2.2.5, we can write sums such as $a+b+c$ without having to worry about which order the numbers are being added together.*

Proposition 1.1.9 (Cancellation law)**.** *Let a, b, c be natural numbers such that $a+b = a+c$. Then we have $b = c$.*

i:2.2.6. We induct on a. For $a = 0$, we have $0 + b = 0 + c$, which by i:2.2.1 implies that $b = c$ as desired. Suppose inductively that we have the cancellation law for a (so that $a + b = a + c$ implies $b = c$); we now have to prove the cancellation law for $a{+}{+}$. In other words, we assume that $(a{+}{+}) + b = (a{+}{+}) + c$ and need to show that $b = c$. By i:2.2.1, we have $(a{+}{+}) + b = (a + b){+}{+}$ and $(a{+}{+}) + c = (a + c){+}{+}$, and so we have $(a + b){+}{+} = (a + c){+}{+}$. By i:2.4, we have $a + b = a + c$. Since we already have the cancellation law for a, we thus have $b = c$ as desired. This closes the induction. □

Definition 1.1.10 (Positive natural numbers)**.** *A natural number n is said to be* positive *iff it is not equal to* 0.

Proposition 1.1.11. *If a is a positive natural number and b is a natural number, then $a+b$ is positive (and hence $b+a$ is also, by i:2.2.4).*

i:2.2.8. We induct on b. If $b=0$, then $a+b=a+0=a$, which is positive, proving the base case. Suppose inductively that $a+b$ is positive. Then $a+(b{+}{+})=(a+b){+}{+}$, which cannot be zero by i:2.3, and is hence positive by i:2.2.7. This closes the induction. □

Corollary 1.1.12. *If a and b are natural numbers such that $a+b=0$, then we have $a=0$ and $b=0$.*

i:2.2.9. Suppose for the sake of contradiction that $a\neq 0$ or $b\neq 0$. If $a\neq 0$, then a is positive, and hence $a+b=0$ is positive by i:2.2.8, a contradiction. Similarly, if $b\neq 0$, then b is positive, and again $a+b=0$ is positive by i:2.2.8, a contradiction. Thus, a and b must both be zero. □

Lemma 1.1.13. *Let a be a positive natural number. Then there exists exactly one natural number b such that $b{+}{+}=a$.*

i:2.2.10. Let $P(n)$ be the statement "either we have $n=0$, or there exists a natural number m, such that $m{+}{+}=n$." We induct on n to show that $P(n)$ is true for any natural number n. Clearly, $P(0)$ is true. So suppose inductively that $P(n)$ is true for some natural number n. We want to show that $P(n{+}{+})$ is true. By i:2.3, we know that $n{+}{+}\neq 0$. So we have to show that there exists a natural number m, such that $m{+}{+}=n{+}{+}$. By i:2.4, we see that $m=n$. Thus, $P(n{+}{+})$ is true, closing the induction.

Now we prove the existence of b. From the first part of the proof, we know that $P(a)$ is true. Since a is a positive natural number, by i:2.2.7, we know that $a\neq 0$. Thus, there must exist a natural number b such that $b{+}{+}=a$.

Finally, we prove the uniqueness of b. Suppose that there exists another natural number c such that $c{+}{+}=a$. But this means $b{+}{+}=c{+}{+}$. Thus, by i:2.4, we have $b=c$. □

Definition 1.1.14 (Ordering of the natural numbers)**.** *Let n and m be natural numbers. We say that n is* greater than or equal to *m, and write $n\geq m$ or $m\leq n$, iff we have $n=m+a$ for some natural number a. We say that n is* strictly greater than *m, and write $n>m$ or $m<n$, iff $n\geq m$ and $n\neq m$.*

Corollary 1.1.15. *We have $n{+}{+}>n$ for any natural number n. Therefore, there is no largest natural number n, because the next number $n{+}{+}$ is always larger.*

i:ac:2.2.3. Let n be a natural number. By i:ac:2.2.2, we have $n{+}{+}=n+1$. Since 1 is a natural number, by i:2.2.11, we have $n{+}{+}\geq n$. To show that $n{+}{+}>n$, by i:2.2.11, we only need to show that $n{+}{+}\neq n$.

We induct on n to show that $n{+}{+}\neq n$ for any natural number n. For $n=0$, we have $0{+}{+}\neq 0$, by i:2.3. Thus, the base case holds. Suppose inductively that $n{+}{+}\neq n$ for some natural number n. Then we have

$$\begin{aligned} & n{+}{+}\neq n \\ \implies & (n{+}{+}){+}{+}\neq n{+}{+}. && i:2.4 \end{aligned}$$

This closes the induction. We conclude that $n{+}{+}>n$ for any natural number n. □

Corollary 1.1.16. *We have $n \geq 0$ for every natural number n. If n is a positive natural number, then we have $n > 0$.*

i:ac:2.2.4. Let n be a natural number. By i:2.2.1, we have $n = 0 + n$. Thus, by i:2.2.11, we have $n \geq 0$.

Now suppose that n is a positive natural number. By i:2.2.7, this means $n \neq 0$. From first paragraph we see that $n \geq 0$. Thus, by i:2.2.11, we have $n > 0$. □

Proposition 1.1.17 (Basic properties of order for natural numbers)**.** *Let a, b, c be natural numbers. Then*

1. *(Order is reflexive) $a \geq a$.*
2. *(Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.*
3. *(Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.*
4. *(Addition preserves order) $a \geq b$ iff $a + c \geq b + c$.*
5. *$a < b$ iff $a{+}{+} \leq b$.*
6. *$a < b$ iff $b = a + d$ for some* positive *number d.*

i:2.2.12(a). We have

$$i : 2.1, i : 2.2.2$$

$\implies a \geq a . i : 2.2.11$
0 is a natural number$a = a + 0 \quad i : 2.1, i : 2.2.2 \implies \quad a \geq a. \quad i : 2.2.11$

□

i:2.2.12(b). Suppose that $a \geq b$ and $b \geq c$. By i:2.2.11, there exist some natural numbers d and e, such that $a = b + d$ and $b = c + e$. Then we have

$$i : 2.2.5, i : ac : 2.2.1$$

$\implies a \geq c . i : 2.2.11$
$a = b + d = (c + e) + d = c + (e + d) e + d$ is a natural number $\quad i : 2.2.5, i : ac : 2.2.1 \implies \quad a \geq c. \quad i : 2.2$

□

i:2.2.12(c). Suppose that $a \geq b$ and $b \geq a$. By i:2.2.11, there exist some natural numbers c and d, such that $a = b + c$ and $b = a + d$. Then we have

$$i : 2.2.2, i : 2.2.5$$

$\implies 0 = d + ci : 2.2.6$
$\implies d = c = 0i : 2.2.9$
$\implies a = b + 0 = b.i : 2.2.2$
$a = a + 0a = b + c = (a + d) + c = a + (d + c) \quad i : 2.2.2, i : 2.2.5 \implies \quad 0 = d + c \quad i : 2.2.6 \implies d = c = 0$

□

i:2.2.12(d). We have

$$\begin{aligned}
& a \geq b && \\
\implies & a = b + d \text{ for some natural number } d && i : 2.2.11 \\
\implies & a + c = c + a = c + (b + d) && i : 2.2.4 \\
& = (c + b) + d = (b + c) + d && i : 2.2.4, i : 2.2.5 \\
\implies & a + c \geq b + c, && i : 2.2.11
\end{aligned}$$

and

$$\begin{aligned}
& a + c \geq b + c && \\
\implies & a + c = (b + c) + d \text{ for some natural number } d && i : 2.2.11 \\
\implies & c + a = (c + b) + d = c + (b + d) && i : 2.2.4, i : 2.2.5 \\
\implies & a = b + d && i : 2.2.6 \\
\implies & a \geq b. && i : 2.2.11
\end{aligned}$$

Thus, we conclude that $a \geq b \iff a + c \geq b + c$. □

i:2.2.12(e). First, suppose that $a < b$. Then we have

$$i : 2.2.11$$

$\implies b = a + c$ for some natural number c
$a \neq bi : 2.2.11$
$\implies c \neq 0i : 2.2.2$
$\implies c$ is a positive natural number$i : 2.2.8$
$\implies$ there exists a natural number d such that $d{+}{+} = ci : 2.2.10$
$\implies b = a + (d{+}{+}) = (a + d){+}{+} = (a{+}{+}) + di : 2.2.1, i : 2.2.3$
$\implies a{+}{+} \leq b.i : 2.2.11$
$a < b \implies \quad a \leq ba \neq b \quad i : 2.2.11 \implies b = a + c$ for some natural number $ca \neq b \quad i : 2.2.11 \implies \quad c \neq 0$

Now suppose that $a{+}{+} \leq b$. Then we have

$$\begin{aligned}
& a{+}{+} \leq b && \\
\implies & b = (a{+}{+}) + c \text{ for some natural number } c && i : 2.2.11 \\
\implies & b = (a{+}{+}) + c = (a + c){+}{+} = a + (c{+}{+}) && i : 2.2.1, i : 2.2.3 \\
\implies & a \leq b. && i : 2.2.11
\end{aligned}$$

By i:2.3, we know that $c{+}{+} \neq 0$. Thus, by i:2.2.6, we must have $b \neq a$. (If $b = a$, then we would have $b = a + (c{+}{+}) = a + 0$, which implies $c{+}{+} = 0$, a contradiction.) By i:2.2.11, this means $a < b$. From all proofs above, we conclude that $a < b \iff a{+}{+} \leq b$. $\square$

i:2.2.12(f). We have

$$\begin{aligned}
& a < b \\
\iff & a{+}{+} \leq b && i : 2.2.12[e] \\
\iff & b = (a{+}{+}) + c \text{ for some natural number } c && i : 2.2.11 \\
\iff & b = (a{+}{+}) + c = (a + c){+}{+} = a + (c{+}{+}) && i : 2.2.1, i : 2.2.3 \\
\iff & b = a + d \text{ for some positive natural number } d. && i : 2.3, i : 2.2.10
\end{aligned}$$

$\square$

Proposition 1.1.18 (Trichotomy of order for natural numbers)**.** *Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b$, $a = b$, or $a > b$.*

i:2.2.13. First, we show that we cannot have more than one of the statements $a < b$, $a = b$, $a > b$ holding simultaneously. If $a < b$, then $a \neq b$ by i:2.2.11, and if $a > b$, then $a \neq b$ by i:2.2.11. If $a > b$ and $a < b$, then by i:2.2.12(c), we have $a = b$, a contradiction. Thus, no more than one of the statements is true.

Now we show that at least one of the statements is true. We keep b fixed and induct on a. When $a = 0$, by i:2.2.1, we have $b = 0 + b$. Thus, by i:ac:2.2.4, we have $0 \leq b$ for any natural number b. So we have either $0 = b$ or $0 < b$, which proves the base case. Suppose we have proven the proposition for a, and now we prove the proposition for $a{+}{+}$. From the trichotomy for a, there are three cases: $a < b$, $a = b$, and $a > b$.

- If $a > b$, then by i:2.2.12(d), we have $a{+}{+} \geq b{+}{+}$. Since $b{+}{+} > b$, by i:2.2.12(b), we have $a{+}{+} \geq b$. Then we must have $a{+}{+} > b$. Otherwise, by i:2.2.11, we would have $a{+}{+} = b$, and by i:2.2.12(e), this implies $a < b$ and contradicts $a > b$ (the first paragraph proves the contradiction).

- If $a = b$, then by i:2.4, we have $a{+}{+} = b{+}{+}$. Since $b{+}{+} > b$, by i:2.2.12(b), we have $a{+}{+} \geq b$. Then we must have $a{+}{+} > b$. Otherwise, by i:2.2.11, we would have $a{+}{+} = b$, and by i:2.2.12(e), this implies $a < b$ and contradicts $a = b$ (the first paragraph proves the contradiction).

- If $a < b$, then by i:2.2.12(e), we have $a{+}{+} \leq b$. Thus, either $a{+}{+} = b$ or $a{+}{+} < b$, and in either case, we are done.

This closes the induction. $\square$

Proposition 1.1.19 (Strong principle of induction)**.** *Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true since, in this case, the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.*

i:2.2.14. Let n be a natural number, and let $Q(n)$ be the statement "$P(m)$ is true for any natural number m satisfying $m_0 \le m < n$." We induct on n to show that $Q(n)$ is true for any natural number n.

For $n = 0$, we want to show that $Q(0)$ is true. However, we know that $0 \le m_0$ for any natural number m_0. Thus, we have either $0 = m_0$ or $0 < m_0$. So we split it into two cases:

- If $0 < m_0$, then the statement "$P(m)$ is true for any natural number m satisfying $m_0 \le m < n$" is vacuously true, since there does not exist a natural number m satisfying $0 < m_0 \le m < n = 0$. Thus, $Q(0)$ is true in this case.

- If $0 = m_0$, then the statement "$P(m)$ is true for any natural number m satisfying $m_0 \le m < n$" is vacuously true, since there does not exist a natural number m satisfying $0 = m_0 \le m < n = 0$. Hence, $Q(0)$ is true in this case.

From all cases above, we see that $Q(0)$ is true. Thus, the base case holds.

Suppose inductively that $Q(n)$ is true for some natural number n. We need to show that $Q(n{+}{+})$ is true. Using the induction hypothesis $Q(n)$ and the hypothesis of P, we see that $P(n)$ is true. Since $n < n{+}{+}$, we know that $P(m)$ is true for any natural number m satisfying $m_0 \le m \le n < n{+}{+}$. So $P(m)$ is true for any natural number m satisfying $m_0 \le m < n{+}{+}$, which in turn implies that $Q(n{+}{+})$ is true. This closes the induction. Hence, we can conclude that $Q(n)$ is true for any natural number n.

Since $Q(n)$ is true for any natural number n, by the hypothesis of P, we know that $P(n)$ is true for any natural number n. In particular, we see that $P(n)$ is true for any natural number n satisfying $n \ge m_0$. □

Remark 1.1.20. *In applications we usually use i:2.2.14 with $m_0 = 0$ or $m_0 = 1$.*

Exercises

2.2.1 Prove i:2.2.5.

i:ex:2.2.1. See i:2.2.5. □

2.2.2 Prove i:2.2.10.

i:ex:2.2.2. See i:2.2.10. □

2.2.3 Prove i:2.2.12.

i:ex:2.2.3. See i:2.2.12. □

2.2.4 Justify the three statements marked in the proof of i:2.2.13.

i:ex:2.2.4. See i:2.2.13. □

2.2.5 Prove i:2.2.14.

i:ex:2.2.5. See i:2.2.14. ☐

2.2.6 [Principle of backwards induction] Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{++})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for any natural numbers $m \leq n$; this is known as the *principle of backwards induction.*

i:ex:2.2.6. We induct on n. For $n = 0$, the only natural number m satisfying $m \leq n = 0$ is 0. By the given hypothesis, $P(0)$ is true. Therefore, the base case holds trivially.

Suppose inductively that for some natural number n, we have the implication "if $P(n)$ is true, then $P(m)$ is true for any natural number m satisfying $m \leq n$." We want to show the implication "if $P(n{++})$ is true, then $P(m)$ is true for any natural number m satisfying $m \leq n{++}$" is also true. But when $P(n{++})$ is true, we know that $P(n)$ is true by the hypothesis of P. Thus, we can apply the induction hypothesis to derive "$P(m)$ is true for any natural number m satisfying $m \leq n$." Combining the statement "$P(n{++})$ is true," we see that the statement "$P(m)$ is true for any natural number m satisfying $m \leq n{++}$" is true. This closes the induction. ☐

2.2.7 Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m)$ is true, $P(m{++})$ is true. Show that if $P(n)$ is true, then $P(m)$ is true for any natural number m satisfying $m \geq n$. This principle is sometimes referred to as *the principle of induction starting from the base case n.*

i:ex:2.2.7. Suppose that $P(n)$ is true. Let $Q(k) = P(n + k)$ for every natural number k. We induct on k to show that $Q(k)$ is true for every natural number k.

For $k = 0$, we have $Q(0) = P(n+0) = P(n)$ by i:2.2.2. Since $P(n)$ is true by the given hypothesis, we know that $Q(0)$ is true. Thus, the base case holds.

Suppose inductively that $Q(k)$ is true for some natural number k. We want to show that $Q(k{++})$ is true. By i:2.2.3, we have $Q(k{++}) = P(n + (k{++})) = P((n + k){++})$. By the induction hypothesis, we know that $Q(k) = P(n+k)$ is true. Thus, we can use the hypothesis of P to show that $P((n+k){++})$ is also true. This closes the induction. We conclude that $P(n + k)$ is true for every natural number k.

By i:2.2.11, we know that for every natural number m, we have $m \geq n \iff m = n+k$ for some natural number k. Thus, $P(m)$ is true for every natural number m satisfying $m \geq n$. ☐