Chapter 1

The Natural Numbers

1.1 The Peano axioms

Axiom 1.1.1. 0 *is a natural number.*

Axiom 1.1.2. *If* n *is a natural number, then* $n{+}{+}$ *is a natural number.*

Axiom 1.1.3. 0 *is not the successor of any natural number; i.e., we have* $n{+}{+} \neq 0$ *for every natural number* n.

Axiom 1.1.4. *Different natural numbers must have different successors; i.e., if* n *and* m *are natural numbers and* $n \neq m$*, then* $n{+}{+} \neq m{+}{+}$*. Equivalently, if* $n{+}{+} = m{+}{+}$*, then we must have* $n = m$.

Axiom 1.1.5 (Principle of mathematical induction)**.** *Let* $P(n)$ *be any property pertaining to a natural number* n*. Suppose that* $P(0)$ *is true, and suppose that whenever* $P(n)$ *is true,* $P(n{+}{+})$ *is also true. Then* $P(n)$ *is true for every natural number* n.

- **Axioms 1.1.1–1.1.5** are known as the *Peano axioms* for the natural numbers.
- "a priori" is Latin for "beforehand" - it refers to what one already knows or assumes to be true before one begins a proof or argument. The opposite is "a posteriori" - what one knows to be true after the proof or argument is concluded.
- **Axiom 1.1.5** should technically be called an *axiom schema* rather than an *axiom* - it is a template for producing an (infinite) number of axioms rather than being a single axiom in its own right.
- A remarkable accomplishment of modern analysis is that by starting from these five very primitive axioms and some additional axioms from set theory, we can build all the other number systems, create functions, and do all the algebra and calculus that we are used to.