1 Finite series

Definition 1.1 (Finite series)**.** *Let m, n be integers, and let $(a_i)_{i=m}^n$ be a finite sequence of real numbers, assigning a real number a_i to each integer i between m and n inclusive (i.e., $m \leq i \leq n$). Then we define the finite sum (or finite series) $\sum_{i=m}^n a_i$ by the recursive formula*

$$\sum_{i=m}^{n} a_i := 0 \text{ whenever } n < m;$$

$$\sum_{i=m}^{n+1} a_i := \left(\sum_{i=m}^{n} a_i\right) + a_{n+1} \text{ whenever } n \geq m-1.$$

we sometimes express $\sum_{i=m}^n a_i$ less formally as

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n.$$

Remark 1.2. *The difference between "sum" and "series" is a subtle linguistic one. Strictly speaking, a series is an* expression *of the form $\sum_{i=m}^n a_i$; this series is mathematically (but not semantically) equal to a real number, which is then the* sum *of that series. For instance, $1+2+3+4+5$ is a series, whose sum is 15; if one were to be very picky about semantics, one would not consider 15 a series and one would not consider $1+2+3+4+5$ a sum, despite the two expressions having the same value. However, we will not be very careful about this distinction as it is purely linguistic and has no bearing on the mathematics; the expressions $1+2+3+4+5$ and 15 are the same number, and thus* mathematically *interchangeable, in the sense of the axiom of substitution, even if they are not semantically interchangeable.*

Remark 1.3. *Note that the variable i (sometimes called the* index of summation*) is a* bound variable *(sometimes called a* dummy variable*); the expression $\sum_{i=m}^n a_i$ does not actually depend on any quantity named i. In particular, one can replace the index of summation i with any other symbol, and obtain the same sum:*

$$\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j.$$

Lemma 1.4.

1. *Let $m \leq n < p$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq p$. Then we have*

$$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{p} a_i.$$

2. *Let $m \leq n$ be integers, k be another integer, and let a_i be a real number assigned to each integer $m \leq i \leq n$. Then we have*

$$\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}.$$

3. *Let $m \leq n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \leq i \leq n$. Then we have*

$$\sum_{i=m}^{n} (a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right).$$

4. *Let $m \leq n$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq n$, and let c be another real number. Then we have*

$$\sum_{i=m}^{n} (c a_i) = c\left(\sum_{i=m}^{n} a_i\right).$$

5. *(Triangle inequality for finite series) Let $m \le n$ be integers, and let a_i be a real number assigned to each integer $m \le i \le n$. Then we have*

$$\left|\sum_{i=m}^{n} a_i\right| \le \sum_{i=m}^{n} |a_i|.$$

6. *(Comparison test for finite series) Let $m \le n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \le i \le n$. Suppose that $a_i \le b_i$ for all $m \le i \le n$. Then we have*

$$\sum_{i=m}^{n} a_i \le \sum_{i=m}^{n} b_i$$

Proof. (a) Let $k = p - m$. By hypothesis we know that $k > 0$. Now we induct on k to show that definition 1.4(a) is true and we start with $k = 1$. For $k = 1$, we have $p = m + 1$ and by definition 1.1 we have

$$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{m} a_i + \sum_{i=m+1}^{p} a_i = a_m + a_{m+1} = \sum_{i=m}^{p} a_i.$$

Thus, the base case holds. Suppose inductively that for some $k \ge 1$ definition 1.4(a) is true. Then for $k + 1 = p - m$, we have $p - 1 = k + m$ and

$$\begin{aligned}
\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i &= \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=n+1}^{p-1} a_i\right) + a_p && i : 7.1.1 \\
&= \left(\sum_{i=m}^{p-1} a_i\right) + a_p \\
&= \sum_{i=m}^{p} a_i. && i : 7.1.1
\end{aligned}$$

This closes the induction. □

Proof. (b) Let $p = n - m$. By hypothesis we know that $p \ge 0$. Now we induct on p to show that definition 1.4(b) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\sum_{j=m+k}^{m+k} a_{j-k} &= \left(\sum_{j=m+k}^{m+k-1} a_{j-k}\right) + a_{m+k-k} && i : 7.1.1 \\
&= 0 + a_{m+k-k} && i : 7.1.1 \\
&= 0 + a_m \\
&= \left(\sum_{i=m}^{m-1} a_i\right) + a_m && i : 7.1.1 \\
&= \sum_{i=m}^{m} a_i. && i : 7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \ge 0$ definition 1.4(b) is true. Then for

$p+1=n-m$, we have $p=n-m-1$ and

$$\begin{aligned}
\sum_{j=m+k}^{n+k} a_{j-k} &= \left(\sum_{j=m+k}^{n+k-1} a_{j-k}\right) + a_{n+k-k} && i: 7.1.1 \\
&= \left(\sum_{j=m+k}^{n+k-1} a_{j-k}\right) + a_n \\
&= \left(\sum_{i=m}^{n-1} a_i\right) + a_n \\
&= \sum_{i=m}^{n} a_i. && i: 7.1.1
\end{aligned}$$

This closes the induction. □

Proof. (c) Let $p=n-m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that definition 1.4(c) is true. For $p=0$, we have $n=m$ and

$$\begin{aligned}
\sum_{i=m}^{m}(a_i+b_i) &= \left(\sum_{i=m}^{m-1}(a_i+b_i)\right) + a_m + b_m && i: 7.1.1 \\
&= 0 + a_m + b_m && i: 7.1.1 \\
&= \left(\sum_{i=m}^{m-1} a_i\right) + \left(\sum_{i=m}^{m-1} b_i\right) + a_m + b_m && i: 7.1.1 \\
&= \left(\sum_{i=m}^{m} a_i\right) + \left(\sum_{i=m}^{m} b_i\right). && i: 7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ definition 1.4(c) is true. Then for $p+1=n-m$, we have $p=n-m-1$ and

$$\begin{aligned}
\sum_{i=m}^{n}(a_i+b_i) &= \left(\sum_{i=m}^{n-1}(a_i+b_i)\right) + a_n + b_n && i: 7.1.1 \\
&= \left(\sum_{i=m}^{n-1} a_i\right) + \left(\sum_{i=m}^{n-1} b_i\right) + a_n + b_n \\
&= \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right). && i: 7.1.1
\end{aligned}$$

This closes the induction. □

Proof. (d) Let $p=n-m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that definition 1.4(d) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\sum_{i=m}^{m} ca_i &= \left(\sum_{i=m}^{m-1} ca_i\right) + ca_m && i : 7.1.1 \\
&= 0 + ca_m && i : 7.1.1 \\
&= c \times 0 + ca_m \\
&= c\left(\sum_{i=m}^{m-1} a_i\right) + ca_m && i : 7.1.1 \\
&= c\left(\left(\sum_{i=m}^{m-1} a_i\right) + a_m\right) \\
&= c\left(\sum_{i=m}^{m} a_i\right). && i : 7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ definition 1.4(d) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\begin{aligned}
\sum_{i=m}^{n} ca_i &= \left(\sum_{i=m}^{n-1} ca_i\right) + ca_n && i : 7.1.1 \\
&= c\left(\sum_{i=m}^{n-1} a_i\right) + ca_n \\
&= c\left(\left(\sum_{i=m}^{n-1} a_i\right) + a_n\right) \\
&= c\left(\sum_{i=m}^{n} a_i\right). && i : 7.1.1
\end{aligned}$$

This closes the induction. □

Proof. (e) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that definition 1.4(e) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\left|\sum_{i=m}^{m} a_i\right| &= \left|\left(\sum_{i=m}^{m-1} a_i\right) + a_m\right| && i : 7.1.1 \\
&= |0 + a_m| && i : 7.1.1 \\
&= 0 + |a_m| \\
&= \left(\sum_{i=m}^{m-1} |a_i|\right) + |a_m| && i : 7.1.1 \\
&= \sum_{i=m}^{m} |a_i|. && i : 7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ definition 1.4(e) is true. Then for $p + 1 =$

$n-m$, we have $p = n-m-1$ and

$$\begin{aligned}
\left|\sum_{i=m}^{n} a_i\right| &= \left|\left(\sum_{i=m}^{n-1} a_i\right) + a_n\right| && i: 7.1.1 \\
&\le \left|\sum_{i=m}^{n-1} a_i\right| + |a_n| \\
&\le \sum_{i=m}^{n-1} |a_i| + |a_n| \\
&= \sum_{i=m}^{n} |a_i|. && i: 7.1.1
\end{aligned}$$

This closes the induction. □

Proof. (f) Let $p = n-m$. By hypothesis we know that $p \ge 0$. Now we induct on p to show that definition 1.4(f) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\sum_{i=m}^{m} a_i &= \left(\sum_{i=m}^{m-1} a_i\right) + a_m && i: 7.1.1 \\
&= 0 + a_m && i: 7.1.1 \\
&\le 0 + b_m && \text{(by hypothesis)} \\
&= \left(\sum_{i=m}^{m-1} b_i\right) + b_m && i: 7.1.1 \\
&= \sum_{i=m}^{m} b_i. && i: 7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \ge 0$ definition 1.4(f) is true. Then for $p+1 = n-m$, we have $p = n-m-1$ and

$$\begin{aligned}
\sum_{i=m}^{n} a_i &= \left(\sum_{i=m}^{n-1} a_i\right) + a_n && i: 7.1.1 \\
&\le \left(\sum_{i=m}^{n-1} b_i\right) + a_n \\
&\le \left(\sum_{i=m}^{n-1} b_i\right) + b_n && \text{(by hypothesis)} \\
&= \sum_{i=m}^{n} b_i. && i: 7.1.1
\end{aligned}$$

This closes the induction. □

Remark 1.5. *In the future we may omit some of the parentheses in series expressions, for instance we may write* $\sum_{i=m}^{n}(a_i + b_i)$ *simply as* $\sum_{i=m}^{n} a_i + b_i$. *This is reasonably safe from being mis-interpreted, because the alternative interpretation* $(\sum_{i=m}^{n} a_i) + b_i$ *does not make any sense (the index* i *in* b_i *is meaningless outside of the summation, since* i *is only a dummy variable).*

Definition 1.6 (Summations over finite sets)**.** *Let X be a finite set with n elements (where $n \in \mathbb{N}$), and let $f : X \to \mathbb{R}$ be a function from X to the real numbers (i.e., f assigns a real number $f(x)$ to each element x of X). Then we can define the finite sum $\sum_{x \in X} f(x)$ as follows. We first select any bijection g from $\{i \in \mathbb{N} : 1 \le i \le n\}$ to X; such a bijection exists since X is assumed to have n elements. We then define*

$$\sum_{x \in X} f(x) := \sum_{i=1}^{n} f(g(i)).$$

In some cases we would like to define the sum $\sum_{x \in X} f(x)$ when $f : Y \to \mathbb{R}$ is defined on a larger set Y than X. In such cases we use exactly the same definition as is given above.

Proposition 1.7 (Finite summations are well-defined)**.** *Let X be a finite set with n elements (where $n \in \mathbb{N}$), let $f : X \to \mathbb{R}$ be a function, and let $g : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$ and $h : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$ be bijections. Then we have*

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i)).$$

Proof. We induct on n; more precisely, we let $P(n)$ be the assertion that "For any set X of n elements, any function $f : X \to \mathbb{R}$, and any two bijections g, h from $\{i \in \mathbb{N} : 1 \le i \le n\}$ to X, we have $\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i))$." (More informally, $P(n)$ is the assertion that definition 1.7 is true for that value of n.) We want to prove that $P(n)$ is true for all natural numbers n.

We first check the base case $P(0)$. In this case $\sum_{i=1}^{0} f(g(i))$ and $\sum_{i=1}^{0} f(h(i))$ both equal to 0, by definition of finite series (definition 1.1), so we are done.

Now suppose inductively that $P(n)$ is true; we now prove that $P(n+1)$ is true. Thus, let X be a set with $n+1$ elements, let $f : X \to \mathbb{R}$ be a function, and let g and h be bijections from $\{i \in N : 1 \le i \le n+1\}$ to X. We have to prove that

$$\sum_{i=1}^{n+1} f(g(i)) = \sum_{i=1}^{n+1} f(h(i)). \tag{i:7.1}$$

Let $x := g(n+1)$; thus x is an element of X. By definition of finite series (definition 1.1), we can expand the left-hand side of eq. (i:7.1) as

$$\sum_{i=1}^{n+1} f(g(i)) = \left(\sum_{i=1}^{n} f(g(i)) \right) + f(x).$$

Now let us look at the right-hand side of eq. (i:7.1). Ideally we would like to have $h(n+1)$ also equal to x - this would allow us to use the inductive hypothesis $P(n)$ much more easily - but we cannot assume this. However, since h is a bijection, we do know that there is *some* index j, with $1 \le j \le n+1$, for which $h(j) = x$. We now use definition 1.4 and the definition of finite series (definition 1.1) to write

$$\begin{aligned}
\sum_{i=1}^{n+1} f(h(i)) &= \left(\sum_{i=1}^{j} f(h(i)) \right) + \left(\sum_{i=j+1}^{n+1} f(h(i)) \right) \\
&= \left(\sum_{i=1}^{j-1} f(h(i)) \right) + f(h(j)) + \left(\sum_{i=j+1}^{n+1} f(h(i)) \right) \\
&= \left(\sum_{i=1}^{j-1} f(h(i)) \right) + f(x) + \left(\sum_{i=j}^{n} f(h(i+1)) \right).
\end{aligned}$$

We now define the function $\tilde{h} : \{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$ by setting $\tilde{h}(i) := h(i)$ when $i < j$ and $\tilde{h}(i) := h(i+1)$ when $i \ge j$. We can thus write the right-hand side of eq. (i:7.1) as

$$= \left(\sum_{i=1}^{j-1} f(\tilde{h}(i)) \right) + f(x) + \left(\sum_{i=j}^{n} f(\tilde{h}(i)) \right) = \left(\sum_{i=1}^{n} f(\tilde{h}(i)) \right) + f(x)$$

where we have used definition 1.4 once again. Thus, to finish the proof of eq. (i:7.1) we have to show that

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(\tilde{h}(i)). \tag{i:7.2}$$

But the function g (when restricted to $\{i \in \mathbb{N} : 1 \leq i \leq n\}$) is a bijection from $\{i \in \mathbb{N} : 1 \leq i \leq n\} \to X - \{x\}$. The function $\tilde{h}$ is also a bijection from $\{i \in \mathbb{N} : 1 \leq i \leq n\} \to X - \{x\}$ (cf. **??**). Since $X - \{x\}$ has n elements (by **??**), the claim eq. (i:7.2) then follows directly from the induction hypothesis $P(n)$. $\square$

Remark 1.8. *The issue is somewhat more complicated when summing over infinite sets; See* **??**.

Remark 1.9. *Suppose that X is a set, that $P(x)$ is a property pertaining to an element x of X, and $f : \{y \in X : P(y) \text{ is true}\} \to \mathbb{R}$ is a function. Then we will often abbreviate*

$$\sum_{x \in \{y \in X : P(y) \text{ is true}\}} f(x)$$

as $\sum_{x \in X : P(x) \text{ is true}} f(x)$ or even as $\sum_{P(x) \text{ is true}} f(x)$ when there is no change of confusion.

Proposition 1.10 (Basic properties of summation over finite sets)**.**

1. *If X is empty, and $f : X \to \mathbb{R}$ is a function (i.e., f is the empty function), we have*

$$\sum_{x \in X} f(x) = 0.$$

2. *If X consists of a single element, $X = \{x_0\}$, and $f : X \to \mathbb{R}$ is a function, we have*

$$\sum_{x \in X} f(x) = f(x_0).$$

3. *(Substitution, part I) If X is a finite set, $f : X \to \mathbb{R}$ is a function, and $g : Y \to X$ is a bijection, then*

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y)).$$

4. *(Substitution, part II) Let $n \leq m$ be integers, and let X be the set $X := \{i \in \mathbb{Z} : n \leq i \leq m\}$. If a_i is a real number assigned to each integer $i \in X$, then we have*

$$\sum_{i=n}^{m} a_i = \sum_{i \in X} a_i.$$

5. *Let X, Y be disjoint finite sets (so $X \cap Y = \emptyset$), and $f : X \cup Y \to \mathbb{R}$ is a function. Then we have*

$$\sum_{z \in X \cup Y} f(z) = \left(\sum_{x \in X} f(x)\right) + \left(\sum_{y \in Y} f(y)\right).$$

6. *(Linearity, part I) Let X be a finite set, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Then*

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x).$$

7. *(Linearity, part II) Let X be a finite set, let $f : X \to \mathbb{R}$ be a function, and let c be a real number. Then*

$$\sum_{x \in X} cf(x) = c \sum_{x \in X} f(x).$$

8. *(Monotonicity) Let X be a finite set, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions such that $f(x) \le g(x)$ for all $x \in \mathbf{X}$. Then we have*

$$\sum_{x \in X} f(x) \le \sum_{x \in X} g(x).$$

9. *(Triangle inequality) Let X be a finite set, and let $f : X \to \mathbb{R}$ be a function, then*

$$\left| \sum_{x \in X} f(x) \right| \le \sum_{x \in X} |f(x)|.$$

Proof. (a) Let $g : \{i \in \mathbb{N} : 1 \le i \le 0\} \to \emptyset$ be a function. Then g is a bijection and

$$\begin{aligned} \sum_{x \in X} f(x) &= \sum_{i=1}^{0} f(g(i)) && i : 7.1.6 \\ &= 0. && i : 7.1.1 \end{aligned}$$

□

Proof. (b) Let $g : \{1\} \to \{x_0\}$ be a function. Then g is a bijection and

$$\begin{aligned} \sum_{x \in X} f(x) &= \sum_{i=1}^{1} f(g(i)) && i : 7.1.6 \\ &= \left(\sum_{i=1}^{0} f(g(i)) \right) + f(g(1)) && i : 7.1.1 \\ &= 0 + f(g(1)) && i : 7.1.1 \\ &= f(x_0). \end{aligned}$$

□

Proof. (c) Let $h : \{i \in \mathbb{N} : 1 \le i \le \#(Y)\} \to Y$ be a bijection. Since X is finite and g is a bijection between X and Y, we know that Y is finite and thus such h is well-defined. Then we know that $g \circ h : \{i \in \mathbb{N} : 1 \le i \le \#(Y)\} \to X$ is also a bijection and

$$\begin{aligned} \sum_{x \in X} f(x) &= \sum_{i=1}^{\#(Y)} f((g \circ h)(i)) && i : 7.1.6 \\ &= \sum_{i=1}^{\#(Y)} f(g(h(i))) \\ &= \sum_{i=1}^{\#(Y)} (f \circ g)(h(i)) \\ &= \sum_{y \in Y} (f \circ g)(y) && i : 7.1.6 \\ &= \sum_{y \in Y} f(g(y)). \end{aligned}$$

□

Proof. (d) Let $f : X \to \{a_i \in \mathbb{R} : n \leq i \leq m\}$ be a function where $f = i \mapsto a_i$. Let $g : \{i \in \mathbb{N} : 1 \leq i \leq m - n + 1\} \to X$ be a function where $g = i \mapsto i + n - 1$. Then g is a bijection and

$$\begin{aligned}
\sum_{i \in X} a_i &= \sum_{i \in X} f(i) \\
&= \sum_{i=1}^{m-n+1} f(g(i)) && i : 7.1.6 \\
&= \sum_{i=1}^{m-n+1} f(i + n - 1) \\
&= \sum_{i=1}^{m-n+1} a_{i+n-1} \\
&= \sum_{i=1+n-1}^{m-n+1+n-1} a_{i+n-1-n+1} && i : 7.1.4[b] \\
&= \sum_{i=n}^{m} a_i.
\end{aligned}$$

□

Proof. (e) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ and $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to Y$ be bijections. Since X, Y are finite, we know that g, h are well-defined and $X \cup Y$ is finite. Let $k : \{i \in \mathbb{N} : 1 \leq i \leq \#(X \cup Y)\} \to X \cup Y$ be a bijection where

$$k(i) = \begin{cases} g(i) & \text{if } 1 \leq i \leq \#(X) \\ h(i - \#(X)) & \text{if } \#(X) + 1 \leq i \leq \#(X) + \#(Y). \end{cases}$$

Since $X \cup Y$ is finite, we know that k is well-defined and $\#(X \cup Y) = \#(X) + \#(Y)$. Then we have

$$\begin{aligned}
\sum_{z \in X \cup Y} f(z) &= \sum_{i=1}^{\#(X \cup Y)} f(k(i)) && i : 7.1.6 \\
&= \sum_{i=1}^{\#(X)} f(k(i)) + \sum_{i=\#(X)+1}^{\#(X \cup Y)} f(k(i)) && i : 7.1.4[a] \\
&= \sum_{i=1}^{\#(X)} f(g(i)) + \sum_{i=\#(X)+1}^{\#(X \cup Y)} f(h(i - \#(X))) \\
&= \sum_{i=1}^{\#(X)} f(g(i)) + \sum_{i=1}^{\#(Y)} f(h(i)) && i : 7.1.4[b] \\
&= \sum_{x \in X} f(x) + \sum_{y \in Y} f(y). && i : 7.1.6
\end{aligned}$$

□

Proof. (f) Let $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that h is

well-defined and

$$\begin{aligned}
\sum_{x\in X}(f(x)+g(x)) &= \sum_{x\in X}(f+g)(x) \\
&= \sum_{i=1}^{\#(X)}(f+g)(h(i)) && i: 7.1.6 \\
&= \sum_{i=1}^{\#(X)}(f(h(i))+g(h(i))) \\
&= \sum_{i=1}^{\#(X)} f(h(i)) + \sum_{i=1}^{\#(X)} g(h(i)) && i: 7.1.4[c] \\
&= \sum_{x\in X} f(x) + \sum_{x\in X} g(x). && i: 7.1.6
\end{aligned}$$

□

Proof. (g) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that g is well-defined and

$$\begin{aligned}
\sum_{x\in X} cf(x) &= \sum_{x\in X}(cf)(x) \\
&= \sum_{i=1}^{\#(X)}(cf)(g(i)) && i: 7.1.6 \\
&= \sum_{i=1}^{\#(X)} cf(g(i)) \\
&= c\sum_{i=1}^{\#(X)} f(g(i)) && i: 7.1.4[d] \\
&= c\sum_{x\in X} f(x). && i: 7.1.6
\end{aligned}$$

□

Proof. (h) Let $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that h is well-defined and

$$\begin{aligned}
\sum_{x\in X} f(x) &= \sum_{i=1}^{\#(X)} f(h(i)) && i: 7.1.6 \\
&\leq \sum_{i=1}^{\#(X)} g(h(i)) && i: 7.1.4[f] \\
&= \sum_{x\in X} g(x). && i: 7.1.6
\end{aligned}$$

□

Proof. (i) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that g is

well-defined and

$$\begin{aligned}\left|\sum_{x\in X} f(x)\right| &= \left|\sum_{i=1}^{\#(X)} f(g(i))\right| && i: 7.1.6\\ &\le \sum_{i=1}^{\#(X)} |f(g(i))| && i: 7.1.4[e]\\ &= \sum_{x\in X} |f(x)|\,. && i: 7.1.6\end{aligned}$$

□

Remark 1.11. *The substitution rule in definition 1.10(c) can be thought of as making the substitution $x := g(y)$ (hence the name). Note that the assumption that g is a bijection is essential. From definition 1.10(c) and (d) we see that*

$$\sum_{i=n}^{m} a_i = \sum_{i=n}^{m} a_{f(i)}$$

for any bijection f from the set $\{i \in \mathbb{Z} : n \le i \le m\}$ to itself. Informally, this means that we can rearrange the elements of a finite sequence at will and still obtain the same value.

Lemma 1.12. *Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then*

$$\sum_{x\in X}\left(\sum_{y\in Y} f(x,y)\right) = \sum_{(x,y)\in X\times Y} f(x,y).$$

Proof. Let n be the number of elements in X. We will use induction on n (cf. definition 1.7); i.e., we let $P(n)$ be the assertion that definition 1.12 is true for any set X with n elements, and any finite set Y and any function $f : X \times Y \to \mathbb{R}$. We wish to prove $P(n)$ for all natural numbers n.

The base case $P(0)$ is easy, following from definition 1.10(a). Now suppose that $P(n)$ is true; we now show that $P(n+1)$ is true. Let X be a set with $n+1$ elements. In particular, by **??**, we can write $X = X' \cup \{x_0\}$, where x_0 is an element of X and $X' := X - \{x_0\}$ has n elements. Then by definition 1.10(e) we have

$$\sum_{x\in X}\left(\sum_{y\in Y} f(x,y)\right) = \sum_{x\in X'}\left(\sum_{y\in Y} f(x,y)\right) + \left(\sum_{y\in Y} f(x_0,y)\right);$$

by the induction hypothesis this is equal to

$$\sum_{(x,y)\in X'\times Y} f(x,y) + \left(\sum_{y\in Y} f(x_0,y)\right).$$

By definition 1.10(c) this is equal to

$$\sum_{(x,y)\in X'\times Y} f(x,y) + \left(\sum_{(x,y)\in \{x_0\}\times Y} f(x,y)\right).$$

By definition 1.10(e) this is equal to

$$\sum_{(x,y)\in X\times Y} f(x,y)$$

as desired. □

Corollary 1.13 (Fubini's theorem for finite series)**.** *Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then*

$$\begin{aligned}\sum_{x\in X}\left(\sum_{y\in Y} f(x,y)\right) &= \sum_{(x,y)\in X\times Y} f(x,y)\\ &= \sum_{(y,x)\in Y\times X} f(x,y)\\ &= \sum_{y\in Y}\left(\sum_{x\in X} f(x,y)\right).\end{aligned}$$

Proof. In light of definition 1.12, it suffices to show that

$$\sum_{(x,y)\in X\times Y} f(x,y) = \sum_{(y,x)\in Y\times X} f(x,y).$$

But this follows from definition 1.10(c) by applying the bijection $h : Y \times X \to X \times Y$ defined by $h(y,x) := (x,y)$. □

Remark 1.14. *We anticipate something interesting to happen when we move from finite sums to infinite sums. However, see* **??**.

Axiomatic Claim 1.15 (Products over finite sets)**.** *Let m, n be integers, and let $(a_i)_{i=m}^n$ be a finite sequence of real numbers, assigning a real number a_i to each integer i between m and n inclusive (i.e., $m \le i \le n$). Then we define the finite product $\prod_{i=m}^n a_i$ by the recursive formula*

$$\begin{aligned}&\prod_{i=m}^{n} a_i := 1 \text{ whenever } n < m;\\ &\prod_{i=m}^{n+1} a_i := \left(\prod_{i=m}^{n} a_i\right) \times a_{n+1} \text{ whenever } n \ge m-1.\end{aligned}$$

Axiomatic Claim 1.16.

1. *Let $m \le n < p$ be integers, and let a_i be a real number assigned to each integer $m \le i \le p$. Then we have*

$$\prod_{i=m}^{n} a_i \times \prod_{i=n+1}^{p} a_i = \prod_{i=m}^{p} a_i.$$

2. *Let $m \le n$ be integers, k be another integer, and let a_i be a real number assigned to each integer $m \le i \le n$. Then we have*

$$\prod_{i=m}^{n} a_i = \prod_{j=m+k}^{n+k} a_{j-k}.$$

3. *Let $m \le n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \le i \le n$. Then we have*

$$\prod_{i=m}^{n} (a_i \times b_i) = \left(\prod_{i=m}^{n} a_i\right) \times \left(\prod_{i=m}^{n} b_i\right).$$

4. *Let $m \le n$ be integers, and let a_i be a real number assigned to each integer $m \le i \le n$, and let c be another real number. Then we have*

$$\prod_{i=m}^{n} (ca_i) = c^{n-m+1}\left(\prod_{i=m}^{n} a_i\right).$$

5. *Let $m \le n$ be integers, and let a_i be a real number assigned to each integer $m \le i \le n$. Then we have*

$$\left| \prod_{i=m}^{n} a_i \right| = \prod_{i=m}^{n} |a_i| .$$

Proof. (a) Let $k = p - m$. By hypothesis we know that $k > 0$. Now we induct on k to show that definition 1.16(a) is true and we start with $k = 1$. For $k = 1$, we have $p = m + 1$ and by definition 1.15 we have

$$\prod_{i=m}^{n} a_i \times \prod_{i=n+1}^{p} a_i = \prod_{i=m}^{m} a_i \times \prod_{i=m+1}^{p} a_i = a_m \times a_{m+1} = \prod_{i=m}^{p} a_i.$$

Thus, the base case holds. Suppose inductively that for some $k \ge 1$ definition 1.16(a) is true. Then for $k + 1 = p - m$, we have $p - 1 = k + m$ and

$$\begin{aligned}
&\prod_{i=m}^{n} a_i \times \prod_{i=n+1}^{p} a_i \\
&= \left(\prod_{i=m}^{n} a_i \right) \times \left(\prod_{i=n+1}^{p-1} a_i \right) \times a_p && i : ac : 7.1.1 \\
&= \left(\prod_{i=m}^{p-1} a_i \right) \times a_p \\
&= \prod_{i=m}^{p} a_i. && i : ac : 7.1.1
\end{aligned}$$

This closes the induction. □

Proof. (b) Let $p = n - m$. By hypothesis we know that $p \ge 0$. Now we induct on p to show that definition 1.16(b) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\prod_{j=m+k}^{m+k} a_{j-k} &= \left(\prod_{j=m+k}^{m+k-1} a_{j-k} \right) \times a_{m+k-k} && i : ac : 7.1.1 \\
&= 1 \times a_{m+k-k} && i : ac : 7.1.1 \\
&= 1 \times a_m \\
&= \left(\prod_{i=m}^{m-1} a_i \right) \times a_m && i : ac : 7.1.1 \\
&= \prod_{i=m}^{m} a_i. && i : ac : 7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \ge 0$ definition 1.16(b) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\begin{aligned}
\prod_{j=m+k}^{n+k} a_{j-k} &= \left(\prod_{j=m+k}^{n+k-1} a_{j-k} \right) \times a_{n+k-k} && i : ac : 7.1.1 \\
&= \left(\prod_{j=m+k}^{n+k-1} a_{j-k} \right) \times a_n \\
&= \left(\prod_{i=m}^{n-1} a_i \right) \times a_n \\
&= \prod_{i=m}^{n} a_i. && i : ac : 7.1.1
\end{aligned}$$

This closes the induction. □

Proof. (c) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that definition 1.16(c) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\prod_{i=m}^{m}(a_i \times b_i) &= \left(\prod_{i=m}^{m-1}(a_i \times b_i)\right) \times a_m \times b_m && i : ac : 7.1.1 \\
&= 1 \times a_m \times b_m && i : ac : 7.1.1 \\
&= \left(\prod_{i=m}^{m-1} a_i\right) \times \left(\prod_{i=m}^{m-1} b_i\right) \times a_m \times b_m && i : ac : 7.1.1 \\
&= \left(\prod_{i=m}^{m} a_i\right) \times \left(\prod_{i=m}^{m} b_i\right). && i : ac : 7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ definition 1.16(c) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\begin{aligned}
\prod_{i=m}^{n}(a_i \times b_i) &= \left(\prod_{i=m}^{n-1}(a_i \times b_i)\right) \times a_n \times b_n && i : ac : 7.1.1 \\
&= \left(\prod_{i=m}^{n-1} a_i\right) \times \left(\prod_{i=m}^{n-1} b_i\right) \times a_n \times b_n \\
&= \left(\prod_{i=m}^{n} a_i\right) \times \left(\prod_{i=m}^{n} b_i\right). && i : ac : 7.1.1
\end{aligned}$$

This closes the induction. □

Proof. (d) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that definition 1.16(d) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\prod_{i=m}^{m} ca_i &= \left(\prod_{i=m}^{m-1} ca_i\right) \times ca_m && i : ac : 7.1.1 \\
&= 1 \times ca_m && i : ac : 7.1.1 \\
&= c \times a_m \\
&= c\left(\prod_{i=m}^{m} a_i\right) && i : ac : 7.1.1 \\
&= c^{m-m+1}\left(\prod_{i=m}^{m} a_i\right).
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ definition 1.16(d) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\begin{aligned}
\prod_{i=m}^{n} ca_i &= \left(\prod_{i=m}^{n-1} ca_i\right) \times ca_n && i : ac : 7.1.1 \\
&= c^{n-1-m+1}\left(\prod_{i=m}^{n-1} a_i\right) \times ca_n \\
&= c^{n-m+1}\left(\left(\prod_{i=m}^{n-1} a_i\right) \times a_n\right) \\
&= c^{n-m+1}\left(\prod_{i=m}^{n} a_i\right). && i : ac : 7.1.1
\end{aligned}$$

This closes the induction. $\square$

Proof. (e) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that definition 1.16(e) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\left|\prod_{i=m}^{m} a_i\right| &= \left|\left(\prod_{i=m}^{m-1} a_i\right) \times a_m\right| && i:ac:7.1.1 \\
&= |1a_m| && i:ac:7.1.1 \\
&= |1|\,|a_m| \\
&= \left(\prod_{i=m}^{m-1} |a_i|\right) \times |a_m| && i:ac:7.1.1 \\
&= \prod_{i=m}^{m} |a_i|\,. && i:ac:7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ definition 1.16(e) is true. Then for $p+1 = n-m$, we have $p = n-m-1$ and

$$\begin{aligned}
\left|\prod_{i=m}^{n} a_i\right| &= \left|\left(\prod_{i=m}^{n-1} a_i\right) \times a_n\right| && i:ac:7.1.1 \\
&= \left|\prod_{i=m}^{n-1} a_i\right| \times |a_n| \\
&= \left(\prod_{i=m}^{n-1} |a_i|\right) \times |a_n| \\
&= \prod_{i=m}^{n} |a_i|\,. && i:ac:7.1.1
\end{aligned}$$

This closes the induction. $\square$

Axiomatic Claim 1.17. *Let X be a finite set with n elements (where $n \in \mathbb{N}$), and let $f : X \to \mathbb{R}$ be a function from X to the real numbers (i.e., f assigns a real number $f(x)$ to each element x of X). Then we can define the finite product $\prod_{x\in X} f(x)$ as follows. We first select any bijection g from $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ to X; such a bijection exists since X is assumed to have n elements. We then define*

$$\prod_{x\in X} f(x) := \prod_{i=1}^{n} f(g(i))$$

Axiomatic Claim 1.18 (Finite products are well-defined)**.** *Let X be a finite set with n elements (where $n \in \mathbb{N}$), let $f : X \to \mathbb{R}$ be a function, and let $g : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to X$ and $h : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to X$ be bijections. Then we have*

$$\prod_{i=1}^{n} f(g(i)) = \prod_{i=1}^{n} f(h(i)).$$

Proof. Let $P(n)$ be the assertion that "For any set X of n elements, any function $f : X \to \mathbb{R}$, and any two bijections g, h from $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ to X, we have $\prod_{i=1}^{n} f(g(i)) = \prod_{i=1}^{n} f(h(i))$." (More informally, $P(n)$ is the assertion that definition 1.18 is true for that value of n.) We induct on n.

We first check the base case $P(0)$. In this case $\prod_{i=1}^{0} f(g(i))$ and $\prod_{i=1}^{0} f(h(i))$ both equal to 1, by definition 1.15, so we are done.

Now suppose inductively that $P(n)$ is true; we now prove that $P(n+1)$ is true. Thus, let X be a set with $n+1$ elements, let $f : X \to \mathbb{R}$ be a function, and let g and h be bijections from $\{i \in N : 1 \leq i \leq n+1\}$

to X. We have to prove that

$$\prod_{i=1}^{n+1} f(g(i)) = \prod_{i=1}^{n+1} f(h(i)). \tag{i:ac:7.1}$$

Let $x := g(n+1)$; thus x is an element of X. By definition 1.15, we can expand the left-hand side of eq. (i:ac:7.1) as

$$\prod_{i=1}^{n+1} f(g(i)) = \left(\prod_{i=1}^{n} f(g(i))\right) \times f(x).$$

Now let us look at the right-hand side of eq. (i:ac:7.1). Since h is a bijection, we do know that there is *some* index j, with $1 \le j \le n+1$, for which $h(j) = x$. We now use **??** 1.15**??** 1.16 to write

$$\begin{aligned}
\prod_{i=1}^{n+1} f(h(i)) &= \left(\prod_{i=1}^{j} f(h(i))\right) \times \left(\prod_{i=j+1}^{n+1} f(h(i))\right) \\
&= \left(\prod_{i=1}^{j-1} f(h(i))\right) \times f(h(j)) \times \left(\prod_{i=j+1}^{n+1} f(h(i))\right) \\
&= \left(\prod_{i=1}^{j-1} f(h(i))\right) \times f(x) \times \left(\prod_{i=j}^{n} f(h(i+1))\right).
\end{aligned}$$

We now define the function $\tilde{h} : \{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$ by setting $\tilde{h}(i) := h(i)$ when $i < j$ and $\tilde{h}(i) := h(i+1)$ when $i \ge j$. We can thus write the right-hand side of eq. (i:ac:7.1) as

$$= \left(\prod_{i=1}^{j-1} f(\tilde{h}(i))\right) \times f(x) \times \left(\prod_{i=j}^{n} f(\tilde{h}(i))\right) = \left(\prod_{i=1}^{n} f(\tilde{h}(i))\right) \times f(x)$$

where we have used definition 1.16 once again. Thus, to finish the proof of eq. (i:ac:7.1) we have to show that

$$\prod_{i=1}^{n} f(g(i)) = \prod_{i=1}^{n} f(\tilde{h}(i)). \tag{i:ac:7.2}$$

But the function g (when restricted to $\{i \in \mathbb{N} : 1 \le i \le n\}$) is a bijection from $\{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$. The function $\tilde{h}$ is also a bijection from $\{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$ (cf. **??**). Since $X - \{x\}$ has n elements (by **??**), the claim eq. (i:ac:7.2) then follows directly from the induction hypothesis $P(n)$. ☐

Axiomatic Claim 1.19 (Basic properties of product over finite sets)**.**

1. *If X is empty, and $f : X \to \mathbb{R}$ is a function (i.e., f is the empty function), we have*

$$\prod_{x \in X} f(x) = 1.$$

2. *If X consists of a single element, $X = \{x_0\}$, and $f : X \to \mathbb{R}$ is a function, we have*

$$\prod_{x \in X} f(x) = f(x_0).$$

3. *(Substitution, part I) If X is a finite set, $f : X \to \mathbb{R}$ is a function, and $g : Y \to X$ is a bijection, then*

$$\prod_{x \in X} f(x) = \prod_{y \in Y} f(g(y)).$$

4. *(Substitution, part II) Let $n \le m$ be integers, and let X be the set $X := \{i \in \mathbb{Z} : n \le i \le m\}$. If a_i is a real number assigned to each integer $i \in X$, then we have*

$$\prod_{i=n}^{m} a_i = \prod_{i \in X} a_i.$$

5. *Let X, Y be disjoint finite sets (so $X \cap Y = \emptyset$), and $f : X \cup Y \to \mathbb{R}$ is a function. Then we have*

$$\prod_{z \in X \cup Y} f(z) = \left(\prod_{x \in X} f(x) \right) \times \left(\prod_{y \in Y} f(y) \right).$$

6. *Let X be a finite set, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Then*

$$\prod_{x \in X} (f(x) \times g(x)) = \prod_{x \in X} f(x) \times \prod_{x \in X} g(x).$$

7. *Let X be a finite set, let $f : X \to \mathbb{R}$ be a function, and let c be a real number. Then*

$$\prod_{x \in X} cf(x) = c^{\#(X)} \prod_{x \in X} f(x).$$

8. *Let X be a finite set, and let $f : X \to \mathbb{R}$ be a function, then*

$$\left| \prod_{x \in X} f(x) \right| = \prod_{x \in X} |f(x)|.$$

Proof. (a) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq 0\} \to \emptyset$ be a function. Then g is a bijection and

$$\begin{aligned} \prod_{x \in X} f(x) &= \prod_{i=1}^{0} f(g(i)) && i : ac : 7.1.3 \\ &= 1. && i : ac : 7.1.1 \end{aligned}$$

□

Proof. (b) Let $g : \{1\} \to \{x_0\}$ be a function. Then g is a bijection and

$$\begin{aligned} \prod_{x \in X} f(x) &= \prod_{i=1}^{1} f(g(i)) && i : ac : 7.1.3 \\ &= \left(\prod_{i=1}^{0} f(g(i)) \right) \times f(g(1)) && i : ac : 7.1.1 \\ &= 1 \times f(g(1)) && i : ac : 7.1.1 \\ &= f(x_0). \end{aligned}$$

□

Proof. (c) Let $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to Y$ be a bijection. Since X is finite and g is a bijection between X and Y, we know that Y is finite and thus such h is well-defined. Then we know that $g \circ h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to X$ is also a bijection and

$$\begin{aligned} \prod_{x \in X} f(x) &= \prod_{i=1}^{\#(Y)} f((g \circ h)(i)) && i : ac : 7.1.3 \\ &= \prod_{i=1}^{\#(Y)} f(g(h(i))) \\ &= \prod_{i=1}^{\#(Y)} (f \circ g)(h(i)) \\ &= \prod_{y \in Y} (f \circ g)(y) && i : ac : 7.1.3 \\ &= \prod_{y \in Y} f(g(y)). \end{aligned}$$

□

Proof. (d) Let $f : X \to \{a_i \in \mathbb{R} : n \leq i \leq m\}$ be a function where $f = i \mapsto a_i$. Let $g : \{i \in \mathbb{N} : 1 \leq i \leq m - n + 1\} \to X$ be a function where $g = i \mapsto i + n - 1$. Then g is a bijection and

$$\begin{aligned}
\prod_{i \in X} a_i &= \prod_{i \in X} f(i) \\
&= \prod_{i=1}^{m-n+1} f(g(i)) && i : ac : 7.1.3 \\
&= \prod_{i=1}^{m-n+1} f(i + n - 1) \\
&= \prod_{i=1}^{m-n+1} a_{i+n-1} \\
&= \prod_{i=1+n-1}^{m-n+1+n-1} a_{i+n-1-n+1} && i : ac : 7.1.2[b] \\
&= \prod_{i=n}^{m} a_i.
\end{aligned}$$

□

Proof. (e) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ and $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to Y$ be bijections. Since X, Y are finite, we know that g, h are well-defined and $X \cup Y$ is finite. Let $k : \{i \in \mathbb{N} : 1 \leq i \leq \#(X \cup Y)\} \to X \cup Y$ be a bijection where

$$k(i) = \begin{cases} g(i) & \text{if } 1 \leq i \leq \#(X) \\ h(i - \#(X)) & \text{if } \#(X) + 1 \leq i \leq \#(X) + \#(Y). \end{cases}$$

Since $X \cup Y$ is finite, we know that k is well-defined and $\#(X \cup Y) = \#(X) + \#(Y)$. Then we have

$$\begin{aligned}
&\prod_{z \in X \cup Y} f(z) \\
&= \prod_{i=1}^{\#(X \cup Y)} f(k(i)) && i : ac : 7.1.3 \\
&= \prod_{i=1}^{\#(X)} f(k(i)) \times \prod_{i=\#(X)+1}^{\#(X \cup Y)} f(k(i)) && i : ac : 7.1.2[a] \\
&= \prod_{i=1}^{\#(X)} f(g(i)) \times \prod_{i=\#(X)+1}^{\#(X \cup Y)} f(h(i - \#(X))) \\
&= \prod_{i=1}^{\#(X)} f(g(i)) \times \prod_{i=1}^{\#(Y)} f(h(i)) && i : ac : 7.1.2[b] \\
&= \prod_{x \in X} f(x) \times \prod_{y \in Y} f(y). && i : ac : 7.1.3
\end{aligned}$$

□

Proof. (f) Let $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that h is

well-defined and

$$\begin{aligned}
&\prod_{x\in X}(f(x)\times g(x)) \\
&= \prod_{x\in X}(f\times g)(x) \\
&= \prod_{i=1}^{\#(X)}(f\times g)(h(i)) && i:ac:7.1.3 \\
&= \prod_{i=1}^{\#(X)}(f(h(i))\times g(h(i))) \\
&= \prod_{i=1}^{\#(X)} f(h(i))\times \prod_{i=1}^{\#(X)} g(h(i)) && i:ac:7.1.2[c] \\
&= \prod_{x\in X} f(x)\times \prod_{x\in X} g(x). && i:ac:7.1.3
\end{aligned}$$

□

Proof. (g) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that g is well-defined and

$$\begin{aligned}
\prod_{x\in X} cf(x) &= \prod_{x\in X}(cf)(x) \\
&= \prod_{i=1}^{\#(X)}(cf)(g(i)) && i:ac:7.1.3 \\
&= \prod_{i=1}^{\#(X)} cf(g(i)) \\
&= c^{\#(X)}\prod_{i=1}^{\#(X)} f(g(i)) && i:ac:7.1.2[d] \\
&= c^{\#(X)}\prod_{x\in X} f(x). && i:ac:7.1.3
\end{aligned}$$

□

Proof. (h) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that g is well-defined and

$$\begin{aligned}
\left|\prod_{x\in X} f(x)\right| &= \left|\prod_{i=1}^{\#(X)} f(g(i))\right| && i:ac:7.1.3 \\
&= \prod_{i=1}^{\#(X)} |f(g(i))| && i:ac:7.1.2[e] \\
&= \prod_{x\in X} |f(x)|. && i:ac:7.1.3
\end{aligned}$$

□

Axiomatic Claim 1.20. *Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then*

$$\prod_{x\in X}\left(\prod_{y\in Y} f(x,y)\right) = \prod_{(x,y)\in X\times Y} f(x,y).$$

Proof. Let n be the number of elements in X. We will use induction on n (cf. definition 1.18); i.e., we let $P(n)$ be the assertion that definition 1.20 is true for any set X with n elements, and any finite set Y and any function $f : X \times Y \to \mathbb{R}$. We wish to prove $P(n)$ for all natural numbers n.

The base case $P(0)$ is easy, following from definition 1.19(a). Now suppose that $P(n)$ is true; we now show that $P(n+1)$ is true. Let X be a set with $n+1$ elements. In particular, by **??**, we can write $X = X' \cup \{x_0\}$, where x_0 is an element of X and $X' := X - \{x_0\}$ has n elements. Then by definition 1.5(e) we have

$$\prod_{x\in X}\left(\prod_{y\in Y} f(x,y)\right) = \prod_{x\in X'}\left(\prod_{y\in Y} f(x,y)\right) \times \left(\prod_{y\in Y} f(x_0,y)\right);$$

by the induction hypothesis this is equal to

$$\prod_{(x,y)\in X'\times Y} f(x,y) \times \left(\prod_{y\in Y} f(x_0,y)\right).$$

By definition 1.10(c) this is equal to

$$\prod_{(x,y)\in X'\times Y} f(x,y) \times \left(\prod_{(x,y)\in\{x_0\}\times Y} f(x,y)\right).$$

By definition 1.10(e) this is equal to

$$\prod_{(x,y)\in X\times Y} f(x,y)$$

as desired. □

Axiomatic Claim 1.21. *Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then*

$$\begin{aligned}\prod_{x\in X}\left(\prod_{y\in Y} f(x,y)\right) &= \prod_{(x,y)\in X\times Y} f(x,y)\\ &= \prod_{(y,x)\in Y\times X} f(x,y)\\ &= \prod_{y\in Y}\left(\prod_{x\in X} f(x,y)\right).\end{aligned}$$

Proof. In light of definition 1.20, it suffices to show that

$$\prod_{(x,y)\in X\times Y} f(x,y) = \prod_{(y,x)\in Y\times X} f(x,y).$$

But this follows from definition 1.19(c) by applying the bijection $h : Y \times X \to X \times Y$ defined by $h(y,x) := (x,y)$. □

Prove definition 1.4.

Proof. See definition 1.4. □

Prove definition 1.10.

Proof. See definition 1.10. □

Form a definition for the finite products $\prod_{i=1}^{n} a_i$ and $\prod_{x\in X} f(x)$. Which of the above result for finite series have analoges for finite products?

Proof. See **??** 1.15–1.21. □

Define the *factorial function* $n!$ for natural numbers n by the recursive definition $0! := 1$ and $(n+1)! := n! \times (n+1)$. If x and y are real numbers, prove the *binomial formula*

$$(x+y)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j}$$

for all natural numbers n.

Proof. We induct on n. For $n = 0$, we have

$$\begin{aligned}
(x+y)^0 &= 1 \\
&= \frac{0!}{0!(0-0)!} x^0 y^{0-0} && \text{(by definition)} \\
&= \sum_{j=0}^{-1} \frac{0!}{j!(0-j)!} x^j y^{0-j} + \frac{0!}{0!(0-0)!} x^0 y^{0-0} && i:7.1.1 \\
&= \sum_{j=0}^{0} \frac{0!}{j!(0-j)!} x^j y^{0-j} && i:7.1.1
\end{aligned}$$

So the base case holds. Suppose inductively that for some $n \geq 0$ the statement holds. Then for $n+1$, we have

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)^n \times (x+y) \\
&= \left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j} \right) \times (x+y) \\
&= \left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} \right) \\
&\quad + \left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) \\
&= \left(\sum_{j=0}^{n-1} \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} \right) && i:7.1.1 \\
&\quad + \left(\frac{n!}{n!0!} x^{n+1} y^0 \right) \\
&\quad + \left(\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) \\
&\quad + \left(\frac{n!}{0!n!} x^0 y^{n+1} \right) \\
&= \left(\sum_{j=0}^{n-1} \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} \right) + x^{n+1} && \text{(by definition)} \\
&\quad + \left(\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) + y^{n+1} \\
&= \left(\sum_{j=1}^{n} \frac{n!}{(j-1)!(n+1-j)!} x^j y^{n+1-j} \right) + x^{n+1} && i:7.1.4[b] \\
&\quad + \left(\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) + y^{n+1}
\end{aligned}$$

and

$$
\begin{aligned}
&\left(\sum_{j=1}^{n} \frac{n!}{(j-1)!(n+1-j)!} x^j y^{n+1-j}\right) \\
&\quad + \left(\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j}\right) \\
&= \sum_{j=1}^{n} \left(\frac{n!}{(j-1)!(n+1-j)!} x^j y^{n+1-j} + \frac{n!}{j!(n-j)!} x^j y^{n+1-j}\right) && i : 7.1.4[c] \\
&= \sum_{j=1}^{n} \left(\frac{j \times n!}{j!(n+1-j)!} x^j y^{n+1-j} + \frac{(n+1-j) \times n!}{j!(n+1-j)!} x^j y^{n+1-j}\right) \\
&= \sum_{j=1}^{n} \left(\frac{j \times n! + (n+1-j) \times n!}{j!(n+1-j)!} x^j y^{n+1-j}\right) \\
&= \sum_{j=1}^{n} \left(\frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}\right).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \\
&= \frac{(n+1)!}{(n+1)!0!} x^{n+1} y^0 + \left(\sum_{j=0}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}\right) && i : 7.1.1 \\
&= x^{n+1} + \left(\sum_{j=0}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}\right) && \text{(by definition)} \\
&= x^{n+1} + \left(\sum_{j=0}^{0} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}\right) && i : 7.1.4[a] \\
&\quad + \left(\sum_{j=1}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}\right) \\
&= x^{n+1} + \frac{(n+1)!}{0!(n+1)!} x^0 y^{n+1} && i : 7.1.1 \\
&\quad + \left(\sum_{j=1}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}\right) \\
&= x^{n+1} + y^{n+1} + \left(\sum_{j=1}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}\right). && \text{(by definition)}
\end{aligned}
$$

Thus, we have

$$(x+y)^{n+1} = \sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}.$$

and this closes the induction. □

Let X be a finite set, let m be an integer, and for each $x \in X$ let $(a_n(x))_{n=m}^{\infty}$ be a convergent sequence of real numbers. Show that the sequence $(\sum_{x\in X} a_n(x))_{n=m}^{\infty}$ is convergent, and

$$\lim_{n\to\infty} \sum_{x\in X} a_n(x) = \sum_{x\in X} \lim_{n\to\infty} a_n(x).$$

Thus, we may always interchange finite sums with convergent limits. Things however get trickier with infinite sums.

Proof. Let $k = \#(X)$. We induct on k. For $k = 0$, we have $X = \emptyset$. So

$$\begin{aligned}
\lim_{n\to\infty} \sum_{x\in X} a_n(x) &= \lim_{n\to\infty} 0 && i : 7.1.11 \\
&= 0 \\
&= \sum_{x\in X} \lim_{n\to\infty} a_n(x). && i : 7.1.11
\end{aligned}$$

Thus, the base case holds. Suppose inductively that for some $k \geq 0$ the statement is true. Then for $k+1$, we have to show that the statement is also true. Let $x_0 \in X$ and $X' = X \setminus \{x_0\}$. So $\#(X') = \#(X) - 1 = n$, and we have

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{x\in X} a_n(x) \\
&= \lim_{n\to\infty} \sum_{x\in \{x_0\}\cup X'} a_n(x) \\
&= \lim_{n\to\infty} \left(\sum_{x\in \{x_0\}} a_n(x) + \sum_{x\in X'} a_n(x) \right) && i : 7.1.11[e] \\
&= \left(\lim_{n\to\infty} \sum_{x\in \{x_0\}} a_n(x) \right) + \left(\lim_{n\to\infty} \sum_{x\in X'} a_n(x) \right) && i : 6.1.19[a] \\
&= \left(\lim_{n\to\infty} a_n(x_0) \right) + \left(\lim_{n\to\infty} \sum_{x\in X'} a_n(x) \right) && i : 7.1.11[b] \\
&= \left(\sum_{x\in \{x_0\}} \lim_{n\to\infty} a_n(x) \right) + \left(\lim_{n\to\infty} \sum_{x\in X'} a_n(x) \right) && i : 7.1.11[b] \\
&= \left(\sum_{x\in \{x_0\}} \lim_{n\to\infty} a_n(x) \right) + \left(\sum_{x\in X'} \lim_{n\to\infty} a_n(x) \right) \\
&= \left(\sum_{x\in \{x_0\}\cup X'} \lim_{n\to\infty} a_n(x) \right) && i : 7.1.11[e] \\
&= \sum_{x\in X} \lim_{n\to\infty} a_n(x).
\end{aligned}$$

This closes the induction. $\square$