# 1 Countability

From **??** we know that the set $\mathbb{N}$ of natural numbers is infinite. The set $\mathbb{N} - \{0\}$ is also infinite, thanks to **??**(a), and is a proper subset of $\mathbb{N}$. However, the set $\mathbb{N} - \{0\}$, despite being "smaller" than $\mathbb{N}$, still has the same cardinality as $\mathbb{N}$, because the function $f : \mathbb{N} \to \mathbb{N} - \{0\}$ defined by $f(n) := n + 1$, is a bijection from $\mathbb{N}$ to $\mathbb{N} - \{0\}$. This is one characteristic of infinite sets.

**Definition 1.1** (Countable sets)**.** *A set $X$ is said to be* countably infinite *(or just* countable*) iff it has equal cardinality with the natural numbers $\mathbb{N}$. A set $X$ is said to be* at most countable *iff it is either countable or finite. We say that a set is* uncountable *if it is infinite but not countable.*

**Remark 1.2.** *Countably infinite sets are also called* denumerable *sets.*

**Example 1.3.** *The even natural numbers $\{2n : n \in \mathbb{N}\}$, since the function $f(n) := 2n$ provides a bijection between $\mathbb{N}$ and the even natural numbers.*

Let $X$ be a countable set. Then, by definition, we know that there exists a bijection $f : \mathbb{N} \to X$. Thus, every element of $X$ can be written in the form $f(n)$ for exactly one natural number $n$. Informally, we thus have

$$X = \{f(0), f(1), f(2), f(3), \dots\}.$$

Thus, a countable set can be arranged in a sequence, so that we have a zeroth element $f(0)$, followed by a first element $f(1)$, then a second element $f(2)$, and so forth, in such a way that all these elements $f(0), f(1), f(2), \dots$ are all distinct, and together they fill out all of $X$. (This is why these sets are called *countable*; because we can literally count them one by one, starting from $f(0)$, then $f(1)$, and so forth.)

**Proposition 1.4** (Well ordering principle)**.** *Let $X$ be a non-empty subset of the natural numbers $\mathbb{N}$. Then there exists exactly one element $n \in X$ such that $n \leq m$ for all $m \in X$. In other words, every non-empty set of natural numbers has a minimum element.*

*Proof.* Suppose for the sake of contradiction that $X$ has no minimum element. Let $n \in \mathbb{N}$ and let $P(n)$ be the statement "$\forall m \in X$, we have $n \leq m$ and $n \notin X$." We now use induction to show that $P(n)$ is true $\forall n \in \mathbb{N}$. For $n = 0$, we have

$$\begin{aligned}
& X \subseteq \mathbb{N} && \\
\implies & \forall m \in X, m \in \mathbb{N} && i : 3.1.15 \\
\implies & \forall m \in X, 0 \leq m && i : 2.3 \\
\implies & 0 \notin X. && (X \text{ has no minimum element})
\end{aligned}$$

Thus, the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then for $n + 1$, we have

$$\begin{aligned}
& \forall m \in X, n \leq m \wedge n \notin X && \\
\implies & \forall m \in X, n < m && i : 2.2.11 \\
\implies & \forall m \in X, n + 1 \leq m && i : 2.2.12[e] \\
\implies & n + 1 \notin X. && (X \text{ has no minimum element})
\end{aligned}$$

This closes the induction.

By hypothesis we know that $X \subseteq \mathbb{N}$ and $X \neq \emptyset$. So let $n \in X$. But $P(n)$ is true, we must have $n \notin X$, a contradiction. Thus, $X$ must have a minimum element $\min(X) \in X$.

Now we show that such $\min(X)$ is unique. Suppose that $\exists n, n' \in X$ such that $\forall m \in X$, we have $n \leq m \wedge n' \leq m$. Since $n, n' \in X$, we have $n \leq n' \wedge n' \leq n$. Thus, $n = n'$. ∎

We will refer to the element $n$ given by the well-ordering principle as the *minimum* of $X$, and write it as $\min(X)$. This minimum is clearly the same as the infimum of $X$, as defined in **??**.

**Proposition 1.5.** *Let $X$ be an infinite subset of the natural numbers $\mathbb{N}$. Then there exists a unique bijection $f : \mathbb{N} \to X$ which is increasing, in the sense that $f(n + 1) > f(n)$ for all $n \in N$. In particular, $X$ has equal cardinality with $\mathbb{N}$ and is hence countable.*

*Proof.* We now define a sequence $a_0, a_1, a_2, \dots$ of natural numbers recursively by the formula

$$a_n := \min\{x \in X : x \neq a_m \text{ for all } m < n\}.$$

Intuitively speaking, $a_0$ is the smallest element of $X$; $a_1$ is the second smallest element of $X$, i.e., the smallest element of $X$ once $a_0$ is removed; $a_2$ is the third smallest element of $X$; and so forth. Observe that in order to define $a_n$, one only needs to know the values of $a_m$ for all $m < n$, so this definition is recursive. Also, since $X$ is infinite, the set $\{x \in X : x \neq a_m \text{ for all } m < n\}$ is infinite, hence non-empty. (If it is finite, then its union with the set $\{a_0, \dots, a_{n-1}\}$ is also finite, but the union is $X$, which contradict to $X$ is infinite.) Thus, by the well-ordering principle (definition 1.5), the minimum, $\min\{x \in X : x \neq a_m \text{ for all } m < n\}$ is always well-defined.

Since $a_{n+1} = \min\{x \in X : x \neq a_m \text{ for all } m < n+1\}$, we know that $a_n < a_{n+1}$. Since $n$ was arbitrary, we see that $a_n$ is an increasing sequence, i.e.

$$a_0 < a_1 < a_2 < \dots,$$

and in particular, that $a_n \neq a_m$ for all $n \neq m$. Also, we have $a_n \in X$ for each natural number $n$ (by definition 1.4).

Now define the function $f : \mathbb{N} \to X$ by $f(n) := a_n$. From the previous paragraph we know that $f$ is one-to-one. Now we show that $f$ is onto. In other words, we claim that for every $y \in X$, there exists an $n$ such that $a_n = y$.

Let $y \in X$. Suppose for the sake of contradiction that $a_n \neq y$ for every natural number $n$. Then this implies that $y$ is an element of the set $\{x \in X : x \neq a_m \text{ for all } m < n\}$ for all $n$. By definition of $a_n$, this implies that $y > a_n$ for every natural number $n$. (If $y < a_n$, then $y = \min\{x \in X : a \neq a_m \text{ for all } m < n\}$ instead of $a_n$, a contradiction) However, since $a_n$ is an increasing sequence, we have $a_n \geq n$, and hence $y \geq n$ for every natural number $n$. In particular, we have $y \geq y+1$, which is a contradiction. Thus, we must have $a_n = y$ for some natural number $n$, and hence $f$ is onto.

Since $f : \mathbb{N} \to X$ is both one-to-one and onto, it is a bijection. We have thus found at least one increasing bijection $f$ from $\mathbb{N}$ to $X$. Now suppose for the sake of contradiction that there was at least one other increasing bijection $g$ from $\mathbb{N}$ to $X$ which was not equal to $f$. Then the set $\{n \in \mathbb{N} : g(n) \neq f(n)\}$ is non-empty, and define $m := \min\{n \in \mathbb{N} : g(n) \neq f(n)\}$, thus in particular, $g(m) \neq f(m) = a_m$, and $g(n) = f(n) = a_n$ for all $n < m$. But we then must have

$$g(m) = \min\{x \in X : x \neq a_t \text{ for all } t < m\} = a_m,$$

a contradiction. Thus, there is no other increasing bijection from $\mathbb{N}$ to $X$ other than $f$. □

**Corollary 1.6.** *All subsets of the natural numbers are at most countable.*

*Proof.* Since finite sets are at most countable by definition, combine with definition 1.5 we thus have all subsets of the natural numbers are at most countable. □

**Corollary 1.7.** *If $X$ is an at most countable set, and $Y$ is a subset of $X$, then $Y$ is at most countable.*

*Proof.* If $X$ is finite then this follows from **??**(c), so assume $X$ is countable. Then there is a bijection $f : X \to \mathbb{N}$ between $X$ and $\mathbb{N}$. Since $Y$ is a subset of $X$, and $f$ is a bijection from $X$ and $\mathbb{N}$, then when we restrict $f$ to $Y$, we obtain a bijection between $Y$ and $f(Y)$. Thus, $f(Y)$ has equal cardinality with $Y$. But $f(Y)$ is a subset of $\mathbb{N}$, and hence at most countable by definition 1.6. Hence $Y$ is also at most countable. □

**Proposition 1.8.** *Let $Y$ be a set, and let $f : \mathbb{N} \to Y$ be a function. Then $f(\mathbb{N})$ is at most countable.*

*Proof.* If $f(\mathbb{N})$ is finite then by definition 1.1 it is at most countable. So assume that $f(\mathbb{N})$ is infinite. Let $A$ be the set

$$A = \{n \in \mathbb{N} : f(m) \neq f(n) \text{ for all } 0 \leq m < n\}.$$

So $A \subseteq \mathbb{N}$ and $A$ is infinite. We now show that $f|_A : A \to f(A)$ is a bijection.

Let $p, q \in A$ and $p \neq q$. By the definition of $A$ we know that $f|_A(p) \neq f|_A(q)$ and thus $f|_A$ is injective. By **??** we also know that $f|_A$ is surjective, thus $f|_A$ is bijective.

Now we show that $\forall y \in f(\mathbb{N})$, $\exists p \in A$ such that $f|_A(p) = y$. Suppose for the sake of contradiction that $\nexists p \in A$ such that $f|_A(p) = y$. Then we have $y \neq f|_A(p)$ for every $p \in A$. Since $y \in f(\mathbb{N})$, we know that $\exists q \in \mathbb{N}$ such that $f(q) = y$ and $q \notin A$. Since $q \notin A$, by the definition of $A$ we know that $\exists 0 \leq m < q$ such that $f(m) = f(q) = y$. Now we let $E$ be the set

$$E = \{m \in \mathbb{N} : f(m) = f(q) = y\}.$$

Since $E \subseteq \mathbb{N}$ and $E \neq \emptyset$, by well ordering principle (definition 1.4) we know that $\min(E)$ exists. This means $\exists p \in E$ such that $\forall 0 \leq m < p$, we have $f(m) \neq f(p) = f(q)$. But then we must have $p \in A$, a contradiction. Thus, $\forall y \in f(\mathbb{N})$, $\exists p \in A$ such that $f|_A(p) = y$. This means $f(\mathbb{N}) \subseteq f(A)$, thus we have $f(\mathbb{N}) = f(A)$.

Since $A \subseteq \mathbb{N}$ and $A$ is infinite, by definition 1.5 $\exists g : \mathbb{N} \to A$ where $g$ is bijective. This means $f|_A \circ g$ is bijective and we have

$$(f|_A \circ g)(\mathbb{N}) = f|_A\big(g(\mathbb{N})\big) = f|_A(A) = f(A) = f(\mathbb{N}).$$

Thus, by definition 1.1 $f(\mathbb{N})$ is countable, and thus at most countable. □

**Corollary 1.9.** *Let $X$ be a countable set, and let $f : X \to Y$ be a function. Then $f(X)$ is at most countable.*

*Proof.* By definition 1.1 $\exists g : \mathbb{N} \to X$ such that $g$ is a bijection. Then we have $f \circ g : \mathbb{N} \to Y$ and by definition 1.8 $(f \circ g)(\mathbb{N})$ is at most countable. But

$$(f \circ g)(\mathbb{N}) = f(g(\mathbb{N})) = f(X).$$

Thus, $f(X)$ is at most countable. □

**Proposition 1.10.** *Let $X$ be a countable set, and let $Y$ be a countable set. Then $X \cup Y$ is a countable set.*

*Proof.* By definition 1.1 $\exists f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$ such that $f$ and $g$ are bijections. Let $h : \mathbb{N} \to X \cup Y$ by setting $h(2n) = f(n)$ and $h(2n+1) = g(n)$ for every natural number $n$. We now show that $h(\mathbb{N}) = X \cup Y$.

$$\begin{aligned}
& z \in h(\mathbb{N}) \\
\iff & \exists k \in \mathbb{N} : h(k) = z \\
\iff & (\exists k \in \mathbb{N} : h(k) = z) \\
& \wedge (\exists n \in \mathbb{N} : k = 2n \vee k = 2n+1) && i : ac : 4.4.2 \\
\iff & \exists n \in \mathbb{N} : z = h(2n) \vee z = h(2n+1) \\
\iff & z = f(n) \vee z = g(n) \\
\iff & z \in X \vee z \in Y \\
\iff & z \in X \cup Y.
\end{aligned}$$

Then by definition 1.9 we have $h(\mathbb{N}) = X \cup Y$ is at most countable. But since $X$ and $Y$ are infinite sets, $X \cup Y$ can not be finite, thus $X \cup Y$ is countable. □

To summarize, any subset or image of a countable set is at most countable, and any finite union of countable sets is still countable.

**Axiomatic Claim 1.11.** *Let $X, Y$ be at most countable sets. Then $X \cup Y$ is at most countable.*

*Proof.* We split into following three cases:

- $X, Y$ are countable. Then by definition 1.10 we know that $X \cup Y$ is countable, thus at most countable.
- $X, Y$ are finite. Then by **??**(b) we know that $X \cup Y$ is finite, thus at most countable.
- $X, Y$ consist of one finite set and one countable set. Without the loss of generality, suppose that $X$ is finite and $Y$ is countable. Since $X$ is finite, there exists a function $f : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$

such that $f$ is bijective. Since $Y$ is countable, by definition 1.1 there exists a function $g : \mathbb{N} \to Y$ such that $g$ is bijective. Now we define a functiton $h : \mathbb{N} \to X \cup Y$ as follow:

$$\forall n \in \mathbb{N}, h(n) = \begin{cases} f(n+1) & \text{if } n < \#(X) \\ g(n - \#(X)) & \text{if } n \geq \#(X) \end{cases}$$

We need to show that $h(\mathbb{N}) = X \cup Y$. Since $h(\mathbb{N}) \subseteq X \cup Y$, it suffices to show that $X \cup Y \subseteq h(\mathbb{N})$.

$$\begin{aligned} &\forall z \in X \cup Y \\ \implies &(z \in X) \vee (z \in Y) \\ \implies &\big(\exists n \in \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} : f(n) = z\big) \\ &\wedge \big(\exists n \in \mathbb{N} : g(n) = z\big) \\ \implies &\big(h(n-1) = f(n) = z\big) \wedge \big(h(n + \#(X)) = g(n) = z\big) \\ \implies &z \in h(\mathbb{N}), \end{aligned}$$

Thus, we have $X \cup Y \subseteq h(\mathbb{N})$. By definition 1.8 $X \cup Y$ is at most countable.

From all cases above, we conclude that $X \cup Y$ is at most countable. $\square$

**Corollary 1.12.** *The integers $\mathbb{Z}$ are countable.*

*Proof.* We already know that the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers are countable. The set $-\mathbb{N}$ defined by

$$-\mathbb{N} := \{-n : n \in \mathbb{N}\} = \{0, -1, -2, -3, \dots\}$$

is also countable, since the map $f(n) := -n$ is a bijection between $\mathbb{N}$ and this set. Since the integers are the union of $\mathbb{N}$ and $-\mathbb{N}$, the claim follows from definition 1.10. $\square$

To establish countability of the rationals, we need to relate countability with Cartesian products. In particular, we need to show that the set $\mathbb{N} \times \mathbb{N}$ is countable.

**Lemma 1.13.** *The set*

$$A := \{(n, m) \in \mathbb{N} \times \mathbb{N} : 0 \leq m \leq n\}$$

*is countable.*

*Proof.* Define the sequence $a_0, a_1, a_2, \dots$ recursively by setting $a_0 := 0$, and $a_{n+1} := a_n + n + 1$ for all natural numbers $n$. Thus

$$a_0 = 0; a_1 = 0 + 1; a_2 = 0 + 1 + 2; a_3 = 0 + 1 + 2 + 3; \dots$$

By induction one can show that $a_n$ is increasing, i.e., that $a_n > a_m$ whenever $n > m$.

Now define the function $f : A \to \mathbb{N}$ by

$$f(n, m) := a_n + m.$$

We claim that $f$ is one-to-one. In other words, if $(n, m)$ and $(n', m')$ are any two distinct elements of $A$, then we claim that $f(n, m) \neq f(n', m')$.

To prove this claim, let $(n, m)$ and $(n', m')$ be two distinct elements of $A$. There are three cases: $n' = n$, $n' > n$, and $n' < n$. First, suppose that $n' = n$. Then we must have $m \neq m'$, otherwise $(n, m)$ and $(n', m')$ would not be distinct. Thus, $a_n + m \neq a_n + m'$, and hence $f(n, m) \neq f(n', m')$, as desired.

Now suppose that $n' > n$. Then $n' \geq n + 1$, and hence

$$f(n', m') = a_{n'} + m' \geq a_{n'} \geq a_{n+1} = a_n + n + 1.$$

But since $(n, m) \in A$, we have $m \leq n < n + 1$, and hence

$$f(n', m') \geq a_n + n + 1 > a_n + m = f(n, m),$$

and thus $f(n', m') \neq f(n, m)$.

The case $n' < n$ is proven similarly, by switching the roles of $n$ and $n'$ in the previous argument. Thus, we have shown that $f$ is one-to-one. Thus, $f$ is a bijection from $A$ to $f(A)$, and so $A$ has equal cardinality with $f(A)$. But $f(A)$ is a subset of $\mathbb{N}$, and hence by definition 1.6 $f(A)$ is at most countable. Therefore $A$ is at most countable. But, $A$ is clearly not finite. (if $A$ was finite, then every subset of $A$ would be finite, and in particular, $\{(n, 0) : n \in \mathbb{N}\}$ would be finite, but this is clearly countably infinite, a contradiction.) Thus, $A$ must be countable. □

**Corollary 1.14.** *The set $\mathbb{N} \times \mathbb{N}$ is countable.*

*Proof.* We already know that the set

$$A := \{(n, m) \in \mathbb{N} \times \mathbb{N} : 0 \leq m \leq n\}$$

is countable. This implies that the set

$$B := \{(n, m) \in \mathbb{N} \times \mathbb{N} : 0 \leq n \leq m\}$$

is also countable, since the map $f : A \to B$ given by $f(n, m) := (m, n)$ is a bijection from $A$ to $B$. We prove $f$ is bijective by showing that $f$ is both injective and surjective.

- To prove that $f$ is injective, suppose that $(n, m), (n', m') \in A$ and $f(n, m) = f(n', m')$. Then we have

$$\begin{aligned} & f(n, m) = f(n', m') & \\ \implies & (m, n) = (m', n') & \text{(by the definition of } f\text{)} \\ \implies & n = n' \wedge m = m' & i : 3.5.1 \\ \implies & (n, m) = (n', m'). & i : 3.5.1 \end{aligned}$$

  Thus, $f$ is injective.

- Since $\forall (n, m) \in B$, we have $n \leq m$, thus $(m, n) \in A$ and $f(m, n) = (n, m)$. So $f$ is surjective.

We now show that $\mathbb{N} \times \mathbb{N} = A \cup B$. By **??** we need to show that $\mathbb{N} \times \mathbb{N} \subseteq A \cup B$ and $A \cup B \subseteq \mathbb{N} \times \mathbb{N}$. It is clearly that $A \cup B \subseteq \mathbb{N} \times \mathbb{N}$. So we only need to show that $\mathbb{N} \times \mathbb{N} \subseteq A \cup B$.

$$\begin{aligned} & \forall (a, b) \in \mathbb{N} \times \mathbb{N} & \\ \implies & (a < b) \vee (a = b) \vee (a > b) & i : 2.2.13 \\ \implies & (a, b) \in A \vee (a, b) \in B & \text{(by the definition of } A \text{ and } B\text{)} \\ \implies & (a, b) \in A \cup B. & i : 3.4 \end{aligned}$$

Thus, by **??** we have $\mathbb{N} \times \mathbb{N} \subseteq A \cup B$.

Since $\mathbb{N} \times \mathbb{N}$ is the union of $A$ and $B$, the claim then follows from definition 1.10. □

**Corollary 1.15.** *If $X$ and $Y$ are countable, then $X \times Y$ is countable.*

*Proof.* By definition 1.1 $\exists f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$ such that $f$ and $g$ are bijections. Let $h : \mathbb{N} \times \mathbb{N} \to X \times Y$ by setting $h(x, y) = (f(x), g(y))$. If $n, n', m, m' \in \mathbb{N}$ and $(n, m) \neq (n', m')$, then

$$h(n, m) = (f(n), g(m)) \neq (f(n'), g(m')) = h(n', m')$$

since $f, g$ are bijections, so $h$ is injective. Again since $f, g$ are bijections, $\forall x \in X \wedge \forall y \in Y$, $\exists n, m \in \mathbb{N}$ such that $x = f(n) \wedge y = g(m)$. So $h$ is surjective, and thus $h$ is bijective.

Since $h$ is bijective, $\mathbb{N} \times \mathbb{N}$ and $X \times Y$ has the same cardinality. But by definition 1.14 we know that $\mathbb{N} \times \mathbb{N}$ is countable. Thus, by definition 1.1 $X \times Y$ is countable. □

**Corollary 1.16.** *The rationals $\mathbb{Q}$ are countable.*

*Proof.* We already know that the integers $\mathbb{Z}$ are countable, which implies that the non-zero integers $\mathbb{Z} - \{0\}$ are countable. (since $\mathbb{Z} - \{0\} \subseteq \mathbb{Z}$, by definition 1.7 we know that $\mathbb{Z} - \{0\}$ is at most countable, and clearly $\mathbb{Z} - \{0\}$ is not finite.) By definition 1.15, the set

$$\mathbb{Z} \times (\mathbb{Z} - \{0\}) = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$$

is thus countable. If one lets $f : \mathbb{Z} \times (\mathbb{Z} - \{0\}) \to \mathbb{Q}$ be the function $f(a, b) := a/b$ (note that $f$ is well-defined since we prohibit $b$ from being equal to 0), we see from definition 1.9 that $f(\mathbb{Z} \times (\mathbb{Z} - \{0\}))$ is at most countable. But we have $f(\mathbb{Z} \times (\mathbb{Z} - \{0\})) = \mathbb{Q}$ (This is basically the definition of the rationals $\mathbb{Q}$). Thus, $\mathbb{Q}$ is at most countable. However, $\mathbb{Q}$ cannot be finite, since it contains the infinite set $\mathbb{N}$. Thus, $\mathbb{Q}$ is countable. ☐

**Remark 1.17.** *Because the rationals are countable, we know* in principle *that it is possible to arrange the rational numbers as a sequence:*

$$\mathbb{Q} = \{a_0, a_1, a_2, a_3, \dots\}$$

*such that every element of the sequence is different from every other element, and that the elements of the sequence exhaust $\mathbb{Q}$ (i.e., every rational number turns up as one of the elements $a_n$ of the sequence). However, it is quite difficult (though not impossible) to actually try and come up with an explicit sequence $a_0, a_1, \dots$ which does this.*

[Dedekind-infinite set] Let $X$ be a set. Show that $X$ is infinite iff there exists a proper subset $Y \subsetneq X$ of $X$ which has the same cardinality as $X$.

*Proof.* We first show that $X$ is infinite implies $\exists Y \subsetneq X$ such that $Y$ has the same cardinality as $X$. Suppose that $X$ is an infinite set. Then we have $X \neq \emptyset$ since by **??** $\#(\emptyset) = 0$.

Let $n \in \mathbb{N}$ and let $P(n)$ be the statement "$\exists A_n \subseteq X$ such that $\#(A_n) = n$." We induct on $n$ to show that $\forall n \in \mathbb{N}$, $P(n)$ is true. For $n = 0$, we have $\emptyset \subseteq X$ by **??**, Thus, the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then we need to show that $P(n+1)$ is true. By the induction hypothesis, $\exists A_n \subseteq X : \#(A_n) = n$. Then by **??**(b) we know that $X \setminus A_n$ is infinite. Since $X \setminus A_n$ is infinite, we know that $X \setminus A_n \neq \emptyset$. Let $x \in X \setminus A_n$. Then we define $A_{n+1} = A_n \cup \{x\}$, and this closes the induction.

By axiom of choice (**??**) the set $\prod_{n \in \mathbb{Z}^+} A_n$ is non-empty since $\forall n \in \mathbb{Z}^+$, $P(n)$ is true. We can now choose an element $(x_n)_{n \in \mathbb{Z}^+}$ from $\prod_{n \in \mathbb{Z}^+} A_n$. In particular, we want to choose a $(x_n)_{n \in \mathbb{Z}^+}$ where $x_i \neq x_j$ for every $i, j \in \mathbb{Z}^+$ and $i \neq j$. This can be done since $\#(A_i) \neq \#(A_j)$ for every $i, j \in \mathbb{Z}^+$ and $i \neq j$. We collect $x_i$ as a set $A = \{x_i : i \in \mathbb{Z}^+\}$. By axiom of choice (**??**) $A$ can be construct as the image of $(x_n)_{n \in \mathbb{Z}^+}$. Now we define a function $f : X \to X \setminus \{x_1\}$ as follow:

$$f(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \text{ for some } x_n \in A, \\ x & \text{if } x \notin A. \end{cases}$$

We show that such $f$ is bijective. We start by showing $f$ is injective. Let $x, x' \in X$ and $x \neq x'$. We split into four cases:

- If $x \in A \wedge x' \in A$, then $\exists n, n' \in \mathbb{Z}^+$ such that $x = x_n \wedge x' = x_{n'}$. By the definition of $x_n$ and $x_{n'}$, we must have $x_n \neq x_{n'} \implies x_{n+1} \neq x_{n'+1}$. Thus, we have $x_{n+1} = f(x) \neq f(x') = x_{n'+1}$.
- If $x \in A \wedge x' \notin A$, then $f(x) \in A \wedge f(x') = x' \notin A$ and thus $f(x) \neq f(x')$.
- If $x \notin A \wedge x' \in A$, then $f(x) = x \notin A \wedge f(x') \in A$ and thus $f(x) \neq f(x')$.
- If $x \notin A \wedge x' \notin A$, then $f(x) = x \neq x' = f(x')$.

From all cases above, we conclude that $x \neq x' \implies f(x) \neq f(x')$, thus $f$ is injective. Now we show that $f$ is surjective. Let $x \in X \setminus \{x_1\}$. We split into two cases:

- If $x \in A$, then $x \neq x_1$ and $\exists n \in \mathbb{Z}^+ \setminus \{1\}$ such that $x = x_n$. Since $n \geq 2$, we have $n - 1 \geq 1$. Thus, by the definition of $A$ we have $x_{n-1} \in A$ and $f(x_{n-1}) = x_n$.
- If $x \notin A$, then we have $f(x) = x$.

Since $x$ was arbitrary, we know that $f$ is surjective. Since $f$ is both injective and surjective, we know that $f$ is bijective, and by **??** $X$ and $X \setminus \{x_1\}$ have the same cardinality. But $x_1 \in X \wedge x_1 \notin X \setminus \{x_1\}$, we have $X \neq X \setminus \{x_1\}$. Thus, by **??** $X \setminus \{x_1\} \subsetneq X$.

Now we show that if $\exists Y \subsetneq X$ where $X$ and $Y$ have the same cardinality, then $X$ is infinite. We prove this by contradiction. Suppose for the sake of contradiction that $X$ is finite. Then by **??**(c) we have $\#(Y) < \#(X)$, a contradiction. Thus, $X$ is infinite. ☐

Prove definition 1.4.

*Proof.* See definition 1.4. ☐

Fill in the gaps marked in definition 1.5.

*Proof.* See definition 1.5. ☐

Prove definition 1.8.

*Proof.* See definition 1.8. ☐

Use definition 1.8 to prove definition 1.9.

*Proof.* See definition 1.9. ☐

Let $A$ be a set. Show that $A$ is at most countable iff there exists an injective map $f : A \to \mathbb{N}$ from $A$ to $\mathbb{N}$.

*Proof.* We first show that if $A$ is at most countable, then there exists an injective map $f : A \to \mathbb{N}$. Suppose that $A$ is at most countable. By definition 1.1 $A$ is either finite or countable.

- If $A$ is finite, then by **??** $\exists g : A \to \{i \in \mathbb{N} : 1 \leq i \leq \#(A)\}$ such that $g$ is bijective. Now let $f : A \to \mathbb{N}$ be the function $f(x) = g(x)$ for every $x \in A$. Since $g$ is a bijection and $\{i \in \mathbb{N} : 1 \leq i \leq \#(A)\} \subseteq \mathbb{N}$, we have $f : A \to \mathbb{N}$ is injective.
- If $A$ is countable, then by definition 1.1 $\exists f : A \to \mathbb{N}$ such that $f$ is a bijection, and hence $f$ is injective.

From all cases above, we conclude that if $A$ is at most countable then there exists an injective map $f : A \to \mathbb{N}$.

Now we show that if there exists an injective map $f : A \to \mathbb{N}$, then $A$ is at most countable. Suppose that $f : A \to \mathbb{N}$ is injective. Since $f(A) \subseteq \mathbb{N}$, by definition 1.6 $f(A)$ is at most countable. Since $f$ is bijective from $A$ to $f(A)$, we know that $A$ and $f(A)$ have equal cardinality, and thus $A$ is at most countable. ☐

Prove definition 1.10.

*Proof.* See definition 1.10. ☐

Use definition 1.14 to prove definition 1.15.

*Proof.* See definition 1.15. ☐

Suppose that $I$ is an at most countable set, and for each $\alpha \in I$, let $A_\alpha$ be an at most countable set. Show that the set $\bigcup_{\alpha \in I} A_\alpha$ is also at most countable. In particular, countable unions of countable sets are countable.

*Proof.* Suppose that $I$ be an at most countable set and $\forall \alpha \in I$ we have $A_\alpha$ is an at most countable set. By definition 1.1 $I$ is either finite or countable.

We first show that if $I$ is finite, then $\bigcup_{\alpha \in I} A_\alpha$ is at most countable. Since $I$ is finite, by **??** $\exists n \in \mathbb{N}$ such that $\#(I) = n$. Let $P(n)$ be the statement "$\#(I) = n$ and $\bigcup_{\alpha \in I} A_\alpha$ is at most countable." We induct on $n$ to show that $P(n)$ is true for every $n \in \mathbb{N}$. For $n = 0$, we have $\#(\emptyset) = 0$ and $\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset$. Thus, the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then we need to show that $P(n+1)$ is also true. Since $\#(I) = n+1 > 0$, we know that $I \neq \emptyset$. Let $i \in I$. Since $\#(I \setminus \{i\}) = n$, by the induction hypothesis we know that the set $\bigcup_{\alpha \in I \setminus \{i\}} A_\alpha$ is at most countable. By **??** we have $\bigcup_{\alpha \in I} A_\alpha = (\bigcup_{\alpha \in I \setminus \{i\}} A_\alpha) \cup A_i$. Then by definition 1.11 we know that $\bigcup_{\alpha\in I} A_\alpha$ is at most countable. This closes the induction. We conclude that finite union of at most countable sets is at most countable.

Now we show the case where $I$ is countable. Let $J = \{\alpha \in I : A_\alpha \neq \emptyset\}$. Since $J \subseteq I$, by definition 1.7 we know that $J$ is at most countable. If $J$ is finite (including the case where $J = \emptyset$), then we already show that finite union of at most countable sets is at most countable. So suppose that $J$ is countable. Then we have

$$\begin{aligned}
&\forall x \in \bigcup_{\alpha\in I} A_\alpha \\
\iff& \exists \alpha' \in I : x \in A'_{\alpha'} \\
\iff& A'_{\alpha'} \neq \emptyset \\
\iff& \alpha' \in J \\
\iff& \exists \alpha' \in J : x \in A'_{\alpha'} \\
\iff& x \in \bigcup_{\alpha\in J} A_\alpha.
\end{aligned}$$

Thus, by **??** we have $\bigcup_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in J} A_\alpha$. To show that $\bigcup_{\alpha\in I} A_\alpha$ is at most countable, it suffices to show that $\bigcup_{\alpha\in J} A_\alpha$ is at most countable.

Since $\forall \alpha \in J$, $A_\alpha$ is at most countable. We split into two cases:

- If $A_\alpha$ is finite, then by **??** $\exists f'_\alpha : \{n \in \mathbb{N} : 1 \leq n \leq \#(A_\alpha)\} \to A_\alpha$ such that $f'_\alpha$ is bijective. We now define a function $f_\alpha : \mathbb{N} \to A_\alpha$ as follow:

$$\forall n \in \mathbb{N} : f_\alpha(n) = \begin{cases} f'_\alpha(n) & \text{if } 1 \leq n \leq \#(A_\alpha), \\ f'_\alpha(1) & \text{if } n = 0 \vee n > \#(A_\alpha). \end{cases}$$

  Thus, $f_\alpha$ is surjective. We can define $F_\alpha$ to be a set of functions

$$F_\alpha = \{f_\alpha : \mathbb{N} \to A_\alpha | f_\alpha \text{ follows the definition above}\}$$

  and $F_\alpha \neq \emptyset$.

- If $A_\alpha$ is countable, then we define $F_\alpha$ to be a set of bijections

$$F_\alpha = \{f_\alpha : \mathbb{N} \to A_\alpha | f_\alpha \text{ is bijective}\}.$$

  Since $A_\alpha$ is countable, we know that $F_\alpha \neq \emptyset$.

Since $\forall \alpha \in J$, $F_\alpha \neq \emptyset$, by axiom of choice (**??**) the set $\prod_{\alpha\in J} F_\alpha \neq \emptyset$. This means we can choose a function $(f_\alpha)_{\alpha\in J}$ from $\prod_{\alpha\in J} F_\alpha$ which maps $\alpha \in J$ to a function $f_\alpha : \mathbb{N} \to A_\alpha$.

We now use axiom of choice (**??**) to choose a function $(f_\alpha)_{\alpha\in J}$ and fix such function. Since $J$ is countable, $\exists g : \mathbb{N} \to J$ such that $g$ is bijective. We now define another function $h : \mathbb{N} \times \mathbb{N} \to \bigcup_{\alpha\in J} A_\alpha$ as follow:

$$\forall (n, m) \in \mathbb{N} \times \mathbb{N} : h(n, m) = f_{g(n)}(m).$$

By definition 1.9 we now that $h(\mathbb{N} \times \mathbb{N})$ is at most countable. If we can show that $h$ is surjective, then we can show that $\bigcup_{\alpha\in J} A_\alpha$ is at most countable. Let $x \in \bigcup_{\alpha\in J} A_\alpha$. We know that $\exists \beta \in J$ such that $x \in A_\beta$. By the definition of $f_\beta$ we know that $f_\beta$ is surjective. Since $f_\beta$ is surjective, $\exists m \in \mathbb{N}$ such that $f_\beta(m) = x$. Since $g$ is bijective, $\exists n \in \mathbb{N}$ such that $g(n) = \beta$. Then we have

$$(n, m) \in \mathbb{N} \times \mathbb{N} \implies h(n, m) = f_{g(n)}(m) = f_\beta(m) = x.$$

Since $x$ was arbitrary, we thus know that $h$ is surjective. We conclude that countable union of at most countable set is at most countable.

Finally we show that countable union of countable set is countable. Let $I$ be a countable set and $\forall \alpha \in I$ let $A_\alpha$ be countable set. From the proof above we know that $\bigcup_{\alpha\in I} A_\alpha$ is at most countable. Suppose for the sake of contradiction that $\bigcup_{\alpha\in I} A_\alpha$ is finite. Let $\beta \in I$. By hypothesis we know that $A_\beta$ is countable, and we have

$$A_\beta \subseteq \bigcup_{\alpha\in I} A_\alpha.$$

But $\bigcup_{\alpha\in I} A_\alpha$ is finite, thus by **??**(c) we know that $A_\beta$ is finite, a contradiction. We conclude that countable union of countable set is countable. $\square$

Find a bijection $f : \mathbb{N} \to \mathbb{Q}$ from the natural numbers to the rationals.

*Proof.* Helped needed. $\square$