

Welcome! Today, we are exploring the conservations of both energy and momentum and how difficult problems can be easily solved through them.

### Idea 1: Energy

We introduce a new idea, energy... but it is quite difficult to define. There are various forms of energy, each with their own formulae. However, two forms of **Energy Transfer** can be more clearly defined: **Heat Transfer and Work Transfer**. Define work transfer as:

$$W = \int F \cdot dx \quad (1)$$

This means that for a small displacement, the component of the force parallel to the displacement times the force is the  $dW$ . Integrating it yields work. This means that the unit of energy, the **Joule** is a  $\frac{kg \cdot m^2}{s^2}$ . You can lift your water bottle about 10 cm, and expend a joule of energy.

All moving objects have energy, known as kinetic energy. The translational kinetic energy of an object is dependent on mass and velocity.

$$K = \frac{1}{2}mv^2 \quad (2)$$

Take a small displacement  $dx$ , and apply a force  $F_p$  parallel to the displacement on a mass  $m$ . Such force produces an acceleration  $\frac{F_p}{m}$ .

$$(V + dv)^2 - (V)^2 = d(V^2) = 2\frac{F_p}{m}dx \quad (3)$$

$$\frac{1}{2}m(d(V^2)) = dK = F_p dx = F \cdot dx = dW \quad (4)$$

$$\text{Therefore, } \Delta K = W \quad (5)$$

Energy can also be **stored**. If an amount of work is done against a conservative force, the amount of work it took to get from an energy level of 0 to that point is the **potential energy**. For example, imagine the ground level to be 0. The minimum force one must apply to counter gravity to lift an object is  $mg$  in the upward direction. The object has a displacement in the upward direction too, so the the work done to lift the object up a height  $h$  is  $mgh$ . This is the potential energy stored in the object.

Heat Transfer and Internal Energy: If work transfer is the transfer of energy based on force, heat transfer is the transfer of energy through a **temperature gradient**. This also uses joules, but is slightly harder to define. All particles have kinetic energy, which changes based on temperature. The formula for internal energy is as follows, with  $U$  being internal energy,  $m$  being mass,  $C$  being the specific heat (constant per material), and  $T$  being Temperature.:

$$U = mCT \quad (6)$$

We will cover this in a later thermodynamics handout, with more on heat transfer. However, heat will not be the focus of this handout.

Adding up all the energies of a system: Kinetic, Potential, Internal... yields total energy  $E$ . This is **conserved** in an isolated system, where energy transfer does not occur. The total

energy of the universe never changes. When energy transfer does occur, the **First Law of Thermodynamics** is used.

$$\Delta E = Q - W \quad (7)$$

where  $Q$  is heat transfer IN to the system, and  $W$  is work transfer OUT of the system. The strange nature of this formatting comes from its invention, where people used heat from burning coal to power industrial sites that did work. They measured work from machine output.

While laws such as Newton's Laws no longer work at high speeds, and one has to switch to relativity, and a whole new system must be created for quantum, this law: **The First Law of Thermodynamics has not been broken** (*Yes, this is thermodynamics advertising, but it is true*).

## Idea 2: Momentum

Another invariant is momentum. Unlike energy, this is a vector, which means that momentum has direction. This quantity can be defined as  $P = mv$ , where  $m$  is mass, and  $v$  is velocity. This quantity does not change in collisions, which means that  $P_{initial} = P_{final}$ . Newton's second law, so far, has only been taught as  $F = ma$ . However, its real definition is:

$$F = \frac{dP}{dt} \quad (8)$$

By the product rule, this becomes:

$$F = m \frac{dv}{dt} + v \frac{dm}{dt} = ma + v \frac{dm}{dt} \quad (9)$$

This means that a changing mass applies a force.

Collisions: Collisions can be described by their elasticity. It is true that in all collisions, momentum is conserved. However, some energy changes form from kinetic energy to internal energy. In an inelastic collision, both masses stick together. The momentum conservation equation is below:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_f \quad (10)$$

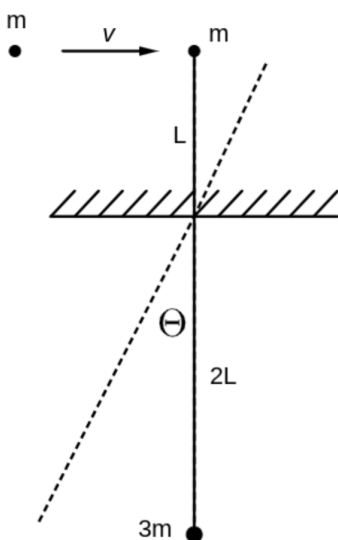
Fully elastic collisions involve both kinetic energy and momentum being conserved. For those,  $P_i = P_f$  and  $K_i = K_f$ .

All collisions are somewhere between fully elastic and fully inelastic, but we will focus on either fully inelastic or fully elastic for simplicity.

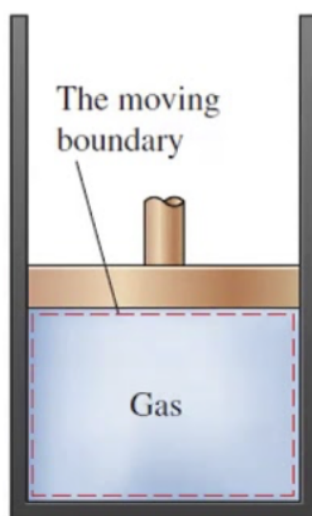
The elastic collision involving an object with a small mass hitting an object with infinite mass (ex. a wall) involves the infinite mass not moving, while the small mass bounces backwards and rebounds at the same speed as before. The momentum transfer into the infinite mass is  $2m_{small}v$ . This is because final kinetic energy must be conserved, and the only way for that to happen is if either the small mass travels into the wall at its initial speed (impossible), or if it rebounds.

Many problems here will include collisions, and the extension of Newton's second law, while including the concepts from before: Forces, Kinematics, and basic calculus to some extent.

- [3] **Problem 1.** For large distances, the gravitational force is defined as  $\frac{GMm}{r^2}$ , where  $G$  is a constant,  $M$  and  $m$  are the masses, and  $r$  is the distance between them. Calculate the potential energy at a distance  $r$  between the masses - remember to use the work done on the system by bringing it from a point of 0 potential energy.
- [4] **Problem 2.** A bungee cord exhibits hooke's spring behavior under tension and typical string behavior under compression. A bungee cord with relaxed length  $x$ , and spring constant  $k$  is attached to a mass  $m$  on one end and a fixed point at height  $h$  above the ground;  $h > x$ . The mass is dropped, and inelastically hits the floor. However, due to the stretching of the bungee rope, the mass rebounds and reaches a new maximum height  $d$ .
1. Find  $d$  in terms of  $k$ ,  $x$  and  $h$ .
  2. All of the energy that is lost due to the inelastic collision gets turned to heat transfer that increases the internal energy of the system. Calculate  $Q$ , the heat transfer in the first inelastic collision.
- [4] **Problem 3.** A mass  $m$  hangs on a hook along a clothesline wire. The wire has an incline, and is an angle of  $\theta$  from the horizontal. There is a frictional coefficient of  $\mu$  between the wire and the hook. The hook is let go, and it slides down the clothesline until it has moved down a vertical height of  $h$ , and the respective horizontal distance  $h \cdot \cot \theta$ . What is the magnitude of the velocity of the mass?
- [3] **Problem 4.** A train, with practically infinite mass, elastically collides with a stationary ball of mass  $m$ . You are excited to watch the ball shoot off at near-relativistic speeds, as the train slows down. However, that is **not** what happens - Calculate the maximum velocity the ball actually reaches, while the train *barely* slows down. Use reference frames (think of the ball rebounding from the wall, but instead its the wall moving towards the ball).
- [5] **Problem 5.** A point mass of mass  $m$  horizontally collides inelastically with the top of a pendulum at velocity  $v$ . The pendulum is vertical, with a mass of  $3m$  at length  $2L$  below the pivot and a mass of  $m$  at  $L$  above the pivot. What will be the maximum angle from the vertical the pendulum will reach, assuming the collided mass and the initial mass stick.



- [6] **Problem 6.** Photons are like packets of energy with energy  $E_p$  per photon. The momentum of a photon is  $\frac{E_p}{C}$ , where  $C$  is the speed of light. A perfect mirror involves perfectly elastic collisions between photon and mirror, but since the mass of a photon is 0, it is essentially a 'ball rebound' scenario.  $I$  is the intensity of a light source, which is energy per a given time in a given area: Joules per square meter second. Light moves at speed  $C$ . What is the radiation pressure (force per unit area) on the mirror, caused by the light?
- [3] **Problem 7.** Consider a cylinder fitted with a frictionless, movable piston of cross-sectional area  $A$ . The cylinder contains an ideal gas. The external pressure on the piston is held constant at  $P_{ext}$ . As the gas expands, the piston moves outward by a small distance  $dx$ . Write the work done by the gas in terms of pressure and volume.



- [4] **Problem 8.** We know about translational kinetic energy. We also know that  $I = \int r^2 dm$ . If an object with moment of inertia  $I$  is rotating at angular velocity  $\omega$ , find the **rotational kinetic energy** of the object.
- [7] **Problem 9.** A rocket has a mass  $M$ , and a fuel mass  $m$ . It is at rest with a full tank in space (no friction), and it starts releasing mass at a rate  $\dot{m}$  in the backwards direction at speed  $v$ . This mass release continues at the same velocity until the fuel tank is fully empty. What is the final velocity of the rocket? *Hint: Look at the  $\frac{dP}{dt}$  part of the momentum section.*
- [8] **Problem 10.** You are a rod with linear mass density  $\rho$  that spins around your center of mass with length  $l$  at angular velocity  $\omega$ . Suddenly, you grow taller instantaneously to length  $nl$ , but your kinetic energy does not change, and the location of your center of mass does not change. Your tip then collides inelastically with a mass  $m$ , and you rotate about your new center of mass. The only kinetic energy loss is during the collision. What is your final angular velocity?