

# Graph matching for connectomics

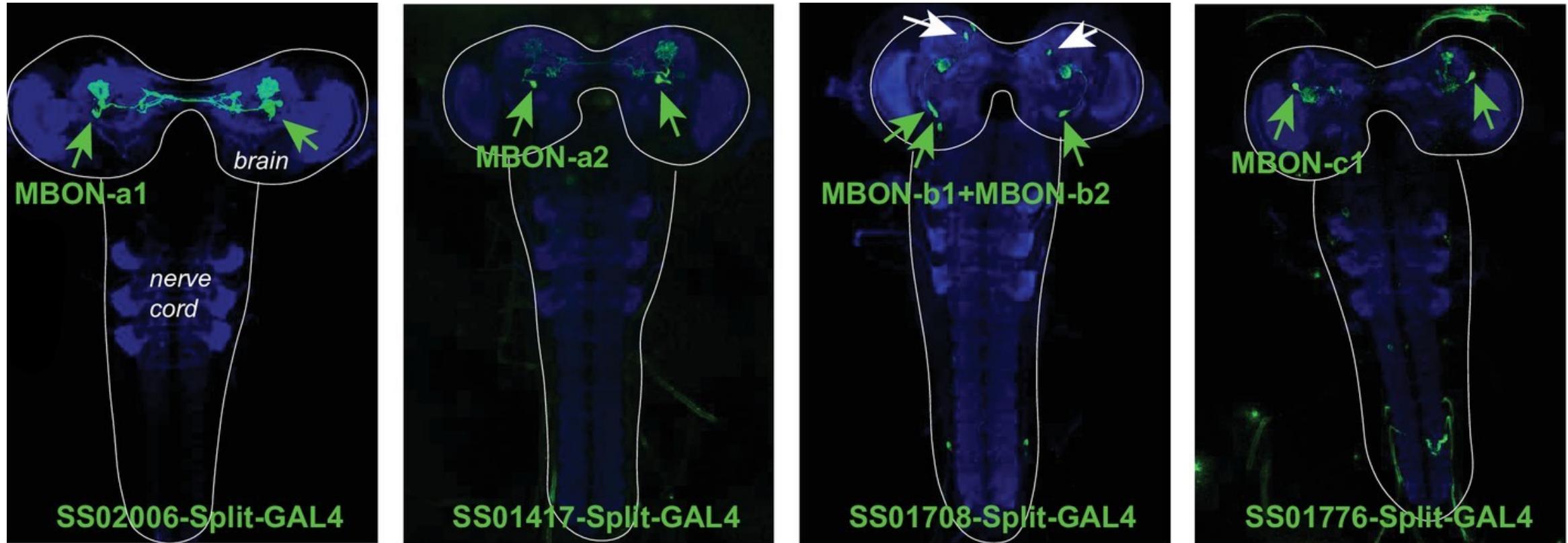
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These slides at:  
<https://tinyurl.com/cambridge-matching>

# Bilaterally homologous neuron pairs

We believe a matching exists!

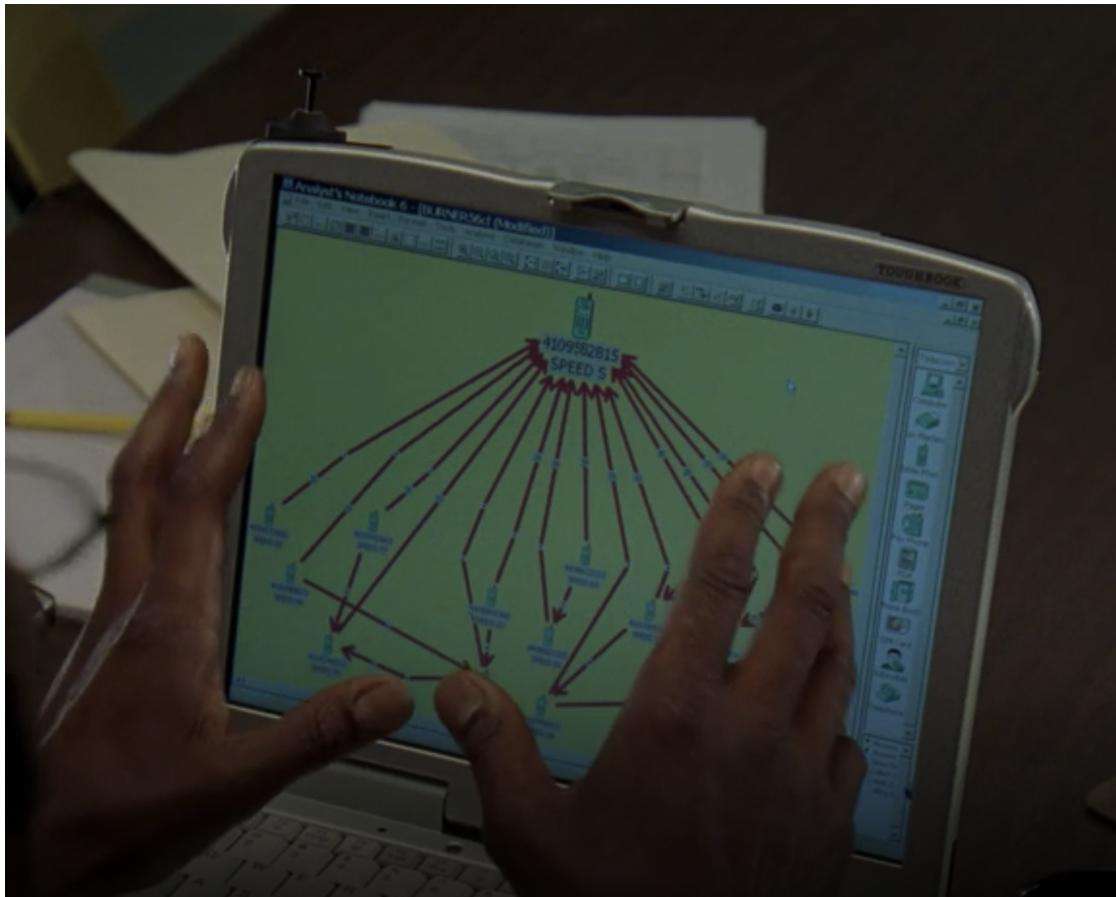


# Why care about the pairs?

- Proofreading
- Understand stereotypy
- Downstream analysis
  - Network embedding, for example
- Useful to collapse complexity by ~2
- Align one connectome to another, in some sense

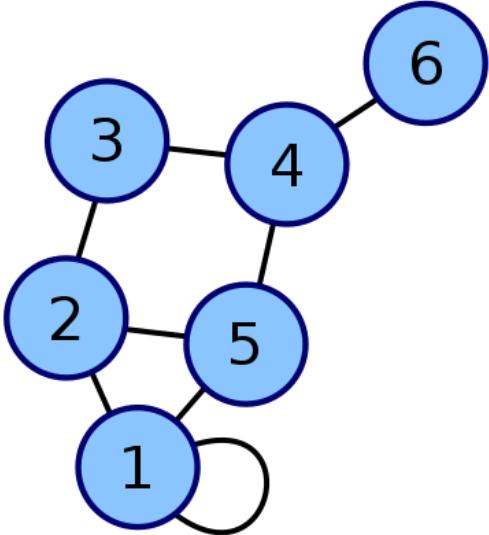
**How can we use the network structure to predict an  
{alignment, matching, pairing} of nodes from one  
network to another?**

# When the node "labels" aren't helpful



- Week 1: observe a network ( $A$ ) of phone #'s and the calls they make to each other
- Week 2: all of the #'s change! But a (noisy) version of that network still exists, with different labels... ( $B$ )
- How to map nodes of network  $A$  to those of network  $B$ ?

# How do we represent a network?

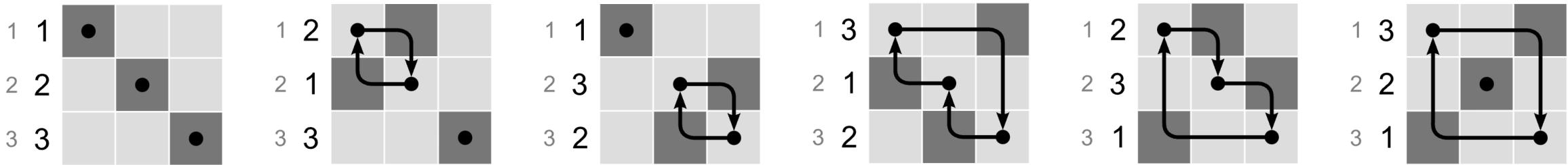


Adjacency matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

# Any permutation of an adjacency matrix is equally valid

## Permutation matrices ( $P$ ) on 3 nodes



## Permuting an adjacency matrix

$PA$  permutes the rows of  $A$

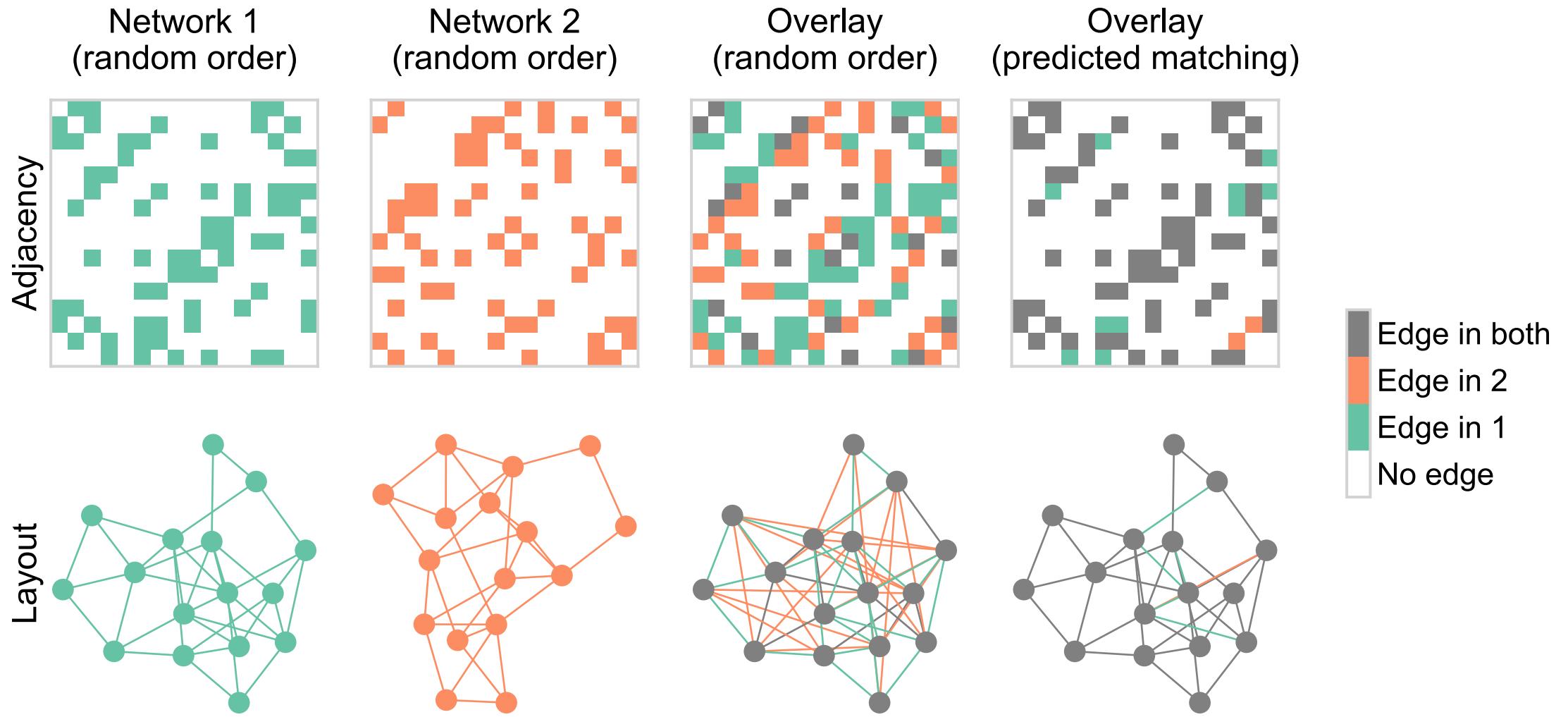
$AP^T$  permutes the columns of  $A$

So for an adjacency matrix, we always want  $PAP^T$ !

**A permutation of one network with respect to another is an {alignment, matching, pairing}**

So we are searching for a permutation of  $B$  with respect to some fixed ordering of  $A$

# What is graph matching?



# Framing the graph matching problem mathematically

$$\min_{P \in \mathcal{P}} \underbrace{\|A - \underbrace{PBP^T}_{\text{reordered } B} \|_F^2}_{\text{distance between adj. mats.}}$$

where  $\mathcal{P}$  is the set of permutation matrices

- Measures the number of edge disagreements for unweighted networks,
- Norm of edge disagreements for weighted networks

# Why is it hard

$$\min_{P \in \mathcal{P}} \|A - PBP^T\|_F^2$$

- Search space is absurdly large
  - $|\mathcal{P}| = n!$
  - $n = 335 \rightarrow |\mathcal{P}| \approx 10^{82}$  (more than number of atoms in universe)
- Search space is not convex
  - $0.5P_1 + 0.5P_2 \notin \mathcal{P}$

# Graph matching - how do we do it?

$$\min_{P \in \mathcal{P}} \|A - PBP^T\|_F^2$$

- Relax to the "birkoff polytope:" doubly stochastic matrices
  - Rows and columns all sum to 1, but not all elements have to be 1
  - Similar to *transport* as opposed to *assignment* problems: we are still mapping nodes between networks, but this is a *soft* mapping
- Apply Frank-Wolfe method
  - Minimize a first-order Taylor series of the objective function over a convex set
  - Requires a gradient  $\nabla f(P)$
- Project back to set of permutations when done

# Let's try it out...

## graspologic

```
from graspologic.match import GraphMatch

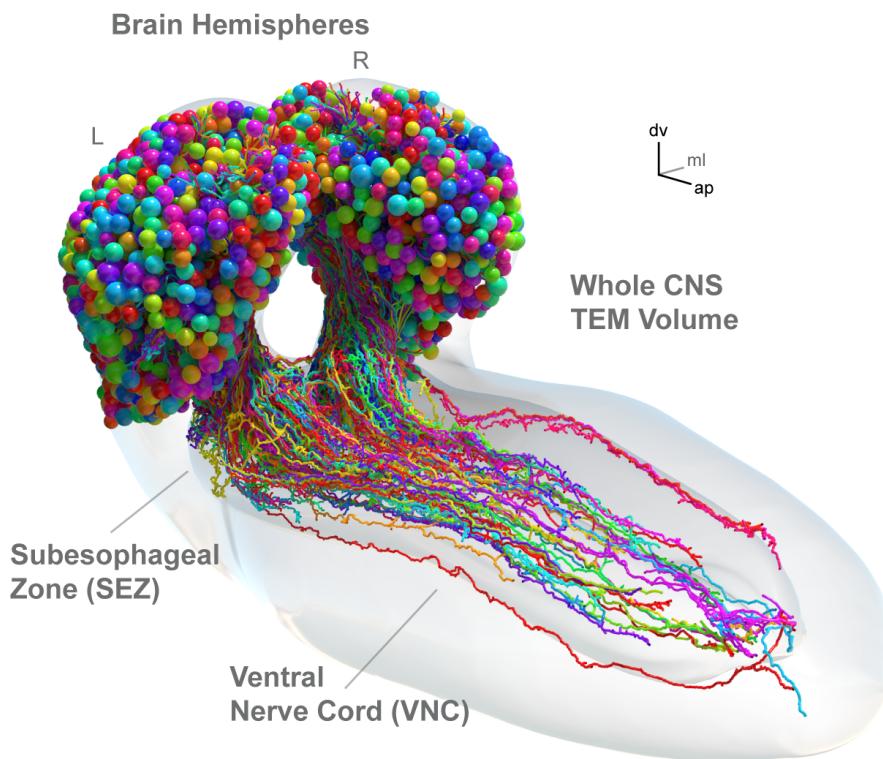
gm = GraphMatch() # parameters go in here
gm.fit(A, B) # adjacency matrices go in here
perm = gm.perm_inds_ # indices representing the permutation
B_permuted = B[perm][:, perm]
```

## SciPy

```
from scipy.optimize import quadratic_assignment

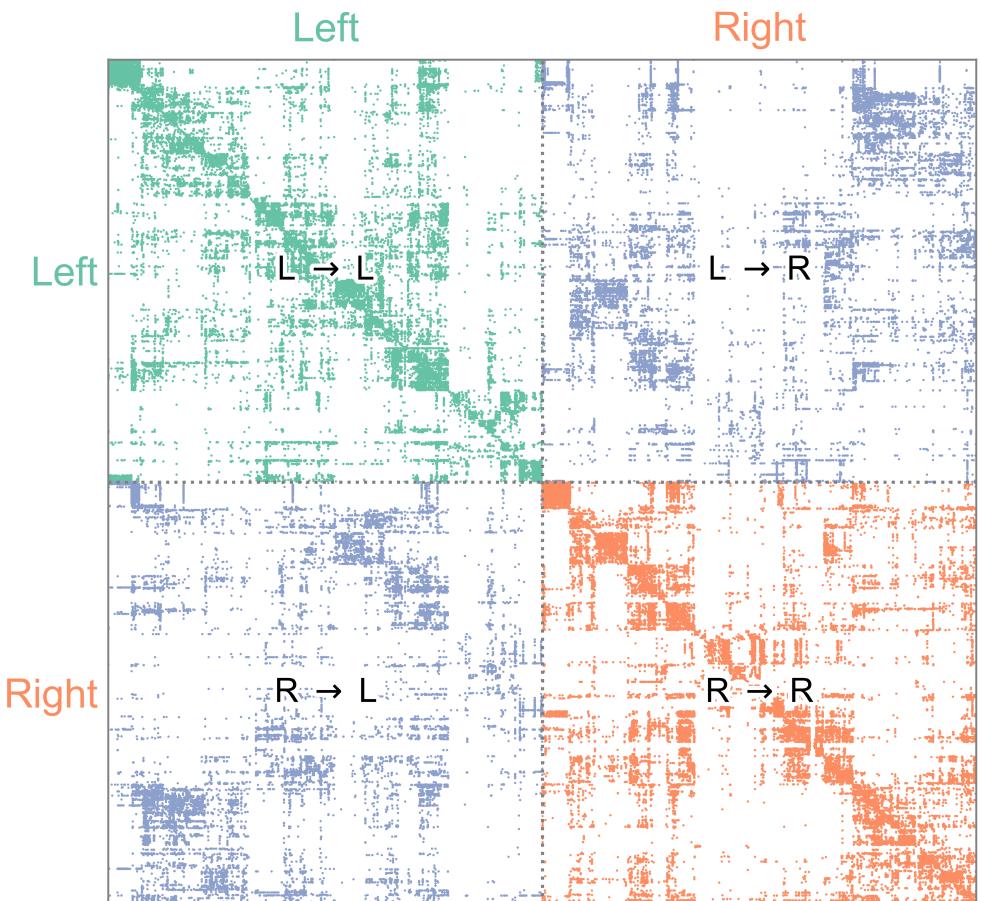
quadratic_assignment(A,B,method='faq',options=dict(maximize=True))
```

# Larval *Drosophila* brain connectome

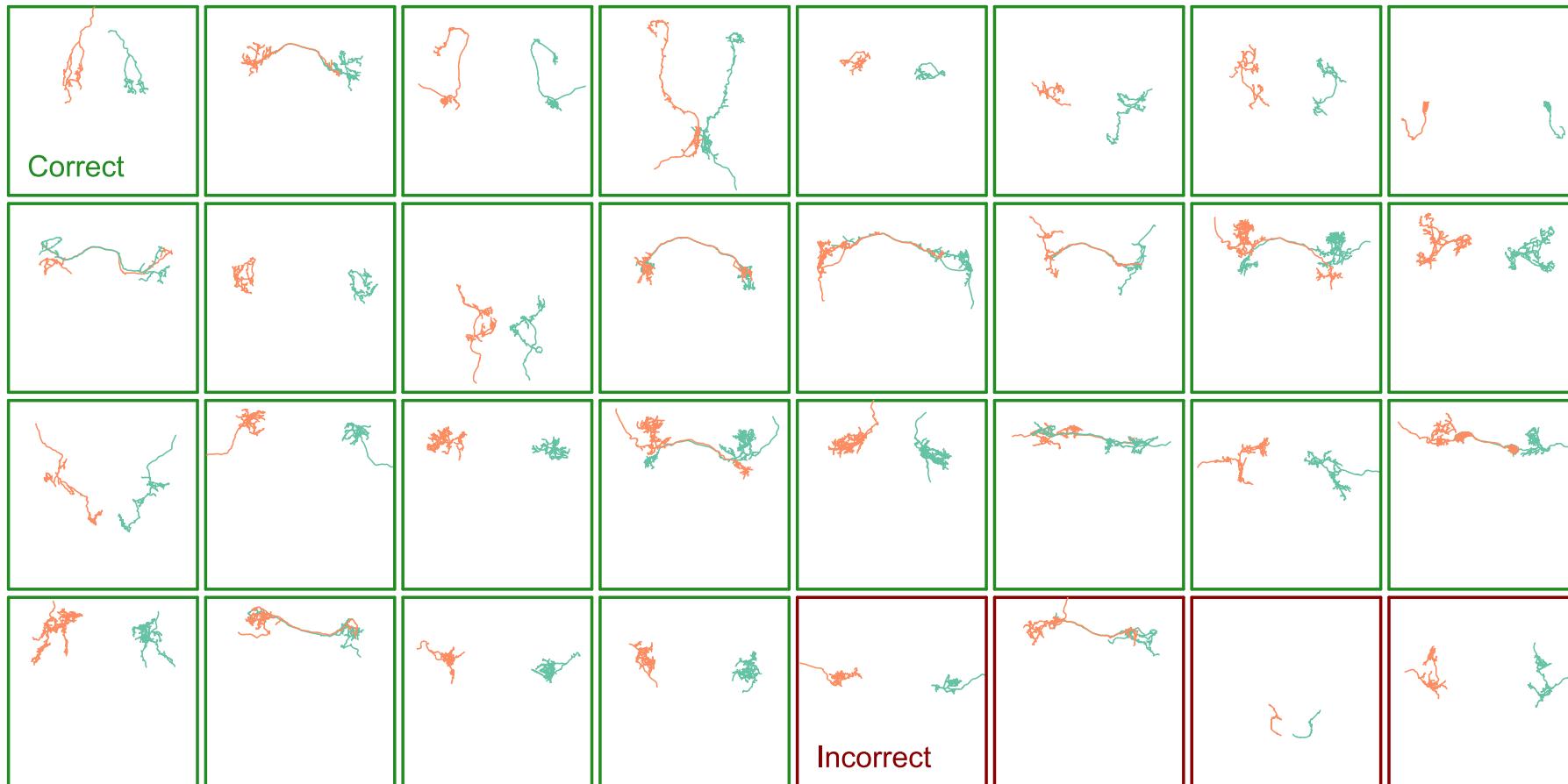


~3k neurons, ~550K synapses

Both hemispheres



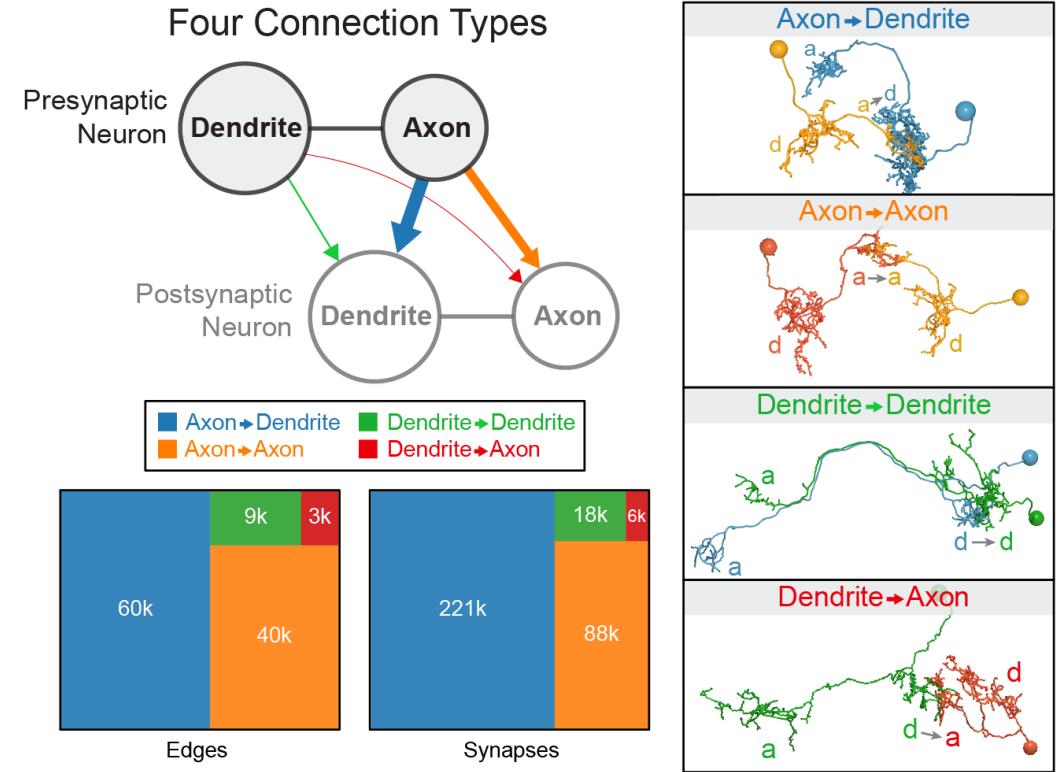
# Matching (by connectivity only) performs fairly well



With "vanilla" graph matching: ~80% correct (according to expert annotator)

# Many ways to try to improve on this...

- Partial knowledge of the matching (seeds)
- Neuron similarity information, (e.g. NBLAST/morphology)
- Edge types allow for "multilayer" graph matching



*Summary of "edge types" based on neuron compartments*

# Seeds (partial known pairing of nodes)

$$\min_{P \in \mathcal{P}_s} \|A - PBP^T\|_F^2$$

where  $\mathcal{P}_s$  is the set of permutations which respect your known, partial pairing

## graspologic

```
from graspologic.match import GraphMatch  
  
gm = GraphMatch()  
gm.fit(A, B, seeds_A=[1,2,3], seeds_B=[4,5,1])
```

## Similarity information (e.g. morphology)

If  $S$  is a matrix representing the similarity of each object in  $A$  (rows) to each object in  $B$  (columns)...

$$\min_{P \in \mathcal{P}} \|A - PBP^T\|_F^2 + \text{trace}(SP^T)$$

### graspologic

```
from graspologic.match import GraphMatch  
  
gm = GraphMatch()  
gm.fit(A, B, S=S)
```

# Edge types (multilayer graph matching)

$$\min_{P \in \mathcal{P}} \|A^{(1)} - PB^{(1)}P^T\|_F^2 + \|A^{(2)} - PB^{(2)}P^T\|_F^2 + \dots$$

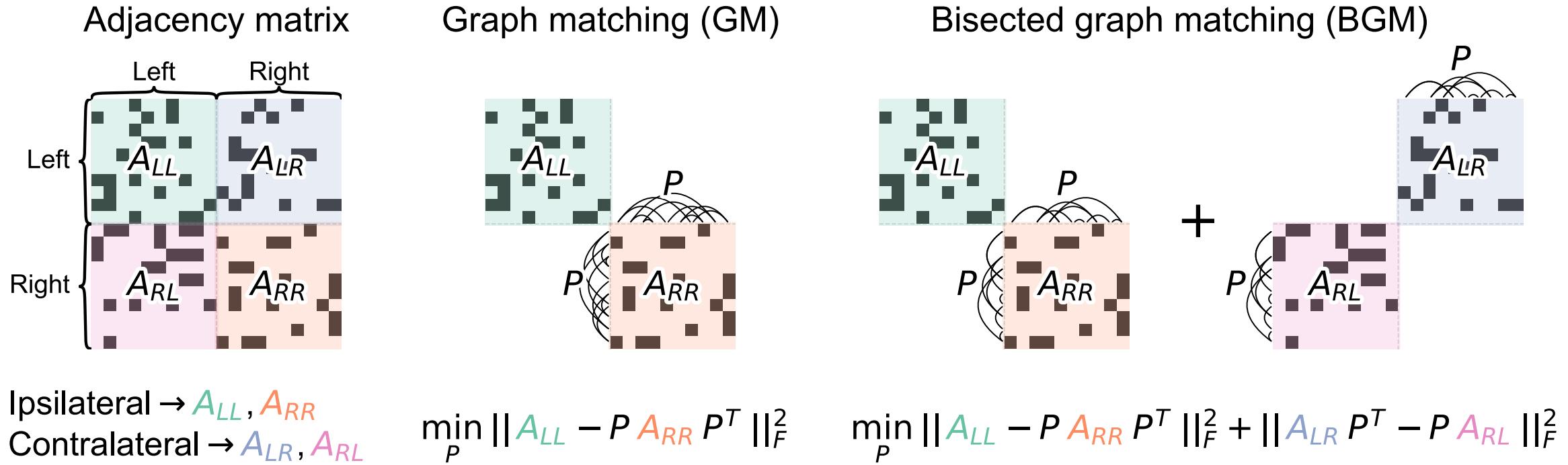
**graspologic** (in development)

```
from graspologic.match import graph_match  
graph_match([A1, A2], [B1, B2])
```

**Thus far, we've not used the contralateral connections**

**These are about 1/3 of the edges in the brain!**

# From graph matching to bisected graph matching

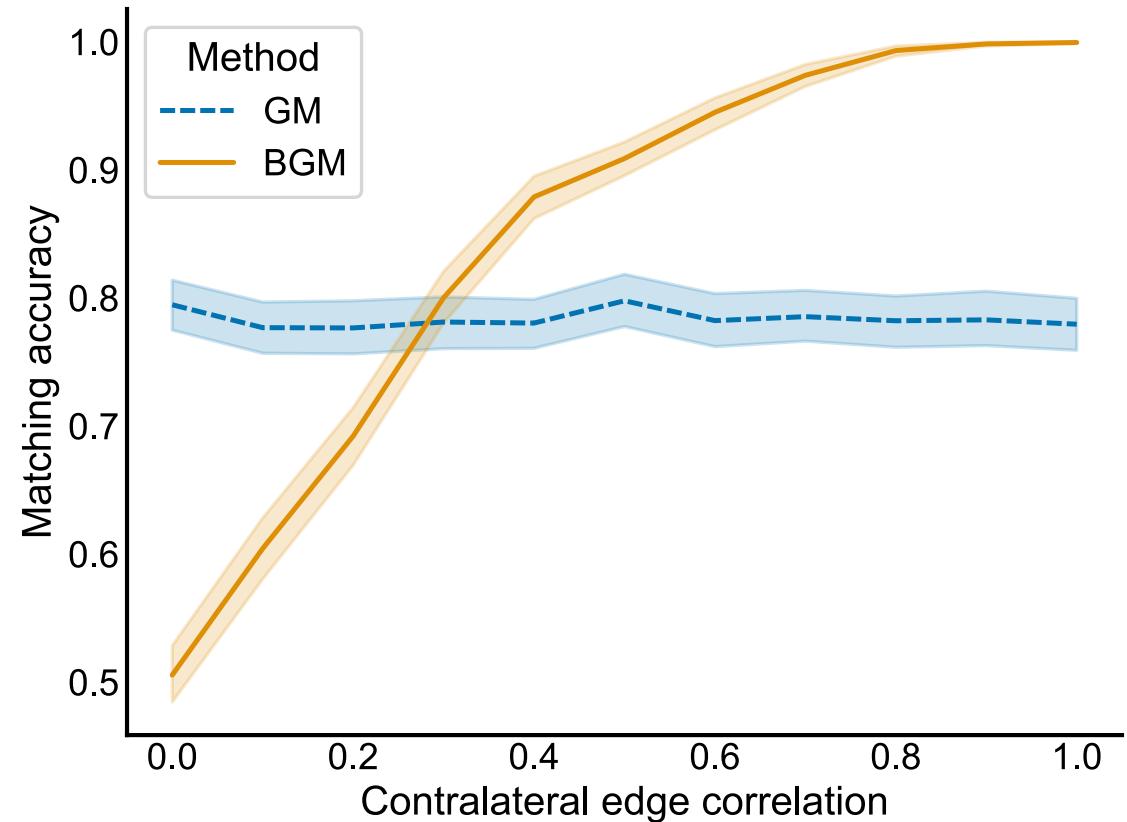


New gradient:

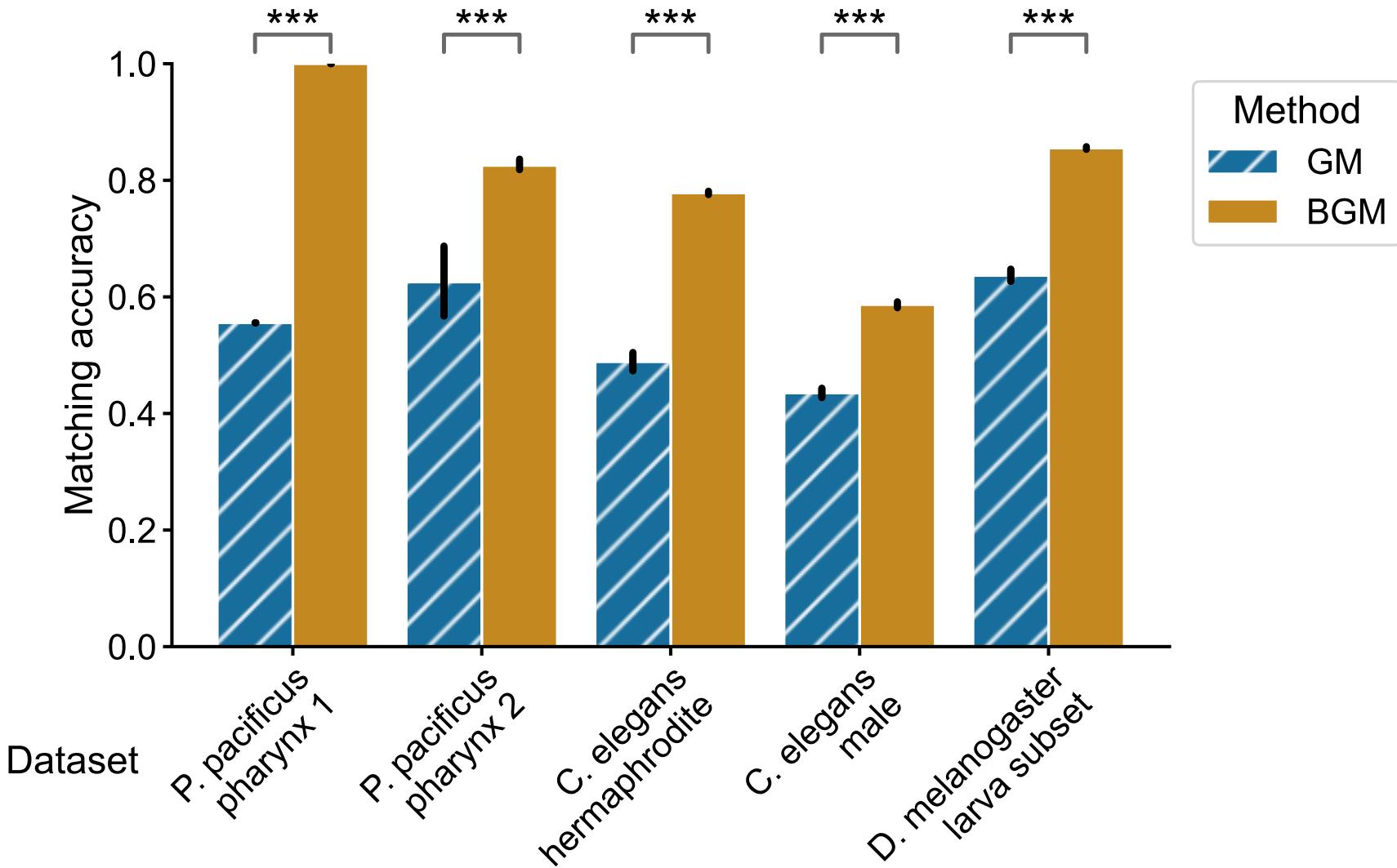
$$\nabla f(P) = -\mathbf{A}_{LL} P \mathbf{A}_{RR}^T + \mathbf{A}_{LL}^T P \mathbf{A}_{RR} + \mathbf{A}_{LR} P^T \mathbf{A}_{RL}^T + \mathbf{A}_{RL}^T P^T \mathbf{A}_{LR}$$

# Simulations

- Created simulated networks with two "sides"
  - Ipsilateral subgraphs had edge correlation of 0.8
  - Contralateral subgraphs had varying correlation (x-axis at right)
- Measured matching accuracy for graph matching (GM) and bisected graph matching (BGM)



# Connectome datasets



# Graph matching for modern connectomics (putting it all together)

$$\min_{P \in \mathcal{P}_s} \left[ \sum_{i=1}^K \|A_{LL}^{(i)} - PA_{RR}^{(i)} P^T\|_F^2 + \sum_{i=1}^K \|A_{LR}^{(i)} P^T - PA_{RL}^{(i)}\|_F^2 + \text{tr}(SP^T) \right]$$

where:

- $\mathcal{P}_s$  is the set of permutation matrices *which respect fixed seeds*,  $s$ , where the seeds give a known correspondence between some of the neurons.
- $A_{LL}$  and  $A_{RR}$  are the within-hemisphere (ipsilateral) subgraphs.
- $A_{LR}$  and  $A_{RL}$  are the between-hemisphere (contralateral) subgraphs.
- $S$  is a matrix of similarity scores between neurons, e.g. morphology (NBLAST).
- $K$  is the number of *layers* or edge types e.g. axo-axonic, axo-dendritic, etc.

# Practical considerations

- Networks not same size (see `padding`)
- Stopping tolerance for the algorithm
  - `max_iter` controls the maximum number of iterations (default is 30). If results are poor and computation is not taking too long, can try increasing
  - `tol` is a stopping tolerance, same considerations as the above
- Initialization
  - Controlled by `init` parameter
  - Default is the "barycenter": can think of as an uninformative initialization
- Number of restarts
  - Algorithm is typically not deterministic, even from the same initialization
  - `n_init` parameter controls number of restarts

# Putting it all together in the (new) code

## graspologic (in development)

```
from graspologic.match import graph_match

indices_A, indices_B, score, misc = graph_match(A_ll, # left-to-left
                                                A_rr, # right-to-right
                                                AB=A_lr, # left-to-right
                                                BA=A_rl, # right-to-left
                                                S=S, # similarity (e.g. morphology)
                                                partial_match=seeds # known alignment)
```

`indices_A` is a selection of the nodes of `A`

`indices_B` is a (at least partial) permutation of `B`

`score` is the final graph matching objective function

`misc` is a dictionary with extra details about the optimization path

## R code

<https://github.com/dpmcsuss/iGraphMatch/>

# More info

- Tutorial on graph matching  
[bdpedigo.github.io/networks-course/graph\\_matching.html](https://bdpedigo.github.io/networks-course/graph_matching.html)
- Code in graspologic  
[microsoft.github.io/graspologic/latest/reference/reference/match.html](https://microsoft.github.io/graspologic/latest/reference/reference/match.html)
- Code in SciPy  
[docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.quadratic\\_assignment.html](https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.quadratic_assignment.html)
- This repo: [github.com/neurodata/bgm](https://github.com/neurodata/bgm)
- Jupyter Book on BGM:  
[docs.neurodata.io/bgm/abstract.html](https://docs.neurodata.io/bgm/abstract.html)
- Manuscript on BGM:  
[biorxiv.org/content/10.1101/2022.05.19.492713](https://biorxiv.org/content/10.1101/2022.05.19.492713)
- New code [WIP]:  
[github.com/microsoft/graspologic/pull/960](https://github.com/microsoft/graspologic/pull/960)

# References / further reading

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# Acknowledgements

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# Questions?

Slides: <https://tinyurl.com/cambridge-matching>

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