

# The Big Data Newsvendor Problem in a Portfolio Optimization Context

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Summer 2015

## Abstract

Following [1], we provide a portfolio optimization method based on machine learning methods.

## 1 Introduction

This document considers a two-asset portfolio, of which one is the risk-free asset, yielding a constant return rate  $R_f$ , and the other being a risky asset  $s$ , typically a stock, yielding a random return rate  $r_{st}$  for each period  $t$ . We suppose that each risky asset  $s$  can be described daily by an *information vector*  $x_{st}$  containing potentially useful information, such as technical, fundamental or news-related information. Furthermore, we assume that the allocation of each asset of the portfolio  $p_{st}$  can be fully determined using a *decision vector*  $q$ . The allocation rule is the following:  $q^T x_{st}$  is allocated to the risky asset and  $1 - q^T x_{st}$  is allocated to the risk-free asset. Over the period  $t$ , the portfolio  $p_{st}$  consisting of asset  $s$  will therefore yield a return rate of:

$$p_{st}(q) = r_{st}q^T x_{st} + (1 - q^T x_{st})R_f. \quad (1)$$

The question we now wish to ask is how the decision vector  $q$  should be chosen. We assume we have access to a training dataset  $S_n$ , comprising of  $s$  different assets over  $t$  periods, such that  $n = s \times t$ .

## 2 Definitions and Bounds

### 2.1 Definitions Notation

Most of the following notation and definitions follow directly from [2].

Let  $S_n$  be a set of  $n$  vectors of  $\mathbf{R}^p \times \mathbf{R}$  of the form:

$$S_n = \{(x_1, r_1), \dots, (x_n, r_n)\}. \quad (2)$$

*Maybe considerations about the length of the period should be added? For example, it's not specified what's the period length of  $R_f$ .*

*What's the difference between having  $n$  points with having  $s \times t$  points? For example, what if  $s \gg t$  or the reverse?*

Each component of  $S_n$  is a tuple  $(x, r)$ , where  $x$  is the information vector and  $r$  is the observed return rate.

Using  $S_n$ , we wish to create a decision vector  $q_{S_n} \in \mathbf{R}^p$  from which we can make an investment decision when confronted with a random draw  $d = (x, r)$ .

**Loss and Cost.** We introduce the loss  $\ell$  and the cost  $c$  of using  $q$  with a random draw  $d = (x, r)$ :

$$\ell(q, d) = c(q(x), r) = c(q^T x, r) \quad (3)$$

Supposing an utility  $U$ , there are two different cost functions we can use. The first one, and perhaps the most obvious one is defined by

$$c(p, r) = -U(pr + (1 - p)R_f). \quad (4)$$

If the utility is piecewise linear, then the cost is not bounded, and so an infinite return would yield a negatively infinite cost. This means that risk-taking will be encouraged since risky position, ie.  $|p| > 1$  can in fact yield a negative cost. On the other hand, with an unbounded utility, the algorithm minimizing the cost over the sample must have  $\lambda > 0$ , otherwise the problem might be unconstrained.

We can also define a new cost function that is always non-negative:

$$c(p, r) = \begin{cases} \lfloor U(r) - U(pr + (1 - p)R_f) \rfloor & \text{if } r > R_f \\ \lfloor U(R_f) - U(pr + (1 - p)R_f) \rfloor & \text{if } r \leq R_f, \end{cases} \quad (5)$$

where by  $\lfloor \cdot \rfloor$  we mean  $\max(\cdot, 0)$ . In the case of exponential utility, this cost function is actually equivalent to the one previously defined, but is quite different in the case of an unbounded utility function, for example the piece-wise linear one. Such a cost does not encourage risky position, since the cost is at least 0. This means that if  $r > 0$ , then there's no point having  $p > 1$  instead of  $p = 1$  since both will yield a zero cost position.

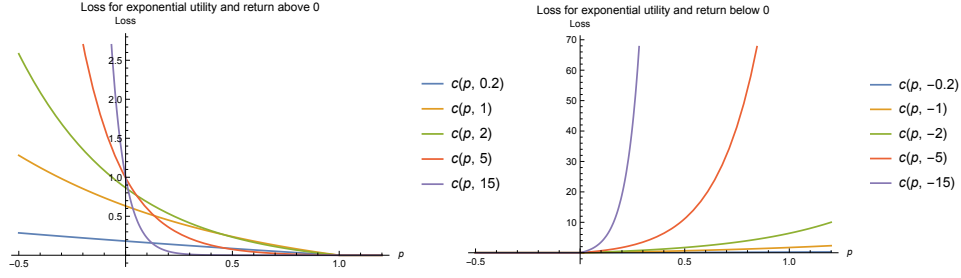
**Utility.** There are two ways we can model our utility, and both are concave shaped, to represent a risk-averse approach. The first utility is the linear utility of the form

$$U(r) = r + \min(0, \beta r), \quad (6)$$

with  $0 < \beta < 1$ . The other utility is exponential:

$$U(r) = -\exp(-\mu r), \quad (7)$$

with  $\mu > 0$ .



**Algorithm.** We will be concerned with probabilistic confidence bounds on results produced using the following algorithm, using dataset  $S_n$ .

$$q^* = \arg \min_{q \in \mathbf{R}^p} \frac{1}{n} \sum_{i=1}^n c(q^T x_i, r_i) + \lambda \|q\|_2^2. \quad (8)$$

**Assumptions.** We will assume that information vectors have been pre-processed and lie in a  $X_{\max}^2$  radius ball. We also assume that the return rates observed are comprised within  $[-\bar{r}, \bar{r}]$ . This last assumption will be relaxed.

**Definition.** An algorithm  $A$  has uniform stability  $\beta$  with respect to the loss function  $\ell$  if, for all  $S_n \in \mathbf{D}^n$  and  $i \in \{1, \dots, n\}$ , the following holds:

*Include reference for definitions and theorems*

$$\|\ell(A_{S_n}, \cdot) - \ell(A_{S_n^{\setminus i}}, \cdot)\|_{\infty} \leq \beta_n, \quad (9)$$

or, equivalently,

$$\sup_{d \in \mathbf{D}} |\ell(A_{S_n}, d) - \ell(A_{S_n^{\setminus i}}, d)| \leq \beta_n. \quad (10)$$

Here,  $S^{\setminus i}$  means the set  $S$  with the  $i$ th data point removed.

Furthermore,  $A$  is stable when  $\beta_n = O(1/n)$ .

**Definition.** A loss function  $\ell$  is  $\sigma$ -admissible if the associated cost function  $c$  is convex with respect to its first argument and the following condition holds for any  $p_1, p_2$  and  $r$ :

$$|c(p_1, r) - c(p_2, r)| \leq \sigma |p_1 - p_2| \quad (11)$$

**Remark.** Our loss function  $\ell$  is  $\sigma$ -admissible with  $\sigma = \bar{r} + R_f$  in the linear case and  $\sigma = (\bar{r} + R_f) \exp(\mu \bar{r})$  in the exponential case.

*Proof.* First, we remark that both forms of  $U$  yield a convex function of  $p$  with  $r$  fixed.

Now we'll suppose that  $c(p_1, r), c(p_2, r) > 0$ . Then the expression  $|c(p_1, r) - c(p_2, r)|$  reduces to

$$|U(p_1 r + (1 - p_1) R_f) - U(p_2 r + (1 - p_2) R_f)|. \quad (12)$$

Now because  $r \in [-\bar{r}, \bar{r}]$ ,  $U$  is Lipschitz continuous on its domain, and so (12) is bounded by

$$\alpha|p_1r + (1 - p_1)R_f - (p_2r + (1 - p_2)R_f)| = \alpha|p_1 - p_2||r - R_f| \quad (13)$$

where

$$\alpha = \sup_{r \in [-\bar{r}, \bar{r}]} |U'(r)|. \quad (14)$$

In the linear case, the derivative is piecewise constant, and is set to 1 on for returns below  $r_c$ , so that  $\alpha = 1$ . In the exponential case,  $U'(r) = \exp \mu r$ , and  $\alpha = \exp \mu \bar{r}$ .

The bound (13) must hold for any  $r$ . The expression  $|r - R_f|$  will reach its largest value at  $r = -\bar{r}$ , since  $R_f$  is assumed to be non-negative.

Finally we consider the case where, without loss of generality,  $c(p_2, r) = 0$ . Then, if  $c$  had not been defined using  $\lfloor \cdot \rfloor$ , then we would have

$$\begin{aligned} |\lfloor c(p_1, r) \rfloor - \lfloor c(p_2, r) \rfloor| &\leq |c(p_1, r) - c(p_2, r)| \\ &\leq \sigma|p_1 - p_2|. \end{aligned} \quad (15) \quad \square$$

**Theorem 1.** *Let  $F$  be a reproducing kernel Hilbert space with kernel  $\kappa$  that  $\forall x \in X$ ,  $\kappa(x, x) \leq \kappa^2 < \infty$ . If  $\ell$  is  $\sigma$ -admissible with respect to  $F$ , then the learning algorithm defined by*

$$A_S = \arg \min_{g \in F} \frac{1}{n} \sum_{i=1}^n \ell(g, d_i) + \lambda \|g\|_k^2 \quad (16)$$

*has uniform stability  $\alpha_n$  with respect to  $\ell$  with*

$$\alpha_n \leq \frac{\sigma^2 \kappa^2}{2\lambda n}. \quad (17)$$

**Remark.** Our proposed algorithm has the form (16), and so has algorithmic stability bounded by

$$\alpha_n \leq \frac{(\bar{r} + R_f)^2 X_{\max}^2}{2\lambda n} \quad (18)$$

with linear utility and

$$\alpha_n \leq \frac{\exp(2\mu \bar{r}) X_{\max}^2}{2\lambda n} \quad (19)$$

in the case of exponential utility.

**Definition.** The *true risk* with respect to algorithm  $A$  and set  $S_n$  is defined as

$$R_{\text{true}}(A, S_n) = E_d[\ell(A_{S_n}, d)], \quad (20)$$

which is, in plain words, the expected loss incurred when applying the algorithm created from training set  $S_n$  in the wild, ie. out of sample.

**Definition.** The *empirical risk* with respect to algorithm  $A$  and set  $S_n$  is defined as

$$\hat{R}(A, S_n) = \frac{1}{n} \sum_{i=1}^n \ell(A_{S_n}, d_i), \quad (21)$$

which is, in plain words, the average cost incurred by our model over all the training set.

**Remark.** The maximum loss we can suffer over a single data point happens when  $r_i = -\bar{r}$  and  $p = 1$ , ie.

$$c(1, -\bar{r}) = U(R_f) - U(\bar{r}). \quad (22)$$

We shall call this quantity  $\gamma$ .

**Theorem 2.** *Let  $A$  be an algorithm with uniform stability  $\alpha_n$  with respect to a loss function  $\ell$  such that  $0 \leq \ell(A_{S_n}, d) \leq M$  for all  $d = (x, r) \sim D$  and all sets  $S_n$  of size  $n$ . Then for any  $n \geq 1$  and any  $\delta \in (0, 1)$ , the following bound holds with probability at least  $1 - \delta$  over the random draw of the sample  $S_n$ :*

$$|R_{\text{true}}(A, S_n) - \hat{R}(A, S_n)| \leq 2\alpha_n + (4n\alpha_n + M) \sqrt{\frac{\log(2/\delta)}{2n}}. \quad (23)$$

**Remark.** Our algorithm has a generalization bound of

$$|R_{\text{true}}(A, S_n) - \hat{R}(A, S_n)| \leq 2\alpha_n + (4n\alpha_n + \gamma) \sqrt{\frac{\log(2/\delta)}{2n}}. \quad (24)$$

### 3 Multi-asset Portfolio

We now consider a multi-asset portfolio, made of  $m$  risky assets and a risk-free asset. Each of these risky assets have information vector  $x_{st}$ , where  $s \in \{1, \dots, m\}$  and  $t \in \{1, \dots, n\}$ . Therefore, we suppose having access to a sample dataset  $S_n = \{X_1, \dots, X_n\}$  where  $X_i \in \mathbf{R}^{m \times p}$ .

There are two ways to approach the multi-asset scenario.

#### 3.1 First approach

On one hand, we can define a single decision vector  $q$  such that the return on a single holding period would be defined by

$$r^T X q + (1 - 1^T X q) R_f, \quad (25)$$

where  $r \in \mathbf{R}^m$  is the return vector of the  $m$  stocks and  $1^T$  is a constant vector of ones in  $\mathbf{R}^m$  whose role is simply to sum over the components of  $Xq$ , ie. sum over the risky allocations of the portfolio. However, this method has the disadvantage that the decision vector is

applied for each stock in the same way, and so all their information vectors are ultimately ‘averaged’ over. Therefore, if some asset is more sensitive to an information component than other assets, this knowledge might be lost if using this method.

Nevertheless, we can again define two different cost functions, which our optimal decision vector would minimize when applied to our sample dataset  $S_n$ . Again, the cost can be simply defined as

$$c(p, r) = -U(r^T p + (1 - 1^T p)R_f). \quad (26)$$

As discussed previously, this definition might have the disadvantage of providing an unconstrained optimization problem when determining the optimal decision vector.

We are not limited to a single cost definition, and in fact, the multi-asset scenario provides many legitimate cost definitions. Here the most ‘aggressive’ one:

$$c(p, r) = \begin{cases} [U(\max r) - U(r^T p + (1 - 1^T p)R_f)] & \text{if } \max r > R_f \\ [U(R_f) - U(r^T p + (1 - 1^T p)R_f)] & \text{if } \max r \leq R_f. \end{cases} \quad (27)$$

This means that we expect (or conversely, that we suffer a lesser loss) when the total return of our portfolio is near the best return achieved by our considered stocks. Instead of  $\max r$ , we also could have chosen the average returns  $\bar{r}$  during the period as the target that we wish to achieve (provided that the average is superior to the risk-free return).

In any case, the algorithm determining the decision vector  $q$  would once be defined as previously, ie. as (8).

## References

- [1] Cynthia Rudin and Gah-Yi Vahn. *The Big Data Newsvendor: Practical Insights from Machine Learning*, Operations Research, 2015.
- [2] Olivier Bousquet and André Elisseeff. *Stability and Generalization*, Journal of Machine Learning Research, 2002.
- [3] Rockafellar, R. T. *Convex Analysis*, Princeton University Press, 1970.