Equations for a deformable, rotating Earth with fluid core

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The variations in rotation of the Earth are governed by the angular momentum conservation law. In a frame attached to the rotating Earth,

$$\frac{\mathrm{d}\vec{H}}{\mathrm{d}t} + \vec{\omega} \times \vec{H} = \vec{\Gamma}, \tag{1}$$

where $\vec{H}$ is the angular momentum of the Earth, $\vec{\omega}$ is the instantaneous angular velocity vector, and $\vec{\Gamma}$ the external torque.

$$\vec{H} = \underline{I}\vec{\omega}, \tag{2}$$

where $\underline{I}$ is the inertia tensor. We use a reference frame with its axes oriented along the Earth's mean axes of inertia. Then the inertia tensor is

$$\underline{I} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \tag{3}$$

where the c_{ij} represent the contributions to the elements of the inertia tensor from the deformations of the Earth due to various causes (e.g., tidal forces, surface loading, centrifugal forces due to Earth rotation variations). Following standard notation, we write the components of the angular velocity vector as $\omega_1 = \Omega m_1$, $\omega_2 = \Omega m_2$, $\omega_3 = \Omega(1 + m_3)$ where Ω is the mean rotation rate of the Earth.

Now consider the fluid core that is rotating inside the mantle with a speed $\vec{\omega}_\mathrm{f}$. We suppose that there are no radial or transversal flows, following the Poincaré hypothesis. Its motion will be governed by an equation similar to Eq. (1):

$$\frac{\mathrm{d}\vec{H}_\mathrm{f}}{\mathrm{d}t} + \vec{\omega} \times \vec{H}_\mathrm{f} = \vec{\Gamma}_\mathrm{f} + \vec{K}, \tag{4}$$

wherein $\vec{\Gamma}_\mathrm{f}$ is the torque due to the gravitational attraction of the celestial bodies and $\vec{K}$ is the core-mantle coupling torque. (If one writes the equations for the rotation of the mantle only, the latter would act with its sign reversed.) Note that the second term of the LHS involves $\vec{\omega}$ which the rotation speed of the rotating frame (and not $\vec{\omega}_\mathrm{f}$!). The core is rotating inside the mantle so that the former shows a small relative angular momentum $\vec{h}_\mathrm{f}$ with respect to the latter. The whole ensemble is rotating at $\vec{\omega}$, so that the angular momentum of the core only is

$$\vec{H}_\mathrm{f} = \underline{I}_\mathrm{f}\vec{\omega} + \vec{h}_\mathrm{f}, \tag{5}$$

with

$$\underline{I}_\mathrm{f} = \begin{pmatrix} A_\mathrm{f} & 0 & 0 \\ 0 & A_\mathrm{f} & 0 \\ 0 & 0 & C_\mathrm{f} \end{pmatrix} + \begin{pmatrix} c_{11,\mathrm{f}} & c_{12,\mathrm{f}} & c_{13,\mathrm{f}} \\ c_{21,\mathrm{f}} & c_{22,\mathrm{f}} & c_{23,\mathrm{f}} \\ c_{31,\mathrm{f}} & c_{32,\mathrm{f}} & c_{33,\mathrm{f}} \end{pmatrix}, \tag{6}$$

and with the angular momentum of the relative core flow, at the first order

$$\vec{h}_\mathrm{f} = \begin{pmatrix} A_\mathrm{f} & 0 & 0 \\ 0 & A_\mathrm{f} & 0 \\ 0 & 0 & C_\mathrm{f} \end{pmatrix} \vec{\omega}_\mathrm{f}, \tag{7}$$

with $\omega_{1,\mathrm{f}} = \Omega m_{1,\mathrm{f}}$, $\omega_{2,\mathrm{f}} = \Omega m_{2,\mathrm{f}}$, $\omega_{3,\mathrm{f}} = \Omega m_{3,\mathrm{f}}$

Putting together Eqs. (1) and (4), one gets the Eqs. (7) and (8) of Sasao et al. (1977):

$$\begin{aligned}\dot{\tilde{m}} - ie\Omega\tilde{m} + \frac{\dot{\tilde{c}} + i\Omega\tilde{c}}{A} + \frac{A_\mathrm{f}}{A}(\dot{\tilde{m}}_\mathrm{f} + i\Omega\tilde{m}_\mathrm{f}) &= \frac{\tilde{\Gamma}}{A\Omega}, &(8)\\ \dot{\tilde{m}} - ie_\mathrm{f}\Omega\tilde{m} + \frac{\dot{\tilde{c}}_\mathrm{f} + i\Omega\tilde{c}_\mathrm{f}}{A_\mathrm{f}} + \dot{\tilde{m}}_\mathrm{f} + i\Omega\tilde{m}_\mathrm{f} &= \frac{\tilde{\Gamma}_\mathrm{f} + \tilde{K}}{A_\mathrm{f}\Omega}, &(9)\end{aligned}$$

where the equatorial component of the relative core flow angular momentum has been replaced by its first order expression $\tilde{h}_\mathrm{f} = A_\mathrm{f}\Omega\tilde{m}_\mathrm{f}$.

The external torque derives from the tesseral part (i.e., degree 2, order 1) of the tide generating potential of the form $\Psi_\mathrm{e} = \phi_1 xz + \phi_2 yz$. By integrating over the ellipsoid, one can show that

$$\tilde{\Gamma} = -iAe\Omega^2\tilde{\phi}, \tag{10}$$

where $\tilde{\phi} = (\phi_1 + i\phi_2)/\Omega^2$.

The same torque arises from the direct action of the tide generating potential on the core masses:

$$\tilde{\Gamma}_\mathrm{f} = -iA_\mathrm{f}e_\mathrm{f}\Omega^2\tilde{\phi}. \tag{11}$$

We also assume in the following that the core is homogeneous (constant density ρ_f), so that isodensity surfaces are equal to equipotential surfaces. The particles of fluid move because of pressure changes in the core, and are also affected by the external potential that includes the gravitational potential of the mantle masses (Ψ_m), of the celestial bodies external to the Earth (Ψ_e), and the centrifugal forces (Ψ_c) due to the rotation of the whole Earth. The Navier-Stokes equation that governs the motion of a fluid particle within the core is therefore written

$$\rho_\mathrm{f}\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\vec{\nabla}p + 2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t} + \rho_\mathrm{f}\vec{\nabla}(\Psi_\mathrm{c} + \Psi_\mathrm{m} + \Psi_\mathrm{e}), \tag{12}$$

but immediately reduces to

$$p = \rho_\mathrm{f}(\Pi + \Psi_\mathrm{m} + \Psi_\mathrm{c} + \Psi_\mathrm{e}) \tag{13}$$

if one notices that, for a solid rotation of the core, $\mathrm{d}^2\vec{r}/\mathrm{d}t^2 = -\vec{\nabla}\Pi$ where Π is the potential of inertia. This potential is basically the difference between the 'pressure potential' and the full external potential made of the gravitational attractions from masses outside the core plus the centrifugal potential. In absence of celestial body excitation, it expresses the hydrostatic equilibrium: pressure forces are exactly balanced by the gravito-elastic forces.

Now, we want to express the torque acting on the core due to the interaction with the mantle at the core-mantle boundary (CMB). This interaction is mainly composed by a pressure exerted on the core's fluid particles (deriving from a 'pressure potential' p) and by a gravitational attraction of the mantle's masses (potential Ψ_m). These two contributions are in opposite sense. The torques arising from the Lorentz force and small-scale topography at the CMB are not considered in this work. The core-mantle coupling torque is

$$\vec{K} = -\int \vec{r}\times\vec{\nabla}(p - \rho_\mathrm{f}\Psi_\mathrm{m})\mathrm{d}V = -\rho_\mathrm{f}\int \vec{r}\times\vec{\nabla}(\Pi + \Psi_\mathrm{c} + \Psi_\mathrm{e})\mathrm{d}V, \tag{14}$$

the second part taking advantage of the expression of the Navier-Stokes equation.

For the equatorial component of $\vec{K}$ involving the potential Ψ_e caused by the external bodies, one has readily

$$\tilde{K}_\mathrm{e} = -\tilde{\Gamma}_\mathrm{f}. \tag{15}$$

Before going further, we introduce spherical harmonic decompositions of some quantities. Excluding the (2,2) part since we consider no triaxiality, the ellipsoidal departure to the sphere as defined by the vector $\vec{r}$ can be written

$$r = r_{20}Y_{20} + r_{21}^R Y_{21}^R + r_{21}^I Y_{21}^I. \tag{16}$$

Then, if we do the same for Ψ_c and for the inertial potential Π, we have

$$\begin{aligned}\Psi_\mathrm{c} &= \psi_{\mathrm{c},20}Y_{20} + \psi_{\mathrm{c},21}^R Y_{21}^R + \psi_{\mathrm{c},21}^I Y_{21}^I, &(17)\\ \Pi &= \pi_{20}Y_{20} + \pi_{21}^R Y_{21}^R + \pi_{21}^I Y_{21}^I, &(18)\end{aligned}$$

and it is possible to show that the equatorial part of the inertial plus centrifugal torque is

$$\tilde{K}_{\mathrm{i+c}} = -ir_{20}(\tilde{\pi} + \tilde{\psi}_{\mathrm{c}}) + i\tilde{r}(\pi_{20} + \psi_{20}), \tag{19}$$

where $\tilde{r} = r_{21}^R + ir_{21}^I$, $\tilde{\pi} = \pi_{21}^R + i\pi_{21}^I$, and $\tilde{\psi}_{\mathrm{c}} = \psi_{\mathrm{c},21}^R + i\psi_{\mathrm{c},21}^I$.

The rotational potential is

$$\Psi_{\mathrm{c}} = \frac{1}{2}(\vec{\omega} \times \vec{r})^2. \tag{20}$$

and results in a centrifugal force due to the rotation of the whole Earth applied on the fluid particles of the core. One can show that the (2,0) and (2,1) parts of this expression are

$$\begin{aligned} \phi_{\mathrm{c},20} &= -\frac{1}{3}\Omega^2 r^2(1 + 2m_3), &(21)\\ \tilde{\phi}_{\mathrm{c}} &= -\frac{1}{3}\Omega^2 r^2 \tilde{m}. &(22) \end{aligned}$$

The derivation of the inertial potential is a bit more complicated. One can make use of the equations of the fluid motion in an inertial frame (in which the Earth rotates with velocity $\vec{\omega}$):

$$-\vec{\nabla}\Pi = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \frac{\partial^2\vec{r}}{\partial t^2} + \frac{\partial\vec{\omega}}{\partial t} \times \vec{r} + 2\vec{\omega} \times \frac{\partial\vec{r}}{\partial t} + \vec{\omega} \times (\vec{\omega} \times \vec{r}). \tag{23}$$

In our simplified model, the solid rotation of the core implies that $\partial\vec{r}/\partial t = \vec{\omega}_{\mathrm{f}} \times \vec{r}$. After some algebra and keeping the (2,0) and (2,1) contribution only, one obtains

$$\begin{aligned} \pi_{20} &= \frac{2}{3}\Omega^2 r^2 \tilde{m}_{3,\mathrm{f}}, &(24)\\ \tilde{\pi} &= \frac{1}{3}\Omega^2 r^2 \tilde{m}_{\mathrm{f}}. &(25) \end{aligned}$$

Putting everything together, it turns out that the total coupling torque acting the fluid core is (Eq. (20) of Sasao et al. 1977)

$$\tilde{K} = -iA_{\mathrm{f}}e_{\mathrm{f}}\Omega^2(\tilde{m} + \tilde{m}_{\mathrm{f}}) - \tilde{\Gamma}_{\mathrm{f}} + i\Omega^2\tilde{c}_{\mathrm{f}}. \tag{26}$$

Taking back Eqs. (8) and (9), one gets

$$\begin{aligned} \dot{\tilde{m}} - ie\Omega\tilde{m} + \frac{\dot{\tilde{c}} + i\Omega\tilde{c}}{A} + \frac{A_{\mathrm{f}}}{A}(\dot{\tilde{m}}_{\mathrm{f}} + i\Omega\tilde{m}_{\mathrm{f}}) &= -ie\Omega\tilde{\phi}, &(27)\\ \dot{\tilde{m}} + \dot{\tilde{m}}_{\mathrm{f}} + i(1 + e_{\mathrm{f}})\Omega\tilde{m}_{\mathrm{f}} + \frac{\dot{\tilde{c}}_{\mathrm{f}}}{A_{\mathrm{f}}} &= 0. &(28) \end{aligned}$$

The MacCullagh formula applied to the various potential expressed above leads to the Sasao et al. (1980)'s solutions for the elastic mass redistribution at the surface and at the CMB:

$$\begin{aligned} \tilde{c} &= -A[\kappa(\tilde{\phi} - \tilde{m}) - \xi\tilde{m}_{\mathrm{f}}], &(29)\\ \tilde{c}_{\mathrm{f}} &= -A_{\mathrm{f}}[\gamma(\tilde{\phi} - \tilde{m}) - \beta\tilde{m}_{\mathrm{f}}]. &(30) \end{aligned}$$

These expressions relate the tesseral deformation of the Earth and of the core to the external and centrifugal potentials, and to the inertial potential due to the core wobble (basically producing changes of pressure at the CMB). The proportionality coefficients are called 'compliances' in the MHB work and can be expressed in terms of the classical Love numbers for κ and γ (see formulae (55) of Sasao et al. 1980 or Folgueira et al. 2007). One has $\kappa = ek_2/k_{\mathrm{s}}$ where k_2 expresses the frequency-dependent deformation at the surface of the Earth due to the action of a tide generating potential on the whole Earth, including the core, and $\gamma = \Omega^2 a^3 h_{\mathrm{f}}/2GM$ where h_{f} expresses the deformation at the CMB. β and ξ express the deformation at the surface and at the CMB, respectively, when the pressure or the potential are applied at the CMB. In the MHB work, these 'elastic' solution are extended to account for anelasticity and dissipative processes within the mantle, the core and in ocean tides.

The equations which emerge after the substitutions are

$$\begin{aligned} (1 + \kappa)\dot{\tilde{m}} - i\Omega(e - \kappa)\tilde{m} + (\xi + A_{\mathrm{f}}/A)\dot{\tilde{m}}_{\mathrm{f}} + i\Omega(\xi + A_{\mathrm{f}}/A)\tilde{m}_{\mathrm{f}} &= \kappa\dot{\tilde{\phi}} - i\Omega(e - \kappa)\tilde{\phi}, &(31)\\ (1 + \gamma)\dot{\tilde{m}} + (1 + \beta)\dot{\tilde{m}}_{\mathrm{f}} + i\Omega(1 + e_{\mathrm{f}})\tilde{m}_{\mathrm{f}} &= \gamma\dot{\tilde{\phi}}. &(32) \end{aligned}$$

This system can be seen at the starting point for our purpose. It is the equivalent of Eq. (52) of Capitaine et al. (2006) with a fluid core, and ignoring the axial part (that I'll treat in a separate note if necessary).

Note that the solution of (27)–(28) in the frequency domain leads to expressing the wobble of the Earth $\tilde{m}$ as a function of the tesseral potential $\tilde{\phi}$ through a frequency-dependent proportionality coefficient, or transfer function (the MHB work divides this proportionality coefficient by the equivalent coefficient relevant to the rigid Earth in order to avoid the calculation of the coefficients of the tide generating potential, but this calculation is however indirectly contained in the chosen rigid Earth theory). The system of equations is rewritten:

$$
\begin{aligned}
(\sigma + \kappa\sigma' - e\Omega)\tilde{m} + (\xi + A_\mathrm{f}/A)\sigma'\tilde{m}_\mathrm{f} &= (\kappa\sigma' - e\Omega)\tilde{\phi}, && (33)\\
(1+\gamma)\sigma\tilde{m} + (\sigma' + \beta\sigma + e_\mathrm{f}\Omega)\tilde{m}_\mathrm{f} &= \gamma\sigma\tilde{\phi}, && (34)
\end{aligned}
$$

where $\sigma' = \sigma + \Omega$ is the frequency in the space-fixed reference frame. A solution without core can be found by dropping γ, β and ξ in the above expressions (in such a case, the second equation is obviously irrelevant). Note that there exists a reciprocity relation $A\xi = A_\mathrm{f}\gamma$, as stated in Sasao et al. (1980). The preceding system also reads:

$$
\begin{aligned}
\left(\sigma - \frac{e-\kappa}{1+\kappa}\Omega\right)\tilde{m} + \left(\xi + \frac{A_\mathrm{f}}{A}\right)\frac{\sigma+\Omega}{1+\kappa}\tilde{m}_\mathrm{f} &= \left(\frac{\kappa}{1+\kappa}\sigma - \frac{e-\kappa}{1+\kappa}\Omega\right)\tilde{\phi}, && (35)\\
\frac{1+\gamma}{1+\beta}\sigma\tilde{m} + \left(\sigma + \frac{1+e_\mathrm{f}}{1+\beta}\Omega\right)\tilde{m}_\mathrm{f} &= \frac{\gamma}{1+\beta}\sigma\tilde{\phi}. && (36)
\end{aligned}
$$

If one designates $\sigma_1 \simeq \Omega(e-\kappa)$ and $\sigma_2 \simeq -\Omega[1+(e_\mathrm{f}-\beta)]$, the system becomes, at a suitable approximation level

$$
\begin{aligned}
(\sigma - \sigma_1)\tilde{m} + \frac{A_\mathrm{f}}{A}(\sigma+\Omega)(1-\kappa)\tilde{m}_\mathrm{f} &= (\kappa\sigma - \sigma_1)\tilde{\phi}, && (37)\\
(1+\gamma)(1-\beta)\sigma\tilde{m} + (\sigma - \sigma_2)\tilde{m}_\mathrm{f} &= \gamma\sigma\tilde{\phi}, && (38)
\end{aligned}
$$

where it is worth noting that σ_2 is close to $-\Omega$ and can therefore be written as $-\Omega + \sigma_2'$. The system admits two eigenvalues that are roots of the polynomial:

$$
(\sigma - \sigma_1)(\sigma - \sigma_2) - \frac{A_\mathrm{f}}{A}(1 - \kappa + \gamma - \beta)(\sigma + \Omega)\sigma = 0. \qquad (39)
$$

One can show that the polynomial can be written as the product $(\sigma - \sigma_\mathrm{CW})(\sigma - \sigma_\mathrm{NDFW})$ where

$$
\begin{aligned}
\sigma_\mathrm{CW} &= \Omega\frac{A}{A_\mathrm{m}}(e - \kappa), && (40)\\
\sigma_\mathrm{NDFW} &= -\Omega - \Omega\frac{A}{A_\mathrm{m}}(e_\mathrm{f} - \beta), && (41)
\end{aligned}
$$

are the frequencies of the Chandler wobble and of the nearly diurnal free wobble associated with the free rotational mode of the liquid core, respectively.

Back to Eqs. (33)–(34), the solutions for the wobbles of the Earth and of the core are

$$
\begin{aligned}
\tilde{m} &= ((\kappa\sigma' - e\Omega)(\sigma' + \beta\sigma + e_\mathrm{f}\Omega) - \gamma(\xi + A_\mathrm{f}/A)\sigma\sigma')\tilde{\phi}/\Delta, && (42)\\
\tilde{m}_\mathrm{f} &= ((\kappa\sigma' - e\Omega)(1+\gamma)\sigma - \gamma(\sigma + \kappa\sigma' - e\Omega)\sigma)\tilde{\phi}/\Delta, && (43)\\
\Delta &= (\sigma + \kappa\sigma' - e\Omega)(\sigma' + \beta\sigma + e_\mathrm{f}\Omega) - (1+\gamma)(\xi + A_\mathrm{f}/A)\sigma\sigma'. && (44)
\end{aligned}
$$

Using the factorization of the determinant in terms of eigenfrequencies, one can set the wobble solutions in the form:

$$
\begin{aligned}
\tilde{m}(\sigma) &= \tilde{T}(\sigma)\tilde{\phi}(\sigma), && (45)\\
\tilde{m}_\mathrm{f}(\sigma) &= \tilde{T}_\mathrm{f}(\sigma)\tilde{\phi}(\sigma), && (46)
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{T}(\sigma) &= \kappa - \frac{A_\mathrm{f}}{A}\gamma - \frac{A_\mathrm{m}}{A}\frac{\sigma_\mathrm{CW}}{\sigma - \sigma_\mathrm{CW}} + \frac{A_\mathrm{f}}{A}(e-\gamma)\frac{\sigma'_\mathrm{NDFW}}{\sigma - \sigma_\mathrm{NDFW}}, && (47)\\
\tilde{T}_\mathrm{f}(\sigma) &= \kappa - \gamma - \frac{A_\mathrm{m}}{A}\frac{\sigma^2_\mathrm{CW}/\Omega}{\sigma - \sigma_\mathrm{CW}} - (e-\gamma)\frac{\Omega}{\sigma - \sigma_\mathrm{NDFW}}. && (48)
\end{aligned}
$$

For a rigid Earth, which flattening is e_{R}, the equation (33) leads to the rigid Earth solution

$$\tilde{m}_{\mathrm{R}}(\sigma) = -\frac{e_{\mathrm{R}}\Omega}{\sigma - e_{\mathrm{R}}\Omega}\tilde{\phi}(\sigma). \tag{49}$$

By diving $\tilde{m}$ by its rigid counterpart, one gets an elegant form for expressing the non-rigid Earth wobble in terms of the results of a rigid Earth theory $\tilde{m}_{\mathrm{R}}$ (like, e.g., REN 2000):

$$\tilde{m}(\sigma) = -\tilde{T}(\sigma)\frac{\sigma - e_{\mathrm{R}}\Omega}{e_{\mathrm{R}}\Omega}\tilde{m}_{\mathrm{R}}(\sigma). \tag{50}$$

The amplitude of the nutation associated with the wobble $\tilde{m}(\sigma)$ is given by

$$\tilde{\eta}(\sigma) = \Delta\epsilon(\sigma) - i\Delta\psi(\sigma)\sin\epsilon_0 = -\frac{\tilde{m}(\sigma)}{\sigma'}, \tag{51}$$

and it is finally worth noting that $\tilde{\eta}/\tilde{\eta}_{\mathrm{R}} = \tilde{m}/\tilde{m}_{\mathrm{R}}$ so that the frequency-dependent coefficient between the rigid and the non-rigid motion quantities is conserved in both parameterizations.

The solution for an elastic Earth is obtained by using Eq. (33) of the Notice in which the term relevant to the core is dropped. It gives:

$$\tilde{m}(\sigma) = \frac{\kappa\sigma' - e\Omega}{\sigma + \kappa\sigma' - e\Omega}\tilde{\phi}(\sigma). \tag{52}$$

Dividing this equation by the rigid Earth solution (Eq. (49) of the Notice) one gets

$$\tilde{m}(\sigma) = -\frac{(\kappa\sigma' - e\Omega)(\sigma - e_{\mathrm{R}}\Omega)}{(\sigma + \kappa\sigma' - e\Omega)e_{\mathrm{R}}\Omega}\tilde{m}_{\mathrm{R}}(\sigma). \tag{53}$$

The rigid Earth flattening in the REN 2000 theory is $e_{\mathrm{R}} = 0.0032845075$. The relation (3) is directly applicable to the sine and cosine terms of the $\Delta\psi$ and $\Delta\epsilon$ quantities.