Linear 1st Order Equation

Normal (Standard) form

$$\begin{cases} y' + a(t)\,y = f(t) \\ y(t_0) = y_0, \text{ IVP} \end{cases}, \quad a(t), f(t) \text{ given}$$

Solution Via Integrating Factor

$$A' = a(t) \quad \text{i.e. } A(t) = \int^t a(s)\,ds$$

$$y(t) = \underbrace{C e^{-A(t)}}_{y(t)\text{ hom.}} + \underbrace{e^{-A(t)} \int^t e^{A(t)} f(s)\,ds}_{y(t)\text{ particular}} \quad \text{Formula}$$

Note: $y(t)$ hom. solves $y' + a(t)\,y = 0$

$y(t)$ particular solves $y' + a(t)\,y = f(t) \Rightarrow$ forcing

Feature of Linear Equations

For IVP Solve for C Equation:

$$y_0 = C e^{-A(t_0)} + e^{-A(t_0)} \int^{t_0} e^{A(s)} f s\,ds$$

Example: $y' + ay = b$, a, b are constants

$y(t_0) = y_0$ IVP $\quad A(t) = \int^t a - at$

Integrating factor is e^{at}, $\int^t e^{as}\, b\,ds$

$= \frac{b}{a} e^{at} + \text{constant}, \quad A' = a$

General Solution: $y(t) = C e^{-at} + e^{-at} \cdot \frac{b}{a} \cdot e^{at}$

$= C e^{-at} + \frac{b}{a}$, IVP, $y(t_0) = y_0$, Solve using given t_0 / y_0

$C = \left(y_0 - \frac{b}{a}\right) e^{at_0}, \quad y(t) = \left(y_0 - \frac{b}{a}\right) e^{-a(t-t_0)} + \frac{b}{a}$

If $a > 0$ then $\lim_{t \to \infty} y(t) = \frac{b}{a}$

Fact: $y = \frac{b}{a}$ is a solution of ODE which does not depend on time, called <u>static</u> or <u>stationary</u>.

Example: $(1+t^2) y' + 4ty = \frac{1}{(1+t^2)^2}$ Not in normal form

$y' + \frac{4t}{1+t^2} \cdot y = \frac{1}{(1+t^2)^2}$ ← Normal form,

$a(t) = \frac{4t}{1+t^2}, \quad f(t) = \frac{1}{(1+t^2)^2}, \quad A' = \frac{4t}{1+t^2}$

$A(t) = \int^t \frac{4t}{1+t^2}\, ds, \quad z = s^2, \quad \frac{1}{2} \int^{t^2} \frac{dz}{1+z}$

$= 2 \ln (1+t^2)$,

$$e^{A(t)} = (1+t^2)^2, \int^t e^{A(s)} f(s)\,ds$$

$$= \int^t \frac{(1+s^2)^2}{(1+s^2)^3}\,ds = \operatorname{atan}(t)$$

$$y(t) = \frac{C}{(1+t^2)^2} + \frac{\operatorname{atan}(t)}{(1+t^2)^2} \rightarrow$$ General Solution

IVP, $y(0)=5$

$$y(0) = \frac{C}{(1+0^2)^2} + \frac{\operatorname{atan}(0)}{(1+0^2)^2}, \quad C=5$$

<u>Existence and Uniqueness Theorem:</u>

↳ For $y' + a(t)y = f(t)$, $y(t_0) = y_0$

Assume $a(t)$, $f(t)$ are continuous in (t_1, t_2) and $t_0 \in (t_L, t_R)$. Then $y(t)$ exists and it is unique in the interval (t_1, t_2).

<u>Example:</u> $y' + \frac{\cos(t)}{\sin(t)} y = \frac{1}{\log(t^2)}$, $y(4)=3$, $t_0=4$, $y_0=3$

Normal/Standard Form, $a(t) = \frac{\cos(t)}{\sin(t)}$, $f(t) = \frac{1}{\log(t^2)}$

$a(t)$, $f(t)$ are continuous except at

$n\pi$, where $n = 0, \pm 1, \pm 2 \ldots$

$t_0 = 4 \in (\pi, 2\pi)$ solution exists in $(\pi, 2\pi)$

If $y(3)=4$ was given, solution exists in $(0,\pi)$

If $y(0)=4$, EUT doesn't apply, not continuous.

<u>Example:</u> $y' - \frac{4}{t}y = 0$, $y(1)=1$, $a(t)=\frac{4}{t}$

$f(t)=0$, Solution exists in $(0,\infty)$

$A(t)=\int^{t} -\frac{4}{s}\,ds = -4\ln(t)$, $e^{A(t)} = \frac{1}{t^4}$

$y(t)=Ct^4$, $y(1)=1 \Rightarrow C=1$

Solution: $y(t)=t^4$, what is wrong with extending solution in $(-\infty,0)$?

If we define $y(t)=\begin{cases} t^4 & \text{for } t>0 \\ Ct^4 & \text{for } t\leq 0 \end{cases}$ $C=$ anything

If you extend to negative infinity, you no longer have unique solutions.

Remark: Example: $y' + \frac{\cos(t)}{\sin(t)}y = \frac{1}{\log(t^2)}$, $y(4)=3$

$a(t)=\frac{\cos(t)}{\sin(t)}$, $A(t)=\int \frac{\cos(s)}{\sin(s)}\,ds = \log(|\sin(t)|)$

Integrating factor: $e^{A(t)} = |\sin(t)|$

Compute: $\int^{t} \frac{\sin(s)}{2\log(s)}\, ds = ?$, Too hard to compute by hand

Application: Tank problems

Separable Equations: General Form: $y' = \underbrace{f(t)\cdot g(y)}_{\text{product}}$,

$y' = \frac{dy}{dt}$, $\frac{dy}{g(y)} = f(t)\,dt$, y and t are separated

Integrate both sides;

$\int \frac{dy}{g(y)} = \int f(t)\,dt$, After computing...

$= G(y) + C = F(t)$, We call $\int \frac{dy}{g(y)} = G(y) + C$,

$\int f(t)\,dt = F(t)$, $\to$ Solution in implicit form

Solving for y we obtain, $y = G^{-1}(F(t) - C)$,

Solving might be difficult or impossible (by formulas). There might be more than one solution.

Example: $\frac{dy}{dt} = ay$, $\int \frac{dy}{y} = \int a\,dt$, $\log(|y|) + C = at$

a = constant

any constant

Example: $\frac{dy}{dt} = -ty$, $\int \frac{dy}{y} = \int -t\,dt$,

$\log(|y|) + C = \frac{-t^2}{2}$, $\underline{\text{Implicit}}$, $y(t) = C e^{-t^2/2}$

Example: $\frac{dy}{dt} = y^2$, $\int \frac{dy}{y^2} = \int dt$, $\boxed{-\frac{1}{y} + C = t}$ $\leftarrow$ Implicit

$\boxed{y = \frac{1}{C - t}}$ $\Rightarrow$ Explicit General Solution