Existence and Uniqueness Theorem:

The theorem concerns the existence and uniqueness of solutions for the Initial Value Problem, IVP.
Consider the following IVP:

$$(1)\quad \frac{dy}{dt} = f(t,y) \qquad (2)\quad y(t_0) = y_0$$

And define the set:

$$D_{good} := \{(t,y) \mid f(t,y) \text{ and } \partial_y \cdot f(t,y) \text{ cont.}\}$$

$$D_{bad} := \text{complement of } D_{good}$$

If $(t_0, y_0) \in D_{good}$ then there exists a unique solution to the IVP problem $y(t)$ for $t \in (a,b)$ with $t_0 \in (a,b)$, $y(t) = y_0$. Moreover, the solution exists as long as $(t, y(t)) \in$ interior of D_{good}.
The theorem above claims that the solution exists and is unique provided that the solution curve

$$(t, y(t)) \in \text{interior of } D_{good}$$

The theorem implies that the solution curves cannot intersect each other because this will violate the uniqueness theorem.

Example #1: $\dfrac{dy}{dt} = \sqrt{1+t^2+y^2}$, so

$$f(t,y) := \sqrt{1+t^2+y^2}$$

$$\partial_y \cdot f(t,y) = \frac{y}{\sqrt{1+t^2+y^2}}$$

Since both $f(t,y)$ and $\partial_y \cdot f(t,y)$ are continuous for all $(t,y) \in \mathbb{R}^2$. Therefore, if we consider the IVP, $y(t_0) = y_0$ then for any choice of (t_0, y_0) there exists a solution for some interval (a,b). As a matter of fact, in this case, $(a,b) = (-\infty, \infty)$.

Example #2: $\frac{dy}{dt} = 3y^{2/3}$, $f(y) := 3y^{2/3}$

$\partial_y \cdot f(y) = 2y^{-1/3}$ which is not continuous at $y = 0$

Consider the initial problem $y(0) = -8$.
We can find the general solution by separating variables: $\frac{1}{3}\int \frac{dy}{y^{2/3}} = \int dt$ or

$y^{1/3} = t - c$ or $y(t) = (t-c)^3$, for $y(0) = -8$ we find: $y(t) = (t-2)^3$ valid for $t \in (-\infty, 2)$

Note we restrict the interval at 2 because

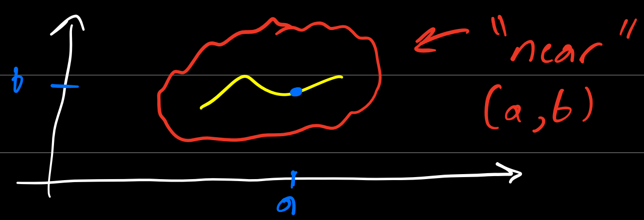
$y(2)=0$ and we know the partial derivative $\partial_y f(y)$ is not continuous at $y=0$. If we continued the solution for $t \geq 2$ there are infinitely many solutions.

☆ Thus, we observe that when the solution curve $(t, y(t))$ hits the "bad" set $y=0$ we stop because if we try to cross the line $y=0$ we lose <u>uniqueness</u>. ☆

<u>Youtube Tutorial:</u>

- When does a solution to an Initial Value Problem Exist? If it exists, is it unique?

→ If f is continuous "near" (a,b) then a solution exists.



$\wedge$

→ If also $\frac{\partial f}{\partial y}$ is continuous near (a,b) then the solution is unique.

Example #3: $x \cdot \frac{dx}{dy} = y$, rewrite as $\frac{dy}{dx} = \frac{y}{x}$

- $\frac{y}{x}$ is continuous near any (a,b) where $a \neq 0$

so a solution with $y(a)=b$ exists when ↑

• $\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}$ is also continuous near (a,b) when $a \neq 0$ so the solution is unique when $a \neq 0$

Notice: $y(x) = Cx$ solves $x\frac{dy}{dx} = y$ for any C.