Separable Equations:

Separable Equations have general form:

$$= \frac{dy}{dt} = f(t) \cdot g(y) \text{ where}$$

$f(t)$, $g(y)$ are given known functions

From here, we separate variables by moving everything with y to the left, everything with t to the right. Such that:

$$\frac{dy}{g(y)} = f(t) \cdot dt$$

Now take the integral of both sides:

$$\int \frac{dy}{g(y)} = \int f(t)\,dt \quad \text{or} \quad G(y) + C = F(t)$$

where

$$G'(y) = \frac{1}{g(y)} \quad \text{and} \quad F'(t) = f(t)$$

Provided we computed the integrals, we now have implicit form:

$$G(y) + C = F(t)$$

Though we might try to solve this equation in terms of y such that, $y(t) = G^{-1}(F(t) - C)$,

this may be difficult or impossible.

Example #1:

Given $\frac{dy}{dt} = ay$, where a = constant,

then, $\frac{dy}{y} = a\,dt$, integrate, $\int \frac{dy}{y} = \int a\,dt$

$\log(|y|) = at - c$, $|y(t)| = e^{at} \cdot e^{-c}$,

$y(t) = e^{at} \cdot C$, where C is an arbitrary constant

Example #2:

Given $\frac{dy}{dt} = -ty$, $\int \frac{dy}{y} = -\int t\,dt$,

$\log(|y|) = -\frac{1}{2}t^2 - c$, $y(t) = e^{-t^2/2} C$,

IVP: $y(0) = y_0$, find $y(t) = y_0 e^{-t^2/2}$, $t \in (-\infty, +\infty)$

Autonomous Separable Equations:

Notice $\frac{dy}{dt} = g(y)$, meaning $f(t) = 1$, this is called autonomous since the right side doesn't depend on t. After integrating:

$$\int \frac{dy}{g(y)} = t - c$$

Example #3: $\frac{dy}{dt} = y^2$, $\int \frac{dy}{y^2} = \int dt$,

$\int \frac{dy}{y^2} = -\frac{1}{y}$, $-\frac{1}{y} = t - c$, $y(t) = \frac{1}{c-t}$

IVP: $y(0) = y_0$, $\frac{1}{c} = y_0$, An interesting feature of these solutions is the fact that the interval of existence depends on the initial data y_0. We can draw the phase portrait of the solutions i.e. on the (t, y) coordinate axes we draw the family of solutions $y(t)$ with initial data $y(0) = y_0$.

Example #4: Consider $y' = 1 + y^2$, solve by

Separating variables: $\frac{dy}{dt} = 1 + y^2$, $\frac{dy}{1+y^2} = dt$,

$\int \frac{dy}{1+y^2} = t - c$, $\arctan(y) = t - c$,

General Solution: $y(t) = \tan(t - c)$

If IVP: $y(0) = y_0$, $c = \arctan(y_0)$ hence

$y(t) = \tan(t + \arctan(y_0))$, interval of existence:

$\left(-\frac{\pi}{2} - \text{atan}(y_0), \frac{\pi}{2} - \text{atan}(y_0)\right)$, Again, the interval of existence depends on the initial data.

Example #5: $\frac{dy}{dt} = \frac{1}{2y}$, $2y\,dy = dt$,

$\int 2y\,dy = \int dt$, $y^2 = t - c$, $y = \pm\sqrt{t - c}$,

$\therefore$ Solution exists if $t < c$

Consider IVP: $y(0) = y_0$, then we find:

$\pm\sqrt{-c} = y_0$ hence $c = -y_0^2$, $y(t) = \pm\sqrt{t + y_0^2}$

Notice we have two choices, in order to make sure our solution satisfies the correct initial condition we choose $y(t) = \text{sign}(y_0)\sqrt{t + y_0^2}$ where

$\text{sign}(y) := \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0 \end{cases}$, where $y = 0$ is not allowed,

$\therefore$ the interval of existence is $(-y_0^2, +\infty)$

Example #6: $\frac{dy}{dt} = -\frac{t}{y}$, $\int y\,dy = \int -t\,dt$,

$y^2 = 2c - t^2$, $y^2 + t^2 = 2c$, $2c > 0$, $\therefore$

$2c = C^2$, $y^2 + t^2 = C^2$, $y(t) = \pm\sqrt{C^2 - t^2}$,

$t \in (-C, C)$, IVP: $y(3) = 4$, $\therefore$ $4^2 + 3^2 = C^2$,

$C = 5$ and $y(t) = \sqrt{25 - t^2}$, $\therefore$ $t \in (-5, 5)$

The implicit form solution for this problem is:

$y^2 + t^2 = C^2$, where C^2 is an arbitrary parameter

We can then draw the family of solutions for various choices of C^2, such level sets,

$H(t, y) := t^2 + y^2$, which we know to be

circles centered at the origin of the coordinate axes

Example #6: $\frac{dy}{dt} = \frac{3t + ty^2}{y + t^2 y} = \frac{3 + y^2}{y} \cdot \frac{t}{1 + t^2}$

Separate variables and integrate:

$\int \frac{y\,dy}{3 + y^2} = \int \frac{dt \cdot t}{1 + t^2}$, $\log(3 + y^2) = \log(1 + t^2) - c$

$3+y^2 = C(1+t^2)$, $C = e^{-c} > 0$,

$y(t) = \pm\sqrt{C(1+t^2)-3}$, Suppose IVP, $y(1) = -3$,

$3+9 = C(1+9) = 12 = C\cdot(1+1^2)$, $C = 6$

thus: $y(t) = -\sqrt{6t^2+3}$, Here we again have $\pm$ yet chose the negative answer as it will satisfy $y(1) = -3$

Example #7: $\frac{dy}{dt} = 4y - y^3$, $\int \frac{dy}{4y-y^3} = \int dt$

By way of Partial Fraction Expansion:

$\frac{1}{8}\log\left(\frac{y^2}{|4-y^2|}\right) = t - c$, $\frac{y^2}{|4-y^2|} = e^{8(t-c)}$

3 cases: i) $y^2 < 4$, ii) $y^2 > 4$, iii) $4y - y^3 = 0$

Example #8: $\frac{dy}{dt} = 3y^{2/3}$, $\int \frac{1}{3} y^{-2/3} = \int dt$

$y^{1/3} = t - c$, $y(t) = (t-c)^3$ = General Solution

Suppose IVP: $y(0) = 0$, we find solution $y(t) = t^3$, however there is more than one solution with initial data $y(0) = 0$, thus we do not have a unique solution to this IVP. We actually have infinitely many solutions that all satisfy $y(0)=0$.

Consider:

$$y(t) = \begin{cases} (t+c)^3 & \text{if } t < -c \\ 0 & \text{if } |t| \le c \\ (t-c)^3 & \text{if } t > c \end{cases}$$

All of these are solutions to $y' = 3y^{2/3}$

and all of them satisfy $y(0)=0$. The reason for this bad behavior of the ODE is because $\partial_y(3y^{2/3}) = 2y^{-1/3}$ is not continuous

at $y=0$. We will see this when we discuss the Existence and Uniqueness Theorem for ODEs.

Example #9: $\dfrac{dy}{dt} = \dfrac{e^y \cos(t)}{1+y}$, separate variables, integrate

$\int (1+y)e^{-y}\,dy = \int \cos(t)\,dy$, $-(2+y)e^{-y} = \sin(t) - c$

Solution in Implicit Form:

$(2+y)e^{-y} + \sin(t) = c$,

but if we are interested in an IVP we have to avoid $y_0 = -1$, why?

$y(t_0) = y_0$; $y_0 \neq -1$,

$c = \sin(t_0) + (2 + y_0) e^{-y_0}$

$$(2+y) \cdot e^{-y} = -\sin(t) + \sin(t_0) + (2+y_0) e^{-y_0}$$

We do not know how to solve for y using simple algebra! However, we can use MATLAB in order to draw the level sets of the function.

$$H(t,y) := (2+y) e^{-y} + \sin(t)$$

Each level set of this function represents a solution curve. This constant c is determined by the initial conditions $y(t_0) = y_0$. See LVRM notes!