Introduction:

1. What is an ODE?

- ODE is an ordinary differential equation

We have an unknown function, $y(t)$

t = independent variable (often time)

Functional relationship between $t, y, \dots y^{N'}$

$F(t, y, y', y'') = 0$, Goal is to find compute $y(t)$

Ex: $y' - y = 0$, $y = e(t)$, since $y'(t) = e^t$

But $y(t) = Ce^t$, C = ordinary constant is also a solution

$F(t, y, y', \dots, y^n) = 0$, solve for $y^{(N)}$

$y^{(N)} = f(t, y, y', y^{(N-1)})$, called an Nth order ODE

Order means the highest order derivative

1st order: $y' = f(t, y)$, f = given

Examples: $y' = y$, $f(t, y) = y$

$y' = \sqrt{1-y^2}$, $f(t, y) = \sqrt{1-y^2}$

$y' = y^2 + t^2$, $f(t, y) = t^2 + y^2$

2nd order: $y'' = f(t, y, y')$

Examples: $y'' + \frac{2}{t} y' + y = 0$

$y'' = \frac{2}{t} y' - y$, $f(t, y, y') = \frac{2}{t} y' - y$

Important Linear ODE's

Linear means that $F(t, y, y', y^{N-1}) = 0$ F is linear function of $y, y', \dots, y^{N-1}$

General Form:

$$a_0(t)y^{(N)} + a(t)y^{(N-1)} + \dots + a(t)y + f(t) = 0$$

Example: $y'' + y' + y + t^2 = 0$, Linear in y, y', y''

Example: $y'' + (y')^2 + y = 0$, Non-linear because $(y')^2$

Also $y'' + y \cdot y' = 0$, Non-linear because of $y \cdot y'$

Also $y' \cdot y'' + y = 0$, Non-linear because of $y' \cdot y''$

Remark: We will write linear ODEs in normal (standard) form:

$$y^{(N)} + a_1(t)y^{(N+1)} + \dots + a_{N-1}(t) + y^2 + a_n(t)y = f(t)$$

Ex: $t^2y'' + 2ty' + e^2y + t^2 + 1 = 0$

Normal: $y'' + \frac{2}{t}y' + ?\,y = ?$

Part I: We will study 1st order ODE's

General form: $y' = f(t, y)$, f = given, find $y(t)$?

Linear: $y' + a(t)y = f(t)$, we will call that forcing

What is a solution of $y' = f(t, y)$?

Solution is a function $y(t)$ such that

(1) $y(t)$, $y'(t)$ are defined in interval (Lt_1, Rt_2)

(2) $f(t, y(t))$ is defined in (t_1, t_2)

(3) $y'(t) = f(t, y(t))$ in (t_1, t_2)

Ex: $y' = \sqrt{1 - y^2}$, $y(t) = \sin(t)$ is a solution of ODE in $(-\pi/2, \pi/2)$, $f = \sqrt{1 - y^2}$

$y' = \cos(t)$, $\sqrt{1-\sin(t)^2} = |\cos(t)|$

$\cos(t) = |\sin(t)|$ only in $(-\pi/2, \pi/2)$

Ex: $y' = \frac{-t}{y}$, $f = -\frac{t}{y}$, $y(t) = \sqrt{1-t^2}$

Solution: $t \in (-1, 1)$, $y' = \frac{-t}{\sqrt{1-t^2}} = \frac{-t}{y}$

$y(t) = \sqrt{1-t^2} = (1-t^2)^{1/2}$, chain rule

Explicit ODE: $y' = f(t)$, where f is given function

Fund. Theorem of Calculus:

$y(t) = \int f + c$, where c is an arbitrary constant, $\int f$ = antiderivative

Example: $y' = \frac{1}{1+t^2}$, $y(t) = \int \frac{1}{1+t^2} + c$

$y(t) = \text{atan}(t) + c$

Initial Value Problem (IVP):

$y' = f(t, y)$, (t_1, t_2)

$y(t_0) = y_0$, $t_0 \in (t_1, t_2)$

Ex: $y' = y$, $y(0) = 5$, $t_0 = 0$, $y_0 = 5$

$y(t) = 5e^t$

Ex: $y' = \frac{1}{1+t^2}$, $y(0) = 5$, This is an IVP

General Solution is $y(t) = \text{atan}(t) + c$

arbitrary constant

I need $y(0)=5 \Rightarrow \underbrace{a\tan(0)}_{\text{equals } 0} + \mathcal{C} = 5$ $\therefore \mathcal{C}=5$

Ex: $y' = \frac{1}{1+t^2}$, $y(1)=5$, $t_0 = 1$

$\therefore \underbrace{a\tan(1)}_{=\pi/4} + \mathcal{C} = 5$, $\therefore \mathcal{C} = 5 - \pi/4$

Ex: $y'(t) = \frac{e^t}{4-t^2}$ explicit ODE,

$y(\underset{t_0}{1}) = 5$, IVP

Calculus: $y(t) = 5 + \int_1^t \frac{e^s}{4-s^2}\,ds$ → I do not know how to compute this!

In general: $y' = f(t)$, $y(t_0) = y_0$

Solution: $y(t) = y_0 + \int_{t_0}^{t_1} f(s)\,ds$

Remark: $\int_1^t \frac{e^s}{4-s^2}\,ds$ is well defined in $(-2,2)$

Solution exists in $(-2,2)$ given $y(1)=5$

II Linear 1st Order Equations

Normal (standard) form

$y' + a(t)y = f(t)$, $\boxed{y(t_0) = y_0, \text{ IVP}}$

Find the general solution → all possible solutions

→ Meaning to contain an arbitrary constant in equation

1st Step: Homogeneous equation, $\boxed{y' + a(t)y = 0}$

Integrating factor: $A(t)$ is to be found

$$\left(e^{A(t)} \cdot y(t)\right)' = e^{A(t)} y' + e^{A(t)} A'(t) \cdot y$$

$$= e^{A(t)} \left(y' + A'(t) + y\right) = 0$$

Choose $A' = a(t)$, $A(t) = \int a(t)$

$e^{A(t)} y(t) = c$ ← constant, $A = a$

$y(t) = c \cdot e^{-A(t)}$

Example: $y' + ty = 0$, $\boxed{a(t) = t}$, $A = \frac{1}{2}t^2$

$y(t) = c\, e^{-t/2}$, $A' = t$

↳ y homogeneous general solution

Now: $y' + a(t)y = f(t)$, multiply with e^t?

$$e^{f(t)}(y' + A'y) = e^{a(t)} \cdot f(t),$$

$$(e^A y)' = e^{A(t)} \cdot f(t)$$

$$e^{f(t)} y(t) = \int e^{A(t)} f(s)\, ds + C$$

$$y(t) = C e^{-A(t)} + e^{-A(t)} \int^{t} e^{A(t)} f(t)\, ds$$