Ex5 TD3 Analyse II.

1. $Y'' + 4Y = x\,e^{x}$.
2. $Y'' + 2Y' = 2x^2$.
3. $Y'' + 2Y' + Y = e^{-x}$.

corr

$(E_1) : y'' + 4y = x e^{x}$

Solution de l'éq. Homogène

$y'' + 4y = 0 \longrightarrow E_c : r^2 + 4 = 0$

$a = 1$ $\quad \Delta = -16 < 0 \quad \sqrt{\Delta} = 4i$

$b = 0$ $\quad r_1 = \frac{-4i}{2} = -2i =$

$c = 4$ $\quad r_2 = \frac{4i}{2} = 2i = 0 + 2i \longrightarrow \begin{cases} \alpha = 0 \\ \beta = 2 \end{cases}$

D'où les solutions de l'équation Homogène

$y_H(x) = A e^{0.x} \cdot \cos(2x) + B e^{0.x} \sin 2x$

$y_H(x) = A \cos 2x + B \sin 2x \quad A, B \in \mathbb{R}$.

Solution particulière

$d(x) = x e^{1x} = P(x) \cdot e^{1.x}$ $\quad$ $s = 1$ $\quad P(x) = x \longrightarrow \deg(P) =$

$s = 1$ n'est pas solution de E_c on prendra alors $y_p(x) = Q(x) e^{x}$
avec $\deg Q = \deg P = 1$

C.à.d $y_p(x) = (\alpha x + \beta) e^{x} \to y_p'(x) = \alpha e^{x} + (\alpha x + \beta) e^{x} = (\alpha + \beta + \alpha x) e^{x}$

y_p est sol $E_1 \iff y_p''(x) + 4y_p(x) = x e^{x}$

$y_p''(x) = (2\alpha + \beta + \alpha x) e^{x}$

$\iff (2\alpha + \beta + \alpha x) e^{x} + 4(\alpha x + \beta) e^{x} = x e^{x}$

$\iff (5\alpha x + \underbrace{2\alpha + 5\beta}) e^{x} = x e^{x}$

Par id : $\begin{cases} 5\alpha = 1 \\ 2\alpha + 5\beta = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{1}{5} \\ \beta = \frac{-2}{25} \end{cases}$

D'où $y_p(x) = \left(\frac{1}{5}x - \frac{2}{25}\right).e^{x}$ est une solution part. de E_1.

c) Les solutions de E_1 sont

$$y_G(x) = \underbrace{A\cos 2x + B\sin 2x}_{y_H''} + \underbrace{\left(\frac{1}{5}x - \frac{2}{25}\right)e^{x}}_{y_p''} \; ; A, B \in \mathbb{R}$$

② $y'' + 2y' = 2x^2 \quad (E_2)$

* Solution Homogène $\longrightarrow$ $E_c : r^2 + 2r = 0$

$a = 1 \; (y'')$

$b = 2 \; (y')$

$c = 0$

$\Leftrightarrow r(r+2) = 0$

$\Leftrightarrow \begin{cases} r_1 = 0 \\ r_2 = -2 \end{cases}$ D'où $y_H(x) = A + Be^{-2x}$; $A, B \in \mathbb{R}$

* Solution particulière

$d(x) = 2x^2 = 2x^2 \cdot e^{0.x}$ $\begin{cases} \lambda = 0 \text{ est une solution simple de } E_c \\ P(x) = 2x^2 \longrightarrow \deg P = 2 \end{cases}$

on cherchera une solution particulière de (E_2) de la forme

$y_p(x) = Q(x) \cdot e^{0.x} = Q(x)$ avec $\left[\begin{array}{l} \deg Q = \deg P + 1 = 3 \\ \mathrm{Val}\, Q = 1 \end{array}\right.$

$y_p(x) = \alpha x^3 + \beta x^2 + \gamma x \rightarrow y_p'(x) = 3\alpha x^2 + 2\beta x + \gamma$

$y_p''(x) = 6\alpha x + 2\beta$

y_p est sol de $E_2 \Leftrightarrow y_p''(x) + 2y_p'(x) = 2x^2$

$\Leftrightarrow 6\alpha x + 2\beta + 2\{3\alpha x^2 + 2\beta x + \gamma\} = 2x^2$

$\Leftrightarrow 6\alpha x^2 + (6\alpha + 4\beta)x + 2\beta + 2\gamma = 2x^2$

$$\Leftrightarrow \begin{cases} 6\alpha = 2 \\ 6\alpha + 4\beta = 0 \\ 2\beta + 2\gamma = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{1}{3} \\ \beta = -\frac{1}{2} \\ \gamma = \frac{1}{2} \end{cases}$$

Cl on prendra $y_p(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x$

Cl final :

Les solutions de (E_2) sont :

$$y_G(x) = \underbrace{A + Be^{-2x}} + \underbrace{\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x} \quad A, B \in \mathbb{R}.$$

3) $y'' + 2y' + y = \boxed{e^{-x}}$.

- Solution de l'équation Homogène :

$E_c : r^2 + 2r + 1 = 0$

$\Delta = 0 \Rightarrow r_1 = r_2 = \frac{-b}{2a} = -1$.

Les solutions de l'éq Hom :

$y_H(x) = A\,.e^{-x} + B.x\,e^{-x}$; $A, B \in \mathbb{R}$.

- Solution particulière.

$d(x) = e^{-x} = \boxed{1} \times e^{(-1)x}$ $\qquad \lambda = -1$ $\qquad P(x) = 1 \rightarrow \deg P = 0$

on a $\lambda = -1$ est une solution double de $r^2+2r+1=0$

on prend ra

$y_p(x) = Q(x)\,.e^{-x}$ avec $\begin{cases} \deg Q = \deg P + 2 = 2 \\ \mathrm{Val}\, Q = 2 \end{cases}$

$(e^{-x})' = -e^{-x}$

c'est à dire $Q(x) = \alpha . x^2$

$y_p(x) = \alpha . x^2 . e^{-x}$ $\qquad \alpha \in \mathbb{R}$??

$\rightarrow y'_p(x) = (2\alpha x - \alpha x^2)e^{-x}$ $\qquad$, $y''_p(x) = (2\alpha - 4\alpha x + \alpha x^2)e^{-x}$

y_p est sol de (E) $\Leftrightarrow y''_p(x) + 2y'_p(x) + y_p(x) = e^{-x}$

$\Leftrightarrow \left((\alpha x^2 - 4\alpha x + 2\alpha) + (4\alpha x - 2\alpha x^2) + \alpha x^2\right)e^{-x} = e^{-x}$

$\Leftrightarrow 2\alpha e^{-x} = e^{-x}$

$\Leftrightarrow 2\alpha = 1$

$\Leftrightarrow \alpha = \frac{1}{2}$

$\underline{cq}$ $\qquad y_p(x) = \frac{1}{2} x^2 e^{-x}$ ☒

D'où les solutions de (E) sont :

$y_G(x) = \left(A + Bx + \frac{1}{2}x^2\right)e^{-x}$; $A, B \in \mathbb{R}$

$$ay''+by'+cy = P(x)\,e^{s\cdot x}$$

$$ay''+by'+cy = P_1(\cos wx+\phi)+P_2\sin(wx+\phi)$$

4) $y''+y = x\sin x$ $\boxed{\times}$

- Solutions de $E_H : y''+y=0$

$E_c : r^2+1=0 \qquad \Delta=-4<0$

$$\begin{cases} r_1 = i = 0+1\cdot i \\ r_2 = -i \end{cases} \qquad \begin{cases} \alpha = 0 \\ \beta = 1 \end{cases}$$

$$y_H(x) = A\,e^{0\cdot x}\cos(1\cdot x)+B\,e^{0\cdot x}\sin(1\cdot x) \quad A,B\in\mathbb{R}$$

$$y_H(x) = A\cos x+B\sin x \quad \boxed{\times}$$

- - Solution particulière : $\leftarrow P_1(x)=0$ $\leftarrow P_2(x)=x$ | $w=1$, $\phi=0$

$$d(x) = x\sin x = \boxed{0}\cdot\cos x + x\sin x$$

$iw = i\cdot 1 = i$ est une solution de (E_c) :

on prend la

$$y_p(x) = x\left(Q_1\cos x+Q_2\sin x\right)$$

avec $\deg Q_1 = \deg Q_2 = \max\{\deg P_1, \deg P_2\}$ ($-\infty$, 1)

$Q_1 = \alpha_1 x+\beta_1 \qquad x\cdot Q_1 = \alpha_1 x^2+\beta_1 x$

$Q_2 = \alpha_2 x+\beta_2 \qquad x\,Q_2 = \alpha_2 x^2+\beta_2 x$

$$y_p(x) = \underbrace{(\alpha_1 x^2+\beta_1 x)}\cos x+\underbrace{(\alpha_2 x^2+\beta_2 x)}\sin x$$

$$y_p'(x) = (\alpha_2\cdot x^2+2\alpha_1\cdot x+\beta_2\cdot x+\beta_1)\cos x+(-\alpha_1\cdot x^2+2\alpha_2 x-\beta_1\cdot x+\beta_2)\cdot\sin x$$

$$y'_p(x) = (\alpha_2 x^2 + (2\alpha_1+\beta_2)\cdot x + \beta_1)\cos x + (-\alpha_1 x^2 + (2\alpha_2-\beta_1)\cdot x + \beta_2)\cdot \sin x$$

$$y''_p(x) = \left(-x^2\alpha_1 + 4x\alpha_2 - x\beta_1 + 2\alpha_1 + 2\beta_2\right)\cos(x) + \left(-x^2\alpha_2 - 4x\alpha_1 - x\beta_2 + 2\alpha_2 - 2\beta_1\right)\sin(x)$$

$$y''_p(x) = \left(-\alpha_1 x^2 + (4\alpha_2-\beta_1)x + 2\alpha_1 + 2\beta_2\right)\cos x + \left(-\alpha_2 x^2 - (4\alpha_1+\beta_2)\cdot x + 2\alpha_2 - 2\beta_1\right)\sin x$$

$$y_p(x) = (\alpha_1 \cdot x^2 + \beta_1 x)\cos x + (\alpha_2 \cdot x^2 + \beta_2 x)\sin x$$

y_p est sol de (E)

$$y''_p(x) + y_p(x) = x\sin x$$

càd : $(4\alpha_2 \cdot x + 2\alpha_1 + 2\beta_2)\cos x + (-4\alpha_1 \cdot x + 2\alpha_2 - 2\beta_1)\cdot\sin x = x\sin x$

$$\Leftrightarrow \begin{cases} 4\alpha_2 x + 2\alpha_1 + 2\beta_2 = 0 \\ -4\alpha_1 \cdot x + 2\alpha_2 - 2\beta_1 = x \end{cases} \Leftrightarrow \begin{cases} 4\alpha_2 = 0 \\ 2\alpha_1 + 2\beta_2 = 0 \\ -4\alpha_1 = 1 \\ 2\alpha_2 - 2\beta_1 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = 0 \\ \beta_1 = 0 \\ \alpha_1 = -\frac{1}{4} \\ \beta_2 = \frac{1}{4} \end{cases}$$

càd $y_p(x) = \left(-\frac{1}{4}x^2 + 0\right)\cos x + \left(0 \cdot x^2 + \frac{1}{4}x\right)\sin x$

$$y_p(x) = -\frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x$$

c) Les solutions de $y'' + y = x\sin x$ sont :

$$y_G(x) = \left(A - \frac{1}{4}x^2\right)\cos x + \left(B + \frac{1}{4}x\right)\cdot\sin x$$

$A, B \in \mathbb{R}$.