VARIABLE ALEATOIRE CONTINUE

Exercice

91. The amount of bread (in hundreds of kilos) that a bakery sells in a day is a random variable with density

$$f(x) = \begin{cases} cx & \text{for} \quad 0 \le x < 3, \\ c(6-x) & \text{for} \quad 3 \le x < 6, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{P}$

3W

(i) Find the value of c which makes f a probability density function.

(ii) What is the probability that the number of kilos of bread that will be sold in a day is, (a) more than 300 kilos? (b) between 150 and 450 kilos?

(iii) Denote by A and B the events in (a) and (b), respectively. Are A and B independent events?

f densité de probabilité $\Leftrightarrow$ $\begin{cases} f(x) \geq 0 \quad \forall x \in \mathbb{R} & i \\ f \ \mathcal{C} & ii \\ \int_{-\infty}^{+\infty} f(x)\,dx = 1 & iii \end{cases}$

$\Leftrightarrow \begin{cases} c \geq 0 \\ f \text{ est continue sur } \mathbb{R} \setminus \{0, 3, 6\} \end{cases}$ ✓ .... continue sur $\mathbb{R}$

$$③\quad \int_{-\infty}^{+\infty} f(x)\,dx = 1 \iff \underbrace{\int_{-\infty}^{0} f(x)dx}_{=0} + \int_{0}^{3} f(x)dx + \int_{3}^{6} f(x)dx + \underbrace{\int_{6}^{+\infty} f(x)dx}_{=0} = 1$$

$$\int_0^3 \boxed{C}\cdot x\,dx + \int_3^6 \boxed{C}(6-x)\,dx = 1$$

$9/2$

$$C\underbrace{\int_0^3 x\,dx}_{} + C\int_3^6 6-x\;dx = 1 \iff$$

$\boxed{36-18}$

$$\cancel{18} - \left[\cancel{18} - \frac{9}{2}\right]$$

$$C\left[\frac{1}{2}x^2\right]_0^3 + \boxed{C}\left[6x - \frac{1}{2}x^2\right]_3^6 = 1 \iff \frac{9}{2}C + \frac{9}{2}C = 1$$

$$\iff 9C = 1 \iff \boxed{C = \frac{1}{3}}$$

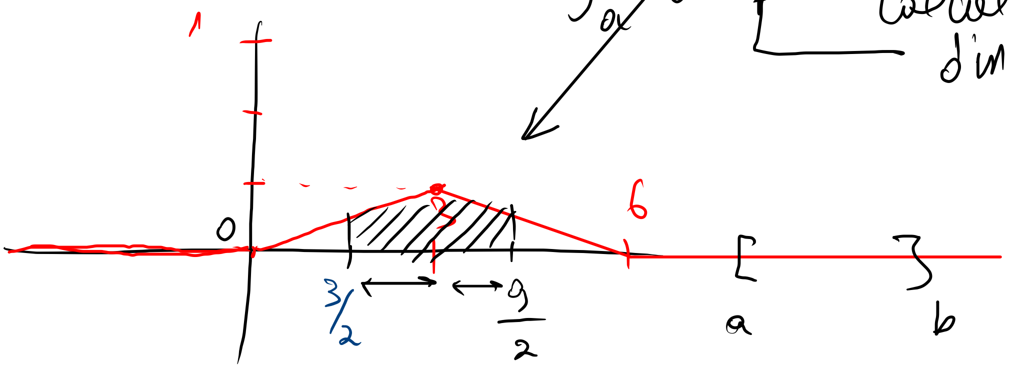
a) f est une densité de prob ssi $\boxed{c = \frac{1}{3}}$ ✓

X_n f $\longrightarrow$ $\mathbb{P}$

$$f(x) = \begin{cases} 0 & \text{si } x \in]-\infty, 0[\\ \frac{1}{9}x & x \in [0,3] \\ \frac{1}{9}(6-x) & x \in [3,6] \\ 0 & \text{si } x \in [6+\infty[\end{cases}$$

$$\mathbb{P}(X \in [a;b]) = \int_a^b f(x)\,dx$$

Calcul d'intégrale



$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$ il faut que

C à d :

$$\underbrace{\int_{-\infty}^{0} f(x)\,dx}_{=0} + \int_{0}^{3} f(x)\,dx + \int_{3}^{6} f(x)\,dx + \underbrace{\int_{6}^{+\infty} f(x)\,dx}_{=0} = 1$$

$$\int_{0}^{3} c\cdot x\,dx + \int_{3}^{6} c\cdot(6-x)\,dx = 1$$

$6c - cx$

$$\left[\frac{1}{2}cx^2\right]_0^3 + \left[6cx - \frac{1}{2}cx^2\right]_3^6 = 1$$

$$\left\{\frac{9c}{2} - 0\right\} + \left[\left\{36c - 18c\right\} - \left\{\cancel{18c} - \frac{9c}{2}\right\}\right] = 1$$

$\cancel{18c}$

$$\frac{9c}{2} + \frac{9c}{2} = 1 \Rightarrow 9c = 1 \Rightarrow c = 1/3$$

CL

f est une densité de prob $\iff$ $C = \frac{1}{9}$.

ii) notons X : la v.a.r de densité f

$1U = (100kg)$ $\qquad A = \{X \geq 3U\}$

$$\mathbb{P}(X \geq 3U) = 1 - \mathbb{P}(X < 3) \qquad f(x) = \begin{cases} \frac{1}{9}x \; ; \text{ si } x \in [0,3[\\ \frac{1}{9}(6-x) \; x \in [3,6[\\ 0 \text{ sinon} \end{cases}$$

$$\int_3^{+\infty} f(x)\,dx =$$

$$\int_3^6 \frac{1}{9}(6-x)\,dx = \frac{9}{2}\cdot\frac{1}{9} = \frac{1}{2}$$

$$\mathbb{P}(X \geq 3)$$

$$= 1 - \int_{-\infty}^{3} f(x)\,dx$$

$$= 1 - \int_0^3 \frac{1}{9}x\,dx = 1 - \left[\frac{1}{18}x^2\right]_0^3$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$\mathbb{P}$ (produire$^{(a)}$ une qt $\geq 3u$) $\qquad u = 100\,kg$

$$X \sim f$$

$$\boxed{\mathbb{P}(X \geq 3u)} = \mathbb{P}(X \in [\underline{\underline{3}}, +\infty[)$$

$$= \int_3^{+\infty} f(x)\,dx = \int_3^6 f(x)\,dx$$

$$= \int_3^6 \frac{1}{9}(6-x)\,dx = \frac{1}{9}\left[6x - \frac{1}{2}x^2\right]_3^6$$

$$= \frac{1}{2}$$

(b) $= \{X \in [1.5u, 4.5u]\}$

$$\mathbb{P}(X \in [\underline{\underline{1,5}}, 4,5]) = \int_{3/2}^{9/2} f(x)\,dx$$

$$\ldots = \int_{\boxed{3/2}}^{\boxed{3}} \frac{1}{9} \cdot x\,dx + \int_{\boxed{3}}^{\boxed{9/2}} \frac{1}{9}(6-x)\,dx$$

$$= \frac{1}{9}\left[\frac{1}{2}x^2\right]_{3/2}^{3} + \frac{1}{9}\left[6x - \frac{1}{2}x^2\right]_{3}^{9/2}$$

$$\overset{\ldots}{=} \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$$

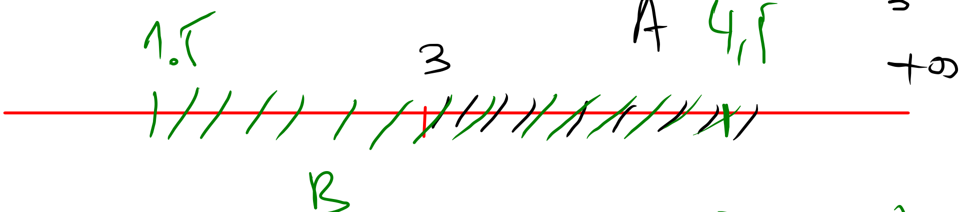
$\mathbb{P}(A) = \frac{1}{2}$ $\qquad A = \{X \geq 3\}$

$\mathbb{P}(B) = \frac{3}{4}$ $\qquad B = \{1.5 \leq X \leq 4.5\}$

A et B sont ils indep ??

$\hookrightarrow \mathbb{P}(A \cap B) \overset{??}{=} \underbrace{\mathbb{P}(A) \cdot \mathbb{P}(B)}$??

$$\mathbb{P}(A \cap B) = \mathbb{P}(X \in [3, 4{,}5]) = \int_3^{4{,}5} f(x)\,dx = \frac{3}{8}$$



$A \cap B = [3; 4{,}5)$

$$\underbrace{\overbrace{\mathbb{P}(A)\cdot\mathbb{P}(B) = 1/2 \times \tfrac{3}{4}}}_{}= \frac{3}{8} = \mathbb{P}(A\cap B)$$

<u>**cl**</u> A et B sont indépendants $\boxtimes$

92. Suppose that the duration in minutes of long-distance telephone conversations follows an exponential density function;

$$f(x) = \tfrac{1}{5}e^{-x/5} \qquad \text{for} \quad x > 0.$$

Find the probability that the duration of a conversation:

(a) will exceed 5 minutes; $\mathbb{P}(X \geq 5) = e^{-1/5 \cdot 5} = e^{-1}$ $\boxed{}$

(b) will be between 5 and 6 minutes; $\mathbb{P}(X \in [5,6]) = e^{-1} - e^{-\frac{6}{5}}$

(c) will be less than 3 minutes;

(d) will be less than 6 minutes given that it was greater than 3 minutes.

$$f(x) = \begin{cases} \frac{1}{5} e^{-x/5} & \text{si } x > 0 \\ 0 & \text{sinon} \end{cases} \sim \mathcal{E}\left(\frac{1}{5}\right)$$

$\boxed{X \sim \mathcal{E}(\lambda)}$

1 $\mathbb{P}(X \geq a) = e^{-\lambda a}$ $a > 0$

2 $\mathbb{P}(X \leq a) = 1 - e^{-\lambda a}$

$\mathbb{P}(X \leq 3) = 1 - e^{-3/5}$

3 $\mathbb{P}(a \leq X < b) = e^{-\lambda a} - e^{-\lambda b}$

$$\mathbb{P}(X \leq 6 \mid X \geq 3) = \frac{\mathbb{P}(X \in [3,6])}{\mathbb{P}(X \geq 3)}$$

$\mathcal{E}(\lambda)$ exp

sans mémoire

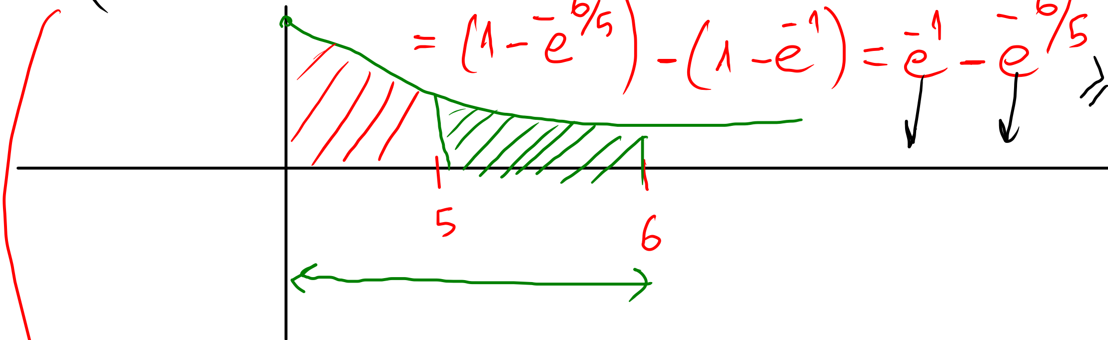
$$= \frac{e^{-3/5} - e^{-6/5}}{e^{-3/5}}$$

$$= 1 - e^{-3/5}$$

$$= \mathbb{P}(X < 3)$$

(b) $\mathbb{P}(5 \leq X \leq 6) = \mathbb{P}(X \leq 6) - \mathbb{P}(X \leq 5)$ Formule

$= (1 - e^{-6/5}) - (1 - e^{-1}) = e^{-1} - e^{-6/5} \geq 0$



$\mathbb{P}(a < X \leq b) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a)$

$X \sim \exp(\lambda)$

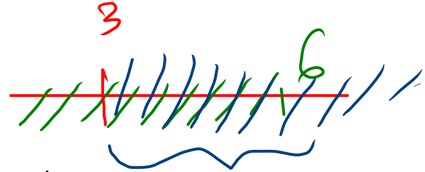
$\mathbb{P}(a < X \leq b) = e^{-\lambda a} - e^{-\lambda b}$ $\boxtimes$

$X \sim \exp(\lambda)$

$a > 0$

$0 \le a \le b$

$\mathbb{P}(X \le a) = 1 - e^{-\lambda a}$



$\mathbb{P}(X \ge a) = e^{-\lambda a}$

$\mathbb{P}(a \le X \le b) = e^{-\lambda a} - e^{-\lambda b}$

(d) will be less than 6 minutes given that it was greater than 3 minutes.

$\lambda = \frac{1}{5}$

$3 \le X \le 6$

$$\mathbb{P}(X \le 6 \mid X \ge 3) = \frac{\mathbb{P}(X \le 6 \cap X \ge 3)}{\mathbb{P}(X \ge 3)}$$

$$= \frac{e^{-3/5} - e^{-6/5}}{e^{-3/5}} = 1 - e^{-3/5} = \mathbb{P}(X \le 3)$$

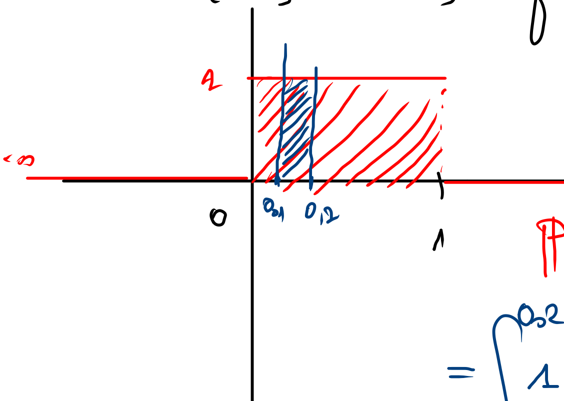
93. A number is randomly chosen from the interval (0, 1). What is the probability that:

(a) its first decimal digit will be a 1;

(b) its second decimal digit will be a 5;

(c) the first decimal digit of its square root will be a 3?

$X \sim U[0;1] \longrightarrow f(x) = \begin{cases} 1 & \text{si } x \in [0;1] \\ 0 & \text{sinon} \end{cases}$



$X = 0.1\#\_\_\_\_ \in [0.1; 0.2[$

$$\mathbb{P}(X \in [0.1; 0.2[) = \int_{0.1}^{0.2} 1 \cdot dx = [x]_{0.1}^{0.2} = 0.2 - 0.1 = 0.1 = \frac{1}{10}$$

$$X \approx 0{,}\#5\# = [0{,}05;0{,}06[\cup [0{,}15, 0{,}16[\cup \cdots$$

$[0{,}05$ $\quad 0{,}16$

$$[0{,}95 \;\; 0{,}96[$$

$[0{,}95$ $\quad 0{,}96$

$$\mathbb{P}(X = 0{,}\#5\#) = \mathbb{P}(X \in \underbrace{[0{,}05, 0{,}06[}_{\subset [0;1]}) + \cdots$$

$$\mathbb{P}(X \in [0{,}95, 0{,}96[)$$

$$= \underbrace{\int_{0{,}05}^{0{,}06} 1\,dx}_{} + \cdots + \int_{0{,}95}^{0{,}96} 1\,dx \qquad \subset [0{,}1]$$

$$= 0{,}01 + 0{,}01 + \text{———} + 0{,}01 = 0{,}1 = \frac{1}{10}$$

$$X \longrightarrow \sqrt{X} = 0.3 \;\#$$

$$\left[0.3 \leq \sqrt{X} < 0.4\right]$$

$$\left[0{,}09 \leq X < 0.16\right]$$

$$\mathbb{P}\left(\sqrt{X} \in [0{,}3, 0{,}4[\right) = \underbrace{\mathbb{P}\left(X \in [0{,}09 ; 0{,}16[\right)}$$

$$= \int_{0{,}09}^{0.16} 1 \, dx = 0{,}16 - 0{,}09 = 0{,}07 = \boxed{\frac{7}{10}}$$

Loi Normale

94. The height of men is normally distributed with mean $\mu = 167$ cm and standard deviation $\sigma = 3$ cm.

(I) What is the percentage of the population of men that have height, (a) greater than 167 cm, (b) greater than 170 cm, (c) between 161 cm and 173 cm?

(II) In a random sample of four men what is the probability that:

(i) all will have height greater than 170 cm;

(ii) two will have height smaller than the mean (and two bigger than the mean)?

$X \sim N(0,1)$

$a = 1.8 \longrightarrow$ Table

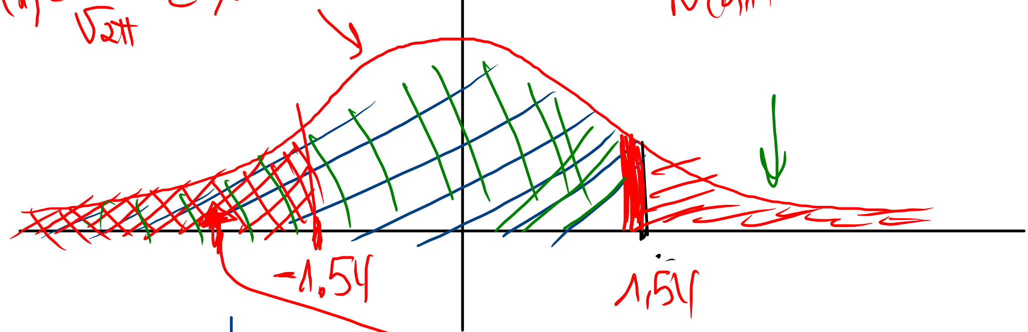
$$P(X \leq 1.8) = \int_{-\infty}^{1.8} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \overset{?}{\cong} \longrightarrow \text{Table}$$

$1{,}89 = 1{,}8 + 0{,}09$

z	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767

$\mathbb{P}(X \leq 1{,}89) \simeq 0{,}9706$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



$$\mathbb{P}(X \leq 1{,}89) = \text{Table [valeurs positives]}$$

$$\boxed{a < 0} \quad \boxed{\mathbb{P}(X \leq a) = 1 - \mathbb{P}(X \leq -a)}$$

$$\mathbb{P}(X \leq -1{,}54) = \mathbb{P}(X > 1{,}54)$$

$$= 1 - \mathbb{P}(X < 1{,}54)$$

$$= 1 - 0{,}9382$$

$$= 0{,}0618$$

$X \sim N(0,1)$

$$\mathbb{P}(a \leqslant X \leqslant b) = \mathbb{P}(X \leqslant \underline{\underline{b}}) - \mathbb{P}(X \leqslant \underline{\underline{a}})$$

$$\mathbb{P}(1{,}25 \leqslant X \leqslant 1{,}38) = \underbrace{\mathbb{P}(X \leqslant 1{,}38)}_{\text{Table}} - \underbrace{\mathbb{P}(X \leqslant 1{,}25)}_{\text{Table}}$$

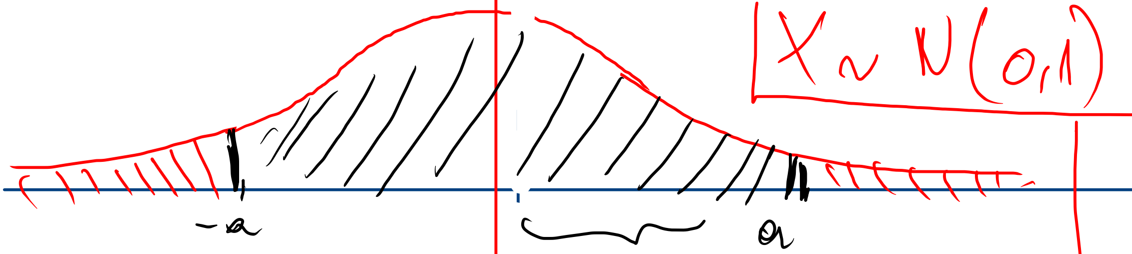
AN

$$\mathbb{P}(-1{,}29 \leqslant X \leqslant 1{,}15) = \underbrace{\mathbb{P}(X \leqslant 1{,}15)}_{\text{Table}} - \underbrace{\mathbb{P}(X \leqslant -1{,}29)}_{= \{1 - \mathbb{P}(X \leqslant 1{,}29)\}}$$

Table

$$\mathbb{P}(-a \leq X \leq a) = \underbrace{2\mathbb{P}(X \leq a) - 1}$$

$$\boxed{X \sim N(0,1)}$$



Table

$$\begin{aligned}
\mathbb{P}(-a \leq X \leq a) &= 1 - 2\mathbb{P}(X \leq -a) \\
&= 1 - 2\{1 - \mathbb{P}(X \leq a)\} \\
&= 1 - 2 + 2\mathbb{P}(X \leq a) \\
&= 2\mathbb{P}(X \leq a) - 1 \quad \boxtimes
\end{aligned}$$

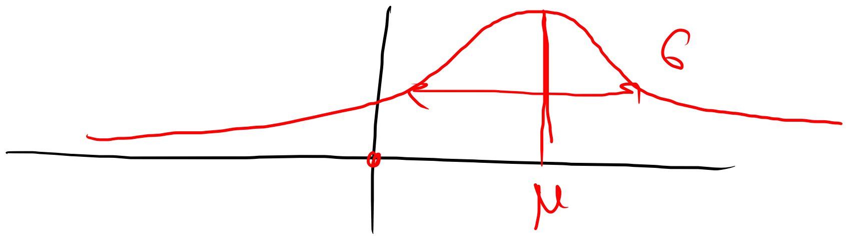
$\underset{\sim}{X} \sim N(\mu, \sigma)$

$\mu \neq 0$

$\sigma \neq 1$

$X \sim \underline{\underline{N(\mu, \sigma)}} \longrightarrow \boxed{\frac{X-\mu}{\sigma}} \sim \underline{\underline{N(0,1)}}$

$$\mathbb{P}(X \leq a) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right)$$



$$\boxed{X \sim N(2,1)} \quad \text{Table} \longrightarrow \boxed{N(0,1)} \longrightarrow \text{Table}$$

$$\mathbb{P}(X \leq 3{,}4) = \mathbb{P}\left(\frac{X-2}{1} \leq \frac{3{,}4-2}{1}\right)$$

$$= \underbrace{\mathbb{P}(X^* \leq 1{,}4)}$$

$$\downarrow$$

Table

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X = taille

$X \sim N(\overset{\mu}{167}, \overset{\sigma}{3})$

$$\mathbb{P}(X > 167) = \mathbb{P}\left(\frac{X - 167}{3} > \overset{0}{\overset{\shortparallel}{\boxed{\frac{167 - 167}{3}}}}\right)$$

$$= \mathbb{P}(X^* > 0) \qquad X^* \sim N(0,1)$$

$$= 50\ \%$$

$$\mathbb{P}(X > 170) = \mathbb{P}\left(X^* > \frac{170 - 167}{3}\right) = \mathbb{P}(X^* > 1)$$

$$\mathbb{P}(X^* > 1) = 1 - \underbrace{\mathbb{P}(X^* < 1)}$$

$X^* \sim N(0,1)$

1,0	0,8413

$$= 1 - 0{,}8413$$

$$= \boxed{0{,}1587}$$

$$\mathbb{P}(161 \leq X \leq 173) = \mathbb{P}\left(\frac{161-167}{3} < X^* < \frac{173-167}{3}\right)$$

$$= \mathbb{P}(-2 < X^* < 2)$$

$$= 2\,\underbrace{\mathbb{P}(X^* < 2)} - 1$$

2.0	.9772

$$= 2 \times 0{,}9772 - 1 = 0{,}9544$$