# Linear Algebra with Python

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## Python Review

## Python

```
print(“Hello, World!”)
```

High-level, easy-to-use programming language

You should already be proficient in programming

- Being proficient with Python is a plus, but not strictly necessary

We’ll cover some basics today

## Variables

```
a = 6
b = 7
string_var = “Hello, World!”
also_a_string_var = ‘Hello, World!’
c = a+b
print(c)
>>> 13

print string_var, a, b
>>> Hello, World! 6 7
```

## Lists

```
empty_list = list()
also_empty_list = []
zeros_list = [0] * 5
print(zeros_list)
>>> [0, 0, 0, 0, 0]

empty_list.append(1)
print(empty_list)
>>> [1]

print(len(empty_list))
>>> 1
```

## List Indexing

```
list_var = range(10)
print(list_var)
>>> [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]

print(list_var[4])
>>> 4

print(list_var[4:7])
>>> [4, 5, 6]

print(list_var[0::3]) # Empty index means to the beginning/end
>>> [0, 3, 6, 9]
```

## List Indexing (cont’d)

```
print(list_var)
>>> [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]

print(list_var[-1])
>>> 9

print(list_var[::-1])
>>> [9, 8, 7, 6, 5, 4, 3, 2, 1, 0]

print(list_var[3:1:-1])
>>> [3, 2]
```

## Dictionaries (similar in function to Map in Java)

```
empty_dict = dict()
also_empty_dict = {}
filled_dict = {3: ‘Hello, ’, 4:‘World!’}
print(filled_dict[3] + filled_dict[4])
>>> Hello, World!

filled_dict[5] = ‘New String’
print(filled_dict)
>>> {3: 'Hello, ', 4: 'World!', 5: 'New String'}
```

## Dictionaries (cont’d)

```
print(filled_dict)
>>> {3: 'Hello, ', 4: 'World!', 5: 'New String'}

del filled_dict[3]
print(filled_dict)
>>> {4: 'World!', 5: 'New String'}

print(len(filled_dict))
>>> 2
```

## Functions, Lambda Functions

```
def add_numbers(a, b):
    return a + b

print(add_numbers(3, 4))
>>> 7

lambda_add_numbers = lambda a, b: a + b
print(lambda_add_numbers(3, 4))
>>> 7
```

## Loops, List and Dictionary Comprehensions

```
for i in range(10):
    print(‘Looping %d’ % i)
>>> Looping 0
...
>>> Looping 9

filled_list = [a/2 for a in range(10)]
print(filled_list)
>>> [0, 0, 1, 1, 2, 2, 3, 3, 4, 4]

filled_dict = {a:a**2 for a in range(5)}
print(filled_dict)
>>> {0: 0, 1: 1, 2: 4, 3: 9, 4: 16}
```

## Linear Algebra Review + How to do it in Python

## Why use Linear Algebra ?

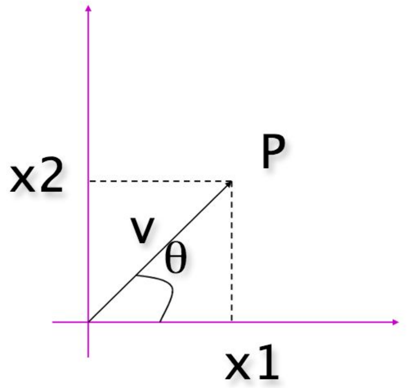
It’s useful to represent many quantities, e.g. 3D points on a scene, 2D points on an image.

Coordinates can be used to perform geometrical transformations and associate 3D points with 2D points (a very common camera operation).

Images are literally matrices filled with numbers.

## Vector Review

$$\mathbf{v} = (x_1, x_2)$$



Magnitude: $\|\mathbf{v}\| = \sqrt{x_1^{\,2} + x_2^{\,2}}$

If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ Is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left( \frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right)$$ Is a unit vector

Orientation: $\theta = \tan^{-1}\left( \frac{x_2}{x_1} \right)$

## Vector Review

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$\mathbf{v} - \mathbf{w} = (x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2)$$

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$

## Matrix Review

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & \boxed{a_{nm}} \end{bmatrix} \longleftrightarrow$$

Pixel's intensity value

Sum: $C_{n\times m} = A_{n\times m} + B_{n\times m}$ $\qquad c_{ij} = a_{ij} + b_{ij}$

A and B must have the same dimensions!

Example: $\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$

## Matrices and Vectors (in Python)

```
import numpy as np
```

A supremely-optimized, well-maintained scientific computing package for Python.

As time goes on, you’ll learn to appreciate NumPy more and more.

Years later I’m **still** learning new things about it!

## Matrices and Vectors (in Python)

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
import numpy as np

M = np.array([[1, 2, 3],
              [4, 5, 6],
              [7, 8, 9]])

v = np.array([[1],
              [2],
              [3]])
```

## Matrices and Vectors (in Python, cont'd)

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
print(M.shape)
>>> (3, 3)

print(v.shape)
>>> (3, 1)

v_single_dim = np.array([1, 2, 3])
print(v_single_dim.shape)
>>> (3,)
```

## Matrices and Vectors (in Python, cont'd)

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
print( v + v)
>>> [[2]
     [4]
     [6]]

print(3*v)
>>> [[3]
     [6]
     [9]]
```

## Other Ways to Create Matrices and Vectors

NumPy provides many convenience functions for creating matrices/vectors.

```
a = np.zeros((2,2))   # Create an array of all zeros
print(a)              # Prints "[[ 0.  0.]
                      #          [ 0.  0.]]"

b = np.ones((1,2))    # Create an array of all ones
print(b)              # Prints "[[ 1.  1.]]"

c = np.full((2,2), 7) # Create a constant array
print(c)               # Prints "[[ 7.  7.]
                       #          [ 7.  7.]]"

d = np.eye(2)         # Create a 2x2 identity matrix
print(d)              # Prints "[[ 1.  0.]
                      #          [ 0.  1.]]"

e = np.random.random((2,2)) # Create an array filled with random values
print(e)                    # Might print "[[ 0.91940167  0.08143941]
                            #               [ 0.68744134  0.87236687]]"
```

## Other Ways to Create Matrices and Vectors (cont'd)

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
v1 = np.array([1, 2, 3])
v2 = np.array([4, 5, 6])
v3 = np.array([7, 8, 9])
M = np.vstack([v1, v2, v3])
print(M)
>>> [[1 2 3]
     [4 5 6]
     [7 8 9]]

# There is also a way to do this horizontally => hstack
```

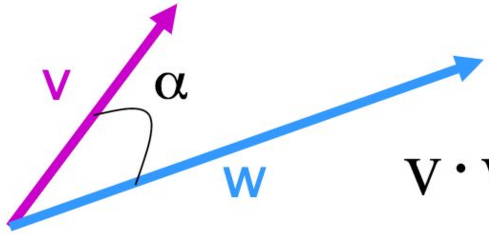
## Matrix Indexing

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
print(M)
>>> [[1 2 3]
     [4 5 6]
     [7 8 9]]

print(M[:2, 1:3])
>>> [[2 3]
     [5 6]]
```

## Dot Product



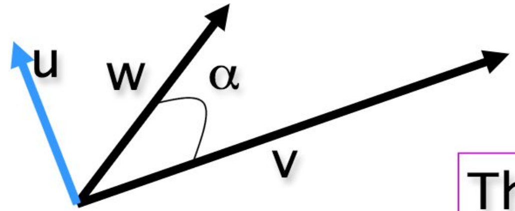
$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2$$

The inner product is a SCALAR!

$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = \| v \| \cdot \| w \| \cos\alpha$$

$$\text{if} \quad v \perp w, \quad v \cdot w = ? = 0$$

## Cross Product



$$u = v \times w$$

The cross product is a VECTOR!

$$\text{Magnitude:} \|u\| = \|v \times w\| = \|v\| \|w\| \sin\alpha$$

Orientation:

$$u \perp v \Rightarrow u \cdot v = (v \times w) \cdot v = 0$$

$$u \perp w \Rightarrow u \cdot w = (v \times w) \cdot w = 0$$

$$\text{if} \quad \mathrm{v} // \mathrm{w}\ ? \quad \rightarrow \mathrm{u} = 0$$

## Cross Product

$$\mathbf{i} = (1,0,0) \qquad \|\mathbf{i}\| = 1 \qquad \mathbf{i} = \mathbf{j} \times \mathbf{k}$$

$$\mathbf{j} = (0,1,0) \qquad \|\mathbf{j}\| = 1 \qquad \mathbf{j} = \mathbf{k} \times \mathbf{i}$$

$$\mathbf{k} = (0,0,1) \qquad \|\mathbf{k}\| = 1 \qquad \mathbf{k} = \mathbf{i} \times \mathbf{j}$$

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = (x_1, x_2, x_3) \times (y_1, y_2, y_3)$$

$$= (x_2 y_3 - x_3 y_2)\mathbf{i} + (x_3 y_1 - x_1 y_3)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}$$

# Matrix Multiplication

$$A_{n\times m} = \begin{bmatrix} \boxed{a_{11} & a_{12} & \dots & a_{1m}} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \mathbf{a}_i \qquad B_{m\times p} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mp} \end{bmatrix}$$

$$\mathbf{b}_j$$

Product:

$$C_{n\times p} = A_{n\times \boxed{m}} B_{\boxed{m}\times p}$$

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = \sum_{k=1}^{m} a_{ik} b_{kj}$$

A and B must have compatible dimensions!

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

## Basic Operations - Dot Multiplication

$$M = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
print(M.dot(v))
>>> [[ 9]
     [-4]
     [ 5]]

print(v.dot(v))
>>> ValueError: shapes (3,1) and (3,1) not aligned: 1 (dim
    1) != 3 (dim 0)

print(v.T.dot(v))
>>> [[14]] # Why these brackets? Because it’s (1,1)-shaped
```

## Basic Operations - Cross Multiplication

$$v_1 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

```
print(v1.cross(v2))
>>> Traceback (most recent call last):
      File "<stdin>", line 1, in <module>
    AttributeError: 'numpy.ndarray' object has no attribute
    'cross'

# Yeah... Slightly convoluted because np.cross() assumes
# horizontal vectors.
print(np.cross(v1, v2, axisa=0, axisb=0).T)
>>> [[-15]
     [ -2]
     [ 39]]
```

## Basic Operations - Element-wise Multiplication

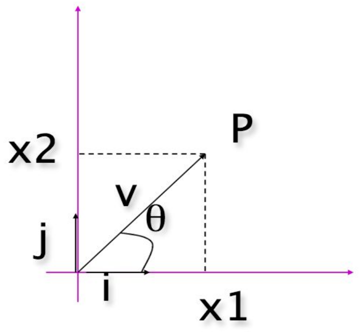
$$M = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
print(np.multiply(M, v))
>>> [[ 3  0  2]
     [ 4  0 -4]
     [ 0  3  3]]

print(np.multiply(v, v))
>>> [[1]
     [4]
     [9]]
```

# Orthonormal Basis

= Orthogonal and Normalized Basis



$$\mathbf{i} = (1,0) \qquad \|\mathbf{i}\| = 1$$

$$\mathbf{j} = (0,1) \qquad \|\mathbf{j}\| = 1$$

$$\mathbf{i} \cdot \mathbf{j} = 0$$

$$\mathbf{v} = (x_1, x_2) \qquad \mathbf{v} = \mathrm{x}_1\mathbf{i} + \mathrm{x}_2\mathbf{j}$$

$$\mathbf{v} \cdot \mathbf{i} = ? \ = (\mathrm{x}_1\mathbf{i} + \mathrm{x}_2\mathbf{j}) \cdot \mathbf{i} = \mathrm{x}_1 1 + \mathrm{x}_2 0 = \mathrm{x}_1$$

$$\mathbf{v} \cdot \mathbf{j} = (\mathrm{x}_1\mathbf{i} + \mathrm{x}_2\mathbf{j}) \cdot \mathbf{j} = \mathrm{x}_1.0 + \mathrm{x}_2.1 = \mathrm{x}_2$$

## Transpose

Definition:

$$\mathbf{C}_{m\times n} = \mathbf{A}^T_{n\times m}$$
$$c_{ij} = a_{ji}$$

Identities:

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$
$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

If $\mathbf{A} = \mathbf{A}^T$, then $\mathbf{A}$ is *symmetric*

# Basic Operations - Transpose

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
print(M.T)
>>> [[1 4 7]
     [2 5 8]
     [3 6 9]]

print(v.T)
>>> [[1 2 3]]

print(M.T.shape, v.T.shape)
>>> (3, 3) (1, 3)
```

## Matrix Determinant

Useful value computed from the elements of a *square* matrix **A**

$$\det \begin{bmatrix} a_{11} \end{bmatrix} = a_{11}$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

## Matrix Inverse

Does not exist for all matrices, necessary (but not sufficient) that the matrix is square

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}, \det \mathbf{A} \neq 0$$

If $\det \mathbf{A} = 0$, $\mathbf{A}$ does not have an inverse.

## Basic Operations - Determinant and Inverse

$$M = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
print(np.linalg.inv(M))
>>> [[ 0.2  0.2  0. ]
     [-0.2  0.3  1. ]
     [ 0.2 -0.3 -0. ]]

Be careful of matrices that are not invertible!

print(np.linalg.det(M))
>>> 10.0 # Thankfully ours is.
```

## Matrix Eigenvalues and Eigenvectors

A eigenvalue $\lambda$ and eigenvector $\mathbf{u}$ satisfies

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

where $\mathbf{A}$ is a square matrix.

- Multiplying $\mathbf{u}$ by $\mathbf{A}$ scales $\mathbf{u}$ by $\lambda$

## Matrix Eigenvalues and Eigenvectors

Rearranging the previous equation gives the system

$$\mathbf{Au} - \lambda\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = 0$$

which has a solution if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

- The eigenvalues are the roots of this determinant which is polynomial in $\lambda$.
- Substitute the resulting eigenvalues back into $\mathbf{Au} = \lambda\mathbf{u}$ and solve to obtain the corresponding eigenvector.

## Basic Operations - Eigenvalues, Eigenvectors

$$M = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

```
eigvals, eigvecs = np.linalg.eig(M)
print(eigvals)
>>> [-1. -2.]

print(eigvecs)
>>> [[ 0.70710678 -0.4472136 ]
     [-0.70710678  0.89442719]]
```

NOTE: Please read the NumPy docs on this function before using it, lots more information about multiplicity of eigenvalues and etc there.

## Singular Value Decomposition

**Singular values**: Non negative square roots of the eigenvalues of $\mathbf{A}^t\mathbf{A}$. Denoted $\sigma_i$, $i=1,\ldots,n$

SVD: If $\mathbf{A}$ is a real $m$ by $n$ matrix then there exist orthogonal matrices $\mathbf{U}$ ($\in \mathbb{R}^{m\times m}$) and $\mathbf{V}$ ($\in \mathbb{R}^{n\times n}$) such that

$$A = U\,\Sigma\,V^{-1} \qquad \mathrm{U}^{-1}\mathrm{AV} = \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & . & \\ & & & \sigma_N \end{bmatrix}$$

## Singular Value Decomposition

Suppose we know the singular values of **A** and we know $r$ are non zero

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq \sigma_{r+1} = \ldots = \sigma_p = 0$$

- Rank(**A**) = $r$.
- Null(**A**) = span$\{\mathbf{v}_{r+1},\ldots,\mathbf{v}_n\}$
- Range(**A**)=span$\{\mathbf{u}_1,\ldots,\mathbf{u}_r\}$

$||\boldsymbol{A}||_F^{\,2} = \sigma_1^{\,2} + \sigma_2^{\,2} + \ldots + \sigma_p^{\,2}$ $\qquad$ $||A||_2 = \sigma_1$

*Numerical rank:* If $k$ singular values of $A$ are larger than a given number $\varepsilon$. Then the $\varepsilon$ rank of A is $k$.

Distance of a matrix of rank $n$ from being a matrix of rank $k = \sigma_{k+1}$

# Singular Value Decomposition

$$M = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
U, S, Vtranspose = np.linalg.svd(M)
print U
>>> [[-0.95123459  0.23048583 -0.20500982]
     [-0.28736244 -0.90373717  0.31730421]
     [-0.11214087  0.36074286  0.92589903]]

print S
>>> [ 3.72021075  2.87893436  0.93368567]

print Vtranspose
>>> [[-0.9215684  -0.03014369 -0.38704398]
     [-0.38764928  0.1253043   0.91325071]
     [ 0.02096953  0.99166032 -0.12716166]]
```

Recall SVD is the factorization of a matrix into the product of 3 matrices, and is formulated like so:

$$M = U\Sigma V^T$$