Robotics 2

Dynamic model of robots:
Algorithm for computing kinetic energy

Prof. Alessandro De Luca

Dipartimento di Ingegneria Informatica Automatica e Gestionale Antonio Ruberti

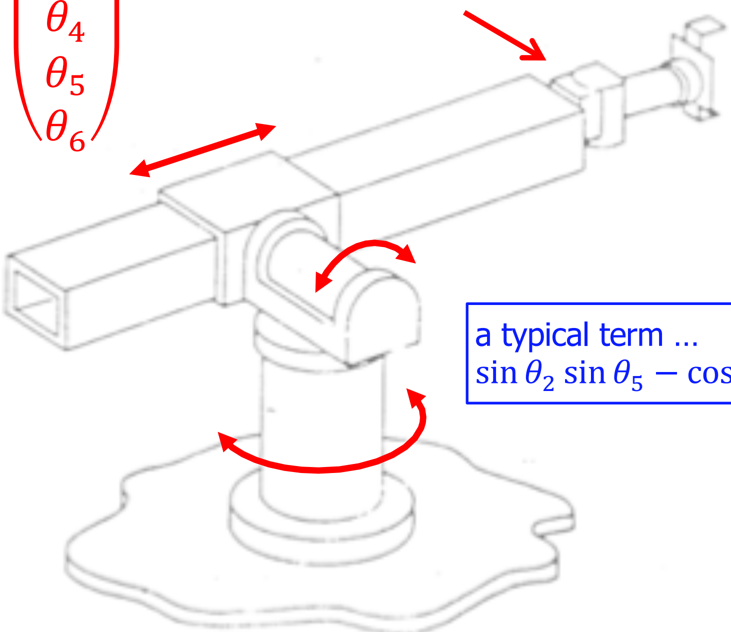
Sapienza Università di Roma

Complexity of robot inertia terms

element $m_{11}(q)$ of Stanford arm

$$q = \begin{pmatrix} \theta_1 \\ \theta_2 \\ d_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{pmatrix}$$

3R spherical wrist



… (derived by hand) in JPL Tech. Memo. 33-669, 1974

$$\begin{aligned}
m_{11} = {} & m_1 k_{122}^2 \\
& + m_2 \left[k_{211}^2 s^2\theta_2 + k_{233}^2 c^2\theta_2 + r_2(2\bar{y}_2 + r_2) \right] \\
& + m_3 \left[k_{322}^2 s^2\theta_2 + k_{333}^2 c^2\theta_2 + r_3(2\bar{z}_3 + r_3) s^2\theta_2 + r_2^2 \right] \\
& + m_4 \left\{ \tfrac{1}{2} k_{411}^2 \left[s^2\theta_2 (2 s^2\theta_4 - 1) + s^2\theta_4 \right] + \tfrac{1}{2} k_{422}^2 (1 + c^2\theta_2 + s^2\theta_4) \right. \\
& \quad \left. + \tfrac{1}{2} k_{433}^2 \left[s^2\theta_2 (1 - 2 s^2\theta_4) - s^2\theta_4 \right] + r_3^2 s^2\theta_2 + r_2^2 - 2\bar{y}_4 r_3 s^2\theta_2 + 2\bar{z}_4 (r_2 s\theta_4 + r_3 s\theta_2 c\theta_2 c\theta_4) \right\} \\
& + m_5 \left\{ \tfrac{1}{2} (-k_{511}^2 + k_{522}^2 + k_{533}^2) \left[(s\theta_2 s\theta_5 - c\theta_2 s\theta_4 c\theta_5)^2 + c^2\theta_4 c^2\theta_5 \right] \right. \\
& \quad + \tfrac{1}{2} (k_{511}^2 - k_{522}^2 + k_{533}^2)(s^2\theta_4 + c^2\theta_2 c^2\theta_4) \\
& \quad + \tfrac{1}{2} (k_{511}^2 + k_{522}^2 - k_{533}^2) \left[(s\theta_2 c\theta_5 + c\theta_2 s\theta_4 s\theta_5)^2 + c^2\theta_4 s^2\theta_5 \right] + r_3^2 s^2\theta_2 + r_2^2 \\
& \quad \left. + 2\bar{z}_5 \left[r_3 (s^2\theta_2 c\theta_5 + s\theta_2 s\theta_4 c\theta_4 s\theta_5) - r_2 c\theta_4 s\theta_5 \right] \right\} \\
& + m_6 \left\{ \tfrac{1}{2} (-k_{611}^2 + k_{622}^2 + k_{633}^2) \left[(s\theta_2 s\theta_5 c\theta_6 - c\theta_2 s\theta_4 c\theta_5 c\theta_6 - c\theta_2 c\theta_4 s\theta_6)^2 + (c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6)^2 \right] \right. \\
& \quad + \tfrac{1}{2} (k_{611}^2 - k_{622}^2 + k_{633}^2) \left[(c\theta_2 s\theta_4 c\theta_5 s\theta_6 - s\theta_2 s\theta_5 s\theta_6 - c\theta_2 c\theta_4 c\theta_6)^2 + (c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6)^2 \right] \\
& \quad + \tfrac{1}{2} (k_{611}^2 + k_{622}^2 - k_{633}^2) \left[(c\theta_2 s\theta_4 s\theta_5 + s\theta_2 c\theta_5)^2 + c^2\theta_4 s^2\theta_5 \right] \\
& \quad + \left[r_6 c\theta_2 s\theta_4 s\theta_5 + (r_6 c\theta_5 + r_3) s\theta_2 \right]^2 + (r_6 c\theta_4 s\theta_5 - r_2)^2 \\
& \quad + 2\bar{z}_6 \left[r_6 (s^2\theta_2 c^2\theta_5 + c^2\theta_4 s^2\theta_5 + c^2\theta_2 s^2\theta_4 s^2\theta_5 + 2 s\theta_2 c\theta_2 s\theta_4 s\theta_5 c\theta_5) \right. \\
& \quad \quad \left. \left. + r_3 (s\theta_2 c\theta_2 s\theta_4 s\theta_5 + s^2\theta_2 c\theta_5) - r_2 c\theta_4 s\theta_5 \right] \right\}
\end{aligned}$$

a typical term … $\sin\theta_2 \sin\theta_5 - \cos\theta_2 \sin\theta_4 \cos\theta_5$

radius of gyration factors k_{ijk}^2 are being used here

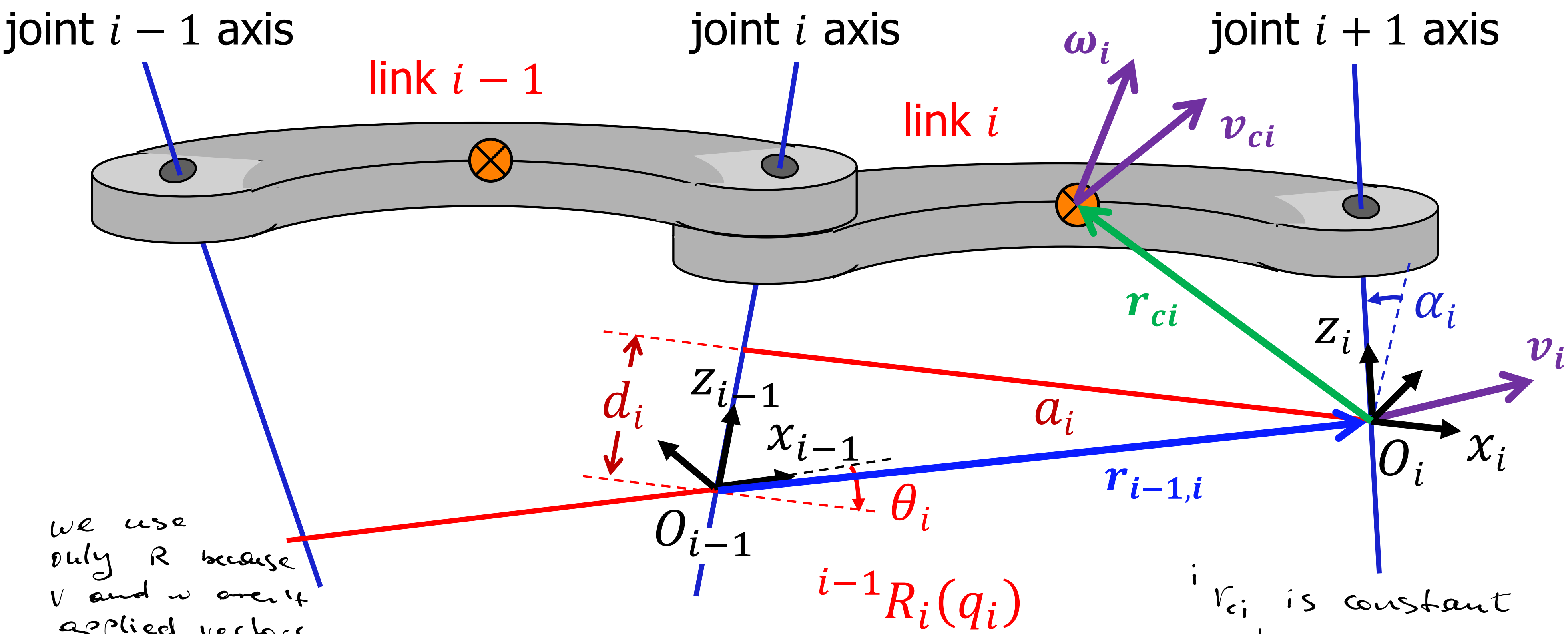
for a body of mass m and moment of inertia I w.r.t. an axis, the radius of gyration k is the distance of a point mass m from the same axis, such that its moment of inertia is I

Expression of v_{ci} and ω_i

- v_{ci} and ω_i can be written using the relations of the robot differential kinematics (partial Jacobians)
- it is useful however to operate in a **recursive** way, expressing each vector quantity related to link i in the "**moving**" **frame** RF$_i$ attached to link i (with the notation **${}^{i}\mathbf{vector}_i$**)
  - particularly convenient when using algebraic/symbolic manipulation languages (Matlab Symbolic Toolbox, Maple, Mathematica, …) for computing the kinetic energy of a (open chain) robot arm, when the number of joints increases (e.g., for $N \geq 4$)

→ Moving Frames

Recall: D-H frames



we use only R because v and ω aren't applied vectors

${}^{i}r_{ci}$ is constant

$$
{}^{i-1}A_i(q_i) = \begin{pmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Moving Frames algorithm

velocity of center of mass of link i → $\boxed{v_{ci} = v_i + \omega_i \times r_{ci}}$ ← position of center of mass of link i w.r.t. O_i

velocity of O_i (origin of RF$_i$)

angular velocity of link i

$$\boxed{\text{set } \sigma_i = \begin{cases} 0 & \text{revolute joint} \\ 1 & \text{prismatic joint} \end{cases}}$$

$${}^{i}\omega_i = {}^{i-1}R_i^T(q_i)\left[{}^{i-1}\omega_{i-1} + (1-\sigma_i)\dot{q}_i\, {}^{i-1}z_{i-1}\right] = {}^{i-1}R_i^T(q_i)\, {}^{i-1}\omega_i$$

z-axis of RF$_{i-1}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$${}^{i}v_i = {}^{i-1}R_i^T(q_i)\left[{}^{i-1}v_{i-1} + \sigma_i\, \dot{q}_i\, {}^{i-1}z_{i-1} + {}^{i-1}\omega_i \times {}^{i-1}r_{i-1,i}\right] = {}^{i-1}R_i^T\, {}^{i-1}v_i$$

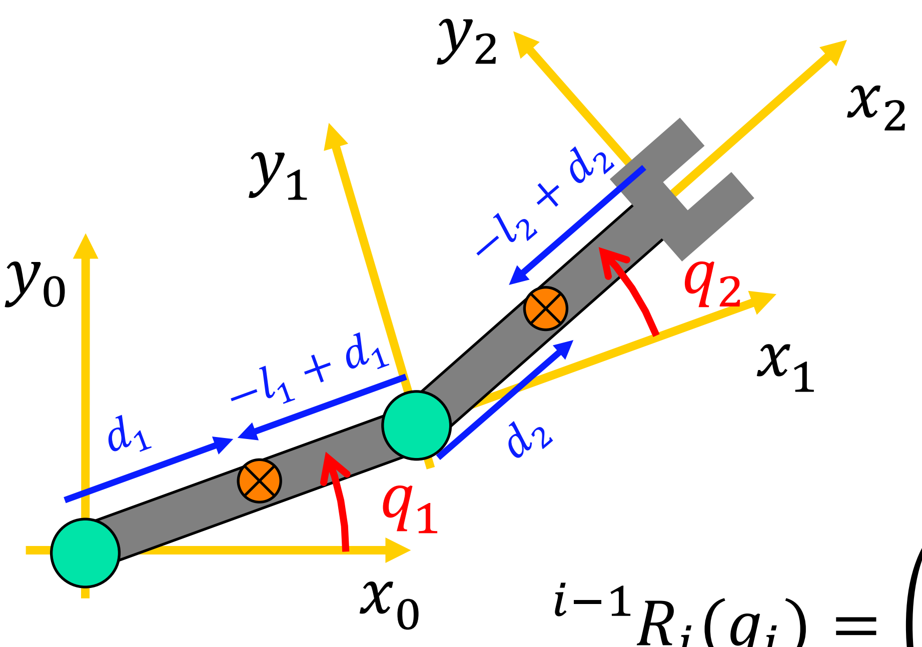
… = ${}^{i}\omega_i$ already computed

prismatic joint addition

… = ${}^{i}r_{i-1,i}$ (constant, if joint i is revolute!)

Dynamic model of a 2R robot

application of the algorithm



$\sigma_i = 0, \qquad i = 1,2$
(revolute joints)

$$g = \begin{pmatrix} 0 \\ -g_0 \\ 0 \end{pmatrix}$$

moving in the vertical plane

$$g_0 = 9.81$$

$$^{i-1}R_i(q_i) = \begin{pmatrix} c_i & -s_i & 0 \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad {}^i r_{ci} = \begin{pmatrix} -l_i + d_i \\ 0 \\ 0 \end{pmatrix}$$

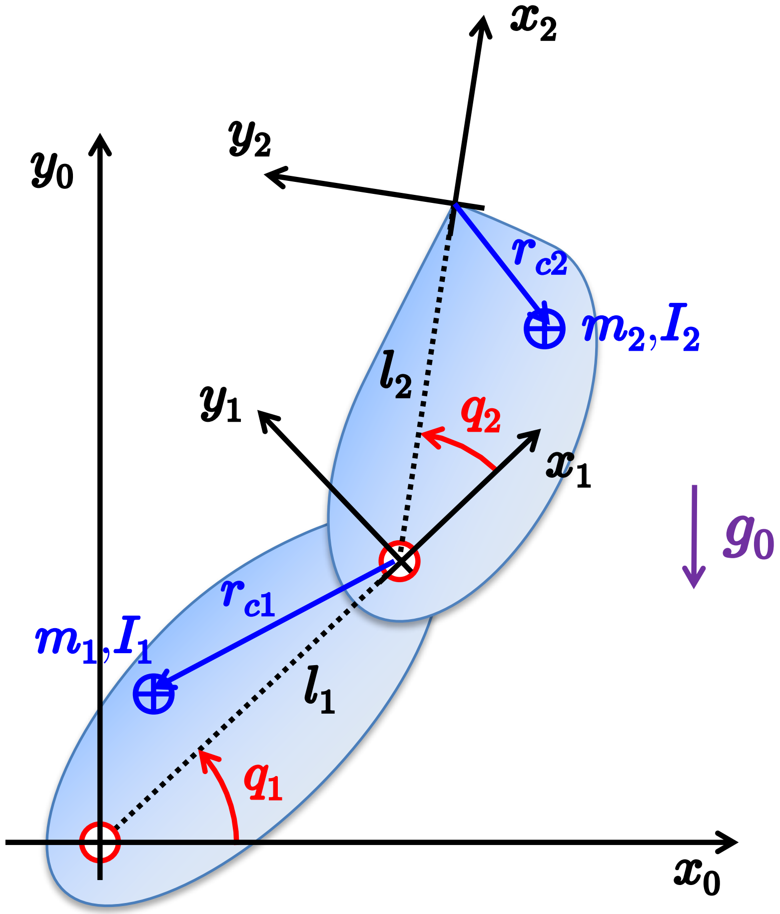
assumption: center of mass of each link is on its kinematic axis

initialization: $i = 0$ $\quad$ $^0\omega_0 = 0$ $\quad$ $^0v_0 = 0$

Dynamic model of a 2R robot

what if the CoM is not on the kinematic axis …

see **Robotics 2 Midterm 2021** (14 April)



$$^{i-1}R_i(q_i) = \begin{pmatrix} \boxed{\begin{matrix} c_i & -s_i \\ s_i & c_i \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

$$^{i}r_{ci} = \begin{pmatrix} \boxed{\begin{matrix} r_{ci,x} \\ r_{ci,y} \end{matrix}} \\ 0 \end{pmatrix}$$

may also work
in 2D (planar case)
$^{i-1}\bar{R}_i(q_i),\ ^{i}\bar{r}_{ci}$

$$T_1 = \frac{1}{2} m_1 \|\boldsymbol{v}_{c1}\|^2 + \frac{1}{2}\boldsymbol{\omega}_1^T \boldsymbol{I}_1 \boldsymbol{\omega}_1$$
$$= \frac{1}{2} m_1 \left((l_1 + r_{c1,x})^2 + r_{c1,y}^2 \right) \dot{q}_1^2 + \frac{1}{2} I_1 \dot{q}_1^2$$

$$^{0}\boldsymbol{p}_{c2} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} + {}^{0}\bar{\boldsymbol{R}}_2(q_1, q_2) \begin{pmatrix} l_2 + r_{c2,x} \\ r_{c2,y} \end{pmatrix}$$

$$^{0}\bar{\boldsymbol{R}}_2(q_1, q_2) = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix}$$ etc …

First step (link 1)

$i = 1$

$$\boxed{{}^{1}\omega_1} = {}^{0}R_1^T(q_1)\left[{}^{0}\omega_0 + \dot{q}_1\begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \boxed{\begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}}$$

$$\boxed{{}^{1}v_1} = {}^{0}R_1^T(q_1)\left[{}^{0}v_0 + \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times\begin{pmatrix}l_1c_1\\l_1s_1\\0\end{pmatrix}\right] = {}^{0}R_1^T(q_1)\begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times{}^{0}R_1^T(q_1)\begin{pmatrix}l_1c_1\\l_1s_1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times\begin{pmatrix}l_1\\0\\0\end{pmatrix} = \boxed{\begin{pmatrix}0\\l_1\dot{q}_1\\0\end{pmatrix}}$$

→ cross product is ⊥ to both vectors

Fundamental kinematic relation

$$\boxed{{}^{1}v_{c1}} = \begin{pmatrix}0\\l_1\dot{q}_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times\begin{pmatrix}-l_1+d_1\\0\\0\end{pmatrix} = \boxed{\begin{pmatrix}0\\d_1\dot{q}_1\\0\end{pmatrix}}$$

Kinetic energy of link 1

$$ {}^{1}\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} \qquad {}^{1}v_{c1} = \begin{pmatrix} 0 \\ d_1 \dot{q}_1 \\ 0 \end{pmatrix} $$

$$ \boxed{T_1 = \frac{1}{2} m_1 d_1^2 \dot{q}_1^2 + \frac{1}{2} I_{c1,zz} \dot{q}_1^2 = \frac{1}{2} \underbrace{\left(I_{c1,zz} + m_1 d_1^2\right)} \dot{q}_1^2} $$

the actual inertia around the rotation axis
of the first joint (parallel axis theorem)

Second step (link 2)

$i = 2$

$$\boxed{{}^2\omega_2} = {}^1R_2^T(q_2)\left[{}^1\omega_1 + \dot{q}_2\begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \boxed{\begin{pmatrix}0\\0\\\dot{q}_1+\dot{q}_2\end{pmatrix}}$$

$$\boxed{{}^2v_2} = {}^1R_2^T(q_2)\left[{}^1v_1 + \begin{pmatrix}0\\0\\\dot{q}_1+\dot{q}_2\end{pmatrix}\times\begin{pmatrix}l_2c_2\\l_2s_2\\0\end{pmatrix}\right] = \begin{pmatrix}l_1s_2\dot{q}_1\\l_1c_2\dot{q}_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot{q}_1+\dot{q}_2\end{pmatrix}\times\begin{pmatrix}l_2\\0\\0\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}l_1s_2\dot{q}_1\\l_1c_2\dot{q}_1+l_2(\dot{q}_1+\dot{q}_2)\\0\end{pmatrix}}$$

Kinetic energy of link 2

$$i = 2$$

$$^2v_{c2} = {}^2v_2 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} -l_2 + d_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2 \dot{q}_1 \\ l_1 c_2 \dot{q}_1 + d_2(\dot{q}_1 + \dot{q}_2) \\ 0 \end{pmatrix}$$

$$T_2 = \frac{1}{2} m_2 \left(l_1^2 \dot{q}_1^2 + d_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2 l_1 d_2 c_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2) \right)$$

$$+ \frac{1}{2} I_{c2,zz} (\dot{q}_1 + \dot{q}_2)^2$$

Robot inertia matrix

$$T = T_1 + T_2 = \frac{1}{2}\begin{pmatrix}\dot q_1 & \dot q_2\end{pmatrix}^T \begin{pmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22} \end{pmatrix} \begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix}$$

$$m_{11}(q) = I_{c1,zz} + m_1 d_1^2 + I_{c2,zz} + m_2 d_2^2 + m_2 l_1^2 + 2 m_2 l_1 d_2 c_2$$

$$= a_1 + 2 a_2 \cos q_2$$

min. 2.07.00

$$m_{12}(q) = m_{21}(q) = I_{c2,zz} + m_2 d_2^2 + m_2 l_1 d_2 c_2 = a_3 + a_2 \cos q_2$$

$$m_{22} = I_{c2,zz} + m_2 d_2^2 = a_3$$

NOTE: introduction of **dynamic coefficients** a_i is a convenient **regrouping** of the dynamic parameters (more on this later → linear parametrization of dynamics)

Centrifugal and Coriolis terms

$$C_1(q) = \frac{1}{2}\left(\frac{\partial M_1}{\partial q} + \left(\frac{\partial M_1}{\partial q}\right)^T - \frac{\partial M}{\partial q_1}\right) = \frac{1}{2}\left(\begin{pmatrix} 0 & -2a_2s_2 \\ 0 & -a_2s_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -2a_2s_2 & -a_2s_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 0 & -a_2s_2 \\ -a_2s_2 & -a_2s_2 \end{pmatrix} \Rightarrow \boxed{c_1(q,\dot{q}) = -a_2s_2(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)}$$

$$C_2(q) = \frac{1}{2}\left(\frac{\partial M_2}{\partial q} + \left(\frac{\partial M_2}{\partial q}\right)^T - \frac{\partial M}{\partial q_2}\right) = \cdots = \begin{pmatrix} a_2s_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{c_2(q,\dot{q}) = a_2s_2\dot{q}_1^2}$$

min 2.14.00

Gravity terms

$$U_1 = -m_1 g^T r_{0,c1} = -\,m_1 (0 \quad -g_0 \quad 0) \begin{pmatrix} * \\ d_1 s_1 \\ * \end{pmatrix} = m_1 g_0\, d_1 s_1$$

$$U_2 = -\,m_2 g^T r_{0,c2} = m_2 g_0\,(l_1 s_1 + d_2 s_{12})$$

$$U = U_1 + U_2$$

$$g(q) = \left(\frac{\partial U}{\partial q}\right)^T = \begin{pmatrix} g_0 (m_1 d_1 c_1 + m_2 l_1 c_1 + m_2 d_2 c_{12}) \\ g_0 m_2 d_2\, c_{12} \end{pmatrix} = \begin{pmatrix} a_4 c_1 + a_5 c_{12} \\ a_5 c_{12} \end{pmatrix}$$

Dynamic model of a 2R robot

$$(a_1 + 2a_2c_2)\ddot{q}_1 + (a_2c_2 + a_3)\ddot{q}_2 - a_2s_2(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2) + a_4c_1 + a_5c_{12} = u_1$$

$$(a_2c_2 + a_3)\ddot{q}_1 + a_3\ddot{q}_2 + a_2s_2\dot{q}_1^2 + a_5c_{12} = u_2$$

$d_2 = 0$ → com at the joint → automatically balanced

Q1: is it $a_2 = 0$ possible? …physical interpretation? …consequences?

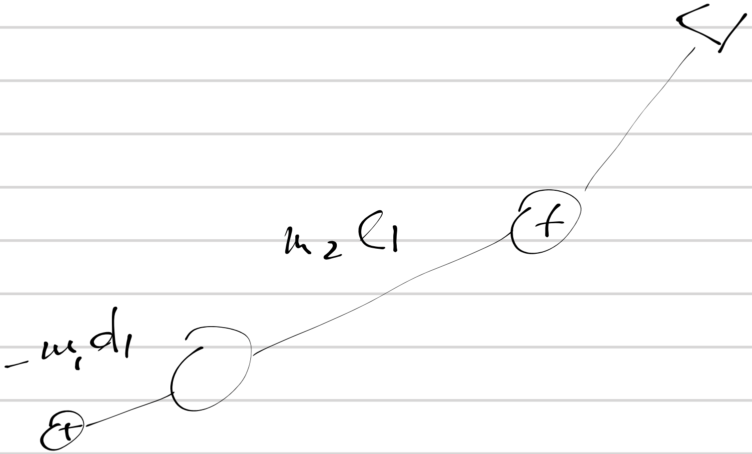
Q2: is it $a_4 = a_5 = 0$ possible as well? …physical interpretation? → CORIOLIS AND CENT DISAPPEAR

Q3: based on the expressions of the dynamic coefficients a_1, a_2, a_3, check that the robot inertia matrix is always positive definite, and in particular that the diagonal elements are always positive ($\forall q$)

Q4: provide two different matrices S' and S'' for the factorization of the quadratic velocity terms, respectively satisfying and not satisfying the skew-symmetry of $\dot{M} - 2S$

$$S = \begin{bmatrix} s_1^T \\ \vdots \\ s_n^T \end{bmatrix} \qquad s_k^T = \dot{q}^T C_k(q)$$

min. 20.30



there's the possibility to self-balance the robot in its structure

by putting com of 1^{st} joint here $a_4 = 0$ → we are eliminating gravity terms

$$\begin{bmatrix} a_1 & a_3 \\ a_3 & a_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} a_4 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$