Robotics 2

Dynamic model of robots: Lagrangian approach

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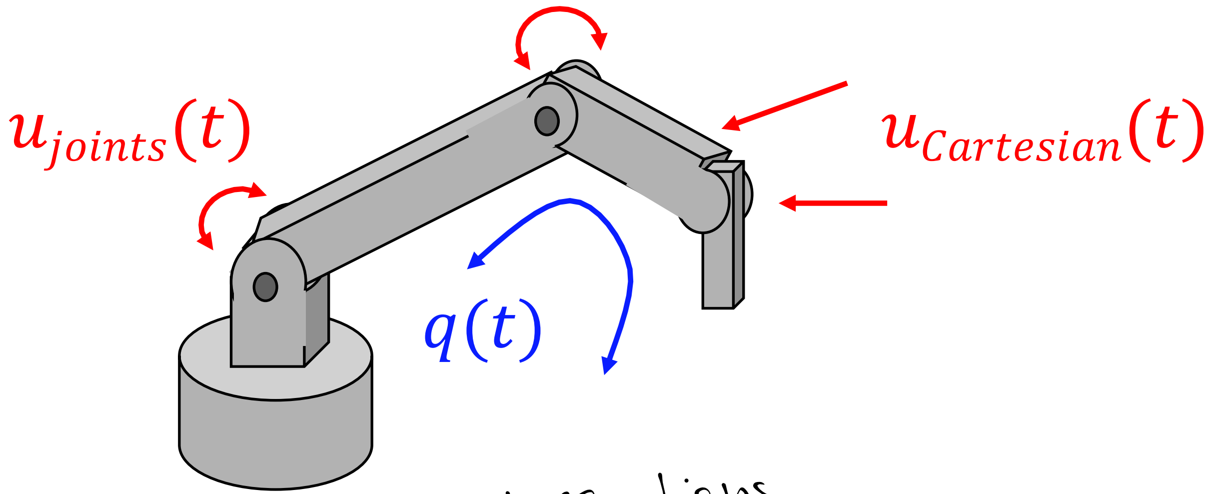
Dynamic model

- provides the relation between

generalized forces $u(t)$ acting on the robot

⇕

robot motion, i.e.,
assumed configurations $q(t)$ over time



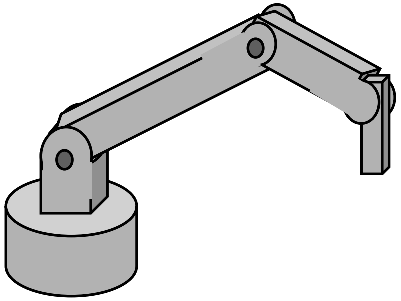
a system of 2nd order differential equations

$$\overbrace{\Phi(q,\dot{q},\ddot{q})}^{n\ \text{equations}} = u$$

Direct dynamics

- direct relation

$$u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \quad \Rightarrow$$



$$\Rightarrow \quad q(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

input for $t \in [0, T]$ + $q(0), \dot{q}(0)$

initial state at $t = 0$

resulting motion

- experimental solution → if you have the robot
  - apply torques/forces with motors and measure joint variables with encoders (with sampling time T_c)
- solution by simulation ⟷ $\boxed{\Phi(q, \dot{q}, \ddot{q}) = u}$ history of movement
  - use dynamic model and integrate numerically the differential equations (with simulation step $T_s \leq T_c$)

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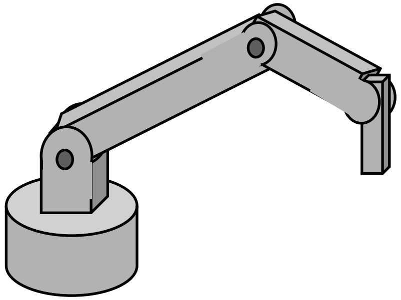
Inverse dynamics

- inverse relation

$q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ →  → $u_d(t)$

desired motion
for $t \in [0, T]$

continuity up
to the acceleration,
in the sense that we
can have a discontinuous
acceleration

required input
for $t \in [0, T]$

- experimental solution
  - repeated motion trials of direct dynamics using $u_k(t)$, with iterative learning of nominal torques updated on trial $k+1$ based on the error in $[0, T]$ measured in trial k: $\lim_{k\to\infty} u_k(t) \Rightarrow u_d(t)$
- analytic solution ⟷ $\boxed{\Phi(q, \dot{q}, \ddot{q}) = u}$
  - use dynamic model and compute algebraically the values $u_d(t)$ at every time instant t

Approaches to dynamic modeling

Euler-Lagrange method
(energy-based approach)

$\rightleftharpoons$

Newton-Euler method
(balance of forces/torques)

- dynamic equations in symbolic/closed form
- best for study of dynamic properties and analysis of control schemes

- dynamic equations in numeric/recursive form
- best for implementation of control schemes (inverse dynamics in real time)

- many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
  - principle of d'Alembert, of Hamilton, of virtual works, Kane's equations …

min 41.30

Euler-Lagrange method
(energy-based approach)

basic assumption: the N links in motion are considered as **rigid bodies**
*(+ later on, include also **concentrated elasticity** at the joints)*

$q \in \mathbb{R}^N$ generalized coordinates (e.g., joint variables, but not only!)

Lagrangian $\boxed{L(q,\dot{q}) = T(q,\dot{q}) - U(q)}$

- T and V are scalars
- They of course are function of the configuration

kinetic energy – potential energy

- principle of least action of Hamilton
- principle of virtual works

minimizing $\int_0^t L(q,\dot{q})\,dt$

n equations

Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \qquad i = 1, \dots, N$$

non-conservative (external or dissipative) generalized forces performing work on q_i

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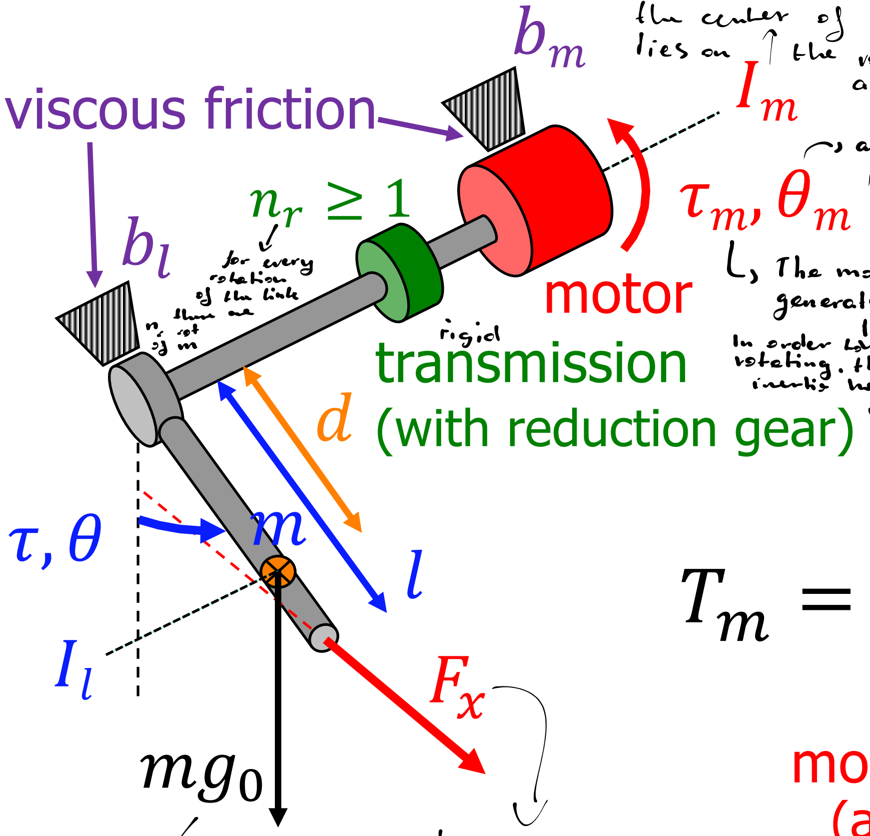
Dynamics of an actuated pendulum

a first example



Irrelevant because the center of mass lies on ↑ the rot axis

↳ angular position
↳ measured by encoders

↳ The motor generates torque
↓
In order to start rotating, the rotor's inertia has to be overcome

n_r ↓ for every rotation of the link there are n_r rot of m

rigid

intrinsic to the configuration

horizontal external force
↓
arbitrarily appliable

$$\dot{\theta}_m = n_r \dot{\theta} \Rightarrow \theta_m = n_r \theta + \cancel{\theta_{m0}} = 0$$

offset between θ_m and θ ↳ can be put to zero

$$\tau = n_r \tau_m$$

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$$\tau \dot{\theta} = \tau_m \dot{\theta}_m = \\ = \tau_m n_r \dot{\theta} \\ \tau = n_r \tau_m$$

$q = \theta$ (or $q = \theta_m$)

$T = T_m + T_l$

(… around the ∥ axis through its base)

$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2 \qquad T_l = \frac{1}{2}(I_l + md^2)\dot{\theta}^2$$

motor inertia (around its spinning axis)

link inertia (around the z-axis through its center of mass…)

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kinetic energy

$$T = \frac{1}{2}(I_l + md^2 + I_m n_r^2)\dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2$$

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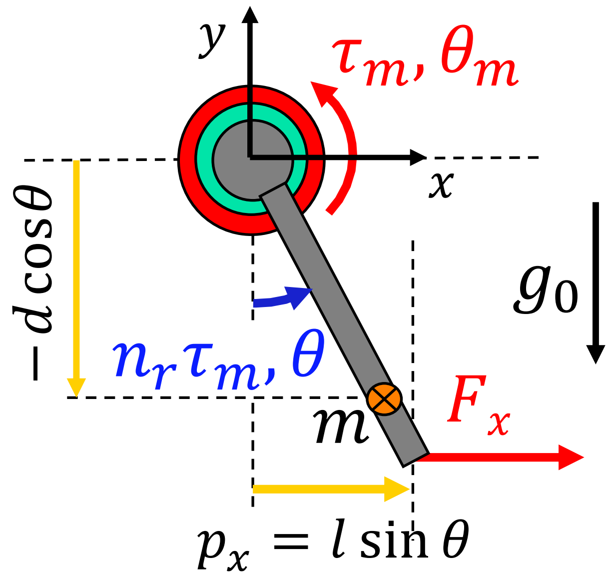
Dynamics of an actuated pendulum (cont)



$p_x = l \sin\theta$

$$\boxed{\dot{p}_x = l\cos\theta \cdot \dot{\theta} = J_x \dot{\theta}}$$

min when $\cos\theta = 1$ => pendulum down or up

$$\boxed{U = U_0 - m g_0 d \cos\theta}$$ potential energy

$$\boxed{L = T - U = \frac{1}{2} I \dot{\theta}^2 + m g_0 d \cos\theta - U_0}$$

no presence of U of the motor => because it is constant

$$\boxed{\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}} \quad \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = I\ddot{\theta}} \quad \boxed{\frac{\partial L}{\partial \theta} = -m g_0 d \sin\theta}$$

Lagrange => $I\ddot{\theta} + mgd\sin\theta = u$ → we still need to compute this → non conservative forces

$$u = n_r\tau_m - b_l\dot{\theta} - n_r b_m \dot{\theta}_m + J_x^T F_x = n_r\tau_m - \underbrace{(b_l + b_m n_r^2)}_{\text{viscous friction}}\dot{\theta} + l\cos\theta\, F_x$$

scalar because the force is only in the x direction

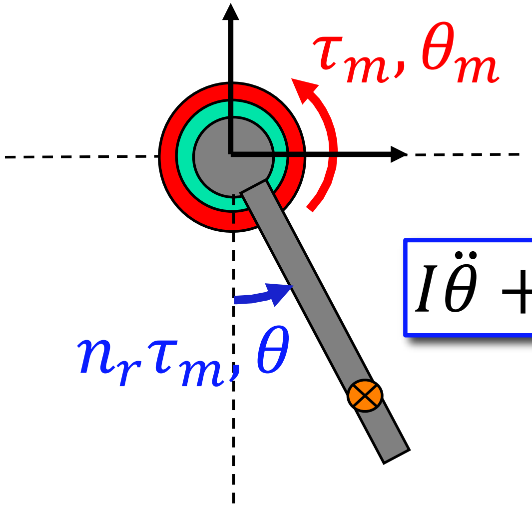
applied or dissipated torques on motor side are multiplied by n_r when moved to the link side

equivalent joint torque due to force F_x applied to the tip at point p_x

no minus because I'm not in the static case anymore

"sum" of non-conservative torques

Dynamics of an actuated pendulum (cont)



dynamic model in $q = \theta$

$$\boxed{I\ddot{\theta} + mg_0 d \sin\theta = n_r \tau_m - (b_l + b_m n_r^2)\dot{\theta} + l\cos\theta \cdot F_x}$$

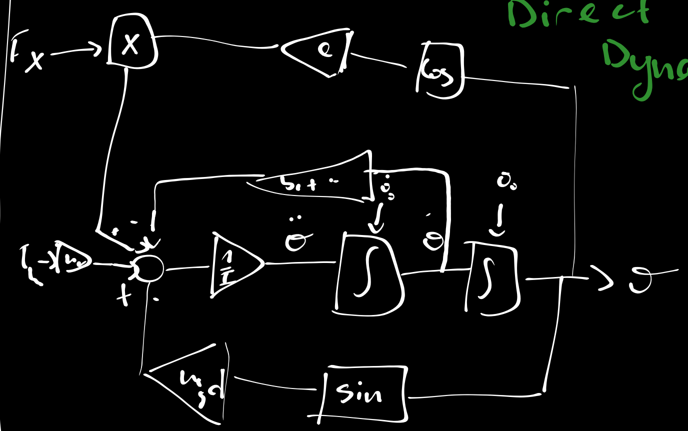
friction link   friction motor

dividing by n_r and substituting $\theta = \theta_m / n_r$

$$\boxed{\frac{I}{n_r^2}\ddot{\theta}_m + \frac{m}{n_r} g_0 d \sin\frac{\theta_m}{n_r} = \tau_m - \left(\frac{b_l}{n_r^2} + b_m\right)\dot{\theta}_m + \frac{l}{n_r}\cos\frac{\theta_m}{n_r} \cdot F_x}$$

dynamic model in $q = \theta_m$

Direct Dynamics



u
$\downarrow$
q

Inverse Dynamics

q
$\downarrow$
u

$\theta_d(t) = A(1-\cos t)$

$\theta_d(0) = 0$

$\dot{\theta}_d(t) = A\sin(t)$

$\dot{\theta}_d(0) = 0$

$\ddot{\theta}_d(t) = A\cos(t)$

We have a redundant input

$t \in [0, T]$

$\uparrow$

10 s

$t = 1.75$

$F_x = 0 \longleftarrow$ First case

$$\tau_{m,d}(t) = \frac{1}{n_r}\left[I\ddot{\theta}_d(t) + mg_0 d\sin\theta_d(t) + (b_\ell + \dots)\dot{\theta}_d(t)\right]$$

- No singularities
- Algebraic substitution → no integration

$\tau_m = 0 \longleftarrow$ Second case

$$F_{x,d}(t) = \frac{1}{l\cos\vartheta_d(t)}\left[I\ddot{\theta}_d(t) + mg_0 d\sin\theta_d(t) + (b_\ell + \dots)\dot{\theta}_d(t)\right]$$

- kinematic singularity at $\pi/2$

Third case

Bad solution

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$n_r\tau_m + l\cos\theta F_x = f(\dots)$

$\min \frac{1}{2}\left\|\begin{matrix}\tau_m \\ F_x\end{matrix}\right\|^2 \longrightarrow$ mixing apples and oranges

Good solution

$\min \frac{1}{2}\left\|\begin{matrix}n_r\tau_m \\ l\cos\theta_d F_x\end{matrix}\right\|$

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$u_1 = \frac{f}{2} \longrightarrow \tau_{md} = \frac{f(\vartheta_d, \dot{\vartheta}_d, \ddot{\vartheta}_d)}{2n_r}$

$u_2 = \frac{f}{2} \longrightarrow \tau_{md} = \frac{f(\dot{\vartheta}_d}{l\cos\vartheta F_x}$

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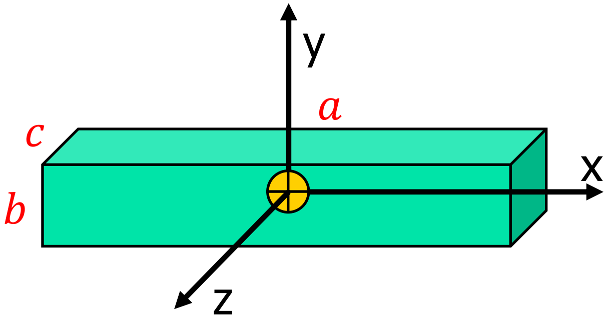
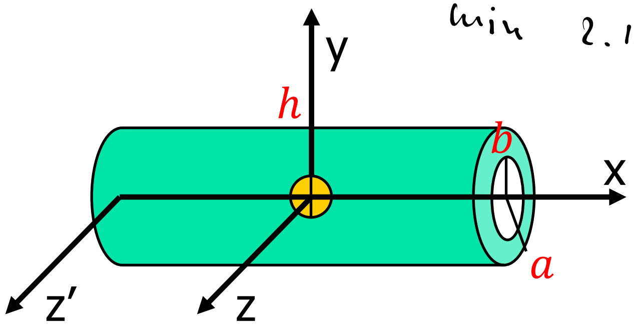
Examples of body inertia matrices

homogeneous bodies of mass m, with axes of symmetry



parallelepiped with sides
a (length/height), b and c (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m(b^2+c^2) & & \\ & \frac{1}{12}m(a^2+c^2) & \\ & & \frac{1}{12}m(a^2+b^2) \end{pmatrix}$$



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empty cylinder with length h,
and external/internal radius a and b

$$I_c = \begin{pmatrix} \frac{1}{2}m(a^2+b^2) & & \\ & \frac{1}{12}m(3(a^2+b^2)+h^2) & \\ & & I_{zz} \end{pmatrix} \qquad I_{zz} = I_{yy}$$

$$I'_{zz} = I_{zz} + m\left(\frac{h}{2}\right)^2$$

(parallel) axis translation theorem

Steiner theorem

displacement of the RF around CoM

$$I = I_c + m(r^Tr \cdot E_{3\times3} - rr^T) = I_c + m\,S^T(r)S(r)$$

body inertia matrix relative to the CoM

identity matrix

Homework: prove last equality

skew-symmetric

… its generalization:
changes on body inertia matrix due to a pure translation r of the reference frame

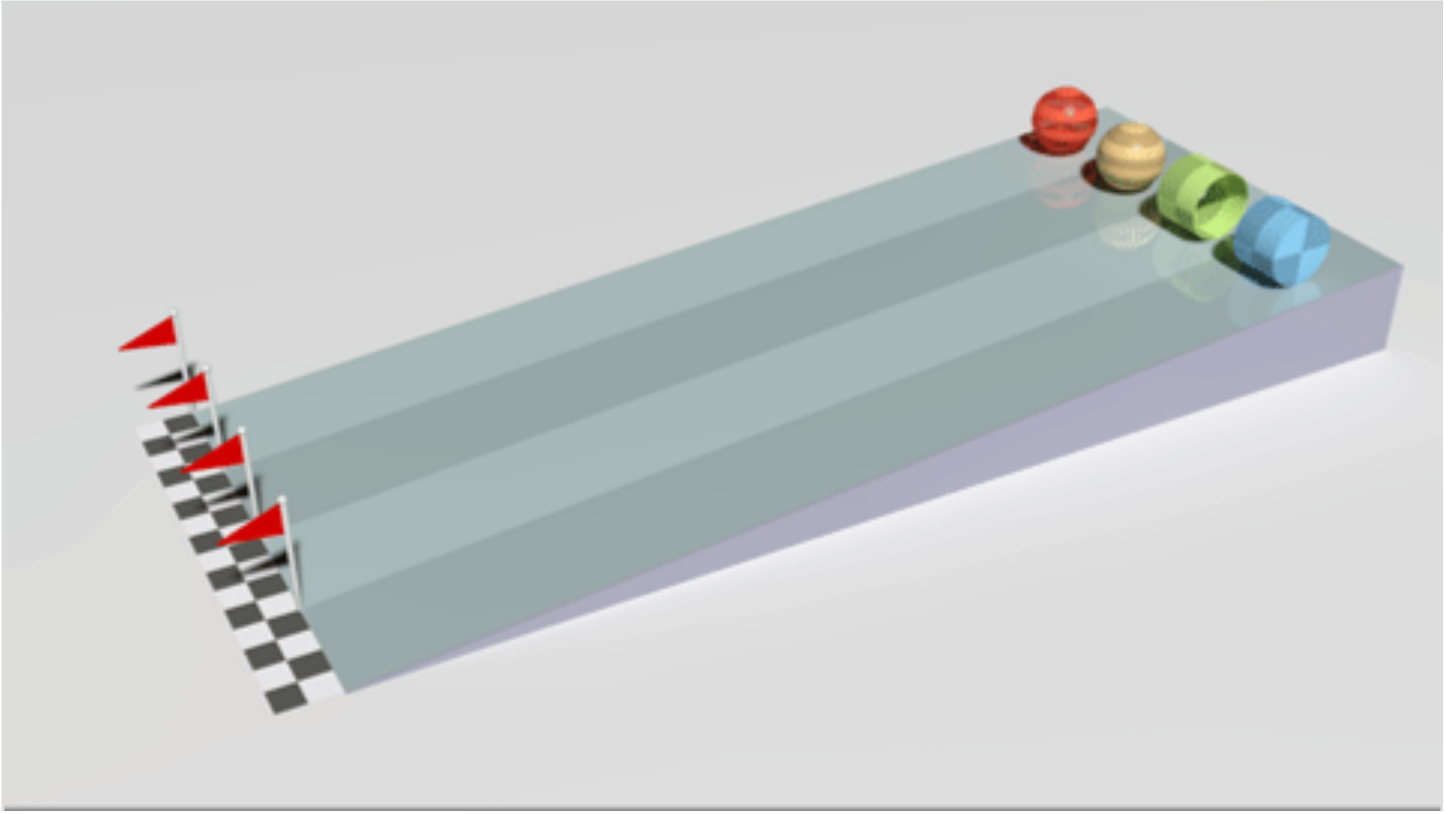
Rolling inertias

https://en.wikipedia.org/wiki/Moment_of_inertia#

4 “circular” bodies
with the same
mass & radius
rolling down
an inclined plane
without slipping

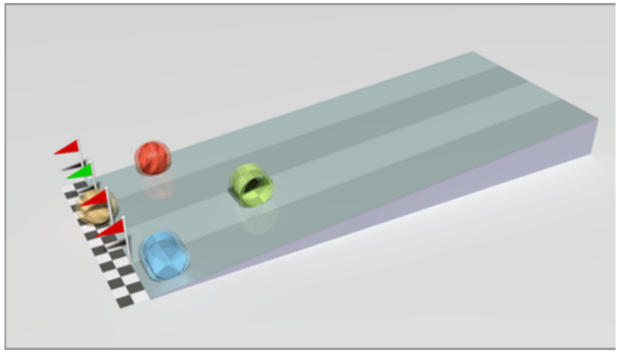
↓

time to reach
the finish line
depends on their
moment of inertia
(about rolling axis!)



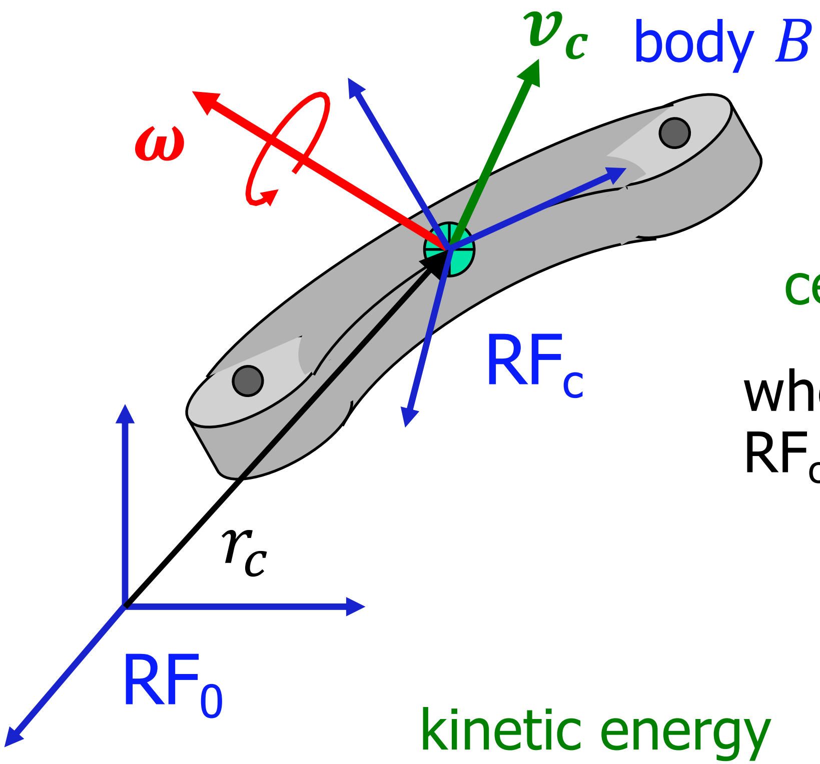
from back to front:

🟥	spherical shell		3rd
🟨	solid sphere	→	1st (smallest)
🟩	cylindrical ring		4th (largest)
🟦	solid cylinder		2nd



• ω pertains to the whole body, while v only to CoM

Kinetic energy of a rigid body



obtained a priori by CAD model

mass density

mass $$m = \int_B \rho(x,y,z)\, dxdydz = \int_B dm$$

position of center of mass (CoM) $$r_c = \frac{1}{m}\int_B r\, dm$$

position of an elementary particle

when all vectors are referred to a body frame $\mathrm{RF_c}$ attached to the CoM, then

$$r_c = 0 \quad \Rightarrow \quad \int_B r\, dm = 0$$

kinetic energy $$T = \frac{1}{2}\int_B v^T(x,y,z)\, v(x,y,z)\, dm$$

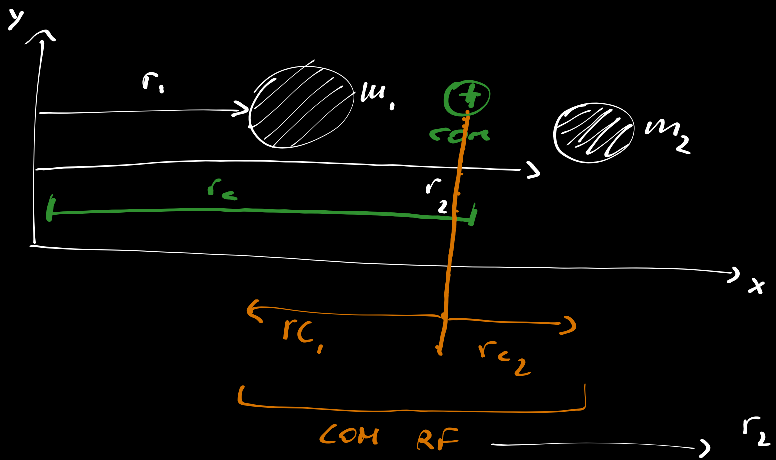
$\frac{1}{2}mv^2$

(fundamental) kinematic relation for a rigid body $$v = v_c + \omega \times r = v_c + S(\omega)\, r$$

distance from CoM

velocity of CoM

skew-symmetric matrix



$$m = m_1 + m_2$$

$$m r_c = m_1 r_1 + m_2 r_2$$

$$r_2 = r_c + r_{c2}$$

$$r_1 = r_c + r_{c1}$$

$$m r_c = m_1 (r_c + r_{c1}) + m_2 (r_c + r_{c2}) =$$

$$= \underbrace{(m_1 + m_2)}_{m} r_c + m_1 r_{c1} + m_2 r_{c2}$$

$$\Downarrow$$

$$\boxed{m_1 r_{c1} + m_2 r_{c2} = 0}$$

$$\boxed{\text{show that } \int r \, dm = 0}$$

Kinetic energy of a rigid body (cont)

$$T = \frac{1}{2}\int_B (v_c + S(\omega)r)^T(v_c + S(\omega)r)\,dm$$

$$= \frac{1}{2}\int_B v_c^T v_c\,dm + \int_B v_c^T S(\omega)\,r\,dm + \frac{1}{2}\int_B r^T S^T(\omega)S(\omega)r\,dm$$

$$\boxed{= \frac{1}{2} m\, v_c^T v_c}$$

$$= v_c^T S(\omega)\int_B r dm = 0$$

$$= \frac{1}{2}\int_B \omega^T S^T(r)S(r)\omega\,dm$$

$$= \frac{1}{2}\omega^T\left(\int_B S^T(r)S(r)\,dm\right)\omega$$

$$\boxed{= \frac{1}{2}\omega^T I_c\,\omega}$$

translational
kinetic energy
(point mass
at CoM)

+

rotational
kinetic energy
(of the whole body) →

body inertia matrix
(around the CoM)

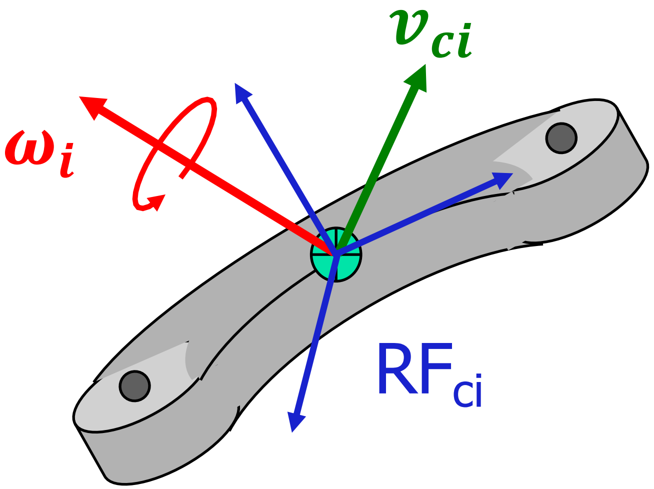
Homework
provide the expression
of the elements
of the inertia matrix I_c

König theorem

Robot kinetic energy

$$T = \sum_{i=1}^{N} T_i$$ ⇐ N rigid bodies (+ fixed base)

$$T_i = T_i(q_j, \dot{q}_j; \underbrace{j \leq i})$$ ⇐ open kinematic chain



i-th link (body)
of the robot

König theorem

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2}\, \omega_i^T I_{ci}\, \omega_i$$

absolute velocity
of the center of mass
(CoM)

absolute
angular velocity
of whole body

Kinetic energy of a robot link

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2}\ \omega_i^T I_{ci}\ \omega_i$$

ω_i, I_{ci} should be expressed in the **same reference frame**, but the product $\omega_i{}^T I_{ci} \omega_i$ is **invariant** w.r.t. any chosen frame

$${}^0\omega_i^T\ {}^0I_{ci}(q)\ {}^0\omega_i = \left(\ {}^0R_i(q)\ \ {}^i\omega_i\ \right)^T\ {}^0I_{ci}(q)\left(\ {}^0R_i(q)\ \ {}^i\omega_i\ \right) = \ {}^i\omega_i^T\left(\ {}^0R_i^T(q)\ {}^0I_{ci}(q)\ {}^0R_i(q)\ \right)\ {}^i\omega_i$$

$$= \ {}^i\omega_i^T\ \ {}^iI_{ci}\ \ {}^i\omega_i$$ ➡ $${}^0I_{ci}(q) = \ {}^0R_i(q)\ \ {}^iI_{ci}\ \ {}^0R_i^T(q)$$

in frame RF$_{ci}$ attached to (the center of mass of) link i

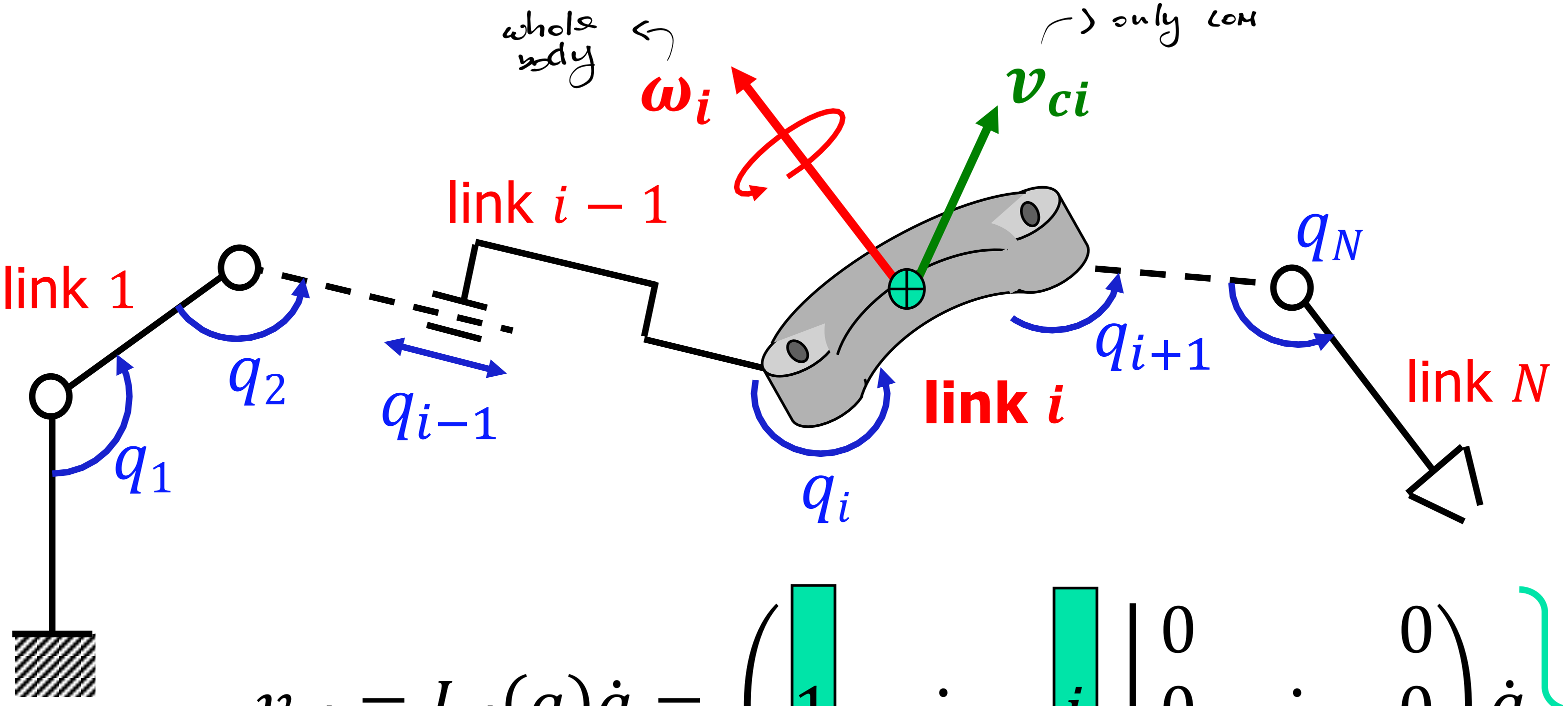
can be computed once for all

and then we use rotation matrices

$${}^iI_{ci} = \begin{pmatrix} \int (y^2+z^2)dm & -\int xy\ dm & -\int xz\ dm \\ & \int (x^2+z^2)dm & -\int yz\ dm \\ symm & & \int (x^2+y^2)dm \end{pmatrix}$$

↑ constant!

Dependence of T from q and $\dot{q}$



$$v_{ci} = J_{Li}(q)\dot{q} = \left(\begin{array}{ccc|ccc} & & & 0 & & 0 \\ 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \end{array}\right)\dot{q} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \text{3 rows}$$

(partial) Jacobians

typically expressed in RF_0

$$\omega_i = J_{Ai}(q)\dot{q} = \left(\begin{array}{ccc|ccc} & & & 0 & & 0 \\ 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \end{array}\right)\dot{q} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \text{3 rows}$$

Final expression of T

$$\boxed{T = \frac{1}{2}\sum_{i=1}^{N}\left(m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci}\,\omega_i\right)}$$

$$= \frac{1}{2}\,\dot{q}^T\left(\sum_{i=1}^{N} m_i\,J_{Li}^T(q) J_{Li}(q) + J_{Ai}^T(q) I_{ci}(q)\,J_{Ai}(q)\right)\dot{q}$$

constant when ω_i is expressed in RF$_{ci}$
else

$$\boxed{{}^0I_{ci}(q) = {}^0R_i(q)\;{}^iI_{ci}\;{}^0R_i^T(q)}$$

$$\boxed{T = \frac{1}{2}\,\dot{q}^T \overset{n\times n}{M(q)}\,\dot{q}}$$

NOTE 1:
in practice, NEVER
use this formula
(or partial Jacobians)
for computing T
⇒ a better method
is available...

NOTE 2:
I used previously
the notation $B(q)$
for the robot
inertia matrix ...
(see past exams!)

robot (generalized) inertia matrix

- symmetric ← belongs to a quadratic form
- positive definite, $\forall q$ ⇒ **always invertible**

↳ 0 only if nothing moves → $\dot{q} = 0$

Robot potential energy

assumption: GRAVITY contribution only

$$U = \sum_{i=1}^{N} U_i$$ ⇐ N rigid bodies (+ fixed base)

$$U_i = U_i(\underbrace{q_j ; j \le i})$$ ⇐ open kinematic chain

$$\boxed{U_i = -m_i g^T r_{0,ci}}$$

{ gravity acceleration vector   position of the center of mass of link i } typically expressed in RF_0

dependence on q

$$\boxed{\begin{pmatrix} r_{0,ci} \\ 1 \end{pmatrix} = {}^{0}A_1(q_1)\,{}^{1}A_2(q_2)\cdots{}^{i-1}A_i(q_i)\begin{pmatrix} r_{i,ci} \\ 1 \end{pmatrix}}$$

constant in RF_i

NOTE: need to work with homogeneous coordinates

Summarizing ...

it's a quadratic form in the velocity, so we can factor out M

Since M is sym $m_{ij} = m_{ji}$

kinetic energy

$$T = \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}\sum_{i,j} m_{ij}(q)\dot{q}_i\dot{q}_j$$

positive definite quadratic form

$$\boxed{\begin{array}{c} T \geq 0, \\ T = 0 \Leftrightarrow \dot{q} = 0 \end{array}}$$

potential energy

$$U = U(q)$$

Lagrangian

$$L = T(q, \dot{q}) - U(q)$$

Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k \qquad k = 1, \dots, N$$

↳ n equations

non-conservative (active/dissipative) generalized forces **performing work** on q_k coordinate

Applying Euler-Lagrange equations

(the scalar derivation – see Appendix for vector format)

Derivations for the k-th eq

$$L(q,\dot{q}) = \frac{1}{2}\sum_{i,j} m_{ij}(q)\dot{q}_i\dot{q}_j - U(q)$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}\dot{q}_j \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial m_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

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(dependences of elements on q are not shown)

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial U}{\partial q_k}$$

LINEAR terms in ACCELERATION $\ddot{q}$

QUADRATIC terms in VELOCITY $\dot{q}$

NONLINEAR terms in CONFIGURATION q

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k-th dynamic equation ...

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

This matrix is in general non symmetric, so we symmetrize it by adding and dividing by 2

$$\sum_j m_{kj}\ddot{q}_j + \sum_{i,j}\left(\frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \frac{\partial U}{\partial q_k} = u_k$$

exchanging "mute" indices i, j

$$\cdots + \sum_{i,j}\underbrace{\frac{1}{2}\left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right)}_{c_{kij} = c_{kji}}\dot{q}_i\dot{q}_j + \cdots$$

Christoffel symbols
of the first kind

K indicates the kth equation (joint)
i and j indicate the product the two velocities

… and interpretation of dynamic terms

rows of M times $\ddot{q}$

$$\boxed{\sum_j m_{kj}(q)\ddot{q}_j} + \boxed{\sum_{i,j} c_{kij}(q)\,\dot{q}_i\dot{q}_j} + \boxed{\frac{\partial U}{\partial q_k}} = u_k \qquad k = 1, \dots, N$$

INERTIAL terms

CENTRIFUGAL ($i = j$) and **CORIOLIS** ($i \neq j$) terms

GRAVITY terms $g_k(q)$

$m_{kk}(q)$ = inertia at joint k when only joint k accelerates ($m_{kk} > 0$!!) POS DEF

$m_{kj}(q)$ = inertia "seen" at joint k when joint j accelerates ($= m_{jk}(q)$)

$c_{kii}(q)$ = coefficient of the centrifugal force at joint k when joint i is moving ($c_{iii} = 0, \forall i$)

$c_{kij}(q)$ = coefficient of the Coriolis force at joint k when joint i and joint j are both moving ($= c_{kji}\ (q)$)

$\frac{\partial M_k}{\partial q}$ is a matrix generally nonsym

↓

we add and divide to symmetrize

Robot dynamic model
in vector formats

1.
$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q)\dot{q}$$

k-th component of vector c

$$C_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$$

↳ only function of q

k-th column of matrix $M(q)$

symmetric matrix!

2.
$$M(q)\ddot{q} + S(q,\dot{q})\dot{q} + g(q) = u$$

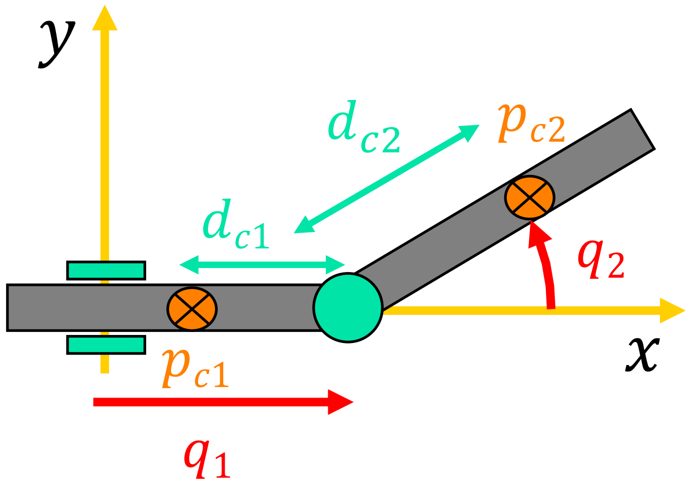
NOTE:
the model
is in the form
$\Phi(q,\dot{q},\ddot{q}) = u$
as expected

NOT a symmetric matrix in general

$$s_{kj}(q,\dot{q}) = \sum_i c_{kij}(q)\dot{q}_i$$

factorization of c by S is **not unique!**

Dynamic model of a PR robot



$$T = T_1 + T_2 \qquad U = \text{constant} \Rightarrow g(q) \equiv 0$$

(on horizontal plane)

$$p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \|v_{c1}\|^2 = \dot{p}_{c1}^T \dot{p}_{c1} = \dot{q}_1^2$$

first link does not rotate, so $\omega = 0$ ←

$$\boxed{T_1 = \frac{1}{2} m_1 \dot{q}_1^2}$$

$$T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2$$

$$p_{c2} = \begin{pmatrix} q_1 + d_{c2} \cos q_2 \\ d_{c2} \sin q_2 \\ 0 \end{pmatrix} \Rightarrow v_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} \sin q_2 \, \dot{q}_2 \\ d_{c2} \cos q_2 \, \dot{q}_2 \\ 0 \end{pmatrix} \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix}$$

ω_2 → direction normal to the plane of motion

parallel axis

$$\boxed{T_2 = \frac{1}{2} m_2 (\dot{q}_1^2 + d_{c2}^2 \, \dot{q}_2^2 - 2 d_{c2} \sin q_2 \, \dot{q}_1 \dot{q}_2) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2}$$

Because the rotation is only in the x-y plane, we are interested only in zz elem ← This is only a scalar, the 3x3 elem

$$M(q)\ddot{q} + c(q,\dot{q}) + \cancel{g(q)} = u$$

$$c_1(q,\dot{q}) = \dot{q}^T C_1(q)\,\dot{q}$$

$$c_2(q,\dot{q}) = \dot{q}^T C_2(q)\,\dot{q}$$

$$\begin{bmatrix} s_1^T(q,\dot{q}) \\ s_2^T(q,\dot{q}) \end{bmatrix} = S(q,\dot{q})\,\dot{q}$$

Dynamic model of a PR robot (cont)

$$M(q) = \left(\underbrace{\boxed{\begin{matrix} m_1 + m_2 \\ -m_2 d_{c2} \sin q_2 \end{matrix}}}_{M_1} \quad \underbrace{\boxed{\begin{matrix} -m_2 d_{c2} \sin q_2 \\ I_{c2,zz} + m_2 d_{c2}^2 \end{matrix}}}_{M_2} \right)$$

$$c(q,\dot{q}) = \begin{pmatrix} c_1(q,\dot{q}) \\ c_2(q,\dot{q}) \end{pmatrix}$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q) \dot{q}$$

where $C_k(q) = \frac{1}{2}\left(\overset{2\times 2}{\frac{\partial M_k}{\partial q}} + \overset{2\times 2}{\left(\frac{\partial M_k}{\partial q}\right)^T} - \overset{2\times 2}{\frac{\partial M}{\partial q_k}} \right)$

$$C_1(q) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

centrifugal force that link 1 feels when link 2 rotates

← centrifugal term ← $c_1(q,\dot{q}) = -m_2 d_{c2} \cos q_2 \, \dot{q}_2^2$

$$C_2(q) = \frac{1}{2}\left(\begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} \right) = 0$$

$c_2(q,\dot{q}) = 0$

Dynamic model of a PR robot (cont)

$$\boxed{M(q)\ddot{q} + c(q,\dot{q}) = u}$$

Q4

$$\underbrace{\begin{bmatrix} 0 & -m_2 d_{c_2} \cos q_2 \, \dot{q}_2 \\ 0 & 0 \end{bmatrix}}_{S} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

S -> these are inf factorizations

$$\begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} \sin q_2 \\ -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d_{c2} \cos q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

-> non cons terms

min 13.20

NOTE: the m_{NN} element (here, for $N = 2$) of $M(q)$ is always constant!

-> the inertia of the last link doesn't depend on the configuration

-> first joint doesn't rotate

Q1: why does variable q_1 not appear in $M(q)$? … this is a general property!

-> q_1 is somehow arbitrary and its choice does not influence the structural properties of the robot

Q2: why Coriolis terms are not present? -> first joint is prismatic

Q3: when applying a force u_1, does the second joint accelerate? … always?

Q4: what is the expression of a factorization matrix S? … is it unique here?

Q5: which is the configuration with "maximum inertia"?

M is posdef => 2 real eigenvalues => min 29.00

$q_d(t)$

1) $q_{1d} = k \quad (\dot{q}_{1d} = \dot{q}_{2d} = 0)$

$q_{2d} = \alpha t \Rightarrow \dot{q}_{2d} = \alpha \, , \ddot{q}_{2d} = 0 \quad t = [0, \dots)$

$$u_d = \begin{pmatrix} u_{1d} \\ u_{2d} \end{pmatrix} = \begin{pmatrix} -m_2 d_{c2} \cos(\alpha t) \alpha^2 \\ 0 \end{pmatrix}$$

Task: keep the first link in its configuration, while rotating the second link

Since we are at initial speed αt without any friction, the second link does not need torque to keep moving, but the first link needs torque in order to stay still

in particular that torque contrasts the centrifugal force from the first joint, in fact there's a minus sign

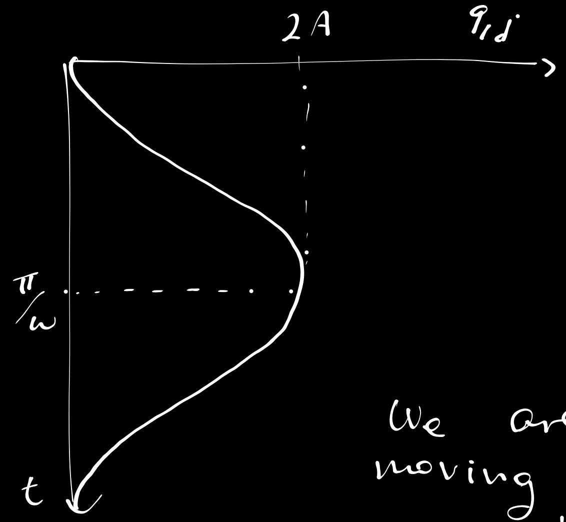
$$q_{1d} = A(1-\cos(\omega t)) \qquad q_{1d}(0) = 0$$

$t \in [0, \dots)$

$$\dot{q}_{1d} = A\omega \sin(\omega t) \qquad \dot{q}_{1d} = 0$$

$$\ddot{q}_{1d} = A\omega^2 \cos(\omega t) \qquad \ddot{q}_{1d}(0) = A\omega^2$$

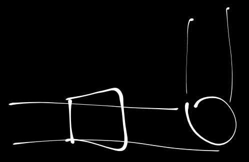
$$q_{2d} = \pi/2 \quad (\dot{q}_2 = \ddot{q}_2 = 0)$$



We are moving up and down the the first link

↓

If I move the first link to the left, the second one falls to the right and viceverse



← Initial configuration

min. 42

$$u_d = \begin{bmatrix} (m_1+m_2) A\omega^2 \cos(\omega t) \\ -m_2 d_{c_2} A\omega^2 \cos(\omega t) \end{bmatrix}$$

→ in order to keep the second joint still, we apply to it a torque with opposite sign to the acceleration

pos acc → clock torque ←

neg acc → counter torque

A structural property

Matrix $\dot{M} - 2S$ is skew-symmetric
(when using Christoffel symbols to define matrix S)

Proof

$$\dot{m}_{kj} = \sum_i \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \qquad 2s_{kj} = \sum_i 2c_{kij}\, \dot{q}_i = \sum_i \left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i$$

⇒ $$\dot{m}_{kj} - 2s_{kj} = \sum_i \left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{ki}}{\partial q_j} \right) \dot{q}_i = n_{kj}$$

$$n_{jk} = \dot{m}_{jk} - 2s_{jk} = \sum_i \left(\frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{ji}}{\partial q_k} \right) \dot{q}_i = -n_{kj}$$

using the symmetry of M

⇒ $$\boxed{x^T \left(\dot{M} - 2S \right) x = 0, \forall x}$$

Energy conservation

- total robot energy

$$E = T + U = \frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q)$$

- its evolution over time (using the dynamic model)

$$\dot{E} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{\partial U}{\partial q}\dot{q}$$

$U =$ scalar
↳ $\frac{\partial U}{\partial q} =$ row

$$= \dot{q}^T\left(u - S(q,\dot{q})\dot{q} - g(q)\right) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T g(q)$$

$$= \dot{q}^T u + \frac{1}{2}\dot{q}^T\left(\dot{M}(q) - 2S(q,\dot{q})\right)\dot{q}$$

i am conservative

here, any factorization of vector c by a matrix S can be used

- if $u \equiv 0$, total energy is constant (no dissipation or increase)

$$\dot{E} = 0 \Rightarrow \boxed{\dot{q}^T\left(\dot{M}(q) - 2S(q,\dot{q})\right)\dot{q} = 0, \forall q, \dot{q}} \Rightarrow \dot{E} = \dot{q}^T u$$

weaker property than skew-symmetry, as the external vector in the quadratic form is the same velocity $\dot{q}$ that appears also inside the two internal matrices $\dot{M}$ and S

in general, the variation of the total energy is equal to the work of non-conservative forces

Appendix
dynamic model: alternative vector format derivation

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)^T - \left(\frac{\partial L}{\partial q}\right)^T = u}$$

$$L = \frac{1}{2}\dot q^T M(q)\dot q - U(q)$$

$(0 \quad \cdots \quad 1 \quad \cdots \quad 0)$

i-th position

$$M(q) = \begin{pmatrix} M_1(q) & \cdots & M_i(q) & \cdots & M_N(q) \end{pmatrix} = \sum_{i=1}^{N} M_i(q) e_i^T$$

dyadic expansion

$$\left(\frac{\partial L}{\partial \dot q}\right)^T = (\dot q^T M(q))^T = M(q)\dot q$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)^T = M(q)\ddot q + \dot M(q)\dot q = M(q)\ddot q + \sum_{i=1}^{N}\left(\frac{\partial M_i}{\partial q}\right)\dot q \dot q_i$$

$$\left(\frac{\partial L}{\partial q}\right)^T = \left(\frac{1}{2}\dot q^T\left(\sum_{i=1}^{N}\frac{\partial M_i(q)}{\partial q}\, e_i^T\right)\dot q - \frac{\partial U(q)}{\partial q}\right)^T = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial M_i}{\partial q}\right)^T \dot q_i\, \dot q - \left(\frac{\partial U}{\partial q}\right)^T$$

this construction gives to $\dot M - 2S$ skew-symmetry

$$\Rightarrow \quad M(q)\ddot q + \underbrace{\left(\sum_{i=1}^{N}\left(\frac{\partial M_i}{\partial q} - \frac{1}{2}\left(\frac{\partial M_i}{\partial q}\right)^T\right)\dot q_i\right)}_{S(q,\dot q)}\dot q + \underbrace{\left(\frac{\partial U}{\partial q}\right)^T}_{g(q)} = u$$

k-th row of matrix S

$$S_k^T(q,\dot q) = \dot q^T C_k(q) \longrightarrow S(q,\dot q)$$