Solutions of Lectures of Calculus

Contents

1. Chapter 1

1.1. Exercises 1.1.

EXERCISE 1.1.1. *Suppose* $f(x) = \frac{x^3+8}{x+2}$, *find* $\lim\limits_{x\to -2} f(x)$.

Sol.

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{(x+2)(x^2-2x+4)}{x+2} = \lim_{x\to -2} \left(x^2-2x+4\right) = 4. \qquad \blacksquare$$

EXERCISE 1.1.2. *Suppose* $f(x) = \frac{|x|}{x}$, *find* $\lim\limits_{x\to 0} f(x)$.

Sol.
$f(x) = \frac{|x|}{x} = \begin{cases} 1, \forall\ x > 1 \\ -1, \forall\ x < 1 \end{cases}$. Since $\lim\limits_{x\to 0^+} f(x) = 1 \neq -1 = \lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0} f(x)$ dose not exist. $\blacksquare$

EXERCISE 1.1.3. *Suppose* $f(x) = 3^{|x|}$, *find* $\lim\limits_{x\to -1} f(x)$.

Sol.

$$\lim_{x\to -1} f(x) = 3^{|-1|} = 3^1 = 3. \qquad \blacksquare$$

EXERCISE 1.1.4. *Suppose* $f(x) = \pi^3$, *find* $\lim\limits_{x\to 3\pi} f(x)$.

Sol.

$$\lim_{x\to 3\pi} f(x) = \pi^3. \qquad \blacksquare$$

1.2. Exercises 1.2.

EXERCISE 1.2.1. *Prove that* $\lim\limits_{x\to -2}(1-x) = 3$.

Sol.
Consider $|(1-x)-3| = |-x-2| = |x+2|$. Given $\varepsilon > 0$, take $\delta = \varepsilon > 0$ such that if $0 < |x-(-2)| = |x+2| < \delta$, then $|(1-x)-3| = |x+2| < \delta = \varepsilon$. Hence $\lim\limits_{x\to -2}(1-x) = 3$. $\blacksquare$

EXERCISE 1.2.2. *Prove that* $\lim\limits_{x\to 3}(4x-5) = 7$.

Sol.
Consider $|(4x-5)-7| = |4x-12| = 4|x-3|$. Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{4} > 0$ such that if $0 < |x-3| < \delta$, then $|(4x-5)-7| = 4|x-3| < 4\delta = 4 \times \frac{\varepsilon}{4} = \varepsilon$. Hence $\lim\limits_{x\to 3}(4x-5) = 7$. ∎

EXERCISE 1.2.3. *Prove that* $\lim\limits_{x\to 2} \frac{x^2-4}{x-2} = 4$.

Sol.
Consider

$$\left|\frac{x^2-4}{x-2} - 4\right| = \left|\frac{(x^2-4)-4(x-2)}{x-2}\right| = \left|\frac{x^2-4x+4}{x-2}\right| = \left|\frac{(x-2)^2}{x-2}\right| = |x-2|,$$

when $x \neq 2$. Given $\varepsilon > 0$, take $\delta = \varepsilon > 0$ such that if $0 < |x-2| < \delta$, then $\left|\frac{x^2-4}{x-2} - 4\right| = |x-2| < \delta = \varepsilon$. Hence $\lim\limits_{x\to 2} \frac{x^2-4}{x-2} = 4$. ∎

EXERCISE 1.2.4. *Prove that* $\lim\limits_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

Sol.
Consider $\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{2|x|}$, so if $0 < |x-2| < 1$ implies $1 < x < 3$, then $\frac{1}{|x|} < 1$. Given $\varepsilon > 0$, take $\delta = \min\{1, 2\varepsilon\} > 0$ such that if $0 < |x-2| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{2|x|} < \frac{|x-2|}{2} < \frac{\delta}{2} \le \frac{2\varepsilon}{2} = \varepsilon.$$

Hence $\lim\limits_{x\to 2} \frac{1}{x} = \frac{1}{2}$. ∎

EXERCISE 1.2.5. *Prove that* $\lim\limits_{x\to 4}(x^2+x-4) = 16$.

Sol.
Consider

$$\left|\left(x^2+x-4\right)-16\right| = \left|x^2+x-20\right| = |(x-4)(x+5)|,$$

so if $0 < |x-4| < 1$ implies $3 < x < 5$, then $|x+5| < 10$. Given $\varepsilon > 0$, take $\delta = \min\left\{1, \frac{\varepsilon}{10}\right\} > 0$ such that if $0 < |x-4| < \delta$, then

$$\left|\left(x^2+x-4\right)-16\right| = |(x-4)(x+5)| < 10|x-4| < 10\delta \le 10 \times \frac{\varepsilon}{10} = \varepsilon.$$

Hence $\lim\limits_{x\to 4}(x^2+x-4) = 16$. ∎

EXERCISE 1.2.6. *Prove that* $\lim\limits_{x\to c} f(x) = 0$ *if and only if* $\lim\limits_{x\to c} |f(x)| = 0$.

Sol.

proof of $(\Rightarrow)$

Given any $\varepsilon > 0$, since $\lim\limits_{x\to c} f(x) = 0$, $\exists\ \delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x)-0| < \varepsilon$. So if $0 < |x-c| < \delta$, then $|f(x)-0| = |f(x)| = ||f(x)|| = ||f(x)|-0| < \varepsilon$. Hence $\lim\limits_{x\to c} |f(x)| = 0$.

proof of $(\Leftarrow)$

Given any $\varepsilon > 0$, since $\lim\limits_{x\to c} |f(x)| = 0$, $\exists\ \delta > 0$ such that if $0 < |x-c| < \delta$, then $||f(x)|-0| < \varepsilon$. So if $0 < |x-c| < \delta$, then $||f(x)|-0| = ||f(x)|| = |f(x)| = |f(x)-0| < \varepsilon$. Hence $\lim\limits_{x\to c} f(x) = 0$. ■

EXERCISE 1.2.7. *True or false:*
(a) If $\lim\limits_{x\to c} f(x) = L$*, then* $\lim\limits_{x\to c} |f(x)| = |L|$*.*
(b) If $\lim\limits_{x\to c} |f(x)| = L$*, then* $\lim\limits_{x\to c} f(x) = L$ *or* $-L$*.*

Sol.

(a) True.

Given any $\varepsilon > 0$, since $\lim\limits_{x\to c} f(x) = L$, $\exists\ \delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x)-L| < \varepsilon$. So if $0 < |x-c| < \delta$, then $||f(x)|-|L|| < |f(x)-L| < \varepsilon$. Hence $\lim\limits_{x\to c} |f(x)| = |L|$. ■

(b) False.

Consider the function $f(x) = \begin{cases} 1\ , \forall\ x \geq 0 \\ -1, \forall\ x < 0 \end{cases}$, then $|f(x)| = 1, \forall\ x \in \mathbb{R}$. So $\lim\limits_{x\to 0} |f(x)| = 1$, but $\lim\limits_{x\to 0} f(x)$ does not exist. ■

EXERCISE 1.2.8. *Suppose* $f(x) = \begin{cases} x, & x \in \mathbb{Q}\ , \\ 0, & x \notin \mathbb{Q}\ , \end{cases}$ *prove that* $\lim\limits_{x\to 0} f(x) = 0$*.*

Sol.

Consider $|f(x)-0| = |f(x)| \leq |x|\ , \forall\ x \in \mathbb{R}$. Given $\varepsilon > 0$, take $\delta = \varepsilon$ such that if $0 < |x-0| = |x| < \delta$, then $|f(x)-0| \leq |x| < \delta = \varepsilon$. Hence $\lim\limits_{x\to 0} f(x) = 0$. ■

EXERCISE 1.2.9. *Suppose* $f(x) = \begin{cases} 1\ , & x \in \mathbb{Q}\ , \\ -1, & x \notin \mathbb{Q}\ , \end{cases}$ *prove that* $\lim\limits_{x\to 0} f(x)$ *does not exist.*

You may use the fact about the density property of rational(irrational) numbers: for any $a, b \in \mathbb{R}$ *and* $a < b$*, then* $\exists\ r \in \mathbb{Q}$ *(* $\exists\ q \notin \mathbb{Q}$ *) such that* $a < r < b$ *(* $a < q < b$ *).*

Sol.

Suppose $\lim\limits_{x\to 0} f(x)$ exists, that is, $\lim\limits_{x\to 0} f(x) = L$ for some $L \in \mathbb{R}$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x-0| = |x| < \delta$, then $|f(x)-L| < \varepsilon$.

So if $0 < |x| < \delta$, then $L - \varepsilon < f(x) < L + \varepsilon$. Now we choose $\frac{1}{2} > 0$, $\exists\ \delta_0 > 0$ such that if $0 < |x| < \delta_0$, then $L - \frac{1}{2} < f(x) < L + \frac{1}{2}$. By the density property of rational numbers and the density property of irrational numbers, we can find a rational number $x_1 \in \mathbb{Q}$ such that $0 < |x_1| < \delta_0$ and an irrational number $x_2 \notin \mathbb{Q}$ such that $0 < |x_2| < \delta_0$. Hence we have $L - \frac{1}{2} < f(x_1) < L + \frac{1}{2}$ and $L - \frac{1}{2} < f(x_2) < L + \frac{1}{2}$. However, since $f(x_1) = 1$ and $f(x_2) = -1$, the length of interval $\left(L - \frac{1}{2}, L + \frac{1}{2}\right)$ is $(L + \frac{1}{2}) - \left(L - \frac{1}{2}\right) = 1$, but $|f(x_1) - f(x_2)| = 2$. This is a contradiction ($f(x_1)$ and $f(x_2)$ could not belong to the interval $\left(L - \frac{1}{2}, L + \frac{1}{2}\right)$ at the same time). Hence $\lim\limits_{x \to 0} f(x)$ does not exist. ∎

Exercise 1.2.10. *Prove that* $\lim\limits_{x \to 0} \frac{|x|}{x}$ *does not exist.*

Sol.

Let $f(x) = \frac{|x|}{x}$. Suppose $\lim\limits_{x \to 0} f(x)$ exists, that is, $\lim\limits_{x \to 0} f(x) = L$, for some $L \in \mathbb{R}$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 0| = |x| < \delta$, then $|f(x) - L| < \varepsilon$. So if $-\delta < x < 0$ and $0 < x < \delta$, then $L - \varepsilon < f(x) < L + \varepsilon$. Now we choose $\frac{1}{2} > 0$, $\exists\ \delta_0 > 0$ such that if for any $-\delta_0 < x < 0$ and $0 < x < \delta_0$, then $L - \frac{1}{2} < f(x) < L + \frac{1}{2}$. Choose a number a with $0 < a < \delta_0$ and a number b with $-\delta_0 < b < 0$ such that $L - \frac{1}{2} < f(a) < L + \frac{1}{2}$ and $L - \frac{1}{2} < f(b) < L + \frac{1}{2}$. Notice that $f(a) = \frac{|a|}{a} = \frac{a}{a} = 1$ and $f(b) = \frac{|b|}{b} = \frac{-b}{b} = -1$, the length of interval $\left(L - \frac{1}{2}, L + \frac{1}{2}\right)$ is $(L + \frac{1}{2}) - \left(L - \frac{1}{2}\right) = 1$, but $|f(a) - f(b)| = 2$. This is a contradiction($f(a)$ and $f(b)$ could not belong to the interval $\left(L - \frac{1}{2}, L + \frac{1}{2}\right)$ at the same time). Hence $\lim\limits_{x \to 0} f(x)$ does not exist. ∎

Exercise 1.2.11. *Does* $\lim\limits_{x \to 0} \sin(\frac{1}{x})$ *exist? If so, find it.*

Sol.

Let $f(x) = \sin(\frac{1}{x})$ and let $a_n = \frac{2}{(4n+1)\pi}$ and $b_n = \frac{2}{(4n+3)\pi}$, $n \in \mathbb{N}$. Then $f(a_n) = \sin(\frac{1}{a_n}) = 1$ and $f(b_n) = \sin(\frac{1}{b_n}) = -1$.

Suppose $\lim\limits_{x \to 0} f(x)$ exists, that is, $\lim\limits_{x \to 0} f(x) = L$ for some $L \in \mathbb{R}$. Then for $\epsilon = \frac{1}{2} > 0$, $\exists\ \delta > 0$ such that if $0 < |x - 0| = |x| < \delta$, then $L - \frac{1}{2} < f(x) < L + \frac{1}{2}$. Now choose $n_0 \in \mathbb{N}$ such that $\frac{(4n+3)\pi}{2} > \frac{(4n+3)\pi}{2} > \frac{1}{\delta}$,then we have $0 < |a_{n_0}| < \delta$ and $0 < |b_{n_0}| < \delta$. Hence $L - \frac{1}{2} < f(a_{n_0}) < L + \frac{1}{2}$ and $L - \frac{1}{2} < f(b_{n_0}) < L + \frac{1}{2}$. However, since $f(a_{n_0}) = 1$ and $f(b_{n_0}) = -1$, the length of interval $\left(L - \frac{1}{2}, L + \frac{1}{2}\right)$ is $(L + \frac{1}{2}) - \left(L - \frac{1}{2}\right) = 1$, but $|f(a_{n_0}) - f(b_{n_0})| = 2$. This is a contradiction ($f(a_{n_0})$ and $f(b_{n_0})$ could not belong to the interval $\left(L - \frac{1}{2}, L + \frac{1}{2}\right)$ at the same time). Hence $\lim\limits_{x \to 0} f(x)$ does not exist. ∎

1.3. Exercises 1.3.

EXERCISE 1.3.1. *Suppose* $f(x) \leq g(x)$, $\forall\, x \in \mathbb{R}$, *and* $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = M$, *prove that* $L \leq M$.

Sol.

Suppose $L > M$. Since $\lim\limits_{x\to c} f(x) = L$, for $\frac{L-M}{2} > 0$, there exists $\delta_1 > 0$, such that if $0 < |x-c| < \delta_1$, then $|f(x) - L| < \frac{L-M}{2}$. So if $0 < |x-c| < \delta_1$, then $\frac{L+M}{2} < f(x) < \frac{3L-M}{2}$. And since $\lim\limits_{x\to c} g(x) = M$, for $\frac{L-M}{2} > 0$, there exists $\delta_2 > 0$, such that if $0 < |x-c| < \delta_2$, then $|g(x) - M| < \frac{L-M}{2}$. So if $0 < |x-c| < \delta_2$, then $\frac{-L+3M}{2} < g(x) < \frac{L+M}{2}$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$, so if $0 < |x-c| < \delta$, then $g(x) < \frac{L+M}{2} < f(x)$. This is a contradiction that $f(x) \leq g(x)$, $\forall\, x \in \mathbb{R}$. Hence $L \leq M$. ■

EXERCISE 1.3.2. *Evaluate the following limits,*

(a) $\lim\limits_{x\to 2}(2x^3 + 4x^2 - x + 6)$.

(b) $\lim\limits_{x\to -1} \frac{x^3-3x+7}{1-2x}$.

(c) $\lim\limits_{x\to 1}(\frac{x^2}{x-1} - \frac{1}{x-1})$.

(d) $\lim\limits_{x\to 16} \frac{\sqrt{x}-4}{x-16}$.

(e) $\lim\limits_{x\to -4} (x+3)^{20}$.

Sol.

(a) Since $\lim\limits_{x\to 2} x^3 = 8$, $\lim\limits_{x\to 2} x^2 = 4$ and $\lim\limits_{x\to 2} x = 2$, by sum rule and constant multiple, we have

$$\begin{aligned} \lim_{x\to 2}(2x^3 + 4x^2 - x + 6) &= 2\lim_{x\to 2} x^3 + 4\lim_{x\to 2} x^2 - \lim_{x\to 2} x + 6 \\ &= 2\cdot 8 + 4\cdot 4 - 2 + 6 = 36. \qquad \blacksquare \end{aligned}$$

(b) Since $\lim\limits_{x\to -1} (x^3 - 3x + 7) = 9$ and $\lim\limits_{x\to -1} (1 - 2x) = 3 \neq 0$, by quotient rule, we have

$$\lim_{x\to -1} \frac{x^3 - 3x + 7}{1 - 2x} = \frac{9}{3} = 3. \qquad \blacksquare$$

(c)

$$\begin{aligned} \lim_{x\to 1}(\frac{x^2}{x-1} - \frac{1}{x-1}) &= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} \\ &= \lim_{x\to 1} (x+1) = 1 + 1 = 2. \qquad \blacksquare \end{aligned}$$

(d)

$$\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}=\lim_{x\to 16}\frac{\sqrt{x}-4}{(\sqrt{x}+4)(\sqrt{x}-4)}=\lim_{x\to 16}\frac{1}{\sqrt{x}+4}=\frac{1}{4+4}=\frac{1}{8}. \qquad \blacksquare$$

(e) Since $\lim\limits_{x\to -4}(x+3)=-1$, by product rule, we have

$$\lim_{x\to -4}(x+3)^{20}=\left(\lim_{x\to -4}(x+3)\right)^{20}=(-1)^{20}=1. \qquad \blacksquare$$

Exercise 1.3.3. *Suppose $f(x)$ and $g(x)$ are real function on $\mathbb{R}$, $c\in\mathbb{R}$. True or false:*

(a) If $\lim\limits_{x\to c}f(x)$ and $\lim\limits_{x\to c}g(x)$ both do not exist, then $\lim\limits_{x\to c}(f(x)+g(x))$ does notexist.

(b) If $\lim\limits_{x\to c}f(x)=L$, for some $L\in\mathbb{R}$, but $\lim\limits_{x\to c}g(x)$ does not exist, then$\lim\limits_{x\to c}(f(x)+g(x))$ does not exist.

(c) If $\lim\limits_{x\to c}f(x)=0$ and $f(x)\neq 0$, for all $x\in\mathbb{R}$, then $\lim\limits_{x\to c}\frac{1}{f(x)}$ does not exist.

(d) If $\lim\limits_{x\to c}f(x)=L$, for some $0\neq L\in\mathbb{R}$, and $\lim\limits_{x\to c}g(x)=0$, and $g(x)\neq 0$, forall $x\in\mathbb{R}$, then $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$ does not exist.

(e) If $\lim\limits_{x\to c}\sqrt{f(x)}=L$, for some $L\in\mathbb{R}$, then $\lim\limits_{x\to c}f(x)=L^2$.

Sol.

(a) False.

Consider fhe functions $f(x)=\begin{cases}-1\ ,\ x\geq 0\\ \ \ 1\ ,\ x<0\end{cases}$ and $g(x)=\begin{cases}\ \ 1\ ,\ x\geq 0\\ -1\ ,\ x<0\end{cases}$. Neither $\lim\limits_{x\to 0}f(x)$ nor $\lim\limits_{x\to 0}g(x)$ exists, but $\lim\limits_{x\to 0}(f(x)+g(x))=\lim\limits_{x\to 0}0=0.$ $\blacksquare$

(b) True.

Suppose $\lim\limits_{x\to c}(f(x)+g(x))=M$ exists. Since $\lim\limits_{x\to c}f(x)=L$ exists, by sum rule, we have $\lim\limits_{x\to c}g(x)=\lim\limits_{x\to c}((f(x)+g(x))-f(x))=\lim\limits_{x\to c}(f(x)+g(x))-\lim\limits_{x\to c}f(x)=M-L$ exists. This is a contradiction that $\lim\limits_{x\to c}g(x)$ does not exist. Hence $\lim\limits_{x\to c}(f(x)+g(x))$ does not exist. $\blacksquare$

(c) True.

Suppose $\lim\limits_{x\to c}\frac{1}{f(x)}=L$ exists. Since $\lim\limits_{x\to c}f(x)=0$ exists, by product rule, we have $1=\lim\limits_{x\to c}1=\lim\limits_{x\to c}\left(f(x)\cdot\frac{1}{f(x)}\right)=\lim\limits_{x\to c}f(x)\cdot\lim\limits_{x\to c}\frac{1}{f(x)}=0\cdot L=0$. This is a contradiction that $0\neq 1$. Hence $\lim\limits_{x\to c}\frac{1}{f(x)}$ does not exist. $\blacksquare$

(d) True.

Suppose $\lim_{x\to c}\frac{f(x)}{g(x)} = M$ exists. Since $\lim_{x\to c} g(x) = 0$ exists, by product rule, we have $L = \lim_{x\to c} f(x) = \lim_{x\to c}\left(\frac{f(x)}{g(x)}\cdot g(x)\right) = \lim_{x\to c}\frac{f(x)}{g(x)}\cdot \lim_{x\to c} g(x) = M\cdot 0 = 0$. This is a contradiction that $L \neq 0$. Hence $\lim_{x\to c}\frac{f(x)}{g(x)}$ does not exist. ■

(e) True.

Since $\lim_{x\to c}\sqrt{f(x)} = L$ exists and we have $f(x) = \sqrt{f(x)}\cdot\sqrt{f(x)}$, by product rule, we get $\lim_{x\to c} f(x) = \lim_{x\to c}\left(\sqrt{f(x)}\cdot\sqrt{f(x)}\right) = \lim_{x\to c}\sqrt{f(x)}\cdot\lim_{x\to c}\sqrt{f(x)} = L\cdot L = L^2$. ■

1.4. Exercises 1.4.

EXERCISE 1.4.1. *Prove that $f(x) = \sqrt{x}$ is continuous on $[0,\infty)$. (Hint: $f(x)$ is right continuous at $x = 0$.)*

Sol.

Claim 1: $f(x)$ is continuous on $(0,\infty)$.

For any $c \in (0,\infty)$, consider

$$|f(x) - f(c)| = \left|\sqrt{x} - \sqrt{c}\right| = \frac{|\sqrt{x} - \sqrt{c}|\,|\sqrt{x} + \sqrt{c}|}{|\sqrt{x} + \sqrt{c}|} = \frac{|x - c|}{|\sqrt{x} + \sqrt{c}|} \leq \frac{|x - c|}{\sqrt{c}}.$$

Given $\varepsilon > 0$, take $\delta_1 = \sqrt{c}\varepsilon > 0$, then if $|x - c| < \delta$, we have

$$|f(x) - f(c)| \leq \frac{|x - c|}{\sqrt{c}} < \frac{1}{\sqrt{c}}\cdot\delta = \frac{1}{\sqrt{c}}\cdot\sqrt{c}\varepsilon = \varepsilon.$$

Hence $f(x)$ is continuous on $(0,\infty)$. ■

Claim 2: $f(x)$ is right continuous at $x = 0$.

Consider $|f(x) - f(0)| = |\sqrt{x} - 0| = \sqrt{x}$. Given $\varepsilon > 0$, take $\delta_2 = \varepsilon^2 > 0$, then if $0 \leq x - 0 < \delta$, we have

$$|f(x) - f(0)| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon.$$

Hence $f(x)$ is right continuous at $x = 0$. ■

Hence by claim 1 and claim 2, $f(x) = \sqrt{x}$ is continuous on $[0,\infty)$. ■

EXERCISE 1.4.2. *Determine the continuity of the following functions at the indicated points.*

(a) $f(x) = \sqrt{2x - 5}$, $c = 4$.

(b) $f(x) = \frac{\sqrt[3]{x+4}}{x-7}$, $c = 7$.

(c) $f(x) = \begin{cases} \sqrt{4-x}, & x \le 4, \\ x-4, & x > 4, \end{cases}$ $c = 4$.
(d) $f(x) = |x|$, $c = 0$.

Sol.
(a) First, we consider

$$\begin{aligned} |f(x) - f(4)| &= \left|\sqrt{2x-5} - \sqrt{3}\right| = \frac{\left|\sqrt{2x-5} - \sqrt{3}\right| \left|\sqrt{2x-5} + \sqrt{3}\right|}{\left|\sqrt{2x-5} + \sqrt{3}\right|} \\ &= \frac{|(2x-5) - 3|}{\left|\sqrt{2x-5} + \sqrt{3}\right|} \le \frac{2}{\sqrt{3}} |x-4|, \end{aligned}$$

for all $x \ge \frac{5}{2}$. Given $\varepsilon > 0$, take $\delta = \frac{\sqrt{3}}{2}\varepsilon > 0$ such that if $|x-4| < \delta$ then

$$|f(x) - f(4)| \le \frac{2}{\sqrt{3}} |x-4| < \frac{2}{\sqrt{3}}\delta = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}\varepsilon = \varepsilon.$$

Hence $f(x)$ is continuous at $x = 4$. ■

(b) Since $x = 7$ is not in the domaian of the function f (f is not defined at $x = 7$), $f(x)$ is not continuous at $x = 7$. ■

(c) First, we consider $|f(x) - f(4)|$. If $x \le 4$, $|f(x) - f(4)| = \left|\sqrt{4-x} - 0\right| = \left|\sqrt{4-x}\right|$. If $x > 4$, $|f(x) - f(4)| = |(x-4) - 0| = |x-4|$. Given $\varepsilon > 0$, take $\delta = \min\{\varepsilon^2, \varepsilon\} > 0$ such that if $|x-4| < \delta$, then $|f(x) - f(4)| < \varepsilon$. Hence $f(x)$ is continuous at $x = 4$.

(d) First, we consider $|f(x) - f(0)| = ||x| - 0| = ||x|| = |x|$, for all $x \in \mathbb{R}$. Given $\varepsilon > 0$, take $\delta = \varepsilon > 0$ such that if $|x - 0| < \delta$ then $|f(x) - f(0)| = |x| < \delta = \varepsilon$. Hence $f(x)$ is continuous at $x = 0$. ■

EXERCISE 1.4.3. *Investigate the continuity of the following functions,*
(a) $f(x) = \frac{2x^4+3x-7}{\sin x}$.
(b) $f(x) = [x]$, *where* [ ] *is the Gauss symbol.*

Sol.
(a) We have known the polynomial $2x^4 + 3x - 7$ is continuous on $\mathbb{R}$, the function $\sin x$ is also continuous on $\mathbb{R}$ and $\sin x = 0$ if $x = n\pi$ for all $n \in \mathbb{Z}$. By quotient rule, $f(x) = \frac{2x^4+3x-7}{\sin x}$ is continuous on $\mathbb{R}\backslash\{n\pi | n \in \mathbb{Z}\}$. ■

(b) Since $\lim\limits_{x\to c^+} f(x) = c$ and $\lim\limits_{x\to c^-} f(x) = c - 1$ for all $c \in \mathbb{Z}$. And $\lim\limits_{x\to c} f(x) = [c] = f(c)$ for all $c \in \mathbb{R}\backslash\mathbb{Z}$, hence $f(x)$ is continuous on $\mathbb{R}\backslash\mathbb{Z}$. ■

EXERCISE 1.4.4. *Supoose* $f(x) = \begin{cases} cx+1, & x \leq 2, \\ cx^2 - 3, & x > 2, \end{cases}$ *determine the value of* c *such that the function* $f(x)$ *will be continuous at* $x = 2$.

Sol.
If we want $f(x)$ is continuous at $x = 2$, it must be $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$. Where $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (cx+1) = 2c+1$ and $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (cx^2 - 3) = 4c - 3$. So, $2c + 1 = 4c - 3$ implies $c = 2$. ■

1.5. Exercises 1.5.

Evaluate the following limits:

EXERCISE 1.5.1. $\lim\limits_{x \to 0} \frac{\sin 3x}{4x}$.

Sol.

$$\lim_{x \to 0} \frac{\sin 3x}{4x} = \lim_{x \to 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3x}{4x} \right) = \frac{3}{4} \cdot \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{4} \cdot 1 = \frac{3}{4}. \qquad \blacksquare$$

EXERCISE 1.5.2. $\lim\limits_{x \to 0} \frac{\sin 3x}{\sin 4x}$.

Sol.

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 3x}{\sin 4x} &= \lim_{x \to 0} \left(\frac{\sin 3x}{3x} \cdot \frac{4x}{\sin 4x} \cdot \frac{3}{4} \right) = \frac{3}{4} \cdot \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \lim_{x \to 0} \frac{4x}{\sin 4x} \\ &= \frac{3}{4} \cdot 1 \cdot 1 = \frac{3}{4}. \qquad \blacksquare \end{aligned}$$

EXERCISE 1.5.3. $\lim\limits_{x \to 0} \frac{x}{\tan x}$.

Sol.

$$\lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \left(\frac{x}{\sin x} \cdot \frac{\cos x}{1} \right) = \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \cos x = 1 \cdot 1 = 1. \qquad \blacksquare$$

EXERCISE 1.5.4. $\lim\limits_{x \to 0} \frac{1 - \cos x}{2 \sin x}$.

Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{2\sin x} &= \lim_{x\to 0}\left(\frac{1}{2}\cdot\frac{1-\cos x}{x}\cdot\frac{x}{\sin x}\right)\\ &= \frac{1}{2}\cdot\lim_{x\to 0}\frac{1-\cos x}{x}\cdot\lim_{x\to 0}\frac{x}{\sin x}=\frac{1}{2}\cdot 0\cdot 1=0. \qquad \blacksquare\end{aligned}$$

EXERCISE 1.5.5. $\lim\limits_{x\to 0}\frac{\sin x}{2x+\tan x}$.

Sol.
Consider

$$\begin{aligned}\lim_{x\to 0}\frac{2x+\tan x}{\sin x} &= \lim_{x\to 0}\left(\frac{2x}{\sin x}+\frac{\tan x}{\sin x}\right)=\lim_{x\to 0}\left(2\frac{x}{\sin x}+\frac{1}{\cos x}\right)\\ &= 2\lim_{x\to 0}\frac{x}{\sin x}+\lim_{x\to 0}\frac{1}{\cos x}=2\cdot 1+1=3\neq 0.\end{aligned}$$

By reciprocal rule, we have

$$\lim_{x\to 0}\frac{\sin x}{2x+\tan x}=\lim_{x\to 0}\frac{1}{\frac{2x+\tan x}{\sin x}}=\frac{1}{\lim\limits_{x\to 0}\frac{2x+\tan x}{\sin x}}=\frac{1}{3}. \qquad \blacksquare$$

EXERCISE 1.5.6. $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$.

Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}=\lim_{x\to 0}\left(\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{1+\cos x}\right)\\ &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\frac{1}{1+\cos x}=1^2\cdot\frac{1}{2}=\frac{1}{2}. \qquad \blacksquare\end{aligned}$$

EXERCISE 1.5.7. $\lim\limits_{x\to 0}\frac{\cos(2x)-1}{x^2}$.

Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{\cos(2x)-1}{x^2} &= \lim_{x\to 0}\frac{1-2\sin^2 x-1}{x^2}=\lim_{x\to 0}\frac{-2\sin^2 x}{x^2}\\ &= \lim_{x\to 0}-2\left(\frac{\sin x}{x}\right)^2=-2. \qquad \blacksquare\end{aligned}$$

1.6. Exercises 1.6.

EXERCISE 1.6.1. *Suppose that* $f(x) = x^3 + x + 1$, *prove that there exists* $c \in \mathbb{R}$ *such that* $f(c) = 100$.

Sol.
Since $f(0) = 1 < 100$, $f(10) = 1011 > 100$ and $f(x)$ is continuous on $[1, 10]$, by intermediate value theorem, there exists $c \in (1, 10)$ such that $f(c) = 100$. ■

EXERCISE 1.6.2. *Suppose that* $f : [0, 1] \to [0, 1]$ *is a continuous function, prove that there exists* $c \in [0, 1]$ *such that* $f(c) = c$.

Sol.
Case1: $f(0) = 0$ or $f(1) = 1$.
In this case, it's nothing further to prove.
Case2: $f(0) \neq 0$ and $f(1) \neq 1$.
Since $f(0) \neq 0$ and $f(1) \neq 1$, $0 < f(0) \leq 1$ and $0 \leq f(1) < 1$. Let $g(x) = f(x) - x$, $\forall\ x \in [0, 1]$. Since $f(x)$ is continuous on $[0, 1]$, so is $g(x)$. We have $g(0) = f(0) - 0 = f(0) > 0$, $g(1) = f(1) - 1 < 0$ and $g(x)$ is continuous on $[0, 1]$. By intermediate value theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$. Where $0 = g(c) = f(c) - c$ implies $f(c) = c$ for $c \in (0, 1)$. ■

2. Chapter 2

2.1. Exercises 2.1.

EXERCISE 2.1.1. *Suppose* $f(x)=\begin{cases} 2x & , \ x\geq 1, \\ x^2+1, & x<1, \end{cases}$ *find* $f'(1)$.

Sol.
By definition, $f'(1)=\lim\limits_{h\to 0}\frac{f(1+h)-f(1)}{h}$, so we have

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{2\,(1+h)-2}{h}=\lim_{h\to 0^+}\frac{2h}{h}=\lim_{h\to 0^+}2=2$$

and

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{\left((1+h)^2+1\right)-2}{h}=\lim_{h\to 0^-}\frac{h^2+2h}{h}=\lim_{h\to 0^+}(h+2)=2.$$

Hence

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{\left((1+h)^2+1\right)-2}{h}=2=f'(1). \qquad \blacksquare$$

EXERCISE 2.1.2. *Suppose* $f(x)=\begin{cases} x \ , & x\in\mathbb{Q}, \\ 0 \ , & x\notin\mathbb{Q}, \end{cases}$ *and* $g(x)=\begin{cases} x^2 & , x\in\mathbb{Q}, \\ 0 & , x\notin\mathbb{Q}. \end{cases}$

(a) Is f differentiable at $x=0$?
(b) Is g differentiable at $x=0$?

Sol.
(a) Let

$$k(x)=\frac{f(x)}{x}=\begin{cases} 1, & x\in\mathbb{Q}\setminus\{0\}, \\ 0, & x\notin\mathbb{Q}. \end{cases}$$

Then by the analogously argumentation of exercise 1.2.9, $\lim\limits_{x\to 0}k(x)$ does not exist.

So we have $\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\frac{f(h)-0}{h}=\lim\limits_{h\to 0}\frac{f(h)}{h}=\lim\limits_{h\to 0}k(x)$ does not exist. Hence f is not differentiable at $x=0$. $\blacksquare$

(b) Compute $\lim\limits_{h\to 0}\frac{g(0+h)-g(0)}{h}=\lim\limits_{h\to 0}\frac{g(h)-0}{h}=\lim\limits_{h\to 0}\frac{g(h)}{h}$. Where $0\leq\frac{g(h)}{h}\leq\frac{h^2}{h}=h$ and $\lim\limits_{h\to 0}0=\lim\limits_{h\to 0}h=0$, by pingching theorem $\lim\limits_{h\to 0}\frac{g(h)}{h}=0$. Hence g is differentiable at $x=0$. $\blacksquare$

EXERCISE 2.1.3. *Determine the value of* P *and* Q *such that the function* $f(x)=\begin{cases} x^2-2 & , \ x\leq 2, \\ Px^2+Qx & , \ x>2, \end{cases}$ *is differentiable at* $x=2$.

Sol.
Since $f(x)$ is differentiable at $x=2$, $\lim\limits_{h\to 0}\frac{f(2+h)-f(2)}{h}$ exists, that is

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h},$$

where

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{\left((2+h)^2-2\right)-2}{h}=\lim_{h\to 0^-}\frac{h^2+4h}{h}=\lim_{h\to 0^-}(h+4)=4$$

and

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^+}\frac{P\,(2+h)^2+Q\,(2+h)-2}{h}\\ &= \lim_{h\to 0^+}\frac{Ph^2+4Ph+4P+Qh+2Q-2}{h}\\ &= \lim_{h\to 0^+}\left(Ph+4P+Q+\frac{4P+2Q-2}{h}\right).\end{aligned}$$

Hence we have $\begin{cases} 4P+Q=4, \\ 4P+2Q=2, \end{cases}$ which impies $\begin{cases} P=\frac{3}{2}, \\ Q=-2. \end{cases}$ ■

EXERCISE 2.1.4. *Is* $f(x)=\begin{cases} x^2\sin(\frac{1}{x}) & , \ x\neq 0, \\ 0 & , \ x=0, \end{cases}$ *differentiable at* $x=0$ *?*

Sol.
Since

$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{f(h)}{h}\\ &= \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h}\right)}{h}=\lim_{h\to 0}\left(h\sin\left(\frac{1}{h}\right)\right),\end{aligned}$$

and since $0\leq h\sin\left(\frac{1}{h}\right)\leq h$ and $\lim\limits_{h\to 0}0=\lim\limits_{h\to 0}h=0$, by pingching theorem $\lim\limits_{h\to 0}\left(h\sin\left(\frac{1}{h}\right)\right)=0$. Hence f is differentiable at $x=0$. ■

EXERCISE 2.1.5. *Is* $f(x)=x\,|x|$ *differentiable at* $x=0$ *?*

Sol.
Since

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{f(h)}{h}=\lim_{h\to 0}\frac{h\,|h|}{h}=\lim_{h\to 0}|h|=0$$

exists, f is differentiable at $x=0$. ■

EXERCISE 2.1.6. *Find the tangent line of the graph $y = x^3 + 2x + 1$ passing through the point $(x, y) = (1, 4)$.*

Sol.
Let $y = f(x) = x^3 + 2x + 1$, we are going to find $f'(1)$. Since

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left((1+h)^3 + 2(1+h) + 1\right) - 4}{h} \\ &= \lim_{h \to 0} \frac{h^3 + 3h^2 + 5h}{h} = \lim_{h \to 0} \left(h^2 + 3h + 5\right) = 5, \end{aligned}$$

the tangent line of the graph $y = x^3 + 2x + 1$ passing through the point $(1, 4)$ is $y - 4 = 5(x - 1)$. ■

2.2. Exercises 2.2.

EXERCISE 2.2.1. *Differentiate the following functions.*

(a) $f(x) = 4$.
(b) $f(x) = \frac{-1}{x}$.
(c) $f(x) = 4x^2$.
(d) $f(x) = (x^6 + \frac{5}{2}x^3)(\frac{1}{3}x^4 - \frac{8}{3}x^2)$.
(e) $f(x) = \frac{4x+3}{4x^2+11x+2}$.
(f) $f(x) = \frac{-x^2+x+1}{x^6+x^4+x^2}$.
(g) $f(x) = x(x+1)(x+2)(x+3)$.

Sol.
(a)

$$f'(x) = (4)' = 0. \qquad \blacksquare$$

(b)

$$f'(x) = \left(\frac{-1}{x}\right)' = \left(-x^{-1}\right)' = x^{-2} = \frac{1}{x^2}. \qquad \blacksquare$$

(c)

$$f'(x) = \left(4x^2\right)' = 8x. \qquad \blacksquare$$

(d)

$$\begin{aligned}
f'(x) &= \left(\left(x^6+\frac{5}{2}x^3\right)\left(\frac{1}{3}x^4-\frac{8}{3}x^2\right)\right)' \\
&= \left(x^6+\frac{5}{2}x^3\right)'\left(\frac{1}{3}x^4-\frac{8}{3}x^2\right)+\left(x^6+\frac{5}{2}x^3\right)\left(\frac{1}{3}x^4-\frac{8}{3}x^2\right)' \\
&= \left(6x^5+\frac{15}{2}x^2\right)\left(\frac{1}{3}x^4-\frac{8}{3}x^2\right)+\left(x^6+\frac{5}{2}x^3\right)\left(\frac{4}{3}x^3-\frac{16}{3}x\right). \qquad \blacksquare
\end{aligned}$$

(e)

$$\begin{aligned}
f'(x) &= \left(\frac{4x+3}{4x^2+11x+2}\right)' \\
&= \frac{(4x+3)'\left(4x^2+11x+2\right)-(4x+3)\left(4x^2+11x+2\right)'}{\left(4x^2+11x+2\right)^2} \\
&= \frac{4\left(4x^2+11x+2\right)-(4x+3)(8x+11)}{\left(4x^2+11x+2\right)^2}. \qquad \blacksquare
\end{aligned}$$

(f)

$$\begin{aligned}
f'(x) &= \left(\frac{-x^2+x+1}{x^6+x^4+x^2}\right)' \\
&= \frac{\left(-x^2+x+1\right)'\left(x^6+x^4+x^2\right)-\left(-x^2+x+1\right)\left(x^6+x^4+x^2\right)'}{\left(x^6+x^4+x^2\right)^2} \\
&= \frac{(-2x+1)\left(x^6+x^4+x^2\right)-\left(-x^2+x+1\right)\left(6x^5+4x^3+2x\right)}{\left(x^6+x^4+x^2\right)^2}. \qquad \blacksquare
\end{aligned}$$

(g)

$$
\begin{aligned}
& f'(x) \\
= \; & (x(x+1)(x+2)(x+3))' \\
= \; & (x)'(x+1)(x+2)(x+3) + x\left((x+1)(x+2)(x+3)\right)' \\
= \; & (x+1)(x+2)(x+3) \\
& + x\left((x+1)'(x+2)(x+3) + (x+1)\left((x+2)(x+3)\right)'\right) \\
= \; & (x+1)(x+2)(x+3) \\
& + x\left((x+2)(x+3) + (x+1)\left((x+2)'(x+3) + (x+2)(x+3)'\right)\right) \\
= \; & (x+1)(x+2)(x+3) \\
& + x\left((x+2)(x+3) + (x+1)\left((x+3) + (x+2)\right)\right) \\
= \; & (x+1)(x+2)(x+3) + x(x+2)(x+3) \\
& + x(x+1)(x+3) + x(x+1)(x+2). \qquad \blacksquare
\end{aligned}
$$

(h)

$$
\begin{aligned}
f'(x) &= \left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}\right)' = \left(\frac{1}{x}\right)' + \left(\frac{2}{x^2}\right)' + \left(\frac{3}{x^3}\right)' \\
&= \left(x^{-1}\right)' + \left(2x^{-2}\right)' + \left(3x^{-3}\right)' = -x^{-2} - 4x^{-3} - 9x^{-4} \\
&= \frac{-1}{x^2} - \frac{4}{x^3} - \frac{9}{x^4}. \qquad \blacksquare
\end{aligned}
$$

EXERCISE 2.2.2. *If $g(x)$ is continuous at $x = 0$ and $f(x) = xg(x)$, prove that $f(x)$ is differentiable at $x = 0$.*

Sol.
Since g is continuous at $x = 0$, that is, $\lim_{h\to 0} g(h) = g(0)$, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - 0 \cdot g(0)}{h} = \lim_{h\to 0} \frac{hg(h)}{h} = \lim_{h\to 0} g(h) = g(0).$$

Hence f is differentiable at $x = 0$. $\blacksquare$

2.3. Exercises 2.3.

2.4. Exercises 2.4.

EXERCISE 2.4.1. $\frac{d}{dx}(2x-1)^{10}$.

Sol.
Let $u = 2x - 1$.

$$\begin{aligned}\frac{d}{dx}(2x-1)^{10} &= \frac{d}{dx}u^{10} = \frac{du^{10}}{du}\cdot\frac{du}{dx}\\ &= 10u^9\cdot\frac{d\,(2x-1)}{dx}\\ &= 10\,(2x-1)^9\cdot 2x. \qquad \blacksquare\end{aligned}$$

EXERCISE 2.4.2. $\frac{d}{dx}(\frac{3x-4}{5x+3})^2$.

Sol.
Let $u = \frac{3x-4}{5x+3}$.

$$\begin{aligned}\frac{d}{dx}(\frac{3x-4}{5x+3})^2 &= \frac{d}{dx}u^2 = \frac{du^2}{du}\cdot\frac{du}{dx} = 2u\cdot\frac{d\left(\frac{3x-4}{5x+3}\right)}{dx}\\ &= 2\left(\frac{3x-4}{5x+3}\right)\cdot\frac{3\,(5x+3)-5\,(3x-4)}{(5x+3)^2}. \qquad \blacksquare\end{aligned}$$

EXERCISE 2.4.3. $\frac{d}{dx}(x\sin(\frac{2}{\pi})+\cos(\frac{2}{\pi}))(x\cos(\frac{2}{\pi})-\sin(\frac{2}{\pi}))$.

Sol.

$$\begin{aligned}&\frac{d}{dx}(x\sin(\frac{2}{\pi})+\cos(\frac{2}{\pi}))(x\cos(\frac{2}{\pi})-\sin(\frac{2}{\pi}))\\ = &\left(\frac{d}{dx}(x\sin(\frac{2}{\pi})+\cos(\frac{2}{\pi}))\right)(x\cos(\frac{2}{\pi})-\sin(\frac{2}{\pi}))\\ &+(x\sin(\frac{2}{\pi})+\cos(\frac{2}{\pi}))\left(\frac{d}{dx}(x\cos(\frac{2}{\pi})-\sin(\frac{2}{\pi}))\right)\\ = &\sin(\frac{2}{\pi})(x\cos(\frac{2}{\pi})-\sin(\frac{2}{\pi}))+\cos(\frac{2}{\pi})(x\sin(\frac{2}{\pi})+\cos(\frac{2}{\pi})). \qquad \blacksquare\end{aligned}$$

EXERCISE 2.4.4. $\frac{d}{dx}\left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}\right)$.

Sol.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}\right) &= \frac{d}{dx}\left(x^{-1}+2x^{-2}+3x^{-3}\right) = -x^{-2}-4x^{-3}-9x^{-4}\\ &= \frac{-1}{x^2}-\frac{4}{x^3}-\frac{9}{x^4}. \qquad \blacksquare\end{aligned}$$

EXERCISE 2.4.5. $\frac{d}{dx}(\cos 2x - 2\sin x)$.

Sol.

$$\frac{d}{dx}(\cos 2x - 2\sin x) = \frac{d}{dx}\left(\cos 2x\right) - \frac{d}{dx}\left(2\sin x\right) = -2\sin 2x - 2\cos x. \qquad \blacksquare$$

EXERCISE 2.4.6. $\frac{d}{dx}(\frac{\sin^2 x}{\sin(x^2)})$.

Sol.

$$\begin{aligned}\frac{d}{dx}(\frac{\sin^2 x}{\sin(x^2)}) &= \frac{\left(\frac{d}{dx}\sin^2 x\right)\left(\sin(x^2)\right) - \left(\sin^2 x\right)\left(\frac{d}{dx}\sin(x^2)\right)}{\left(\sin(x^2)\right)^2} \\ &= \frac{\left(2\sin x\cos x\right)\left(\sin(x^2)\right) - \left(\sin^2 x\right)\left(\cos\left(x^2\right)\cdot 2x\right)}{\sin^2(x^2)}. \qquad \blacksquare\end{aligned}$$

EXERCISE 2.4.7. $\frac{d}{dx}(\tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan(x^5))$.

Sol.

$$\begin{aligned}&\frac{d}{dx}(\tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan(x^5)) \\ = \ &\sec^2 x - \frac{1}{3}\cdot 3\tan^2 x\cdot\sec^2 x + \frac{1}{5}\sec^2\left(x^5\right)\cdot 5x^4. \qquad \blacksquare\end{aligned}$$

EXERCISE 2.4.8. $\frac{d}{dx}(\sin(\cos^2(\tan^3(x^4))))$.

Sol.

$$\begin{aligned}&\frac{d}{dx}(\sin(\cos^2(\tan^3(x^4)))) \\ = \ &\cos\left(\cos^2\left(\tan^3\left(x^4\right)\right)\right)\cdot 2\cos\left(\tan^3\left(x^4\right)\right) \\ &\cdot - \sin\left(\tan^3\left(x^4\right)\right)\cdot 3\tan^2\left(x^4\right)\cdot\sec^2\left(x^4\right)\cdot 4x^3. \qquad \blacksquare\end{aligned}$$

EXERCISE 2.4.9. $\frac{d}{dx}(\frac{1}{\cos^3 x})$.

Sol.

$$\begin{aligned}\frac{d}{dx}(\frac{1}{\cos^3 x}) &= \frac{\left(\frac{d}{dx}1\right)(\cos^3 x) - 1\cdot\left(\frac{d}{dx}\cos^3 x\right)}{\left(\cos^3 x\right)^2}\\ &= \frac{-3\cos^2 x\cdot -\sin x}{\cos^6 x} = \frac{3\sin x}{\cos^4 x}. \qquad \blacksquare\end{aligned}$$

EXERCISE 2.4.10. $\frac{d}{dx}(\sec^2(\frac{x}{2}) + \csc^2(\frac{x}{2}))$.

Sol.

$$\begin{aligned} &\frac{d}{dx}(\sec^2(\frac{x}{2}) + \csc^2(\frac{x}{2}))\\ = \;& 2\sec\left(\frac{x}{2}\right)\cdot\tan\left(\frac{x}{2}\right)\sec\left(\frac{x}{2}\right)\\ &\cdot\frac{1}{2} + 2\csc\left(\frac{x}{2}\right)\cdot -\cot\left(\frac{x}{2}\right)\csc\left(\frac{x}{2}\right)\cdot\frac{1}{2}\\ = \;& \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right) - \cot\left(\frac{x}{2}\right)\csc^2\left(\frac{x}{2}\right). \qquad \blacksquare\end{aligned}$$

2.5. Exercises 2.5.

EXERCISE 2.5.1. $x^2 + 2xy - y^2 = 2x$, $\frac{dy}{dx} =$?

Sol.

$$x^2 + 2xy - y^2 = 2x$$

$$\begin{aligned} &\Rightarrow \frac{d}{dx}\left(x^2 + 2xy - y^2\right) = \frac{d}{dx}(2x)\\ &\Rightarrow 2x + 2\left(y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx} = 2\\ &\Rightarrow (x-y)\frac{dy}{dx} = 1 - x - y\\ &\Rightarrow \frac{dy}{dx} = \frac{1-x-y}{x-y}. \qquad \blacksquare\end{aligned}$$

EXERCISE 2.5.2. $y^2 = 2x$, $\frac{dy}{dx} =$?

Sol.

$$y^2 = 2x$$

$$\Rightarrow \quad \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(2x)$$
$$\Rightarrow \quad 2y\frac{dy}{dx} = 2$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{y}. \qquad \blacksquare$$

Exercise 2.5.3. $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $\frac{dy}{dx} =$?

Sol.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{x^2}{4} + \frac{y^2}{9}\right) = \frac{d}{dx}(1)$$
$$\Rightarrow \quad \frac{2x}{4} + \frac{2y\frac{dy}{dx}}{9} = 0$$
$$\Rightarrow \quad \frac{2y}{9} \cdot \frac{dy}{dx} = \frac{-x}{2}$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{-9x}{4y}. \qquad \blacksquare$$

Exercise 2.5.4. $\sqrt{x} + \sqrt{y} = \sqrt{2}$, $\frac{dy}{dx} =$?

Sol.

$$\sqrt{x} + \sqrt{y} = \sqrt{2}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) = \frac{d}{dx}\left(\sqrt{2}\right)$$
$$\Rightarrow \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$
$$\Rightarrow \quad \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = \frac{-1}{2\sqrt{x}}$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}}. \qquad \blacksquare$$

Exercise 2.5.5. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$, $\frac{dy}{dx} =$?

Sol.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

$$\begin{aligned}
&\Rightarrow \quad \frac{d}{dx}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right) = \frac{d}{dx}(4) \\
&\Rightarrow \quad \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \\
&\Rightarrow \quad \frac{2}{3\sqrt[3]{y}}\frac{dy}{dx} = \frac{-2}{3\sqrt[3]{x}} \\
&\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt[3]{y}}{\sqrt[3]{x}} = \sqrt[3]{\frac{y}{x}}. \qquad \blacksquare
\end{aligned}$$

EXERCISE 2.5.6. $\frac{d}{dx}(x + \sqrt{x} + \sqrt[3]{x})$.

Sol.

$$\begin{aligned}
\frac{d}{dx}(x + \sqrt{x} + \sqrt[3]{x}) &= \frac{d}{dx}\left(x + x^{\frac{1}{2}} + x^{\frac{1}{3}}\right) = 1 + \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} \\
&= 1 + \frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}.
\end{aligned}$$

EXERCISE 2.5.7. $\frac{d}{dx}(\frac{1}{x} + \frac{1}{\sqrt{x}} + \frac{1}{\sqrt[3]{x}})$.

Sol.

$$\begin{aligned}
\frac{d}{dx}(\frac{1}{x} + \frac{1}{\sqrt{x}} + \frac{1}{\sqrt[3]{x}}) &= \frac{d}{dx}(x^{-1} + x^{-\frac{1}{2}} + x^{-\frac{1}{3}}) = -x^{-2} - \frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{3}x^{-\frac{4}{3}} \\
&= \frac{-1}{x^2} - \frac{1}{2\sqrt{x^3}} - \frac{1}{3\sqrt[3]{x^4}}. \qquad \blacksquare
\end{aligned}$$

EXERCISE 2.5.8. $\frac{d}{dx}\sqrt{x + \sqrt{x + \sqrt{x}}}$.

Sol.

$$
\begin{aligned}
& \frac{d}{dx}\sqrt{x+\sqrt{x+\sqrt{x}}} \\
= \quad & \frac{d}{dx}\left(x+\left(x+x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
= \quad & \frac{1}{2}\left(x+\left(x+x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \left(1+\frac{1}{2}\left(x+x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\right) \cdot \left(1+\frac{1}{2}x^{-\frac{1}{2}}\right) \\
= \quad & \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\right) \cdot \left(1+\frac{1}{2\sqrt{x}}\right). \qquad \blacksquare
\end{aligned}
$$

EXERCISE 2.5.9. $\frac{d}{dx}\sqrt[3]{x+\sqrt{x}+\sqrt[3]{x}}$.

Sol.

$$
\begin{aligned}
& \frac{d}{dx}\sqrt[3]{x+\sqrt{x}+\sqrt[3]{x}} \\
= \quad & \frac{d}{dx}\left(x+x^{\frac{1}{2}}+x^{\frac{1}{3}}\right)^{\frac{1}{3}} \\
= \quad & \frac{1}{3}\left(x+x^{\frac{1}{2}}+x^{\frac{1}{3}}\right)^{-\frac{2}{3}} \cdot \left(1+\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{3}x^{-\frac{2}{3}}\right) \\
= \quad & \frac{1}{3\sqrt[3]{(x+\sqrt[2]{x}+\sqrt[3]{x})^2}} \cdot \left(1+\frac{1}{2\sqrt{x}}+\frac{1}{3\sqrt[3]{x^2}}\right).
\end{aligned}
$$

EXERCISE 2.5.10. $\frac{d}{dx}3(\sqrt[3]{\cot^2 x}+\sqrt[3]{\cot^8 x})$.

Sol.

$$
\begin{aligned}
& \frac{d}{dx}3(\sqrt[3]{\cot^2 x}+\sqrt[3]{\cot^8 x}) \\
= \quad & \frac{d}{dx}3\left((\cot x)^{\frac{2}{3}}+(\cot x)^{\frac{8}{3}}\right) \\
= \quad & 3\left(\frac{2}{3}(\cot x)^{-\frac{1}{3}} \cdot -\csc^2 x+\frac{8}{3}(\cot x)^{\frac{5}{3}} \cdot -\csc^2 x\right) \\
= \quad & \frac{-2\csc^2 x}{\sqrt[3]{\cot x}}-8\csc^2 x\sqrt[3]{\cot^5 x}. \qquad \blacksquare
\end{aligned}
$$

3. Chapter 3

3.1. Exercises 3.1.

EXERCISE 3.1.1. *Let $f(x) = x + \frac{1}{x}$. Show that f satisfies the conditions of the mean-value theorem on the interval $[1, 9]$, and find all numbers c which satisfy the conditions of the mean-value theorem.*

Sol.

Since f is differentiable on $\mathbb{R}$, f is continuous on $[1, 9]$ and is differentiable on $(1, 9)$. Since $f'(x) = 1 - \frac{1}{x^2}$ and $\frac{f(9)-f(1)}{9-1} = \frac{8}{9}$, if $c \in (1, 9)$ and $f'(c) = \frac{f(9)-f(1)}{9-1}$, then $1 - \frac{1}{c^2} = \frac{8}{9}$.So $c = 3$. ($-3 \notin (1, 9)$) $\blacksquare$

EXERCISE 3.1.2. *Let $f(x) = \sin \pi x$. Show that f satisfies the conditions of Rolle's theorem on the interval $[0, 1]$, and find all numbers c which satisfy the conditions of the Rolle's theorem.*

Sol.

Since f is differentiable on $\mathbb{R}$, f is continuous on $[0, 1]$ and is differentiable on $(0, 1)$. Morover, $f(0) = 0 = f(1)$. Since $f'(x) = \pi \cos \pi x$, if $c \in (0, 1)$ and $f'(c) = 0$, then $\pi \cos \pi c = 0$.So $c = \frac{1}{2}$. $\blacksquare$

EXERCISE 3.1.3. *Prove that the equation $2x^3 + 6x^2 + 7x - 10 = 0$ has at most one root.*

Sol.

We are going to prove by contradiction.

Let $f(x) = 2x^3 + 6x^2 + 7x - 10$. If the equation had at least two roots, then $\exists$ $a_1,\ a_2,\ a_1 < a_2$, such that $f(a_1) = f(a_2) = 0$. Then since f is differentiable on $\mathbb{R}$, f is continuous on $[a_1, a_2]$ and is differentiable on (a_1, a_2). Thus by Rolle's theorem, $\exists$ $c \in (a_1, a_2)$ such that $f'(c) = 0$. However,

$$f'(x) = 6x^2 + 12x + 7 = 6\left(x+1\right)^2 + 1 > 0,\ \forall\ x \in \mathbb{R}.$$

This leads to a contradiction. $\blacksquare$

EXERCISE 3.1.4. *Prove that the equation $x^4 + 4x^3 + 7x^2 - 20x - 1 = 0$ has at most two roots.*

Sol.

We are going to prove by contradiction.

Let $g(x) = x^4 + 4x^3 + 7x^2 - 20x - 1$. If the equation had at least three roots, then $\exists$ $a_1, a_2,\ a_3,\ a_1 < a_2 < a_3$, such that $g(a_1) = g(a_2) = g(a_3) = 0$. Then since g is differentiable on $\mathbb{R}$, g is continuous on $[a_1, a_2],\ [a_2, a_3]$ and is differentiable on

(a_1, a_2), (a_2, a_3). Thus by Rolle's theorem, $\exists\, c_1 \in (a_1, a_2)$ and $\exists\, c_2 \in (a_2, a_3)$ such that $g'(c_1) = g'(c_2) = 0$. However, since

$$g'(x) = 4x^3 + 12x^2 + 14x - 20 = 2\left(2x^3 + 6x^2 + 7x - 10\right),$$

by exercise 3.1.3, $g'(x)$ has at most one root. This leads to a contradiction. ■

EXERCISE 3.1.5. *Prove that* $|\tan a - \tan b| \geq |a - b|$, $\forall\, a, b \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

Sol.
If $a = b$, then $|\tan a - \tan b| = 0 = |a - b|$.
If $a \neq b$, then without loss of generality, we may assume $a < b$. Let $f(x) = \tan x$. Then since f is differentiable on $\mathbb{R}$, f is continuous on $[a, b]$ and is differentiable on (a, b). Then by the mean value theorem, $\exists\, c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, that is, $\sec^2 c = \frac{\tan b - \tan a}{b-a}$. Then since $\sec^2 c \geq 1$, we have

$$|a - b| = \sec^2 c\, |\tan a - \tan b| \leq |\tan a - \tan b|. \quad ■$$

3.2. Exercises 3.2.

EXERCISE 3.2.1. *Find the intervals on which f is increasing and the intervals on which f is*

decreasing.
(a) $f(x) = \begin{cases} x^3 - x + 2, & x \leq 0, \\ x^2 - 2x + 2, & x > 0. \end{cases}$
(b) $f(x) = \frac{x+3}{x^2+2x+2}$.
(c) $f(x) = \tan x - 2\sec x$, $0 \leq x \leq 2\pi$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}$.
Sol.
(a) Since $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = 2 = f(2)$, f is continuous on $\mathbb{R}$. Then since for $x < 0$, $f'(x) = 3x^2 - 1$, and for $x > 0$, $f'(x) = 2x - 2$, we have

x	$\sim$	$-\frac{1}{\sqrt{3}}$	$\sim$	0	$\sim$	1	$\sim$
$f'(x)$	$+$	0	$-$	$\times$	$-$	0	$+$

Thus by theorem 3.2.5, f is increasing on $(-\infty, -\frac{1}{\sqrt{3}}]$ and $[1, \infty)$, and is decreasing on $[-\frac{1}{\sqrt{3}}, 0]$ and $[0, 1]$, that is, f is decreasing on $[-\frac{1}{\sqrt{3}}, 1]$. ■

(b) Since $x^2 + 2x + 2 > 0$, $\forall\, x \in \mathbb{R}$, f is continuous on $\mathbb{R}$. Then since

$$f'(x) = \frac{(x^2 + 2x + 2) \cdot 1 - (x + 3)(2x + 2)}{(x^2 + 2x + 2)^2} = \frac{-x^2 - 6x - 4}{(x^2 + 2x + 2)^2},$$

we have

x	$\sim$	$-3-\sqrt{5}$	$\sim$	$-3+\sqrt{5}$	$\sim$
$f'(x)$	$-$	0	$+$	0	$-$

Thus by theorem 3.2.5, f is increasing on $[-3-\sqrt{5},-3+\sqrt{5}]$ and is decreasing on $(-\infty,-3-\sqrt{5}]$ and $[-3+\sqrt{5},\infty)$. ■

(c) f is continuous on $[0,\frac{\pi}{2})$, $(\frac{\pi}{2},\frac{3\pi}{2})$ and $(\frac{3\pi}{2},2\pi]$. Then since

$$f'(x)=\sec^2 x-2\sec x\tan x=\frac{1-2\sin x}{\cos^2 x},$$

we have

x	0	$\sim$	$\frac{\pi}{6}$	$\sim$	$\frac{\pi}{2}$	$\sim$	$\frac{5\pi}{6}$	$\sim$	$\frac{3\pi}{2}$	$\sim$	2π
$f'(x)$		$+$	0	$-$	$\times$	$-$	0	$+$	$\times$	$+$	

Thus by theorem 3.2.5, f is increasing on $[0,\frac{\pi}{6}]$, $[\frac{5\pi}{6},\frac{3\pi}{2})$ and $(\frac{3\pi}{2},2\pi]$, and is decreasing on $[\frac{\pi}{6},\frac{\pi}{2})$ and $(\frac{\pi}{2},\frac{5\pi}{6}]$. ■

EXERCISE 3.2.2. *Prove that* $f(x)=\cos x^2+x^2+2$ *is increasing on* $\left[0,\sqrt{\frac{\pi}{2}}\right]$.

Sol.
Since f is differentiable on $\mathbb{R}$, f is continuous on $[0,\sqrt{\frac{\pi}{2}}]$ and is differentiable on $(0,\sqrt{\frac{\pi}{2}})$. Then since

$$f'(x)=-2x\sin x^2+2x=2x\left(1-\sin x^2\right)>0,\ \forall\ x\in(0,\sqrt{\frac{\pi}{2}}),$$

by theorem 3.2.5, f is increasing on $[0,\sqrt{\frac{\pi}{2}}]$. ■

EXERCISE 3.2.3. *Prove that* $\tan x\geq x^2$, $\forall\ x\in\left[0,\frac{\pi}{8}\right]$.

Sol.
Let $f(x)=\tan x-x^2,\ x\in\left[0,\frac{\pi}{8}\right]$. Then f is continuous on $[0,\frac{\pi}{8}]$ and is differentiable on $(0,\frac{\pi}{8})$. Then since

$$f'(x)=\sec^2 x-2x\geq 1-\frac{2\pi}{8}>0,\ \forall\ x\in(0,\frac{\pi}{8}),$$

by theorem 3.2.5, f is increasing on $[0,\frac{\pi}{8}]$. So we have $f(x)\geq f(0)=0,\ \forall\ x\in[0,\frac{\pi}{8}]$, that is, $\tan x\geq x^2$, $\forall\ x\in\left[0,\frac{\pi}{8}\right]$. ■

3.3. Exercises 3.3.

EXERCISE 3.3.1. *Find the critical points and the local extreme points of f.*

(a) $f(x) = \frac{x}{x^2+x+1}$.
(b) $f(x) = \frac{x^2+1}{\sqrt{x}}$, $x > 0$.
(c) $f(x) = \sin^2 x - \sin x + 1$, $0 \le x \le 2\pi$.
(d) $f(x) = \frac{1}{1+\sin^2 x}$, $0 \le x \le 2\pi$.

Sol.

(a)Since $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$, $\forall\ x \in \mathbb{R}$, f is differentiable on $\mathbb{R}$. Then since

$$f'(x) = \frac{(x^2 + x + 1) \cdot 1 - x\,(2x + 1)}{(x^2 + x + 1)^2} = \frac{-x^2 + 1}{(x^2 + x + 1)^2} = \frac{-\,(x + 1)\,(x - 1)}{(x^2 + x + 1)^2},$$

the critical points are 1 and -1. Then since we have

x	$\sim$	-1	$\sim$	1	$\sim$
$f'(x)$	$-$	0	$+$	0	$-$

,

by the first derivative test, 1 is the local maximum point and -1 is the local minimum point. ■

(b) The domain of f is $(0, \infty)$ and f is differentiable on $(0, \infty)$. Since

$$f'(x) = \frac{\sqrt{x} \cdot 2x - \frac{1}{2\sqrt{x}}\,(x^2 + 1)}{\sqrt{x}^2} = \frac{3x^2 - 1}{2x\sqrt{x}} = \frac{(\sqrt{3}x + 1)(\sqrt{3}x - 1)}{(x^2 + x + 1)^2},$$

the critical point is $\frac{1}{\sqrt{3}}$. Then since we have

$$\begin{array}{cccccc} x & 0 & \sim & \frac{1}{\sqrt{3}} & \sim \\ f'(x) & & - & 0 & + \end{array},$$

by the first derivative test, $\frac{1}{\sqrt{3}}$ is the local minimum point. ■

(c) f is differentiable on $(0, 2\pi)$. Then since

$$f'(x) = 2\sin x \cos x - \cos x = \cos x(2\sin x - 1),$$

the critical points are $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$ and $\frac{3\pi}{2}$. Then since

$$f''(x) = 2\cos^2 x - 2\sin^2 x + \sin x,$$

we have

x	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{3\pi}{2}$
$f''(x)$	$+$	$-$	$+$	$-$

,

by the second derivative test, $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are the local maximum points and $\frac{\pi}{6}$ and $\frac{3\pi}{2}$ are the local minimum points. ■

(d) Since $1+\sin^2 x > 0$, $\forall\ x \in (0, 2\pi)$, f is differentiable on $(0, 2\pi)$. Then since

$$f'(x) = -\frac{2\sin x\cos x}{\left(1+\sin^2 x\right)^2},$$

the critical points are $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$. Then since we have

x	0	$\sim$	$\frac{\pi}{2}$	$\sim$	π	$\sim$	$\frac{3\pi}{2}$	$\sim$	2π
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	

,

by the first derivative test, π is the local maximum point and $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are the local minimum points. ■

EXERCISE 3.3.2. *Suppose f is continuous on $[a, b]$ and $f(a) = f(b)$. Show that f has at least a critical point in (a, b).*

Sol.
If f is not differentiable on (a, b), then f has a critical point in (a, b). Then there is nothing further to prove.

If f is differentiable on (a, b), then let $g(x) = f(x) - f(a)$. Then since f is continuous on $[a, b]$ and is differentiable on (a, b), so is g. Moreover, we have $g'(x) = f'(x)$. Then since $g(a) = f(a) - f(a) = 0$ and $g(b) = f(b) - f(a) = 0$, by Rolle's theorem, $\exists\ c \in (a, b)$ such that $g'(c) = 0$. Hence we have $f'(c) = 0$, that is, c is a critical point of f. ■

3.4. Exercises 3.4.

EXERCISE 3.4.1. *Find the local and absolute extreme points of f.*

(a) $f(x) = x - \frac{1}{x}$, $1 \le x \le 10$.
(b) $f(x) = |x^2 - x|$, $-1 \le x \le 5$.
(c) $f(x) = x + \sin x$, $-2\pi \le x \le 2\pi$.
(d) $f(x) = \sqrt{1+x^2}$.

Sol.
(a) Since $f'(x) = 1 + \frac{1}{x^2} > 0$, $\forall\ x \in (1, 10)$, there are no critical points and no local extreme points. Then since $f(1) = 0$ and $f(10) = \frac{99}{10}$, 10 is the absolute maximum point and 1 is the absolute minimum point. ■

(b) Since

$$f(x) = |x(x-1)| = \begin{cases} x^2 - x, & -1 \le x < 0, \\ x - x^2, & 0 \le x < 1, \\ x^2 - x, & 1 \le x \le 5, \end{cases}$$

we have

$$f'(x) = \begin{cases} 2x-1, & -1<x<0, \\ 1-2x, & 0<x<1, \\ 2x-1, & 1<x<5. \end{cases}$$

Hence the critical points are 0, $\frac{1}{2}$ and 1. Then since we have

x	-1	$\sim$	0	$\sim$	$\frac{1}{2}$	$\sim$	1	$\sim$	5
$f'(x)$		$-$	$\times$	$+$	0	$-$	$\times$	$+$	
$f(x)$	2		0		$\frac{1}{2}$		0		20

by the first derivative test, $\frac{1}{2}$ is the local maximum point, 0 and 1 are the local minimum points, 5 is the absolute maximum point and 0 and 1 are the absolute minimum point. ■

(c) Since $f'(x) = 1+\cos x$, the critical points are $-\pi$ and π. Then since we have

x	-2π	$\sim$	$-\pi$	$\sim$	π	$\sim$	2π
$f'(x)$		$+$	0	$+$	0	$+$	
$f(x)$	-2π		$-\pi$		π		2π

by the first derivative test, $-\pi$ and π are saddle points, -2π is the absolute maximum point and 2π is the absolute minimum point. ■

(d) Since $f'(x) = \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$,the critical point is 0. Then since we have

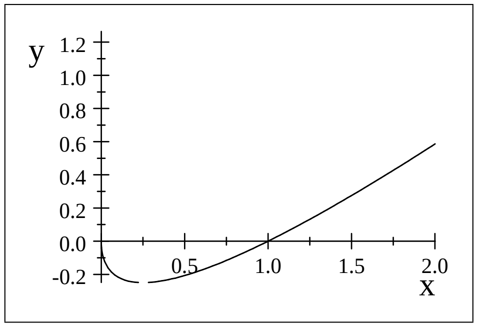
x	$\sim$	0	$\sim$
$f'(x)$	$-$	0	$+$
$f(x)$		1	

0 is the local and absolute minimum point and there is no local and absolute maximum point. ■

3.5. Exercises 3.5.

Exercise 3.5.1. *Draw the grapf of $y = f(x)$.*

(a) $f(x) = x - \sqrt{x}$, $0 \le x \le 2$.
(b) $f(x) = x^3 - 6x^2 + 9x - 2$, $0 \le x \le 5$.
(c) $f(x) = 2x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(d) $f(x) = \sin^2 x$, $0 \le x \le \pi$.
(e) $f(x) = \sin^2 x + 2\sin x - 1$, $0 \le x \le \pi$.
Sol.
(a) $f(x) = x - \sqrt{x}$, $0 \le x \le 2$.



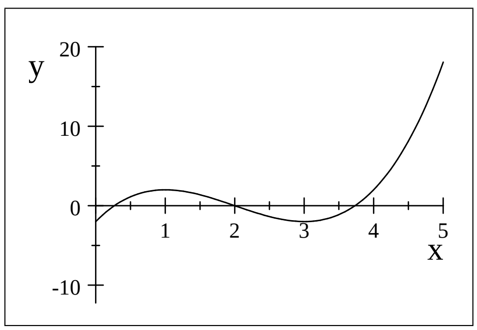
Since $f'(x) = 1 - \frac{1}{2\sqrt{x}}$, the critical point is $\frac{1}{4}$. Then since $f''(x) = \frac{1}{4\sqrt{x^3}} > 0\,,\ \forall\, x \in (0, 2)$, there is no point of inflection. So we have

x	0	$\sim$	$\frac{1}{4}$	$\sim$	2
$f''(x)$		$+$	$+$	$+$	
$f'(x)$		$-$	0	$+$	
$f(x)$	0		$-\frac{1}{2}$		$2-\sqrt{2}$

.

Therefore, f is concave up on $(0, 2)$, is decreasing on $[0, \frac{1}{4}]$, is increasing on $[\frac{1}{4}, 2]$, $\frac{1}{4}$ is the local and absolute minimum point, and 2 is the absolute maximum point. ■

(b) $f(x) = x^3 - 6x^2 + 9x - 2$, $0 \le x \le 5$.



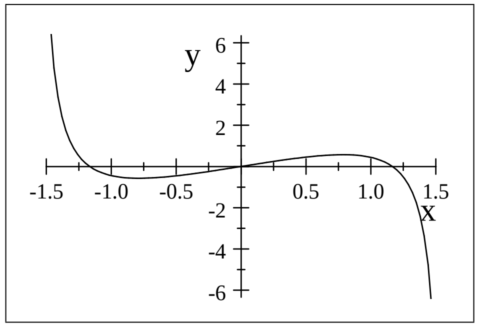
Since $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, the critical points are 1 and 3. Then since $f''(x) = 6x - 12 = 6(x-2)\,,\ f''(x) = 0$ at 2. So we have

x	0	$\sim$	1	$\sim$	2	$\sim$	3	$\sim$	5
$f''(x)$		$-$	$-$	$-$	0	$+$	$+$	$+$	
$f'(x)$		$+$	0	$-$	$-$	$-$	0	$+$	
$f(x)$	-2		2		0		-2		18

.

Therefore, f is concave down on $(0, 2)$, is concave up on $(2, 5)$, is decreasing on $[1, 3]$, is increasing on $[0, 1]$ and $[3, 5]$, 3 is the local minimum point, 1 is the local maximum point, 0 and 3 are the absolute minimum points, and 5 is the absolute maximum point. ■

(c) $f(x) = 2x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



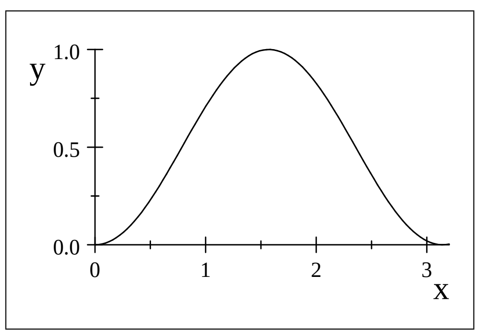
Since $f'(x) = 2 - \sec^2 x$, the critical points are $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. Then since $f''(x) = -2\sec^2 x \tan x$, $f''(x) = 0$ at 0. So we have

x	$-\frac{\pi}{2}$	$\sim$	$-\frac{\pi}{4}$	$\sim$	0	$\sim$	$\frac{\pi}{4}$	$\sim$	$\frac{\pi}{2}$
$f''(x)$		$+$	$+$	$+$	0	$-$	$-$	$-$	
$f'(x)$		$-$	0	$+$	$+$	$+$	0	$-$	
$f(x)$	∞		$-\frac{\pi}{2}+1$		0		$\frac{\pi}{2}-1$		$-\infty$

.

Therefore, f is concave down on $(0, \frac{\pi}{2})$, is concave up on $(-\frac{\pi}{2}, 0)$, is decreasing on $[-\frac{\pi}{2}, -\frac{\pi}{4}]$ and $[\frac{\pi}{4}, \frac{\pi}{2}]$, is increasing on $[-\frac{\pi}{4}, \frac{\pi}{4}]$, $-\frac{\pi}{4}$ is the local minimum point, $\frac{\pi}{4}$ is the local maximum point, and there are no absolute extreme points. ■

(d) $f(x) = \sin^2 x$, $0 \le x \le \pi$.



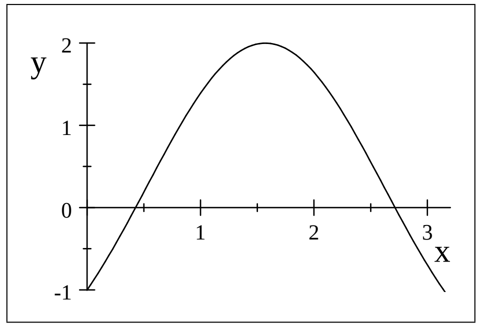
Since $f'(x) = 2\sin x \cos x$, the critical point is $\frac{\pi}{2}$. Then since $f''(x) = 2\cos^2 x - 2\sin^2 x$, $f''(x) = 0$ at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. So we have

x	0	$\sim$	$\frac{\pi}{4}$	$\sim$	$\frac{\pi}{2}$	$\sim$	$\frac{3\pi}{4}$	$\sim$	$\pi.$
$f''(x)$		$+$	0	$-$	$-$	$-$	0	$+$	
$f'(x)$		$+$	$+$	$+$	0	$-$	$-$	$-$	
$f(x)$	0		$\frac{1}{2}$		1		$\frac{1}{2}$		0

.

Therefore, f is concave down on $(\frac{\pi}{4}, \frac{3\pi}{4})$, is concave up on $(0, \frac{\pi}{4})$ and $(\frac{3\pi}{4}, \pi)$, is decreasing on $[\frac{\pi}{2}, \pi]$, is increasing on $[0, \frac{\pi}{2}]$, $\frac{\pi}{2}$ is the local maximum point, 0 and π are the absolute minimum points, and $\frac{\pi}{2}$ is the absolute maximum point. ■

(e) $f(x) = \sin^2 x + 2\sin x - 1$, $0 \le x \le \pi$.



Since $f'(x) = 2\sin x\cos x + 2\cos x = 2\cos x(\sin x + 1)$, the critical point is $\frac{\pi}{2}$. Then since

$$\begin{aligned} f''(x) &= 2\cos^2 x - 2\sin^2 x - 2\sin x \\ &= 2 - 4\sin^2 x - 2\sin x \\ &= -2(2\sin x - 1)(\sin x + 1), \end{aligned}$$

$f''(x) = 0$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. So we have

x	0	$\sim$	$\frac{\pi}{6}$	$\sim$	$\frac{\pi}{2}$	$\sim$	$\frac{5\pi}{6}$	$\sim$	π.
$f''(x)$		$+$	0	$-$	$-$	$-$	0	$+$	
$f'(x)$		$+$	$+$	$+$	0	$-$	$-$	$-$	
$f(x)$	-1		$\frac{1}{4}$		2		$\frac{1}{4}$		-1

Therefore, f is concave down on $(\frac{\pi}{6}, \frac{5\pi}{6})$, is concave up on $(0, \frac{\pi}{6})$ and $(\frac{5\pi}{6}, \pi)$, is decreasing on $[\frac{\pi}{2}, \pi]$, is increasing on $[0, \frac{\pi}{2}]$, $\frac{\pi}{2}$ is the local maximum point, 0 and π are the absolute minimum points, and $\frac{\pi}{2}$ is the absolute maximum point. ■

3.6. Exercises 3.6.

EXERCISE 3.6.1. *Find the asymptotes of f.*

(a) $f(x) = \frac{x^2}{1+x^2}$.
(b) $f(x) = \frac{1-x}{1+x}$.
Sol.
(a) Since $1 + x^2 > 0,\ \forall\ x \in \mathbb{R}$, there is no vertical asymptotes. Then since $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$, we have

$$\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^2}+1} = \frac{1}{0+1} = 1$$

and

$$\lim_{x\to-\infty} \frac{x^2}{1+x^2} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x^2}+1} = \frac{1}{0+1} = 1.$$

So the horizontal asymptotes is $y = 1$. ■

(b) Since

$$\begin{aligned}\lim_{x\to-1^+}\frac{1-x}{1+x} &= \lim_{x\to-1^+}(\frac{2}{1+x}-1)=\infty,\\ \lim_{x\to-1^-}\frac{1-x}{1+x} &= \lim_{x\to-1^-}(\frac{2}{1+x}-1)=-\infty,\end{aligned}$$

the vertical asymptotes is $x=-1$. Then since

$$\begin{aligned}\lim_{x\to\infty}\frac{1-x}{1+x} &= \lim_{x\to\infty}\frac{1-x}{1+x}=\frac{0-1}{0+1}=-1,\\ \lim_{x\to-\infty}\frac{1-x}{1+x} &= \lim_{x\to-\infty}\frac{1-x}{1+x}=\frac{0-1}{0+1}=-1,\end{aligned}$$

the horizontal asymptotes is $y=-1$. ■

4. Chapter 4

4.1. Exercises 4.1.

EXERCISE 4.1.1. *Given* $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$*, use the definition of the definite integral to prove that* $\int_1^2 x^2 dx = \frac{7}{3}$.

Sol.

$$\begin{aligned}
\int_1^2 x^2 dx &= \lim_{n\to\infty} \frac{2-1}{n}((1+\frac{1}{n})^2 + (1+\frac{2}{n})^2 + ... + (1+\frac{n}{n})^2) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(1+\frac{2k}{n}+\frac{k^2}{n^2}) \\
&= \lim_{n\to\infty} \frac{1}{n}(n + \frac{2}{n}\frac{n(n+1)}{2} + \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}) \\
&= \lim_{n\to\infty}(1 + \frac{n+1}{n} + \frac{n(n+1)(2n+1)}{6n^3}) = 1 + 1 + \frac{2}{6} = \frac{7}{3}. \qquad \blacksquare
\end{aligned}$$

EXERCISE 4.1.2. *Given* $\sum_{k=1}^{n} k^3 = \frac{[n(n+1)]^2}{4}$*, use the definition of the definite integral to prove that* $\int_{-1}^0 x^3 dx = -\frac{1}{4}$.

Sol.

$$\begin{aligned}
\int_{-1}^0 x^3 dx &= \lim_{n\to\infty} \frac{0-(-1)}{n}((-1+\frac{1}{n})^3 + (-1+\frac{2}{n})^3 + ... + (-1+\frac{n}{n})^3) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(-1+\frac{3k}{n}-\frac{3k^2}{n^2}+\frac{k^3}{n^3}) \\
&= \lim_{n\to\infty} \frac{1}{n}(-n + \frac{3}{n}\frac{n(n+1)}{2} - \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\frac{[n(n+1)]^2}{4}) \\
&= \lim_{n\to\infty}(-1 + \frac{3(n+1)}{2n} - \frac{3n(n+1)(2n+1)}{6n^3} + \frac{[n(n+1)]^2}{4n^4}) \\
&= -1 + \frac{3}{2} - 1 + \frac{1}{4} = -\frac{1}{4}. \qquad \blacksquare
\end{aligned}$$

EXERCISE 4.1.3. *Find* $\lim_{n\to\infty}(-\frac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k})$.

Sol.

$$\lim_{n\to\infty}(-\frac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k}) = -\lim_{n\to\infty}(\frac{1}{n}\sum_{k=1}^{n}\frac{\sqrt{k}}{\sqrt{n}}) = -\int_0^1 \sqrt{x}dx = -\frac{2}{3}. \qquad \blacksquare$$

EXERCISE 4.1.4. *Find* $\lim\limits_{n\to\infty}(n\sum_{k=1}^{n}\frac{1}{(n+k)^2})$.

Sol.

$$\begin{aligned}\lim_{n\to\infty}(n\sum_{k=1}^{n}\frac{1}{(n+k)^2}) &= \lim_{n\to\infty}(\frac{1}{n}\sum_{k=1}^{n}\frac{n^2}{(n+k)^2}) = \lim_{n\to\infty}(\frac{1}{n}\sum_{k=1}^{n}(\frac{1}{1+\frac{k}{n}})^2) \\ &= \int_0^1(\frac{1}{1+x})^2dx = -\frac{1}{1+x}\Big|_0^1 = \frac{1}{2}. \qquad \blacksquare\end{aligned}$$

4.2. Exercises 4.2.

EXERCISE 4.2.1. *For* $x \in R$, *set* $F(x) = \int_0^x t\sqrt{1+\sin t}dt$.

(a) Find $F(0)$.
(b) Find $F'(x)$.
(c) Find $F'(\frac{\pi}{2})$.
Sol.
(a) $F(0) = \int_0^0 t\sqrt{1+\sin t}dt = 0$. $\blacksquare$

(b) By the Fundamental Theorem of Calculus I, we have $F'(x) = x\sqrt{1+\sin x}$. $\blacksquare$

(c) $F'(\frac{\pi}{2}) = \frac{\pi}{2}\sqrt{1+\sin\frac{\pi}{2}} = \frac{\pi\sqrt{2}}{2}$. $\blacksquare$

EXERCISE 4.2.2. *For* $x \in R$, *set* $F(x) = \int_{-1}^x \sqrt{t^2+1}dt$.

(a) Find $F(-1)$.
(b) Find $F'(x)$.
(c) Find $F'(-6)$.
(d) Find $F''(x)$.
Sol.
(a) $F(-1) = \int_{-1}^{-1}\sqrt{t^2+1}dt = 0$. $\blacksquare$

(b) By the Fundamental Theorem of Calculus I, we have $F'(x) = \sqrt{x^2+1}$. $\blacksquare$

(c) $F'(-6) = \sqrt{(-6)^2+1} = \sqrt{37}$. $\blacksquare$

(d) $F''(x) = \frac{d}{dx}\sqrt{x^2+1} = \frac{x}{\sqrt{x^2+1}}$. $\blacksquare$

EXERCISE 4.2.3. *For* $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *set* $F(x) = \int_0^x \sqrt{\sec t}dt$.

(a) Find $F(0)$.
(b) Find $F'(x)$.
(c) Find $F'(\frac{\pi}{4})$.
Sol.
(a) $F(0) = \int_0^0 \sqrt{\sec t}dt = 0$. ∎

(b) By the Fundamental Theorem of Calculus I, we have $F'(x) = \sqrt{\sec x}$. ∎

(c) $F'(\frac{\pi}{4}) = \sqrt{\sec \frac{\pi}{4}} = \sqrt{\sqrt{2}} = 2^{\frac{1}{4}}$. ∎

EXERCISE 4.2.4. *Let* $F(x) = \int_1^x \sin t^2 dt$.

(a) Find $F'(\sqrt{\frac{\pi}{2}})$.
(b) Find $F''(x)$.
Sol.
(a) By the Fundamental Theorem of Calculus I, we have $F'(x) = \sin x^2$, so $F'(\sqrt{\frac{\pi}{2}}) = \sin(\sqrt{\frac{\pi}{2}})^2 = \sin \frac{\pi}{2} = 1$. ∎

(b) $F''(x) = \frac{d}{dx} \sin x^2 = 2x \cos x^2$. ∎

EXERCISE 4.2.5. *Compute* $\frac{d}{dx} \int_0^{\sin x} \sqrt{t^3+1}dt$.

Sol.
$\frac{d}{dx} \int_0^{\sin x} \sqrt{t^3+1}dt = (\frac{d}{d\sin x} \int_0^{\sin x} \sqrt{t^3+1}dt)(\frac{d\sin x}{dx}) = \sqrt{\sin^3 x + 1} \cdot \cos x$. ∎

EXERCISE 4.2.6. *Compute* $\frac{d}{ds} \int_0^{\sqrt{s}} \frac{\sqrt{t^6+1}}{t^2+1}dt$.

Sol.
$\frac{d}{ds} \int_0^{\sqrt{s}} \frac{\sqrt{t^6+1}}{t^2+1}dt = (\frac{d}{d\sqrt{s}} \int_0^{\sqrt{s}} \frac{\sqrt{t^6+1}}{t^2+1}dt)(\frac{d\sqrt{s}}{ds}) = \frac{\sqrt{s^3+1}}{s+1} \cdot \frac{1}{2\sqrt{s}}$. ∎

EXERCISE 4.2.7. *Compute* $\frac{d}{dx} \int_1^{x^2} \frac{\sin t}{t}dt$, *for* $x \geq 1$.

Sol.
$\frac{d}{dx} \int_1^{x^2} \frac{\sin t}{t}dt = (\frac{d}{dx^2} \int_1^{x^2} \frac{\sin t}{t}dt)(\frac{dx^2}{dx}) = \frac{\sin x^2}{x^2} \cdot 2x = \frac{2\sin x^2}{x}$. ∎

EXERCISE 4.2.8. *Suppose* f *is continuous on* R, *let* $F(x) = \int_0^x \sqrt{t}(\int_{\sqrt{2}}^{\sec t} f(u)du)dt$.

(a) Compute $F(0)$.
(b) Compute $F'(\frac{\pi}{4})$.
(c) Find $F''(x)$.
Sol.
(a) $F(x) = \int_0^0 \sqrt{t}(\int_{\sqrt{2}}^{\sec t} f(u)du)dt = 0$. ∎

(b) By the Fundamental Theorem of Calculus I, we have $F'(x) = \sqrt{x}(\int_{\sqrt{2}}^{\sec x} f(u)du)$, so $F'(\frac{\pi}{4}) = \sqrt{\frac{\pi}{4}}(\int_{\sqrt{2}}^{\sqrt{2}} f(u)du) = 0$. ■

(c) $F''(x) = \frac{d}{dx}\sqrt{x}(\int_{\sqrt{2}}^{\sec x} f(u)du) = \sqrt{x}f(\sec x) + \frac{1}{2\sqrt{x}}\int_{\sqrt{2}}^{\sec x} f(u)du$. ■

4.3. Exercises 4.3.

Compute the following integrals.

EXERCISE 4.3.1. $\int_1^{10}(\frac{1}{x^2} + x^2)dx$

Sol.

$$\int_1^{10}(\frac{1}{x^2} + x^2)dx = (-\frac{1}{x} + \frac{1}{3}x^3)\Big|_1^{10} = \frac{3339}{10}. \qquad \blacksquare$$

EXERCISE 4.3.2. $\int_1^3(3x^2 - \frac{1}{x^2})dx$

Sol.

$$\int_1^3(3x^2 - \frac{1}{x^2})dx = (x^3 + \frac{1}{x})\Big|_1^3 = \frac{76}{3}. \qquad \blacksquare$$

EXERCISE 4.3.3. $\int_0^2 \sqrt{x}dx$

Sol.

$$\int_0^2 \sqrt{x}dx = \int_0^2 x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 = \frac{4\sqrt{2}}{3}. \qquad \blacksquare$$

EXERCISE 4.3.4. $\int_{-1}^1(x^2 - 2)^2dx$

Sol.

$$\begin{aligned}\int_{-1}^1(x^2 - 2)^2dx &= \int_{-1}^1(x^4 - 4x^2 + 4)dx \\ &= (\frac{1}{5}x^5 - \frac{4}{3}x^3 + 4x)\Big|_{-1}^1 = \frac{86}{15}. \qquad \blacksquare\end{aligned}$$

EXERCISE 4.3.5. $\int_0^{\frac{\pi}{4}} \tan^2 x dx$

Sol.

$$\int_0^{\frac{\pi}{4}} \tan^2 x dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)dx = (\tan x - x)|_0^{\frac{\pi}{4}} = 1 - \frac{\pi}{4}. \qquad \blacksquare$$

EXERCISE 4.3.6. $\int_0^{\frac{\pi}{4}} \frac{1}{(1-\sin x)(1+\sin x)}dx$

Sol.

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1}{(1-\sin x)(1+\sin x)}dx &= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin^2 x}dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x}dx \\ &= \int_0^{\frac{\pi}{4}} \sec^2 x dx = \tan x|_0^{\frac{\pi}{4}} = 1. \qquad \blacksquare\end{aligned}$$

EXERCISE 4.3.7. $\int_0^1 (1-x^2)\sqrt{x}dx$

Sol.

$$\int_0^1 (1-x^2)\sqrt{x}dx = \int_0^1 (x^{\frac{1}{2}} - x^{\frac{5}{2}})dx = (\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{7}x^{\frac{7}{2}})\Big|_0^1 = \frac{8}{21}. \qquad \blacksquare$$

4.4. Exercises 4.4.

EXERCISE 4.4.1. *Let $f(x) = (1+\cos x)^2 + \sin^2 x$, $x \in [-\pi, \pi]$. Find the area between the graph of f and the x-axis.*

Sol.
Note that $f(x) \geq 0, \forall\, x \in [-\pi, \pi]$. The area is

$$\begin{aligned}\int_{-\pi}^{\pi} [(1+\cos x)^2 + \sin^2 x]dx &= \int_{-\pi}^{\pi} (1 + 2\cos x + \cos^2 x + \sin^2 x)dx \\ &= \int_{-\pi}^{\pi} (2 + 2\cos x)dx = (2x + 2\sin x)_{-\pi}^{\pi} = 4\pi. \qquad \blacksquare\end{aligned}$$

EXERCISE 4.4.2. *Let $f(x) = x^3$, $x \in [-1, 1]$. Find the area between the graph of f and the y-axis.*

Sol.
Solution 1:
Since the area between the graph of f and the y-axis is equal to

$$2 \cdot (1 \cdot 1) - \textit{the area between the graph of } f \textit{ and the } x - \textit{axis},$$

and the area between the graph of f and the x-axis is equal to

$$\int_0^1 x^3 dx + \int_{-1}^0 (0 - x^3) dx = \frac{1}{2}.$$

(Note that $f(x) \geq 0, \ \forall \ x \in [0,1], \ f(x) \leq 0, \forall \ x \in [-1,0]$.) We have that the area between the graph of f and the y-axis is equal to $\frac{3}{2}$. ■

Solution 2:

Consider $x = g(y) = y^{\frac{1}{3}}, y \in [-1,1]$. The area between the graph of f and the y-axis is equal to the area between the graph of g and the y-axis, so the area is

$$\int_0^1 y^{\frac{1}{3}} dy + \int_{-1}^0 (0 - y^{\frac{1}{3}}) dy = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}.$$

(Note that $g(y) \geq 0, \ \forall \ y \in [0,1], \ g(y) \leq 0, \forall \ y \in [-1,0]$.) ■

EXERCISE 4.4.3. *Let $f(x) = \tan^2 x, \ x \in [0, \frac{\pi}{4}]$. Find the area between the graph of f and the x-axis.*

Sol.

Note that $f(x) \geq 0, \forall \ x \in [0, \frac{\pi}{4}]$. The area is $\int_0^{\frac{\pi}{4}} \tan^2 x dx = 1 - \frac{\pi}{4}$. (see Exercise 4.3.5) ■

EXERCISE 4.4.4. *Let $f(x) = \sec x \tan x, \ x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. Find the area between the graph of f and the x-axis.*

Sol.

Note that $f(x) \geq 0, \forall \ x \in [0, \frac{\pi}{4}], \ f(x) \leq 0, \forall \ x \in [-\frac{\pi}{4}, 0]$. The area is

$$\begin{aligned} & \int_0^{\frac{\pi}{4}} \sec x \tan x dx + \int_{-\frac{\pi}{4}}^0 (0 - \sec x \tan x) dx \\ = \ & \sec x|_0^{\frac{\pi}{4}} - \sec x|_{-\frac{\pi}{4}}^0 = 2\sqrt{2} - 2. \end{aligned}$$ ■

EXERCISE 4.4.5. *Sketch the region bounded by the curves and find its area.*

(a) $y = \sqrt{x}, \ y = x^{\frac{1}{4}}$.
(b) $x = y^2, \ y = x^2$.
(c) $y = 36x, \ y = x^3$.
(d) $y = \sin(\pi x), \ x = 0.5y$.

Sol.

(a) Since $\sqrt{x} = x^{\frac{1}{4}}$ at $x = 0$ or 1, and since $\sqrt{x} \leq x^{\frac{1}{4}}$ for $x \in [0,1]$, the area is

$$\int_0^1 (x^{\frac{1}{4}} - \sqrt{x}) dx = (\frac{4}{5} x^{\frac{5}{4}} - \frac{2}{3} x^{\frac{3}{2}}) \Big|_0^1 = \frac{2}{15}.$$ ■

(b) Consider $x = y^2 \Leftrightarrow y = \pm\sqrt{x}$. Since $\sqrt{x} = x^2$ at $x = 0$ or 1, and since $\sqrt{x} \geq x^2$ for $x \in [0, 1]$, the area is

$$\int_0^1 (\sqrt{x} - x^2)dx = (\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3)\Big|_0^1 = \frac{1}{3}. \qquad \blacksquare$$

(c) Since $36x = x^3$ at $x = -6, 0$ or 6, and since $36x \geq x^3$ for $x \in [0, 6]$, $36x \leq x^3$ for $x \in [-6, 0]$, the area is

$$\int_0^6 (36x - x^3)dx + \int_{-6}^0 (x^3 - 36x)dx = 648. \qquad \blacksquare$$

(d) Consider $x = 0.5y \Leftrightarrow y = 2x$. Since $\sin(\pi x) = 2x$ at $x = 0$ or $\frac{1}{2}$, and since $\sin(\pi x) \geq 2x$ for $x \in [0, \frac{1}{2}]$. the area is

$$\int_0^{\frac{1}{2}} (\sin(\pi x) - 2x)dx = (-\frac{1}{\pi}\cos(\pi x) - x^2)\Big|_0^{\frac{1}{2}} = \frac{1}{\pi} - \frac{1}{4}. \qquad \blacksquare$$

4.5. Exercises 4.5.

Compute the following integrals.

Exercise 4.5.1. $\int \frac{3s^2}{\sqrt{1-s^3}}ds$

Sol.
Let $u = 1 - s^3, du = -3s^2 ds$.

$$\int \frac{3s^2}{\sqrt{1-s^3}}ds = \int \frac{-1}{\sqrt{u}}du = -2\sqrt{u} + C = -2\sqrt{1-s^3} + C. \qquad \blacksquare$$

Exercise 4.5.2. $\int \frac{1}{(1+t)^2}dt$

Sol.
Let $u = 1 + t, du = dt$.

$$\int \frac{1}{(1+t)^2}dt = \int \frac{1}{u^2}du = -\frac{1}{u} + C = -\frac{1}{1+t} + C. \qquad \blacksquare$$

Exercise 4.5.3. $\int \frac{1}{3}x^2 \tan x^3 dx$

Sol.
Let $u = x^3, du = 3x^2 dx$.

$$\int \frac{1}{3}x^2 \tan x^3 dx = \int \frac{1}{9}\tan u du = \frac{1}{9}\ln|\sec u| + C = \frac{1}{9}\ln|\sec x^3| + C. \qquad \blacksquare$$

EXERCISE 4.5.4. $\int_2^4 \frac{1}{\sqrt{x+1}}dx$

Sol.
Let $u = x+1, du = dx$.

$$\int_2^4 \frac{1}{\sqrt{x+1}}dx = \int_3^5 \frac{1}{\sqrt{u}}du = \left(2\sqrt{u}\right)\Big|_3^5 = 2\sqrt{5} - 2\sqrt{3}. \qquad \blacksquare$$

EXERCISE 4.5.5. $\int x^2\sqrt{x-1}dx$

Sol.
Let $u = x-1, du = dx$.

$$\begin{aligned}\int x^2\sqrt{x-1}dx &= \int (u+1)^2\sqrt{u}du = \int (u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}})du \\ &= \frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{7}(x-1)^{\frac{7}{2}} + \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 4.5.6. $\int \frac{x^2}{\sqrt{x+1}}dx$

Sol.
Let $u = x+1, du = dx$.

$$\begin{aligned}\int \frac{x^2}{\sqrt{x+1}}dx &= \int \frac{(u-1)^2}{\sqrt{u}}du = \int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}})du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}} + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 4.5.7. $\int(9x+4)(3x-1)^9dx$

Sol.
Let $u = 3x-1, du = 3dx$.

$$\begin{aligned}\int (9x+4)(3x-1)^9dx &= \int \frac{1}{3}(3u+7)u^9du = \int (u^{10} + \frac{7}{3}u^9)du \\ &= \frac{1}{11}u^{11} + \frac{7}{30}u^{10} + C \\ &= \frac{1}{11}(3x-1)^{11} + \frac{7}{30}(3x-1)^{10} + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 4.5.8. $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$

Sol.
Let $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx = \int 2 \sec^2 u du = 2 \tan u + C = 2 \tan \sqrt{x} + C. \qquad \blacksquare$$

EXERCISE 4.5.9. $\int \frac{x}{\csc x^2} dx$

Sol.
Let $u = x^2, du = 2x dx$.

$$\int \frac{x}{\csc x^2} dx = \int \frac{1}{2} \sin u du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos x^2 + C. \qquad \blacksquare$$

EXERCISE 4.5.10. $\int (\tan^3 x)(\sec^3 x) dx$

Sol.
Consider

$$\int (\tan^3 x)(\sec^3 x) dx = \int (\sec^2 x - 1)(\sec^2 x)(\tan x \sec x) dx.$$

Let $u = \sec x, du = \sec x \tan x dx$.

$$\begin{aligned} \int (\sec^2 x - 1)(\sec^2 x)(\tan x \sec x) dx &= \int (u^2 - 1) u^2 du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\ &= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C. \qquad \blacksquare \end{aligned}$$

EXERCISE 4.5.11. $\int (\sin^3 x + \cos^3 x) dx$

Sol.
Consider

$$\begin{aligned} \int (\sin^3 x + \cos^3 x) dx &= \int \sin^3 x dx + \int \cos^3 x dx \\ &= \int \sin x (1 - \cos^2 x) dx + \int \cos x (1 - \sin^2 x) dx. \end{aligned}$$

Let $u = \cos x, du = -\sin x dx, v = \sin x, dv = \cos x dx$.

$$\begin{aligned} & \int \sin x (1 - \cos^2 x) dx + \int \cos x (1 - \sin^2 x) dx \\ = & \int -(1 - u^2) du + \int (1 - v^2) dv = \frac{1}{3} u^3 - u + C_1 + v - \frac{1}{3} v^3 + C_2 \\ = & \frac{1}{3} \cos^3 x - \cos x + \sin x - \frac{1}{3} \sin^3 x + C. \qquad \blacksquare \end{aligned}$$

Exercise 4.5.12. $\int (\sin^3 x)(\cos^3 x)dx$

Sol.
Let $u = \sin x, du = \cos x dx$.

$$\begin{aligned}\int (\sin^3 x)(\cos^3 x)dx &= \int (\sin^3 x)(1-\sin^2 x)(\cos x)dx \\ &= \int (u^3 - u^5)du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C \\ &= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C. \qquad \blacksquare\end{aligned}$$

Exercise 4.5.13. *Suppose the following definite integrals both exist, and* $-f(x) = f(-x), \forall\ x$. *Prove that* $\int_{-a}^{a} f(x)dx = 0$.

Sol.
Consider

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{-a}^{0} f(x)dx.$$

For $\int_{-a}^{0} f(x)dx$, let $u = -x, du = -dx$. Then we have

$$\begin{aligned}\int_{-a}^{0} f(x)dx &= \int_{a}^{0} -f(-u)du = \int_{0}^{a} f(-u)du \\ &= \int_{0}^{a} -f(u)du = -\int_{0}^{a} f(u)du = -\int_{0}^{a} f(-x)dx.\end{aligned}$$

Thus

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{0}^{a} f(x)dx + \int_{-a}^{0} f(x)dx = \int_{-a}^{a} f(x)dx \\ &= \int_{0}^{a} f(x)dx + \left(-\int_{0}^{a} f(-x)dx\right) = 0. \qquad \blacksquare\end{aligned}$$

Exercise 4.5.14. *Suppose* $f'(x) > 0, \forall\ x \in R$, *and* $f(-1) = -1,\ f(1) = 1$.*Set* $F(x) = \int_{0}^{2x} f(3t)dt$. *Prove that*

(a) F is twice differentiable.
(b) F has a critical point on the interval $(-1, 1)$, and this critical point is a local minimum point.
Sol.

(a) By the Fundamental Theorem of Calculus I, we have

$$F'(x) = \left(\frac{d}{d(2x)} \int_0^{2x} f(3t)dt \right) \frac{d(2x)}{dx} = 2f(6x).$$

Then since $f'(x) > 0, \forall\ x \in R,$ we know that $f'(x)$ exists, $\forall\ x \in R$. So F is twice differentiable and $F''(x) = 12f'(6x)$. ■

(b) Since $f'(x)$ exists, $\forall\ x \in R$, $f(x)$ is continuous on R. Then since we have $f(-1) = -1$, $f(1) = 1$, by the intermediate value theorem (Thm 1.6.1), $\exists\ c \in (-1, 1)$ such that $f(c) = 0$. Therefore, $\exists\ d = \frac{1}{6}c \in (-\frac{1}{6}, \frac{1}{6}) \subseteq (-1, 1)$ such that $f(6d) = f(c) = 0$. Then since $F'(d) = 2f(6d) = 0$, d is a critical point of F. Moreover, since $f'(x) > 0, \forall\ x \in R$, $F''(d) = 12f'(6d) > 0$. Then by the second derivative test (Thm 3.3.9), d is a local minimum point of F. ■

5. Chapter 5

5.1. Exercises 5.1.

For the following problems, find the area of the region enclosed by the given graphs.

EXERCISE 5.1.1. $y = \frac{x}{3}$, $y = x^{\frac{2}{3}}$.

Sol.

Since the solutions of $\begin{cases} y = \frac{x}{3} \\ y = x^{\frac{2}{3}} \end{cases}$ are $(0, 0)$ and $(27, 9)$, the area of the enclosed region is

$$\int_0^{27} (x^{\frac{2}{3}} - \frac{x}{3})dx = (\frac{3}{5}x^{\frac{5}{3}} - \frac{1}{6}x^2)\Big|_0^{27} = \frac{243}{10}. \qquad \blacksquare$$

EXERCISE 5.1.2. $y^2 = x - 4$, $y^2 = \frac{x}{2}$.

Sol.

Since the solutions of $\begin{cases} y^2 = x - 4 \\ y^2 = \frac{x}{2} \end{cases}$ are $(8, 2)$ and $(8, -2)$, the area of the enclosed region is

$$\int_{-2}^{2} (y^2 + 4 - 2y^2)dy = (-\frac{1}{3}y^3 + 4y)\Big|_{-2}^{2} = 16. \qquad \blacksquare$$

EXERCISE 5.1.3. $4y = x$, $y = x^2$, $x > 0$.

Sol.

Since the solutions of $\begin{cases} 4y = x \\ y = x^2 \end{cases}$ are $(0, 0)$ and $(\frac{1}{4}, \frac{1}{16})$, the area of the enclosed region is

$$\int_0^{\frac{1}{4}} (\frac{1}{4}x - x^2)dx = (\frac{1}{8}x^2 - \frac{1}{3}x^3)\Big|_0^{\frac{1}{4}} = \frac{1}{384}. \qquad \blacksquare$$

EXERCISE 5.1.4. *Find the number c such that the area of the region bounded by these graphs $y + x^2 = c^2$ and $y - x^2 = -c^2$ is 576.*

Sol.

Since the solutions of $\begin{cases} y + x^2 = c^2 \\ y - x^2 = -c^2 \end{cases}$ are $(c, 0)$ and $(-c, 0)$, the area of the enclosed region is

$$\int_{-c}^{c} [(-x^2 + c^2) - (x^2 - c^2)]dx = (-\frac{2}{3}x^3 + 2c^2x)\Big|_{-c}^{c} = \frac{8}{3}c^3 = 576.$$

So $c = 6$. $\blacksquare$

5.2. Exercises 5.2.

For the following problems, find the volume of the solid obtained by revolving the region bounded by the graphs about the given line.

EXERCISE 5.2.1. $y = \frac{x}{2}$, $x = y^2$ *about the* y*-axis.*

Sol.

Since the solutions of $\begin{cases} y = \frac{1}{2}x \\ y^2 = x \end{cases}$ are $(0,0)$ and $(4,2)$, the volume of the bounded solid revolving about y-axis is

$$\int_0^2 \pi[(2y)^2 - (y^2)^2]dx = [\pi(\frac{4}{3}y^3 - \frac{1}{5}y^5)]\Big|_0^2 = \frac{64}{15}\pi. \qquad \blacksquare$$

EXERCISE 5.2.2. $y = \sqrt{x}$, $y = x$ *about* $y = 1$.

Sol.

Since the solutions of $\begin{cases} y = \sqrt{x} \\ y = x \end{cases}$ are $(0,0)$ and $(1,1)$, the volume of the bounded solid revolving about $y = 1$ is

$$\begin{aligned} & \int_0^1 \pi[(x-1)^2 - (x^{\frac{1}{2}} - 1)^2]dx \\ = \quad & [\pi(\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}})]\Big|_0^1 = \frac{1}{6}\pi. \qquad \blacksquare \end{aligned}$$

EXERCISE 5.2.3. $y^2 = x$, $y = x^2$ *about* $x = -1$.

Sol.

Since the solutions of $\begin{cases} y^2 = x \\ y = x^2 \end{cases}$ are $(0,0)$ and $(1,1)$, the volume of the bounded solid revolving about $x = -1$ is

$$\begin{aligned} & \int_0^1 \pi[(y^{\frac{1}{2}} + 1)^2 - (y^2 + 1)^2]dx \\ = \quad & [\pi(-\frac{1}{5}y^5 - \frac{2}{3}y^3 + \frac{1}{2}y^2 + \frac{4}{3}y^{\frac{3}{2}})]\Big|_0^1 = \frac{29}{30}\pi. \qquad \blacksquare \end{aligned}$$

EXERCISE 5.2.4. $y = x^2 - 4x + 3$, $y = x - 1$ *about* $y = 3$.

Sol.

Since the solutions of $\begin{cases} y = x^2 - 4x + 3 \\ y = x - 1 \end{cases}$ are $(1,0)$ and $(4,3)$, the volume of the bounded solid revolving about $y = 3$ is

$$\begin{aligned} & \int_1^4 \pi[(x^2 - 4x + 3 - 3)^2 - (x - 1 - 3)^2]dx \\ = \quad & [\pi(\frac{1}{5}x^5 - 2x^4 + 5x^3 + 4x^2 - 16x)]\Big|_1^4 = \frac{108}{5}\pi. \qquad \blacksquare \end{aligned}$$

EXERCISE 5.2.5. *$y = 4x + 6$, $y = x^3 - x^2 + 2x - 6$ about the x-axis.*

Sol.

Since the solutions of $\begin{cases} y = 4x - 6 \\ y = x^3 - x^2 + 2x - 6 \end{cases}$ are $(1,0)$ and $(4,3)$, the volume of the bounded solid revolving about $y = 2$ is

$$\begin{aligned} & \int_0^2 \pi[(x^3 - x^2 + 2x - 6 - 2)^2 - (4x - 6 - 2)^2]dx \\ & + \int_{-1}^0 \pi[(4x - 6 - 2)^2 - (x^3 - x^2 + 2x - 6 - 2)^2]dx \\ = \quad & [\pi(\frac{1}{7}x^7 - \frac{1}{3}x^6 + x^5 - 5x^4 + \frac{4}{3}x^3 + 16x^2)]\Big|_0^2 \\ & + [\pi(-\frac{1}{7}x^7 + \frac{1}{3}x^6 - x^5 + 5x^4 - \frac{4}{3}x^3 - 16x^2)]\Big|_{-1}^0 \\ = \quad & \frac{766}{21}\pi. \qquad \blacksquare \end{aligned}$$

5.3. Exercises 5.3.

For the following problems, use the shell method to find the volume generated by revolving the region bounded by the given graphs about the given line.

EXERCISE 5.3.1. *$y = 3$, $y = 4x - x^2$ about $x = 3$.*

Sol.

Since the solutions of $\begin{cases} y = 3 \\ y = 4x - x^2 \end{cases}$ are $(1,3)$ and $(3,3)$, the volume of the bounded shell revolving about $x = 3$ is

$$\begin{aligned} & \int_1^3 2\pi\,(-x + 3)\,[(4x - x^2) - 3]dx \\ = \quad & [2\pi(\frac{1}{4}x^4 - \frac{7}{3}x^3 + \frac{15}{2}x^2 - 9x)]\Big|_1^3 = \frac{8}{3}\pi. \qquad \blacksquare \end{aligned}$$

EXERCISE 5.3.2. $(y-2)^2 = x-1$, $x=2$ *about the x-axis.*

Sol.

Since the solutions of $\begin{cases} (y-2)^2 = x-1 \\ x = 2 \end{cases}$ are $(2,3)$ and $(2,1)$, the volume of the bounded shell revolving about x-axis is

$$\begin{aligned} & \int_1^3 2\pi y[2-((y-2)^2+1)]dx \\ = & [2\pi(-\frac{1}{4}x^4 + \frac{4}{3}x^3 - \frac{3}{2}x^2)]\Big|_1^3 = \frac{16}{3}\pi. \qquad \blacksquare \end{aligned}$$

EXERCISE 5.3.3. $(y-1)^2 = 1-x^2$ *about the y-axis.*

Sol.

Since $(y-1)^2 = 1-x^2$ is a circle centered at $(0,1)$ with radius 1, consider $\begin{cases} y = 1+\sqrt{1-x^2} \\ y = 1-\sqrt{1-x^2} \end{cases}$, then the volume of the bounded shell revolving about y-axis is

$$\int_0^1 2\pi x[(1+\sqrt{1-x^2}) - (1-\sqrt{1-x^2})]dx.$$

Let $u = 1-x^2$, $du = -2xdx$, then

$$\begin{aligned} & \int_0^1 2\pi x[(1+\sqrt{1-x^2}) - (1-\sqrt{1-x^2})]dx \\ = & \int_0^1 4\pi x\sqrt{1-x^2}dx \\ = & \int_1^0 (-2\pi)u^{\frac{1}{2}}du = \frac{-4}{3}\pi u^{\frac{3}{2}}\Big|_1^0 = \frac{4}{3}\pi. \qquad \blacksquare \end{aligned}$$

EXERCISE 5.3.4. $y = x^3$, $y = 8x$ *about the y-axis.*

Sol.

Since the solutions of $\begin{cases} y = x^3 \\ y = 8x \end{cases}$ are $(2\sqrt{2}, 16\sqrt{2})$, $(0,0)$ and $(-2\sqrt{2}, -16\sqrt{2})$, the volume of the bounded shell revolving about y-axis is

$$\begin{aligned} & \int_0^{2\sqrt{2}} 2\pi x[8x - x^3]dx + \int_{-2\sqrt{2}}^0 2\pi(-x)[x^3-8x]dx \\ = & [2\pi(\frac{8}{3}x^3 - \frac{1}{5}x^5)]\Big|_0^{2\sqrt{2}} + [2\pi(\frac{8}{3}x^3 - \frac{1}{5}x^5)]\Big|_{-2\sqrt{2}}^0 \\ = & \frac{512\sqrt{2}}{15}\pi. \qquad \blacksquare \end{aligned}$$

EXERCISE 5.3.5. $y = 2$, $y = 4$, $y = 4x^2$, $x = 0$ *about the y-axis.*

Sol.

Since when $x > 0$, the solutions of $\left\{ \begin{array}{c} y = 4 \\ y = 4x^2 \end{array} \right.$ and $\left\{ \begin{array}{c} y = 2 \\ y = 4x^2 \end{array} \right.$ are $(1, 4)$ and $(\frac{1}{\sqrt{2}}, 2)$, the volume of the bounded shell revolving about y-axis is

$$\begin{aligned} & \int_0^1 2\pi x[4-2]dx - \int_{\frac{1}{\sqrt{2}}}^1 2\pi x(4x^2-2)dx \\ = \quad & [2\pi(x^2)]\Big|_0^1 - [2\pi(x^4 - x^2)]\Big|_{\frac{1}{\sqrt{2}}}^1 = \frac{3}{2}\pi. \qquad \blacksquare \end{aligned}$$

6. Chapter 6

6.1. Exercises 6.1.

EXERCISE 6.1.1. *Let* $f(x) = \tan\left(\frac{\pi x}{2}\right) + x^2 + 3$, $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$. *Show that* f *is an 1-1 function and find* $\left(f^{-1}\right)'(3)$.

Sol.
Since f is continuous on $[-\frac{\pi}{4}, \frac{\pi}{4}]$, is differentiable on $(-\frac{\pi}{4}, \frac{\pi}{4})$, and since

$$f'(x) = \frac{\pi}{2}\sec^2(\frac{\pi x}{2}) + 2x > \frac{\pi}{2}\cdot 1 + 2\cdot(-\frac{\pi}{4}) = 0,\ \forall\ x \in (-\frac{\pi}{4}, \frac{\pi}{4}),$$

by theorem 3.2.5, f is increasing on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Hence f is 1-1.
Then since $f(0) = 3$ and $f'(0) = \frac{\pi}{2}$, by theorem 6.1.4,

$$\left(f^{-1}\right)'(3) = \frac{1}{f'(0)} = \frac{2}{\pi}. \qquad \blacksquare$$

EXERCISE 6.1.2. *Let* $f(x) = \int_{10}^{x} \sqrt{1+t^3}dt$. *Show that* f *is an 1-1 function and find* $\left(f^{-1}\right)'(0)$.

Sol.
Since $\sqrt{1+t^3}$ is continuous on $[-1, \infty)$, by the fundamental theorem of calculus I, f is continuous on $[-1, \infty)$, is differentiable on $(-1, \infty)$, and

$$f'(x) = \sqrt{1+x^3} > 0,\ \forall\ x \in (-1, \infty).$$

Hence by theorem 3.2.5, f is increasing on $[-1, \infty)$. So f is 1-1.
Then since $f(10) = 0$ and $f'(10) = \sqrt{1001}$, by theorem 6.1.4,

$$\left(f^{-1}\right)'(0) = \frac{1}{f'(10)} = \frac{1}{\sqrt{1001}}. \qquad \blacksquare$$

6.2. Exercises 6.2.

6.3. Exercises 6.3.

EXERCISE 6.3.1. *Find* $\lim\limits_{x\to 1}[\ln(1-x^2) - \ln(1-x)]$.

Sol.
$\lim\limits_{x\to 1}[\ln(1-x^2) - \ln(1-x)] = \lim\limits_{x\to 1}\ln\frac{1-x^2}{1-x} = \lim\limits_{x\to 1}\ln(1+x) = \ln 2.$ $\blacksquare$

EXERCISE 6.3.2. *Use the definition of derivative to prove that* $\lim\limits_{x\to 0}\frac{\ln(x+1)}{x}$.

Sol.
Let $f(x) = \ln(x+1)$. Then we have

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(x+1)}{x} &= \lim_{h\to 0}\frac{\ln(h+1)-\ln(1)}{h}\\ &= \lim_{h\to 0}\frac{f(h)-f(0)}{h}\\ &= f'(0) = \frac{1}{0+1} = 1. \qquad \blacksquare\end{aligned}$$

EXERCISE 6.3.3. $\int_6^e \frac{1}{x\ln x}dx =$ *?*

Sol.
Let $u = \ln x,\ du = \frac{1}{x}dx$, then we have

$$\int_6^e \frac{1}{x\ln x}dx = \int_{u=\ln 6}^{u=\ln e} \frac{1}{u}du = \ln|u|\big|_{\ln 6}^{1} = \ln 1 - \ln|\ln 6| = -\ln|\ln 6|. \qquad \blacksquare$$

EXERCISE 6.3.4. $\int \frac{2\sin x\cos x}{1+\cos^2 x}dx =$ *?*

Sol. Let $u = 1+\cos^2 x,\ du = -2\sin x\cos x dx$, then we have

$$\int \frac{2\sin x\cos x}{1+\cos^2 x}dx = \int -\frac{1}{u}du = -\ln|u| + C = -\ln(1+\cos^2 x) + C. \qquad \blacksquare$$

EXERCISE 6.3.5. $\frac{d}{dx}\ln|\tan 2x| =$ *?*

Sol. $\frac{d}{dx}\ln|\tan 2x| = \frac{1}{\tan 2x}\cdot\frac{d}{dx}\tan 2x = \frac{1}{\tan 2x}\cdot\sec^2(2x)\cdot 2.$ $\blacksquare$

EXERCISE 6.3.6. *Find the tangent line of the graph* $y = \sin(\ln x^2)$ *at the point* $(1,0)$.

Sol. Since $\frac{dy}{dx} = \frac{d}{dx}\sin(\ln x^2) = \cos(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x = \frac{2\cos(\ln x^2)}{x}$, the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{2\cos(\ln 1^2)}{1} = 2.$$

Hence the tangent line is $y = 2(x-1)$. $\blacksquare$

6.4. Exercises 6.4.

EXERCISE 6.4.1. $\frac{d}{dx}e^{e^x} =$?

Sol. $\frac{d}{dx}e^{e^x} = e^{e^x} \cdot \frac{d}{dx}e^x = e^{e^x} \cdot e^x.$ ■

EXERCISE 6.4.2. *Find* y' *if* $x - y = e^{\frac{x}{y}}$.

Sol. Differentiate both sides with respect to x, we have

$$1 - y' = e^{\frac{x}{y}} \cdot \frac{d}{dx}\frac{x}{y} = e^{\frac{x}{y}} \cdot \frac{y \cdot 1 - xy'}{y^2}.$$

So $y' = \frac{y^2 - ye^{\frac{x}{y}}}{y^2 - xe^{\frac{x}{y}}}$. ■

EXERCISE 6.4.3. *Find the local extreme points of* $f(x) = -e^x + x$.

Sol.
Since $f'(x) = -e^x + 1$, the critical point is 0. Then since $f''(x) = -e^x$, $f''(0) = -1 < 0$. Hence by the second derivative test, 0 is a local maximum point. ■

EXAMPLE 1. $\int_0^2 e^{-\pi x}dx =$?

Sol.

$$\int_0^2 e^{-\pi x}dx = \int_0^2 \frac{1}{-\pi}e^{-\pi x}d(-\pi x) = -\frac{1}{\pi}e^{-\pi x}\Big|_0^2 = -\frac{1}{\pi}e^{-2\pi} + \frac{1}{\pi}.$$ ■

EXERCISE 6.4.4. $\int e^x\sqrt{1 + e^x}dx =$?

Sol.
Let $u = 1 + e^x$, $du = e^x dx$, then we have

$$\int e^x\sqrt{1 + e^x}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1 + e^x)^{\frac{3}{2}} + C.$$ ■

EXERCISE 6.4.5. $\int e^{\tan x}\sec^2 x dx =$?

Sol.
Let $u = \tan x$, $du = \sec^2 x dx$, then we have

$$\int e^{\tan x}\sec^2 x dx = \int e^u du = e^u + C = e^{\tan x} + C.$$ ■

EXERCISE 6.4.6. $\int_1^2 x^{-2}e^{\frac{1}{x}}dx =$?

Sol.

$$\int_1^2 x^{-2}e^{\frac{1}{x}}dx = \int_1^2 -e^{\frac{1}{x}}d(\frac{1}{x}) = -e^{\frac{1}{x}}\Big|_1^2 = -e^{\frac{1}{2}} + e. \qquad \blacksquare$$

6.5. Exercises 6.5.

EXERCISE 6.5.1. *Find* $\frac{d}{dx}a^x$ *if* $a > 0$.

Sol.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot\frac{d}{dx}(x\ln a) = e^{x\ln a}\cdot\ln a = a^x\cdot\ln a. \qquad \blacksquare$$

EXERCISE 6.5.2. $\frac{d}{dx}\tan(4^{x^2}) =$ *?*

Sol.
By exercise 6.5.1,

$$\begin{aligned}\frac{d}{dx}\tan(4^{x^2}) &= \sec^2(4^{x^2})\cdot\frac{d}{dx}(4^{x^2}) \\ &= \sec^2(4^{x^2})\cdot 4^{x^2}\cdot\ln 4\cdot\frac{d}{dx}x^2 \\ &= \sec^2(4^{x^2})\cdot 4^{x^2}\cdot\ln 4\cdot 2x. \qquad \blacksquare\end{aligned}$$

EXERCISE 6.5.3. $\frac{d}{dx}x^{\sin x} =$ *?*

Sol.

$$\begin{aligned}\frac{d}{dx}x^{\sin x} &= \frac{d}{dx}e^{\sin x\ln x} = e^{\sin x\ln x}\cdot\frac{d}{dx}(\sin x\ln x) \\ &= x^{\sin x}\cdot(\cos x\ln x + \sin x\cdot\frac{1}{x}). \qquad \blacksquare\end{aligned}$$

EXERCISE 6.5.4. $\frac{d}{dx}(\cos x)^x =$ *?*

Sol.

$$\begin{aligned}\frac{d}{dx}(\cos x)^x &= \frac{d}{dx}e^{x\ln(\cos x)} = e^{x\ln(\cos x)}\cdot\frac{d}{dx}[x\ln(\cos x)] \\ &= (\cos x)^x\cdot[\ln(\cos x) + x\cdot\frac{1}{\cos x}\cdot(-\sin x)]. \qquad \blacksquare\end{aligned}$$

EXERCISE 6.5.5. *Find* y' *if* $y^x = x^y$.

Sol.
Since
$$\frac{d}{dx}y^x = \frac{d}{dx}e^{x\ln y} = e^{x\ln y}\cdot\frac{d}{dx}(x\ln y) = y^x\cdot(\ln y + \frac{x}{y}\cdot y')$$
and
$$\frac{d}{dx}x^y = \frac{d}{dx}e^{y\ln x} = e^{y\ln x}\cdot\frac{d}{dx}(y\ln x) = x^y\cdot(y'\ln x + \frac{y}{x}),$$
we have
$$y' = \frac{\frac{yx^y}{x} - y^x\ln y}{\frac{xy^x}{y} - x^y\ln x}. \qquad \blacksquare$$

EXERCISE 6.5.6. $\int_1^2 10^x dx =$ *?*

Sol. By exercise 6.5.1,
$$\int_1^2 10^x dx = \frac{1}{\ln 10}10^x\Big|_1^2 = \frac{100}{\ln 10} - \frac{10}{\ln 10} = \frac{90}{\ln 10}. \qquad \blacksquare$$

EXERCISE 6.5.7. $\int 3^{\sin x}\cos x dx =$ *?*

Sol.
Let $u = \sin x,\ du = \cos x dx$, then we have
$$\int 3^{\sin x}\cos x dx = \int 3^u du = \frac{1}{\ln 3}3^u + C = \frac{1}{\ln 3}3^{\sin x} + C. \qquad \blacksquare$$

6.6. Exercises 6.6.

EXERCISE 6.6.1. *Show that* $\cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}$ *and* $\sec\left(\tan^{-1}x\right) = \sqrt{1+x^2}$. *Find* $\tan\left(\sec^{-1}x\right)$*.*

Sol.
Since $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2},\ \cos\left(\sin^{-1}x\right) \ge 0$. Then since
$$1 = \cos^2\left(\sin^{-1}x\right) + \sin^2\left(\sin^{-1}x\right) = \cos^2\left(\sin^{-1}x\right) + x^2,$$
we have $\cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}$.
Since $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2},\ \sec\left(\tan^{-1}x\right) > 0$. Then since
$$\sec^2\left(\tan^{-1}x\right) = 1 + \tan^2\left(\tan^{-1}x\right) = 1 + x^2,$$
we have $\sec\left(\tan^{-1}x\right) = \sqrt{1+x^2}$.
If $x \ge 1$, then $0 \le \sec^{-1}x < \frac{\pi}{2}$ and $\tan\left(\sec^{-1}x\right) \ge 0$. Then since
$$1 + \tan^2\left(\sec^{-1}x\right) = \sec^2\left(\sec^{-1}x\right) = x^2,$$
we have $\tan\left(\sec^{-1}x\right) = \sqrt{x^2-1}$.

If $x \le -1$, then $\frac{\pi}{2} < \sec^{-1} x \le \pi$ and $\tan(\sec^{-1} x) \le 0$. Then since

$$1 + \tan^2\left(\sec^{-1} x\right) = \sec^2\left(\sec^{-1} x\right) = x^2,$$

we have $\tan(\sec^{-1} x) = -\sqrt{x^2 - 1}$. ■

Exercise 6.6.2. *Let* $f(x) = \sqrt{16 - x^2} + x \sin^{-1}(\frac{x}{4})$. *Find* $f'(2)$.

Sol.
Since

$$\begin{aligned} f'(x) &= \frac{1}{2}(16 - x^2)^{-\frac{1}{2}} \cdot (-2x) + \sin^{-1}(\frac{x}{4}) + \frac{x}{\sqrt{1 - (\frac{x}{4})^2}} \cdot \frac{1}{4} \\ &= -\frac{x}{\sqrt{16 - x^2}} + \sin^{-1}(\frac{x}{4}) + \frac{x}{\sqrt{16 - x^2}} \\ &= \sin^{-1}(\frac{x}{4}), \end{aligned}$$

we have $f'(2) = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$. ■

Exercise 6.6.3. $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx =$ *?*

Sol.
Let $u = \sin^{-1} x,\ du = \frac{1}{\sqrt{1-x^2}} dx$, then we have

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx = \int_{u=\sin^{-1} 0}^{u=\sin^{-1} \frac{1}{2}} u du = \frac{1}{2} u^2 \bigg|_0^{\frac{\pi}{6}} = \frac{\pi^2}{72}. \quad ■$$

Exercise 6.6.4. $\int \frac{x+1}{x^2+1} dx =$ *?*

Sol.

$$\begin{aligned} \int \frac{x + 1}{x^2 + 1} dx &= \int \frac{x}{x^2 + 1} dx + \int \frac{1}{x^2 + 1} dx \\ &= \int \frac{\frac{1}{2}}{x^2 + 1} d(x^2 + 1) + \tan^{-1} x \\ &= \frac{1}{2} \ln\left|x^2 + 1\right| + \tan^{-1} x + C. \quad ■ \end{aligned}$$

Exercise 6.6.5. $\int \frac{1}{\sqrt{x}(x+1)} dx =$ *? (Hint: Let* $u = \sqrt{x}$.*)*

Sol.
Let $u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}} dx$, then we have

$$\int \frac{1}{\sqrt{x}(x + 1)} dx = \int \frac{2}{u^2 + 1} du = 2 \tan^{-1} u + C = 2 \tan^{-1} \sqrt{x} + C. \quad ■$$

Exercise 6.6.6. *$\int \frac{1}{\sqrt{a^2-x^2}}dx =$? $(a > 0)$*

Sol.

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}dx = \int \frac{1}{\sqrt{1-\frac{x^2}{a^2}}}d(\frac{x}{a}) = \sin^{-1}(\frac{x}{a}) + C. \qquad \blacksquare$$

Exercise 6.6.7. *$\int \frac{1}{a^2+x^2}dx =$? $(a > 0)$*

Sol.

$$\int \frac{1}{a^2+x^2}dx = \int \frac{1}{a(1+\frac{x^2}{a^2})}d(\frac{x}{a}) = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C. \qquad \blacksquare$$

Exercise 6.6.8. *$\int \frac{1}{|x|\sqrt{x^2-a^2}}dx =$? $(a > 0)$*

Sol.

$$\begin{aligned}\int \frac{1}{|x|\sqrt{x^2-a^2}}dx &= \int \frac{1}{a^2\left|\frac{x}{a}\right|\sqrt{\frac{x^2}{a^2}-1}}dx = \int \frac{1}{a\left|\frac{x}{a}\right|\sqrt{\frac{x^2}{a^2}-1}}d(\frac{x}{a}) \\ &= \frac{1}{a}\sec^{-1}(\frac{x}{a}) + C. \qquad \blacksquare\end{aligned}$$

7. Chapter 7

7.1. Exercises 7.1.

EXERCISE 7.1.1. $\int x^3 e^{-x^4} dx =$?

Sol.
Let $u = -x^4,\ du = -4x^3 dx$, then we have

$$\int x^3 e^{-x^4} dx = \int -\frac{1}{4} e^u du = -\frac{1}{4} e^u + C = -\frac{1}{4} e^{-x^4} + C. \qquad \blacksquare$$

EXERCISE 7.1.2. $\int \frac{\ln x}{x} dx =$?

Sol.

$$\int \frac{\ln x}{x} dx = \int \ln x d(\ln x) = \frac{1}{2} (\ln x)^2 + C. \qquad \blacksquare$$

EXAMPLE 2.

EXERCISE 7.1.3. $\int \frac{2x+1}{x^2+x+1} dx =$?

Sol.

$$\int \frac{2x+1}{x^2+x+1} dx = \int \frac{1}{x^2+x+1} d(x^2+x+1) = \ln \left| x^2+x+1 \right| + C. \qquad \blacksquare$$

EXERCISE 7.1.4. $\int \sin^5 x \cos x dx =$?

Sol.

$$\int \sin^5 x \cos x dx = \int \sin^5 x d(\sin x) = \frac{1}{6} \sin^6 x + C. \qquad \blacksquare$$

EXERCISE 7.1.5. $\int \frac{\tan x}{\sqrt{1-(\ln|\sec x|)^2}} dx =$?

Sol.
Let $u = \ln|\sec x|,\ du = \frac{\sec x \tan x}{\sec x} dx = \tan x dx$, then we have

$$\int \frac{\tan x}{\sqrt{1-(\ln|\sec x|)^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1}(\ln|\sec x|) + C. \qquad \blacksquare$$

EXERCISE 7.1.6. $\int \frac{\tan^{-1} x}{1+x^2} dx =$?

Sol.

$$\int \frac{\tan^{-1} x}{1+x^2} dx = \int \tan^{-1} x d(\tan^{-1} x) = \frac{1}{2} (\tan^{-1} x)^2 + C. \qquad \blacksquare$$

EXERCISE 7.1.7. $\int \frac{\sec(\sqrt{x})}{\sqrt{x}}dx =$?

Sol.

$$\int \frac{\sec(\sqrt{x})}{\sqrt{x}}dx = \int 2\sec(\sqrt{x})d(\sqrt{x}) = 2\ln\left|\sec(\sqrt{x}) + \tan(\sqrt{x})\right| + C. \qquad \blacksquare$$

EXERCISE 7.1.8. $\int x^3\sqrt{1-x^2}dx =$?

Sol.
Let $u = 1 - x^2,\ du = -2xdx$, then we have

$$\begin{aligned}\int x^3\sqrt{1-x^2}dx &= \int -\frac{1}{2}(1-u)\sqrt{u}du \\ &= \int -\frac{u^{\frac{1}{2}}}{2} + \frac{u^{\frac{3}{2}}}{2}du \\ &= -\frac{1}{3}u^{\frac{3}{2}} + \frac{1}{5}u^{\frac{5}{2}} + C \\ &= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{1}{5}(1-x^2)^{\frac{5}{2}} + C. \qquad \blacksquare\end{aligned}$$

7.2. Exercises 7.2.

EXAMPLE 3. $\int \tan^{-1} xdx =$?

Sol.
Let $u = \tan^{-1} x,\ du = \frac{1}{1+x^2}dx,\ dv = dx,\ v = x$, then we have

$$\begin{aligned}\int \tan^{-1} xdx &= x\tan^{-1} x - \int \frac{x}{1+x^2}dx \\ &= x\tan^{-1} x - \int \frac{\frac{1}{2}}{1+x^2}d(1+x^2) \\ &= x\tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.2.1. $\int e^{2x}\sin 3xdx =$?

Sol.
Let $u = e^{2x},\ du = 2e^{2x}dx,\ dv = \sin 3xdx,\ v = -\frac{1}{3}\cos 3x$, then we have

$$\int e^{2x}\sin 3xdx = -\frac{1}{3}e^{2x}\cos 3x + \int \frac{2}{3}e^{2x}\cos 3xdx.$$

Once again, let $u = e^{2x},\ du = 2e^{2x}dx,\ dv = \cos 3xdx,\ v = \frac{1}{3}\sin 3x$, then we have

$$\int e^{2x}\cos 3xdx = \frac{1}{3}e^{2x}\sin 3x - \int \frac{2}{3}e^{2x}\sin 3xdx.$$

Hence we have

$$\begin{aligned}\int e^{2x}\sin 3xdx &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3xdx \\ &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}\int e^{2x}\sin 3xdx.\end{aligned}$$

Thus

$$\frac{13}{9}\int e^{2x}\sin 3xdx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x + C,$$

that is,

$$\int e^{2x}\sin 3xdx = -\frac{3}{13}e^{2x}\cos 3x + \frac{2}{13}e^{2x}\sin 3x + C. \qquad \blacksquare$$

EXERCISE 7.2.2. $\int (\ln x)^2\,dx =$ *?*

Sol.
Let $u = (\ln x)^2,\ du = \frac{2\ln x}{x}dx,\ dv = dx,\ v = x$, then by example 7.2.6, we have

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln xdx = x(\ln x)^2 - 2x\ln x + 2x + C. \qquad \blacksquare$$

EXERCISE 7.2.3. $\int (\ln x)^3\,dx =$ *?*

Sol.
Let $u = (\ln x)^3,\ du = \frac{3(\ln x)^2}{x}dx,\ dv = dx,\ v = x$, then by exercise 7.2.3, we have

$$\begin{aligned}\int (\ln x)^3\,dx &= x(\ln x)^3 - \int 3(\ln x)^2dx \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.2.4. $\int x^2\sin xdx =$ *?*

Sol.
Let $u = x^2,\ du = 2xdx,\ dv = \sin xdx,\ v = -\cos x$, then we have

$$\int x^2\sin xdx = -x^2\cos x + \int 2x\cos xdx.$$

Once again, let $u = x,\ du = dx,\ dv = \cos x dx,\ v = \sin x$, then we have

$$\begin{aligned}\int x^2 \sin x dx &= -x^2 \cos x + 2\int x \cos x dx \\ &= -x^2 \cos x + 2(x \sin x - \int \sin x dx) \\ &= -x^2 \cos x + 2x \sin x + 2\cos x + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.2.5. $\int x^3 \ln x dx =$?

Sol.
Let $u = \ln x,\ du = \frac{1}{x}dx,\ dv = x^3 dx,\ v = \frac{1}{4}x^4$, then we have

$$\int x^3 \ln x dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C. \qquad \blacksquare$$

EXERCISE 7.2.6. $\int x^2 (\ln x)^2 dx =$?

Sol.
Let $u = (\ln x)^2,\ du = \frac{2\ln x}{x}dx,\ dv = x^2 dx,\ v = \frac{1}{3}x^3$, then we have

$$\int x^2 (\ln x)^2 dx = \frac{1}{3}x^3 (\ln x)^2 - \int \frac{2}{3}x^2 \ln x dx.$$

Once again, let $u = \ln x,\ du = \frac{1}{x}dx,\ dv = x^2 dx,\ v = \frac{1}{3}x^3$, then we have

$$\begin{aligned}\int x^2 (\ln x)^2 dx &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\int x^2 \ln x dx \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}(\frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 dx) \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.2.7. $\int \frac{xe^x}{(x+1)^2}dx =$?

Sol.
Let $u = xe^x,\ du = (xe^x + e^x)dx,\ dv = \frac{1}{(x+1)^2}dx,\ v = -\frac{1}{x+1}$, then we have

$$\begin{aligned}\int \frac{xe^x}{(x+1)^2} &= -\frac{xe^x}{x+1} + \int \frac{xe^x + e^x}{x+1}dx \\ &= -\frac{xe^x}{x+1} + \int e^x dx \\ &= -\frac{xe^x}{x+1} + e^x + C. \qquad \blacksquare\end{aligned}$$

7.3. Exercises 7.3.

Exercise 7.3.1. $\int \cos^3 x dx =$?

Sol.

$$\begin{aligned}\int \cos^3 x dx &= \int \cos x(1-\sin^2 x)dx \\ &= \int (1-\sin^2 x)d(\sin x) \\ &= \sin x - \frac{1}{3}\sin^3 x + C. \qquad \blacksquare\end{aligned}$$

Exercise 7.3.2. $\int \sin^4 x dx =$?

Sol.

$$\begin{aligned}\int \sin^4 x dx &= \int (\frac{1-\cos 2x}{2})^2 dx = \int \frac{1-2\cos 2x+\cos^2 2x}{4}dx \\ &= \int (\frac{1}{4} - \frac{\cos 2x}{2} + \frac{1+\cos 4x}{8})dx \\ &= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C. \qquad \blacksquare\end{aligned}$$

Exercise 7.3.3. $\int \sin^3 x \cos^2 x dx =$?

Sol.

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin x(1-\cos^2 x)\cos^2 x dx \\ &= \int -(\cos^2 x - \cos^4 x)d(\cos x) \\ &= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C. \qquad \blacksquare\end{aligned}$$

Exercise 7.3.4. $\int \sin^4 x \cos^2 x dx =$?

Sol.

$$\begin{aligned}\int \sin^4 x \cos^2 x dx &= \int \frac{1-\cos 2x}{2} \cdot \frac{\sin^2 2x}{4} dx \\ &= \int \frac{\sin^2 2x}{8} dx - \int \frac{\cos 2x \sin^2 2x}{8} dx \\ &= \int \frac{1-\cos 4x}{16} dx - \int \frac{\sin^2 2x}{16} d(\sin 2x) \\ &= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.3.5. $\int \tan^3 x \sec^3 x dx =$ *?*

Sol.

$$\begin{aligned}\int \tan^3 x \sec^3 x dx &= \int \tan x \left(\sec^2 x - 1\right) \sec^3 x dx \\ &= \int \tan x \sec^5 x - \tan x \sec^3 x dx \\ &= \int \sec^4 x - \sec^2 x d(\sec x) \\ &= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.3.6. $\int \tan^2 x \sec^4 x dx =$ *?*

Sol.

$$\begin{aligned}\int \tan^2 x \sec^4 x dx &= \int \tan^2 x \left(\tan^2 x + 1\right) \sec^2 x dx \\ &= \int \tan^4 x + \tan^2 x d(\tan x) \\ &= \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.3.7. $\int \tan^4 x dx =$ *?*

Sol.

$$\begin{aligned}\int \tan^4 x dx &= \int \tan^2 x \left(\sec^2 x - 1\right) dx \\ &= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx \\ &= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C. \qquad \blacksquare\end{aligned}$$

Exercise 7.3.8. $\int \tan^2 x \sec x dx =$ *?*

Sol.
Let $u = \tan x,\ du = \sec^2 x dx,\ dv = \tan x \sec x dx,\ v = \sec x$, then we have

$$\begin{aligned}\int \tan^2 x \sec x &= \tan x \sec x - \int \sec^3 x dx \\ &= \tan x \sec x - \int (1 + \tan^2 x) \sec x dx \\ &= \tan x \sec x - \int \sec x dx - \int \tan^2 x \sec x dx \\ &= \tan x \sec x - \ln|\sec x + \tan x| - \int \tan^2 x \sec x dx.\end{aligned}$$

Hence we have

$$2\int \tan^2 x \sec x = \tan x \sec x - \ln|\sec x + \tan x| + C,$$

that is,

$$\int \tan^2 x \sec x = \frac{1}{2}\tan x \sec x - \frac{1}{2}\ln|\sec x + \tan x| + C. \qquad \blacksquare$$

Exercise 7.3.9. $\int \sin 3x \sin 5x dx =$ *? (Hint:* $\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha-\beta)}{2}$*)*

Sol.

$$\int \sin 3x \sin 5x dx = \int \frac{\cos 2x - \cos 8x}{2} dx = \frac{\sin 2x}{4} - \frac{\sin 8x}{16} + C. \qquad \blacksquare$$

7.4. Exercises 7.4.

Exercise 7.4.1. $\int \frac{\sqrt{9-x^2}}{x} dx =$ *?*

Sol.
Let $x = 3\sin u,\ dx = 3\cos u du,\ -\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then we have

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x}dx &= \int \frac{\sqrt{9-9\sin^2 u}}{3\sin u}\cdot 3\cos u du \\
&= \int \frac{3\cos^2 u}{\sin u}du \\
&= \int \frac{3-3\sin^2 u}{\sin u}du \\
&= \int 3\csc u du - \int 3\sin u du \\
&= -3\ln|\csc u + \cot u| + 3\cos u + C.
\end{aligned}$$

Then since

$$\sin u = \frac{x}{3},\ \cos u = \sqrt{1-\sin^2 u} = \sqrt{1-\frac{x^2}{9}},$$

$$\csc u = \frac{1}{\sin u} = \frac{3}{x},\ \cot u = \frac{\cos u}{\sin u} = \frac{3}{x}\cdot\sqrt{1-\frac{x^2}{9}} = \frac{\sqrt{9-x^2}}{x},$$

we have

$$\int \frac{\sqrt{9-x^2}}{x}dx = -3\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + 3\sqrt{1-\frac{x^2}{9}} + C. \qquad \blacksquare$$

EXERCISE 7.4.2. $\int \sqrt{1-9x^2}dx =$ *?*

Sol.
Let $x = \frac{1}{3}\sin u,\ dx = \frac{1}{3}\cos u du,\ -\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then we have

$$\begin{aligned}
\int \sqrt{1-9x^2}dx &= \int \sqrt{1-\sin^2 u}\cdot\frac{1}{3}\cos u du \\
&= \int \frac{\cos^2 u}{3}du \\
&= \int \frac{1+\cos 2u}{6}du \\
&= \frac{u}{6} + \frac{\sin 2u}{12} + C.
\end{aligned}$$

Then since

$$\sin u = 3x,\ \cos u = \sqrt{1-\sin^2 u} = \sqrt{1-9x^2},$$

$$\sin 2u = 2\sin u\cos u = 6x\sqrt{1-9x^2},$$

we have

$$\int \sqrt{1-9x^2}dx = \frac{\sin^{-1}(3x)}{6} + \frac{x\sqrt{1-9x^2}}{2} + C. \qquad \blacksquare$$

EXERCISE 7.4.3. $\int x^2 \left(4-4x^2\right)^{-\frac{3}{2}} dx =$ *?*

Sol.

Let $x = \sin u,\ dx = \cos u du,\ -\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then we have

$$\begin{aligned}\int x^2 \left(4-4x^2\right)^{-\frac{3}{2}} dx &= \int \sin^2 u(4-4\sin^2 u)^{-\frac{3}{2}} \cdot \cos u du \\ &= \int \frac{\sin^2 u \cos u}{8\cos^3 u} du \\ &= \int \frac{1}{8}\tan^2 u du \\ &= \int (\frac{1}{8}\sec^2 u - \frac{1}{8}) du \\ &= \frac{1}{8}\tan u - \frac{u}{8} + C.\end{aligned}$$

Then since

$$\sin u = x,\ \cos u = \sqrt{1-\sin^2 u} = \sqrt{1-x^2},$$
$$\tan u = \frac{\sin u}{\cos u} = \frac{x}{\sqrt{1-x^2}},$$

we have

$$\int x^2 \left(4-4x^2\right)^{-\frac{3}{2}} dx = \frac{x}{8\sqrt{1-x^2}} - \frac{\sin^{-1} x}{8} + C. \qquad \blacksquare$$

EXERCISE 7.4.4. $\int x\sqrt{1+x^2}dx =$ *?*

Sol.

Let $x = \tan u,\ dx = \sec^2 u du,\ -\frac{\pi}{2} < u < \frac{\pi}{2}$. Then we have

$$\begin{aligned}\int x\sqrt{1+x^2}dx &= \int \tan u\sqrt{1+\tan^2 u} \cdot \sec^2 u du \\ &= \int \tan u \sec^3 u du \\ &= \int \sec^2 u d(\sec u) \\ &= \frac{1}{3}\sec^3 u + C.\end{aligned}$$

Then since

$$\tan u = x,\ \sec u = \sqrt{1+\tan^2 u} = \sqrt{1+x^2},$$

we have

$$\int x\sqrt{1+x^2}dx = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C. \qquad \blacksquare$$

EXERCISE 7.4.5. $\int \frac{1}{x\sqrt{25x^2+49}}dx =$?

Sol.
Let $x = \frac{7}{5}\tan u,\ dx = \frac{7}{5}\sec^2 u du,\ -\frac{\pi}{2} < u < \frac{\pi}{2}$. Then we have

$$\begin{aligned}\int \frac{1}{x\sqrt{25x^2+49}}dx &= \int \frac{1}{\frac{7}{5}\tan u\sqrt{49\tan^2 u+49}} \cdot \frac{7}{5}\sec^2 u du \\ &= \int \frac{\sec u}{7\tan u}du \\ &= \int \frac{1}{7}\csc u du \\ &= -\frac{1}{7}\ln|\csc u + \cot u| + C.\end{aligned}$$

Then since

$$\tan u = \frac{5}{7}x,\ \sec u = \sqrt{1+\tan^2 u} = \sqrt{1+\frac{25x^2}{49}},$$

$$\cot u = \frac{1}{\tan u} = \frac{7}{5x},\ \csc u = \frac{\sec u}{\tan u} = \frac{\sqrt{1+\frac{25x^2}{49}}}{\frac{5}{7}x} = \frac{\sqrt{49+25x^2}}{5x},$$

we have

$$\int \frac{1}{x\sqrt{25x^2+49}}dx = -\frac{1}{7}\ln\left|\frac{\sqrt{49+25x^2}}{5x} + \frac{7}{5x}\right| + C. \qquad \blacksquare$$

EXERCISE 7.4.6. $\int \frac{x}{\sqrt{x^2-1}}dx =$?

Sol.
Let $u = x^2-1,\ du = 2xdx$. Then we have

$$\int \frac{x}{\sqrt{x^2-1}}dx = \int \frac{1}{2\sqrt{u}}du = \sqrt{u} + C = \sqrt{x^2-1} + C. \qquad \blacksquare$$

EXERCISE 7.4.7. $\int \frac{1}{\sqrt{4x^2-25}}dx =$?

Sol.

Let $x = \frac{5}{2}\sec u,\ dx = \frac{5}{2}\tan u \sec u du,\ 0 \le u < \frac{\pi}{2}$ if $x > 0, \frac{\pi}{2} < u \le \pi$ if $x < 0$. Then we have

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2-25}}dx &= \int \frac{1}{\sqrt{25\sec^2 u - 25}} \cdot \frac{5}{2}\tan u \sec u du \\ &= \int \frac{\sec u}{2} du \\ &= \frac{1}{2}\ln|\sec u + \tan u| + C.\end{aligned}$$

Then since

$$\sec u = \frac{2}{5}x,\ \tan u = \sqrt{\sec^2 u - 1} = \sqrt{\frac{4x^2}{25} - 1},$$

we have

$$\int \frac{1}{\sqrt{4x^2-25}}dx = \frac{1}{2}\ln\left|\frac{2}{5}x + \sqrt{\frac{4x^2}{25}-1}\right| + C. \qquad \blacksquare$$

Exercise 7.4.8. $\int_{3/2}^{3/\sqrt{2}} \frac{\sqrt{4x^2-9}}{x^2}dx =$?

Sol.
Let $x = \frac{3}{2}\sec u,\ dx = \frac{3}{2}\tan u \sec u du,\ 0 \le u \le \frac{\pi}{4}$ since $\frac{3}{2} \le x \le \frac{3}{\sqrt{2}}$. Then we have

$$\begin{aligned}\int_{3/2}^{3/\sqrt{2}} \frac{\sqrt{4x^2-9}}{x^2}dx &= \int_0^{\frac{\pi}{4}} \frac{\sqrt{9\sec^2 u - 9}}{\frac{9}{4}\sec^2 u} \cdot \frac{3}{2}\tan u \sec u du \\ &= \int_0^{\frac{\pi}{4}} \frac{2\tan^2 u}{\sec u} du \\ &= \int_0^{\frac{\pi}{4}} \frac{2\sec^2 u - 2}{\sec u} du \\ &= \int_0^{\frac{\pi}{4}} 2\sec u du - \int_0^{\frac{\pi}{4}} 2\cos u du \\ &= [2\ln|\sec u + \tan u| - 2\sin u]_0^{\frac{\pi}{4}} \\ &= 2\ln\left|\sqrt{2}+1\right| - \sqrt{2}. \qquad \blacksquare\end{aligned}$$

Exercise 7.4.9. $\int \sec^{-1} x dx =$?

Sol.
Let $u = \sec^{-1} x,\ du = \frac{1}{|x|\sqrt{x^2-1}}dx,\ dv = dx,\ v = x$, then we have

$$\int \sec^{-1} x dx = x\sec^{-1} x - \int \frac{x}{|x|\sqrt{x^2-1}}dx.$$

If $x > 0$, then $\int \frac{x}{|x|\sqrt{x^2-1}}dx = \int \frac{1}{\sqrt{x^2-1}}dx$. Let $x = \sec u,\ dx = \tan u \sec u du,\ 0 \le u < \frac{\pi}{2}$. Then we have

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-1}}dx &= \int \frac{1}{\sqrt{\sec^2 u - 1}} \cdot \tan u \sec u du \\ &= \int \sec u du \\ &= \ln|\sec u + \tan u| + C \\ &= \ln\left|x + \sqrt{x^2-1}\right| + C.\end{aligned}$$

So

$$\int \sec^{-1} x dx = x \sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\right| + C.$$

If $x < 0$, then $\int \frac{x}{|x|\sqrt{x^2-1}}dx = \int -\frac{1}{\sqrt{x^2-1}}dx$. Let $x = \sec u,\ dx = \tan u \sec u du,\ \frac{\pi}{2} < u \le \pi$. Then we have

$$\begin{aligned}\int -\frac{1}{\sqrt{x^2-1}}dx &= \int -\frac{1}{\sqrt{\sec^2 u - 1}} \cdot \tan u \sec u du \\ &= \int \sec u du \\ &= \ln|\sec u + \tan u| + C \\ &= \ln\left|x - \sqrt{x^2-1}\right| + C.\end{aligned}$$

So

$$\int \sec^{-1} x dx = x \sec^{-1} x - \ln\left|x - \sqrt{x^2-1}\right| + C. \qquad \blacksquare$$

7.5. Exercises 7.5.

Exercise 7.5.1. $\int \frac{1}{x^2-5x+6}dx =$?

Sol.

Since $x^2 - 5x + 6 = (x-3)(x-2)$, assume $\frac{1}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}$. Then $A = 1$ and $B = -1$. So we have

$$\int \frac{1}{x^2-5x+6}dx = \int (\frac{1}{x-3} - \frac{1}{x-2})dx = \ln|x-3| - \ln|x-2| + C. \qquad \blacksquare$$

Exercise 7.5.2. $\int \frac{x^2+4x-33}{x^2+2x-3}dx =$?

Sol.

Since $\frac{x^2+4x-33}{x^2+2x-3} = 1 + \frac{2x-30}{x^2+2x-3}$ and $x^2+2x-3 = (x+3)(x-1)$, assume $\frac{2x-30}{x^2+2x-3} = \frac{A}{x+3} + \frac{B}{x-1}$. Then $A = 9$ and $B = -7$. So we have

$$\begin{aligned}\int \frac{x^2+4x-33}{x^2+2x-3}dx &= \int (1 + \frac{9}{x+3} - \frac{7}{x-1})dx \\ &= x + 9\ln|x+3| - 7\ln|x-1| + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.5.3. $\int \frac{14}{x^3-x}dx =$ *?*

Sol.
Since $x^3 - x = x(x+1)(x-1)$,assume $\frac{14}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$. Then $A = -14,\ B = 7$ and $C = 7$. So we have

$$\begin{aligned}\int \frac{14}{x^3-x}dx &= \int (-\frac{14}{x} + \frac{7}{x+1} + \frac{7}{x-1})dx \\ &= -14\ln|x| + 7\ln|x+1| + 7\ln|x-1| + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.5.4. $\int \frac{11x+6}{(x-1)^2}dx =$ *?*

Sol.
Assume $\frac{11x+6}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$. Then $A = 11$ and $B = 17$. So we have

$$\begin{aligned}\int \frac{11x+6}{(x-1)^2}dx &= \int (\frac{11}{x-1} + \frac{17}{(x-1)^2})dx \\ &= 11\ln|x-1| - \frac{17}{x-1} + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.5.5. $\int \frac{x^2+x-3}{(x-1)^2(x-2)}dx =$ *?*

Sol.
Assume $\frac{x^2+x-3}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$. Then $A = -2,\ B = 1$ and $C = 3$. So we have

$$\begin{aligned}\int \frac{x^2+x-3}{(x-1)^2(x-2)}dx &= \int (-\frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{3}{x-2})dx \\ &= -2\ln|x-1| - \frac{1}{x-1} + 3\ln|x-2| + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.5.6. $\int \frac{x^2-2}{x(x^2+2)}dx =$ *?*

Sol.
Assume $\frac{x^2-2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$. Then $A = -1,\ B = 1$ and $C = 0$. So we have

$$\begin{aligned}\int \frac{x^2-2}{x\left(x^2+2\right)}dx &= \int (-\frac{1}{x} + \frac{x}{x^2+2})dx \\ &= -\ln|x| + \frac{1}{2}\ln\left|x^2+2\right| + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.5.7. $\int \frac{1}{(x^2+a^2)^2}dx =$? $(a > 0)$

Sol.
Let $x = a\tan u,\ dx = a\sec^2 u du,\ -\frac{\pi}{2} < u < \frac{\pi}{2}$. Then we have

$$\begin{aligned}\int \frac{1}{\left(x^2+a^2\right)^2}dx &= \int \frac{1}{\left(a^2\tan^2 u + a^2\right)^2} \cdot a\sec^2 u du \\ &= \int \frac{\cos^2 u}{a^3}du \\ &= \int \frac{1+\cos 2u}{2a^3}du \\ &= \frac{u}{2a^3} + \frac{\sin 2u}{4a^3} + C.\end{aligned}$$

Then since

$$\tan u = \frac{x}{a},\ \sec u = \sqrt{1+\tan^2 u} = \sqrt{1+\frac{x^2}{a^2}},\ \cos u = \frac{1}{\sec u} = \frac{a}{\sqrt{a^2+x^2}},$$

$$\sin u = \frac{\tan u}{\sec u} = \frac{x}{\sqrt{a^2+x^2}},\ \sin 2u = 2\sin u\cos u = \frac{2ax}{a^2+x^2},$$

we have

$$\int \frac{1}{\left(x^2+a^2\right)^2}dx = \frac{\tan^{-1}(\frac{x}{a})}{2a^3} + \frac{x}{2a^2(a^2+x^2)} + C. \qquad \blacksquare$$

EXERCISE 7.5.8. $\int \frac{x^5}{(x^2+4)^2}dx =$?

Sol.
Since $\frac{x^5}{(x^2+4)^2} = x + \frac{-8x^3-16x}{(x^2+4)^2}$,assume $\frac{-8x^3-16x}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$. Then $A = -8,\ B = 0,\ C = 16$ and $D = 0$. So we have

$$\begin{aligned}\int \frac{x^5}{(x^2+4)^2}dx &= \int (x - \frac{8x}{x^2+4} + \frac{16x}{(x^2+4)^2})dx \\ &= \frac{1}{2}x^2 - 4\ln\left|x^2+4\right| - \frac{8}{x^2+4} + C. \qquad \blacksquare\end{aligned}$$

EXERCISE 7.5.9. $\int \frac{x^4+13x^2-5x+17}{(x^2+1)^2(x-3)}dx =$?

Sol. Assume $\frac{x^4+13x^2-5x+17}{(x^2+1)^2(x-3)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$. Then $A = 2,\ B = -1,\ C = -3,\ D = 1$ and $E = -2$. So by exercise 7.5.7, we have

$$\begin{aligned}
&\int \frac{x^4+13x^2-5x+17}{(x^2+1)^2\,(x-3)}dx \\
= &\int (\frac{2}{x-3} - \frac{x}{x^2+1} - \frac{3}{x^2+1} + \frac{x}{(x^2+1)^2} - \frac{2}{(x^2+1)^2})dx \\
= &\ 2\ln|x-3| - \frac{1}{2}\ln\left|x^2+1\right| - 3\tan^{-1}x \\
&-\frac{1}{2(x^2+1)} - \tan^{-1}x - \frac{x}{x^2+1} + C. \qquad \blacksquare
\end{aligned}$$

EXERCISE 7.5.10. $\int \frac{5x+2}{x^3-8}dx =$?

Sol.
Since $x^3 - 8 = (x-2)(x^2+2x+4)$, assume $\frac{5x+2}{x^3-8} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$. Then $A = 1,\ B = -1$, and $C = 1$. So we have

$$\begin{aligned}
\int \frac{5x+2}{x^3-8}dx &= \int (\frac{1}{x-2} + \frac{-x+1}{x^2+2x+4})dx \\
&= \int \frac{1}{x-2}dx + \int \frac{-x+1}{(x+1)^2+3}dx.
\end{aligned}$$

Let $u = x+1,\ du = dx$, then by exercise 6.6.7 we have

$$\begin{aligned}
\int \frac{5x+2}{x^3-8}dx &= \int \frac{1}{x-2}dx + \int \frac{-(u-1)+1}{u^2+3}du \\
&= \ln|x-2| - \int \frac{u}{u^2+3}du + \int \frac{2}{u^2+3}du \\
&= \ln|x-2| - \frac{1}{2}\ln\left|u^2+3\right| + \frac{2}{\sqrt{3}}\tan^{-1}(\frac{u}{\sqrt{3}}) + C \\
&= \ln|x-2| - \frac{1}{2}\ln\left|(x+1)^2+3\right| + \frac{2}{\sqrt{3}}\tan^{-1}(\frac{x+1}{\sqrt{3}}) + C. \qquad \blacksquare
\end{aligned}$$

8. Chapter 8

8.1. Exercises 8.1.

EXERCISE 8.1.1. *Find the least upper bound ($\sup S$) and the greatest lower bound ($\inf S$) of the set S.*

(a) $S = [0, 1]$
(b) $S = \{x \mid x^4 \leq 81\}$
(c) $S = \{x \mid x^3 \geq 8\}$
(d) $S = \{x \mid \ln x < 1\}$
(e) $S = \{x \mid x^2 + x + 2 \geq 0\}$
Sol.
(a) $\sup S = 1,\ \inf S = 0.$ ■

(b) Since $S = \{x \mid x^4 \leq 81\} = \{x \mid -3 \leq x \leq 3\} = [-3, 3],\ \sup S = 3,\ \inf S = -3.$ ■

(c) Since $S = \{x \mid x^3 \geq 8\} = \{x \mid x \geq 2\} = [2, \infty)$, $\sup S$ does not exist, $\inf S = 2.$ ■

(d) Since $S = \{x \mid \ln x < 1\} = \{x \mid 0 < x < e\} = (0, e),\ \sup S = e,\ \inf S = 0.$ ■

(e) Since $x^2 + x + 2 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0,\ \forall\, x \in \mathbb{R},\ S = \mathbb{R}$. So $\sup S$ and $\inf S$ do not exist. ■

EXERCISE 8.1.2. *Suppose M is an upper bound of a set S of real numbers. Show that if $M \in S$, then $\sup S = M$.*

Sol.
Since M is an upper bound of S, $\sup S \leq M$. On the other hand, since $M \in S$, $M \leq \sup S$. So $\sup S = M$. ■

EXERCISE 8.1.3. *Suppose S is a nonempty bounded set of real numbers and T is a nonempty subset of S.*

(a) Show that T is bounded.
(b) Show that $\inf S \leq \inf T \leq \sup T \leq \sup S$.
Sol.
(a) Since S is bounded, let M be an upper bound of S and m be a lower bound of S. Then $\forall\, x \in T \subseteq S,\ m \leq x \leq M$. So T is also bounded. ■

(b) It is obviously that $\inf T \le \sup T$. Then since $T \subseteq S$ and since $\sup S$ is an upper bound of S, $\sup S$ is also an upper bound of T. So $\sup T \le \sup S$. Similarly, since $\inf S$ is a lower bound of S, $\inf S$ is also a lower bound of T. So $\inf S \le \inf T$. Hence we have $\inf S \le \inf T \le \sup T \le \sup S$. ■

EXERCISE 8.1.4. *Let S be a nonempty set of real numbers and $T = \{|x| \mid x \in S\}$. Show that S is bounded if and only if T is bounded above.*

Sol.

If S is bounded, then let M be an upper bound of S and m be a lower bound of S. Then $\forall\ x \in S,\ m \le x \le M$, that is, $\forall\ |x| \in T$, $|x| \le |m| + |M|$. So $|m| + |M|$ is an upper bound of T, that is, T is bounded above.

On the other hand, if T is bounded above, then let K be an upper bound of T. Then $\forall\ |x| \in T,\ |x| \le K$, that is, $\forall\ x \in S,\ -K \le x \le K$. Hence S is bounded. ■

EXERCISE 8.1.5. *Suppose the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are bounded. Show that $\{c_n = a_n \times b_n\}_{n=1}^{\infty}$ is also bounded.*

Sol.

Since $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are bounded, by exercise 8.1.4, $\{|a_n|\}_{n=1}^{\infty}$ and $\{|b_n|\}_{n=1}^{\infty}$ are bounded above. Then let M_1 be an upper bound of $\{|a_n|\}_{n=1}^{\infty}$ and M_2 be an upper bound of $\{|b_n|\}_{n=1}^{\infty}$. Then since $\forall\ n \in \mathbb{N},\ |a_n| \le M_1$ and $|b_n| \le M_2$, we have $|c_n| = |a_n \times b_n| \le M_1 M_2$. Hence $\{|c_n|\}_{n=1}^{\infty}$ is bounded above. Then by exercise 8.1.4 once again, $\{c_n\}_{n=1}^{\infty}$ is bounded. ■

8.2. Exercises 8.2.

EXERCISE 8.2.1. *For each of the following sequence, find an upper bound and a lower bound, and determine whether the sequence is increasing or decreasing.*

(a) $a_n = \frac{10^{23}}{n}$, $n \in \mathbb{N}$.
(b) $a_n = (1.000001)^n$, $n \in \mathbb{N}$.
(c) $a_n = \frac{4n}{\sqrt{4n^2+1}}$, $n \in \mathbb{N}$.
(d) $a_n = \ln(\frac{2n}{n+2})$, $n \in \mathbb{N}$.

Sol.

(a) Since $\forall\ n \in \mathbb{N},\ a_n = \frac{10^{23}}{n} > \frac{10^{23}}{n+1} = a_{n+1},\ \{a_n\}_{n=1}^{\infty}$ is decreasing. Then since $\forall\ n \in \mathbb{N}$, $10^{23} = a_1 \ge a_n > 0,\ 10^{23}$ is an upper bound and 0 is a lower bound. ■

(b) Since $\forall\ n \in \mathbb{N},\ a_n = (1.000001)^n < (1.000001)^{n+1} = a_{n+1},\ \{a_n\}_{n=1}^{\infty}$ is increasing. Then since $\forall\ n \in \mathbb{N}$, we have

$$\begin{aligned} a_n &= (1.000001)^n \\ &= (1 + 0.000001)^n \\ &= C_0^n + C_1^n \cdot 0.000001 + \cdots + C_n^m \cdot (0.000001)^n \\ &\geq 1 + n \cdot 0.000001, \end{aligned}$$

$\{a_n\}_{n=1}^{\infty}$ is not bounded above. Finally, since $\forall\ n \in \mathbb{N}$, $a_n > 0$, 0 is a lower bound. ■

(c) Since $\forall\ n \in \mathbb{N}$,

$$a_n = \frac{4n}{\sqrt{4n^2+1}} = \frac{4}{\sqrt{4+\frac{1}{n^2}}} < \frac{4}{\sqrt{4+\frac{1}{(n+1)^2}}} = \frac{4(n+1)}{\sqrt{4(n+1)^2+1}} = a_{n+1},$$

$\{a_n\}_{n=1}^{\infty}$ is increasing. Then since $\forall\ n \in \mathbb{N}$, $0 < a_n = \frac{4n}{\sqrt{4n^2+1}} < \frac{4n}{\sqrt{4n^2}} = \frac{4n}{2n} = 2,\ 2$ is an upper bound and 0 is a lower bound. ■

(d) Since $\forall\ n \in \mathbb{N}$,

$$a_n = \ln(\frac{2n}{n+2}) = \ln(\frac{2}{1+\frac{2}{n}}) < \ln(\frac{2}{1+\frac{2}{n+1}}) = \ln(\frac{2(n+1)}{(n+1)+2}) = a_{n+1},$$

$\{a_n\}_{n=1}^{\infty}$ is increasing. Then since $\forall\ n \in \mathbb{N}$, $0 < a_n = \ln(\frac{2n}{n+2}) < \ln(\frac{2n}{n}) = \ln 2,\ \ln 2$ is an upper bound and 0 is a lower bound. ■

EXERCISE 8.2.2. *Let $p \in \mathbb{N}$. Show that $\left\{\frac{p^n}{n!}\right\}_{n=1}^{\infty}$ is decreasing for $n \geq p$.*

Sol.
Since $\forall\ n \geq p,\ \frac{p}{n+1} < 1$, hence we have

$$\frac{p^n}{n!} > \frac{p^n}{n!} \cdot \frac{p}{n+1} = \frac{p^{n+1}}{(n+1)!} = a_{n+1},$$

that is, $\left\{\frac{p^n}{n!}\right\}_{n=1}^{\infty}$ is decreasing for $n \geq p$. ■

8.3. Exercises 8.3.

EXERCISE 8.3.1. *Let $a_1 = 2,\ a_{n+1} = \frac{a_n^2+2}{2a_n},\ n \geq 1$.*

(a) Show that $a_n \geq \sqrt{2}$, $\forall\ n \geq 1$.
(b) Show that $\{a_n\}_{n=1}^{\infty}$ converges.
(c) Show that $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

(d) Show that the greatest lower bound of $\{a_n\}_{n=1}^{\infty}$ is $\sqrt{2}$.

Sol.

(a) $a_1 = 2 > \sqrt{2}$. Then by the arithmetic-geometric mean inequality, $\forall\ n \geq 1$, we have

$$a_{n+1} = \frac{a_n^2+2}{2a_n} = \frac{a_n^2}{2a_n} + \frac{2}{2a_n} = \frac{a_n}{2} + \frac{1}{a_n} \geq 2\sqrt{\frac{a_n}{2}\cdot\frac{1}{a_n}} = \sqrt{2}. \qquad \blacksquare$$

(b) Since by (a), $\forall\ n \geq 1$, $\frac{a_n}{2} \geq \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ and $\frac{1}{a_n} \leq \frac{1}{\sqrt{2}}$, we have

$$a_n - a_{n+1} = a_n - \frac{a_n^2+2}{2a_n} = \frac{a_n^2-2}{2a_n} = \frac{a_n}{2} - \frac{1}{a_n} \geq \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0.$$

Hence $\{a_n\}_{n=1}^{\infty}$ is nonincreasng and bounded below. Then by theorem 8.3.8, $\{a_n\}_{n=1}^{\infty}$ converges to its greatst lower bound. $\blacksquare$

(c) Since by (b), $\{a_n\}_{n=1}^{\infty}$ converges to its greatst lower bound, let $\lim_{n\to\infty} a_n = L$. Then since $\sqrt{2}$ is a lower bound of $\{a_n\}_{n=1}^{\infty}$ and L is the greatst lower bound, $L \geq \sqrt{2} > 0$. Thus we have

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n^2+2}{2a_n} = \frac{L^2+2}{2L},$$

that is, $L = \sqrt{2}$. $\blacksquare$

(d) This follows immediately from (c). $\blacksquare$

Exercise 8.3.2. $\lim_{n\to\infty} \frac{n+(-1)^n}{n} =$ *?*

Sol.

Since

$$\frac{n-1}{n} \leq \frac{n+(-1)^n}{n} \leq \frac{n+1}{n},$$

and since

$$\begin{aligned} \lim_{n\to\infty} \frac{n-1}{n} &= \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1} = 1, \\ \lim_{n\to\infty} \frac{n+1}{n} &= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1, \end{aligned}$$

by the pinching theorem, $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = 1$. $\blacksquare$

Exercise 8.3.3. $\lim_{n\to\infty} \frac{1}{n^2} =$ *?*

Sol.
Since $\lim_{n\to\infty} \frac{1}{n} = 0$, by the product rule, $\lim_{n\to\infty} \frac{1}{n^2} = 0$. $\blacksquare$

EXERCISE 8.3.4. $\lim_{n\to\infty} (-1)^n \sqrt{n} =$ *?*

Sol.
Assume $\lim_{n\to\infty} (-1)^n \sqrt{n}$ converges, that is, $\lim_{n\to\infty} (-1)^n \sqrt{n} = L$. Then by remark 8.3.11, $|L| = \lim_{n\to\infty} |(-1)^n \sqrt{n}| = \lim_{n\to\infty} \sqrt{n}$. However, since $\{\sqrt{n}\}_{n=1}^{\infty}$ is not bounded above, by theorem 8.3.8, $\lim_{n\to\infty} \sqrt{n}$ diverges. This leads to a contradiction. $\blacksquare$

EXERCISE 8.3.5. $\lim_{n\to\infty} \sin(\frac{\pi}{2n}) =$ *?*

Sol.
Since $\lim_{n\to\infty} \frac{\pi}{2n} = 0$, by theorem 8.3.14,

$$\lim_{n\to\infty} \sin(\frac{\pi}{2n}) = \sin(\lim_{n\to\infty} \frac{\pi}{2n}) = \sin 0 = 0. \qquad \blacksquare$$

EXERCISE 8.3.6. $\lim_{n\to\infty} \ln(\frac{2n}{n+2}) =$ *?*

Sol.
Since

$$\lim_{n\to\infty} \frac{2n}{n+2} = \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}} = 2,$$

by theorem 8.3.14,

$$\lim_{n\to\infty} \ln(\frac{2n}{n+2}) = \ln(\lim_{n\to\infty} \frac{2n}{n+2}) = \ln 2. \qquad \blacksquare$$

EXERCISE 8.3.7. $\lim_{n\to\infty} \frac{3^n}{n!} =$ *?*

Sol.
Since $\forall\ n \geq 4$, $\frac{3}{n} \leq 1$,

$$0 < \frac{3^n}{n!} = \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdot \cdots \cdot \frac{3}{n} \leq \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{n} = \frac{27}{2n},$$

and since $\lim_{n\to\infty} \frac{27}{2n} = 0$, by the pinching theorem, $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

EXERCISE 8.3.8. $\lim_{n\to\infty} 2^{\frac{1}{n}} =$ *?*

Sol.

Since $2^{\frac{1}{n}} = e^{\frac{1}{n}\ln 2}$, by theorem 8.3.14,

$$\lim_{n\to\infty} 2^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln 2} = e^{\lim\limits_{n\to\infty}\frac{1}{n}\ln 2} = e^0 = 1. \qquad \blacksquare$$

EXERCISE 8.3.9. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, $b_n = a_{2n-1}$, $n \geq 1$, and $c_n = a_{2n}$, $n \geq 1$. Show that $\lim\limits_{n\to\infty} a_n = L$ if and only if $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = L$.*

Sol.

If $\lim\limits_{n\to\infty} a_n = L$, then $\forall\ \epsilon > 0,\ \exists\ n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|a_n - L| < \epsilon$. So if $n \geq n_0$, then $2n-1 \geq n_0$, hence

$$|b_n - L| = |a_{2n-1} - L| < \epsilon,$$

and if $n \geq n_0$, then $2n \geq n_0$, hence

$$|c_n - L| = |a_{2n} - L| < \epsilon.$$

Therefore, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = L$.

On the other hand, if $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = L$, then $\forall\ \epsilon > 0,\ \exists\ n_1 \in \mathbb{N}$ such that if $n \geq n_1$, then $|b_n - L| < \epsilon$, and $\exists\ n_2 \in \mathbb{N}$ such that if $n \geq n_2$, then $|c_n - L| < \epsilon$. So if $n \geq 2n_1 + 2n_2 + 1$ and n is odd, then

$$\frac{n+1}{2} \geq \frac{2n_1 + 2n_2 + 2}{2} \geq n_1$$

and

$$|a_n - L| = \left|b_{\frac{n+1}{2}} - L\right| < \epsilon,$$

and if $n \geq 2n_1 + 2n_2 + 1$ and n is even, then

$$\frac{n}{2} \geq \frac{2n_1 + 2n_2 + 1}{2} \geq n_2$$

and

$$|a_n - L| = \left|c_{\frac{n}{2}} - L\right| < \epsilon.$$

Hence if $n \geq 2n_1 + 2n_2 + 1$, $|a_n - L| < \epsilon$. So $\lim\limits_{n\to\infty} a_n = L$. $\blacksquare$

8.4. Exercises 8.4.

EXERCISE 8.4.1. $\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 3x} =$ *?*

Sol.

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2\cos 2x}{3\cos 3x} = \frac{2}{3}. \qquad \blacksquare$$

EXERCISE 8.4.2. $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} =$ *?*

Sol.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}. \qquad \blacksquare$$

EXERCISE 8.4.3. $\lim_{x\to 0}\frac{\tan x - x}{x-\sin x} = ?$

Sol.

$$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\sec^2 x - 1}{1-\cos x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2\tan x\sec^2 x}{\sin x} = \lim_{x\to 0} 2\sec^3 x = 2. \qquad \blacksquare$$

EXERCISE 8.4.4. $\lim_{x\to 0}\frac{3\tan 4x - 12\tan x}{3\sin 4x - 12\sin x} = ?$

Sol.

$$\begin{aligned}
&\lim_{x\to 0}\frac{3\tan 4x - 12\tan x}{3\sin 4x - 12\sin x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{12\sec^2 4x - 12\sec^2 x}{12\cos 4x - 12\cos x}\\
&\overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{96\tan 4x\sec^2 4x - 24\tan x\sec^2 x}{-48\sin 4x + 12\sin x}\\
&\overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{384\sec^4 4x + 768\tan^2 4x\sec^2 4x - 24\sec^4 x - 48\tan^2 x\sec^2 x}{-192\cos 4x + 12\cos x}\\
= \quad &\frac{384-24}{-192+12} = -2. \qquad \blacksquare
\end{aligned}$$

EXERCISE 8.4.5. $\lim_{x\to 0}\frac{e^x-1}{x^2} = ?$

Sol.

$$\lim_{x\to 0}\frac{e^x-1}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{e^x}{2x} \quad \text{diverges.} \qquad \blacksquare$$

EXERCISE 8.4.6. $\lim_{x\to \frac{\pi}{4}}\frac{\sqrt[3]{\tan x}-1}{2\sin^2 x - 1} = ?$

Sol.

$$\lim_{x\to \frac{\pi}{4}}\frac{\sqrt[3]{\tan x}-1}{2\sin^2 x - 1} \overset{\frac{0}{0}}{=} \lim_{x\to \frac{\pi}{4}}\frac{\frac{1}{3}\tan^{-\frac{2}{3}} x\sec^2 x}{4\sin x\cos x} = \frac{\frac{1}{3}\cdot 1^{\frac{2}{3}}\cdot 2}{4\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}} = \frac{1}{3}. \qquad \blacksquare$$

EXERCISE 8.4.7. $\lim_{x\to 0}\frac{x(e^x+1)-2(e^x-1)}{x^3} = ?$

Sol.

$$
\begin{aligned}
& \lim_{x\to 0}\frac{x\left(e^x+1\right)-2\left(e^x-1\right)}{x^3} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{e^x+xe^x+1-2e^x}{3x^2} \\
& \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{e^x+e^x+xe^x-2e^x}{6x} \\
= & \lim_{x\to 0}\frac{e^x}{6}=\frac{1}{6}. \qquad \blacksquare
\end{aligned}
$$

EXERCISE 8.4.8. $\lim\limits_{x\to 0}\frac{1-\cos\left(x^2\right)}{x^2\sin(x^2)} = ?$

Sol.

$$
\begin{aligned}
& \lim_{x\to 0}\frac{1-\cos\left(x^2\right)}{x^2\sin\left(x^2\right)} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2x\sin\left(x^2\right)}{2x\sin\left(x^2\right)+2x^3\cos\left(x^2\right)} \\
= & \lim_{x\to 0}\frac{\sin\left(x^2\right)}{\sin\left(x^2\right)+x^2\cos\left(x^2\right)} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2x\cos\left(x^2\right)}{2x\cos\left(x^2\right)+2x\cos\left(x^2\right)-2x^3\sin\left(x^2\right)} \\
= & \lim_{x\to 0}\frac{\cos\left(x^2\right)}{2\cos\left(x^2\right)-x^2\sin\left(x^2\right)}=\frac{1}{2}. \qquad \blacksquare
\end{aligned}
$$

EXERCISE 8.4.9. $\lim\limits_{x\to 0}\frac{\sin^{-1}(2x)-2\sin^{-1}x}{x^3} = ?$

Sol.

$$
\begin{aligned}
& \lim_{x\to 0}\frac{\sin^{-1}\left(2x\right)-2\sin^{-1}x}{x^3} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\frac{2}{\sqrt{1-4x^2}}-\frac{2}{\sqrt{1-x^2}}}{3x^2} \\
& \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\frac{8x}{(1-4x^2)^{\frac{3}{2}}}-\frac{2x}{(1-x^2)^{\frac{3}{2}}}}{6x} \\
= & \lim_{x\to 0}\frac{4}{3(1-4x^2)^{\frac{3}{2}}}-\frac{1}{3(1-x^2)^{\frac{3}{2}}}=1. \qquad \blacksquare
\end{aligned}
$$

EXERCISE 8.4.10. $\lim\limits_{x\to 0}\frac{3^x-2^x}{x^2} = ?$

Sol.

$$
\lim_{x\to 0}\frac{3^x-2^x}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\ln 3\cdot 3^x-\ln 2\cdot 2^x}{2x} \quad \text{diverges.} \qquad \blacksquare
$$

8.5. Exercises 8.5.

EXERCISE 8.5.1. $\lim\limits_{x\to\infty}\frac{2x^3-x^2+3x+1}{3x^3+2x^2-x-1} = ?$

Sol.

$$\lim_{x\to\infty}\frac{2x^3-x^2+3x+1}{3x^3+2x^2-x-1}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{6x^2-2x+3}{9x^2+4x-1}$$
$$\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{12x-2}{18x+4}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{12}{18}=\frac{2}{3}. \qquad \blacksquare$$

EXERCISE 8.5.2. $\lim_{x\to\infty}\frac{\ln x}{x^{0.1}} =$ *?*

Sol.

$$\lim_{x\to\infty}\frac{\ln x}{x^{0.1}}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{\frac{1}{x}}{0.1\cdot x^{-0.9}}=\lim_{x\to\infty}\frac{1}{0.1\cdot x^{0.1}}=0. \qquad \blacksquare$$

EXERCISE 8.5.3. $\lim_{x\to\infty}\frac{x^{100}}{e^x} =$ *?*

Sol.

$$\lim_{x\to\infty}\frac{x^{100}}{e^x}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{100x^{99}}{e^x}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{100\cdot 99x^{98}}{e^x}\overset{\frac{\infty}{\infty}}{=}\cdots\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{100!}{e^x}=0. \qquad \blacksquare$$

EXERCISE 8.5.4. $\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\ln|\cos x|} =$ *?*

Sol.

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\ln|\cos x|}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x}{-\frac{\sin x}{\cos}}=\lim_{x\to\frac{\pi}{2}}-\frac{1}{\sin x\cos x}\quad\text{diverges.} \qquad \blacksquare$$

EXERCISE 8.5.5. $\lim_{x\to 0^+} x(\ln x)^2 =$ *?*

Sol.

$$\lim_{x\to 0^+} x(\ln x)^2\overset{0\cdot\infty}{=}\lim_{x\to 0^+}\frac{(\ln x)^2}{\frac{1}{x}}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to 0^+}\frac{2\ln x\cdot\frac{1}{x}}{-\frac{1}{x^2}}$$
$$=\lim_{x\to 0^+}\frac{2\ln x}{-\frac{1}{x}}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to 0^+}\frac{\frac{2}{x}}{\frac{1}{x^2}}=\lim_{x\to 0^+}2x=0. \qquad \blacksquare$$

EXERCISE 8.5.6. $\lim_{x\to 0^+} x^{x^2} =$ *?*

Sol.
Since $x^{x^2}=e^{x^2\ln x}$ and since

$$\lim_{x\to 0^+} x^2\ln x\overset{0\cdot\infty}{=}\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^2}}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{2}{x^3}}=\lim_{x\to 0^+}-\frac{x^2}{2}=0,$$

we have

$$\lim_{x\to 0^+} x^{x^2} = \lim_{x\to 0^+} e^{x^2 \ln x} = e^0 = 1. \qquad \blacksquare$$

EXERCISE 8.5.7. $\lim_{x\to 0^+} (\sin x)^{\tan x} =$ *?*

Sol.
Since $(\sin x)^{\tan x} = e^{\tan x \ln(\sin x)}$ and since

$$\lim_{x\to 0^+} \tan x \ln(\sin x) \overset{0\cdot\infty}{=} \lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to 0^+} -\sin x \cos x = 0,$$

we have

$$\lim_{x\to 0^+} (\sin x)^{\tan x} = \lim_{x\to 0^+} e^{\tan x \ln(\sin x)} = e^0 = 1. \qquad \blacksquare$$

EXERCISE 8.5.8. $\lim_{n\to\infty} \sqrt[n]{n} =$ *?*

Sol.
Since $\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\frac{1}{n}\ln n}$ and since

$$\lim_{n\to\infty} \frac{\ln n}{n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0,$$

by theorem 8.3.14,

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^{(\lim_{n\to\infty} \frac{\ln n}{n})} = e^0 = 1. \qquad \blacksquare$$

EXERCISE 8.5.9. $\lim_{n\to\infty} (1+\frac{a}{n})^n =$ *?*

Sol.
Since $(1+\frac{a}{n})^n = e^{n\ln(1+\frac{a}{n})}$ and since

$$\lim_{n\to\infty} n\ln(1+\frac{a}{n}) \overset{\infty\cdot 0}{=} \lim_{n\to\infty} \frac{\ln(1+\frac{a}{n})}{\frac{1}{n}} \overset{\frac{0}{0}}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{a}{n}}\cdot(-\frac{a}{n^2})}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{a}{1+\frac{a}{n}} = a,$$

by theorem 8.3.14,

$$\lim_{n\to\infty} (1+\frac{a}{n})^n = \lim_{n\to\infty} e^{n\ln(1+\frac{a}{n})} = e^{(\lim_{n\to\infty} n\ln(1+\frac{a}{n}))} = e^a. \qquad \blacksquare$$

EXERCISE 8.5.10. *Suppose* $\lim_{x\to 0} f(x) = \lim_{x\to 0} f'(x) = \lim_{x\to 0} f''(x) = \lim_{x\to 0} f'''(x) = 0$ *and* $\lim_{x\to 0} \frac{x^2 f'''(x)}{f''(x)} = 2$. *Find* $\lim_{x\to 0} \frac{x^2 f'(x)}{f(x)}$.

Sol.

$$\lim_{x\to 0}\frac{x^2f'(x)}{f(x)} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2xf'(x)+x^2f''(x)}{f'(x)} = \lim_{x\to 0}2x + \lim_{x\to 0}\frac{x^2f''(x)}{f'(x)} = \lim_{x\to 0}\frac{x^2f''(x)}{f'(x)}$$
$$\overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2xf''(x)+x^2f'''(x)}{f''(x)} = \lim_{x\to 0}2x + \lim_{x\to 0}\frac{x^2f'''(x)}{f''(x)} = 2. \qquad \blacksquare$$

EXERCISE 8.5.11. *Suppose f is twice differentiable at the point c. Find* $\lim\limits_{h\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2}$.

Sol.

Since f is twice differentiable at c, $f'(x)$ exists on $(c-\delta, c+\delta)$ for some $\delta > 0$, hence $f'(c+h)$ and $f'(c-h)$ exist when h approaches to 0. So we have

$$\lim_{h\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2} \overset{\frac{0}{0}}{=} \lim_{h\to 0}\frac{f'(c+h)-f'(c-h)}{2h}.$$

(Note: We can't apply L'Hospital rule twice since $f''(c+h)$ and $f''(c-h)$ may not exist.)

Then since

$$f''(c) = \lim_{h\to 0}\frac{f'(c+h)-f'(c)}{h},$$

let $s = -h$, we have

$$\lim_{h\to 0}\frac{f'(c)-f'(c-h)}{h} = \lim_{s\to 0}\frac{f'(c)-f'(c+s)}{-s} = \lim_{s\to 0}\frac{f'(c+s)-f'(c)}{s} = f''(c).$$

Thus we have

$$\lim_{h\to 0}\frac{f'(c+h)-f'(c-h)}{2h} = \lim_{h\to 0}\frac{f'(c+h)-f'(c)+f'(c)-f'(c-h)}{2h}$$
$$= \lim_{h\to 0}\frac{f'(c+h)-f'(c)}{2h} + \lim_{h\to 0}\frac{f'(c)-f'(c-h)}{2h} = \frac{f''(c)}{2} + \frac{f''(c)}{2} = f''(c). \qquad \blacksquare$$

8.6. Exercises 8.6.

EXERCISE 8.6.1. $\int_0^1 \frac{1}{x^2}dx =$ *?*

Sol.

$$\int_0^1 \frac{1}{x^2}dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}dx = \lim_{a\to 0^+} -\frac{1}{x}\Big|_a^1 = \lim_{a\to 0^+}(-1+\frac{1}{a}) \text{ diverges.} \qquad \blacksquare$$

EXERCISE 8.6.2. $\int_0^\infty \frac{1}{1+x^2}dx =$ *?*

Sol.

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^2}dx &= \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}dx = \lim_{b\to\infty} \tan^{-1} x\Big|_0^b \\ &= \lim_{b\to\infty}(\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2}. \qquad \blacksquare\end{aligned}$$

EXERCISE 8.6.3. $\int_0^5 \frac{1}{5-x}dx =$ *?*

Sol.

$$\begin{aligned}\int_0^5 \frac{1}{5-x}dx &= \lim_{b\to 5^-}\int_0^b \frac{1}{5-x}dx = \lim_{b\to 5^-} -\ln|5-x|\Big|_0^b \\ &= \lim_{b\to 5^-}\left(-\ln|5-b| + \ln 5\right) \text{ diverges.} \qquad \blacksquare\end{aligned}$$

EXERCISE 8.6.4. $\int_1^\infty e^{-x}dx =$ *?*

Sol.

$$\int_1^\infty e^{-x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty}(-e^{-b}+e^{-1}) = \frac{1}{e}. \qquad \blacksquare$$

EXERCISE 8.6.5. $\int_0^2 \frac{1}{\sqrt[3]{x-1}}dx =$ *?*

Sol.

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt[3]{x-1}}dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}dx \\ &= \lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt[3]{x-1}}dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{\sqrt[3]{x-1}}dx \\ &= \lim_{b\to 1^-}\frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_0^b + \lim_{a\to 1^+}\frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_a^2 \\ &= \lim_{b\to 1^-}[\frac{3}{2}(b-1)^{\frac{2}{3}} - \frac{3}{2}] + \lim_{a\to 1^+}[\frac{3}{2} - \frac{3}{2}(a-1)^{\frac{2}{3}}] = 0. \qquad \blacksquare\end{aligned}$$

EXERCISE 8.6.6. $\int_0^5 \frac{x}{(x^2-1)^2}dx =$ *?*

Sol.

Assume $\frac{x}{(x^2-1)^2} = \frac{x}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$. Then $A = 0$, $B = \frac{1}{4}$, $C = 0$, and $D = -\frac{1}{4}$. Hence we have

$$\begin{aligned}
&\int_0^5 \frac{x}{(x^2-1)^2} dx \\
= \;& \int_0^5 \left(\frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) dx \\
= \;& \lim_{b\to 1^-} \int_0^b \left(\frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) dx + \lim_{a\to 1^+} \int_a^5 \left(\frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) dx \\
= \;& \lim_{b\to 1^-} \left(-\frac{1}{4(x-1)} + \frac{1}{4(x+1)}\right)\Big|_0^b + \lim_{a\to 1^+} \left(-\frac{1}{4(x-1)} + \frac{1}{4(x+1)}\right)\Big|_a^5 \\
= \;& \lim_{b\to 1^-} \left[-\frac{1}{4(b-1)} + \frac{1}{4(b+1)} - \frac{1}{4} - \frac{1}{4}\right] + \lim_{a\to 1^+} \left[-\frac{1}{16} + \frac{1}{24} + \frac{1}{4(a-1)} - \frac{1}{4(a+1)}\right]
\end{aligned}$$

diverges. ■

Exercise 8.6.7. *Does* $\int_1^\infty \frac{3x}{x^3+1} dx$ *converge or diverge?*

Sol.

Since $\forall\, x \in [1, \infty)$, $0 \le \frac{3x}{x^3+1} \le \frac{3x}{x^3} = \frac{3}{x^2}$, and since by example 8.6.7, $\int_1^\infty \frac{3}{x^2} dx = 3\int_1^\infty \frac{1}{x^2} dx$ converges, by the comparison test, $\int_1^\infty \frac{3x}{x^3+1} dx$ converges. ■

Exercise 8.6.8. *Does* $\int_2^\infty \frac{3x}{x^2-1} dx$ *converge or diverge?*

Sol.

Since $\forall\, x \in [2, \infty)$, $\frac{3x}{x^2-1} \ge \frac{3x}{x^2} = \frac{3}{x} \ge 0$, and since by example 8.6.7, $\int_2^\infty \frac{3}{x} dx$ diverges, by the comparison test, $\int_2^\infty \frac{3x}{x^2-1} dx$ diverges. ■

9. Chapter 9

9.1. Exercises 9.1.

EXERCISE 9.1.1. *Determine the following series converges or diverges, and find its value if it converges.*

(a) $\sum\limits_{k=1}^{\infty} 1$.
(b) $\sum\limits_{k=1}^{\infty} \frac{6}{(k+2)(k+3)}$.
(c) $\sum\limits_{k=1}^{\infty} \frac{2^{k+1}+(-3)^k}{5^{k+2}}$.
(d) $\sum\limits_{k=1}^{\infty} \frac{k^2-2k+3}{2k^2+k+1}$.
(e) $\sum\limits_{k=2}^{\infty} \ln\left(\frac{(k-1)(k+1)}{k^2}\right)$.

Sol.

(a) Since $s_n = \sum\limits_{k=1}^{n} 1 = n$ and $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} n = \infty$ diverges, by definition, $\sum\limits_{k=1}^{\infty} 1$ diverges. ■

(b) Since

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{6}{(k+2)(k+3)} = \sum_{k=1}^{n} 6\left(\frac{1}{k+2} - \frac{1}{k+3}\right) \\ &= 6\left(\left(\frac{1}{1+2} - \frac{1}{1+3}\right) + \left(\frac{1}{2+2} - \frac{1}{2+3}\right) + \cdots + \left(\frac{1}{n+2} - \frac{1}{n+3}\right)\right) \\ &= 6\left(\frac{1}{3} - \frac{1}{n+3}\right) \end{aligned}$$

and $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} 6\left(\frac{1}{3} - \frac{1}{n+3}\right) = 2$, by definition, $\sum\limits_{k=1}^{\infty} \frac{6}{(k+2)(k+3)} = 2$ converges. ■

(c) First, we have

$$\frac{2^{k+1}+(-3)^k}{5^{k+2}} = \frac{2^{k+1}}{5^{k+2}} + \frac{(-3)^k}{5^{k+2}} = \frac{2}{25}\left(\frac{2}{5}\right)^k + \frac{1}{25}\left(\frac{-3}{5}\right)^k$$

for all $k \in \mathbb{N}$. Then since $\sum\limits_{k=1}^{\infty}\left(\frac{2}{5}\right)^k = \frac{\frac{2}{5}}{1-\frac{2}{5}} = \frac{2}{3}$ and $\sum\limits_{k=1}^{\infty}\left(\frac{-3}{5}\right)^k = \frac{\frac{-3}{5}}{1-\left(\frac{-3}{5}\right)} = \frac{-3}{8}$ converge, by the sum rule and constant multiple, we get

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{2^{k+1}+(-3)^k}{5^{k+2}} &= \frac{2}{25}\sum_{k=1}^{\infty}\left(\frac{2}{5}\right)^k + \frac{1}{25}\sum_{k=1}^{\infty}\left(\frac{-3}{5}\right)^k \\ &= \frac{2}{25}\cdot\frac{2}{3}+\frac{1}{25}\cdot\frac{-3}{8} = \frac{23}{600}\end{aligned}$$

converges. ■

(d) Since $\lim\limits_{k\to\infty}\frac{k^2-2k+3}{2k^2+k+1} = \frac{1}{2} \neq 0$, $\sum\limits_{k=1}^{\infty}\frac{k^2-2k+3}{2k^2+k+1}$ diverges by theorem 9.1.6. ■

(e) Since

$$\begin{aligned} s_n &= \sum_{k=2}^{n}\ln\left(\frac{(k-1)(k+1)}{k^2}\right) \\ &= \ln\left(\frac{(2-1)(2+1)}{2\cdot 2}\right)+\ln\left(\frac{(3-1)(3+1)}{3\cdot 3}\right)+\cdots+\ln\left(\frac{(n-1)(n+1)}{n\cdot n}\right) \\ &= \ln\left(\frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\cdots\cdot\frac{(n-1)(n+1)}{n\cdot n}\right) = \ln\left(\frac{1}{2}\cdot\frac{n+1}{n}\right),\end{aligned}$$

and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\ln\left(\frac{1}{2}\cdot\frac{n+1}{n}\right) = \ln\left(\frac{1}{2}\right),$$

by definition, $\sum\limits_{k=2}^{\infty}\ln\left(\frac{(k-1)(k+1)}{k^2}\right) = \ln\left(\frac{1}{2}\right)$ converges. ■

9.2. Exercises 9.2.

EXERCISE 9.2.1. *Determine the following series converges or diverges.*

(a) $\sum\limits_{k=1}^{\infty}\frac{2k+3}{k^2+3k+2}$.

(b) $\sum\limits_{k=1}^{\infty}\frac{\ln k}{k}$.

(c) $\sum\limits_{k=1}^{\infty}\frac{1}{\sqrt{k}}$.

(d) $\sum\limits_{k=1}^{\infty}\frac{1}{k\sqrt{k}}$.

(e) $\sum\limits_{k=1}^{\infty}\frac{k-1}{k^3}$.

(f) $\sum\limits_{k=1}^{\infty} \frac{\ln(k+1)}{(k+1)^3}$.

(g) $\sum\limits_{k=1}^{\infty} \frac{k+1}{k \cdot 2^k}$.

(h) $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}$.

(i) $\sum\limits_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k}$.

(j) $\sum\limits_{k=1}^{\infty} \frac{3k}{\sqrt[3]{k^5+1}}$.

Sol.

(a) Suppose $f(x) = \frac{2x+3}{x^2+3x+2}$, then $f(x)$ is continuous, positive and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^{\infty} f(x)dx &= \lim_{b\to\infty}\left(\int_1^b \frac{2x+3}{x^2+3x+2}dx\right) = \lim_{b\to\infty}\left(\ln\left|x^2+3x+2\right| \Big|_1^b\right) \\ &= \lim_{b\to\infty}\left(\ln\left|b^2+3b+2\right| - \ln|6|\right) = \infty,\end{aligned}$$

$\int_1^{\infty} f(x)dx$ diverges. Hence $\sum\limits_{k=1}^{\infty} f(k) = \sum\limits_{k=1}^{\infty} \frac{2k+3}{k^2+3k+2}$ diverges by the integral test (Theorem 9.2.3). ∎

(b) Suppose $f(x) = \frac{\ln x}{x}$, $f'(x) = \frac{1-\ln x}{x^2} < 0$, $\forall\ x \geq 3$, then $f(x)$ is continuous, positive and decreasing on $[3, \infty)$. Since

$$\begin{aligned}\int_3^{\infty} f(x)dx &= \lim_{b\to\infty}\left(\int_3^b \frac{\ln x}{x}dx\right) = \lim_{b\to\infty}\left(\ln|\ln x| \Big|_3^b\right) \\ &= \lim_{b\to\infty}\left(\ln|\ln b| - \ln|\ln 3|\right) = \infty,\end{aligned}$$

$\int_3^{\infty} f(x)dx$ diverges. Hence by the integral test (Theorem 9.2.3), $\sum\limits_{k=3}^{\infty} f(k) = \sum\limits_{k=3}^{\infty} \frac{\ln k}{k}$ diverges. Therefore

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} = \frac{\ln 1}{1} + \frac{\ln 2}{2} + \sum_{k=3}^{\infty} \frac{\ln k}{k} = \frac{\ln 2}{2} + \sum_{k=3}^{\infty} \frac{\ln k}{k}$$

also diverges. ∎

(c) Since $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum\limits_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$, where $p = \frac{1}{2} < 1$, $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by p-series (Example 9.2.5). ∎

(d) Since $\sum\limits_{k=1}^{\infty} \frac{1}{k\sqrt{k}} = \sum\limits_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$, where $p = \frac{3}{2} > 1$, $\sum\limits_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$ converges by p-series (Example 9.2.5). ■

(e) First, we have

$$k^2(k-1) = k^3 - k^2 \leq k^3, \ \forall\ k \geq 1,$$

implies

$$0 \leq \frac{k-1}{k^3} \leq \frac{1}{k^2}, \ \forall\ k \geq 1.$$

Then since $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ converges (p-series, $p = 2$), $\sum\limits_{k=1}^{\infty} \frac{k-1}{k^3}$ converges by the basic comparison test (Theorem 9.2.8). ■

(f) First, we have

$$\ln(k+1) \leq k+1, \ \forall\ k \geq 1,$$

implies

$$0 < \frac{\ln(k+1)}{(k+1)^3} \leq \frac{(k+1)}{(k+1)^3} = \frac{1}{(k+1)^2}, \ \forall\ k \geq 1.$$

Then since $\sum\limits_{k=1}^{\infty} \frac{1}{(k+1)^2} = \sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ converges, $\sum\limits_{k=1}^{\infty} \frac{\ln(k+1)}{(k+1)^3}$ converges by the basic comparison test (Theorem 9.2.8). ■

(g) First, we have

$$k+1 \leq k+k = 2k, \ \forall\ k \geq 1,$$

implies

$$0 < \frac{k+1}{k \cdot 2^k} \leq \frac{2k}{k \cdot 2^k} = \frac{1}{2^{k-1}}, \ \forall\ k \geq 1.$$

THen since $\sum\limits_{k=1}^{\infty} \frac{1}{2^{k-1}}$ converges (geometric series), $\sum\limits_{k=1}^{\infty} \frac{k+1}{k \cdot 2^k}$ converges by the basic comparison test (Theorem 9.2.8). ■

(h) Since $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges (p-series, $p = 1$) and since

$$\lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{\sqrt{k(k+1)}}} = \lim_{k\to\infty} \frac{\sqrt{k(k+1)}}{k} = \lim_{k\to\infty} \sqrt{\frac{k^2+k}{k^2}} = 1,$$

by the limit comparison test (Theorem 9.2.11), $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}$ also diverges. ■

(i) Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (p-series, $p = 1$) and since

$$\lim_{k\to\infty} \frac{\frac{\sqrt[k]{k}}{k}}{\frac{1}{k}} = \lim_{k\to\infty} \sqrt[k]{k} = 1,$$

by the limit comparison test (Theorem 9.2.11), $\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k}$ also diverges. ■

(j) Since $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{2}{3}}}$ diverges (p-series, $p = \frac{2}{3}$) and since

$$\lim_{k\to\infty} \frac{\frac{3k}{\sqrt[3]{k^5+1}}}{\frac{1}{k^{\frac{2}{3}}}} = \lim_{k\to\infty} \frac{3k^{\frac{5}{3}}}{\sqrt[3]{k^5+1}} = \lim_{k\to\infty} \frac{3}{\sqrt[3]{1+\frac{1}{k^5}}} = 3,$$

by the limit comparison test (Theorem 9.2.11), $\sum_{k=1}^{\infty} \frac{3k}{\sqrt[3]{k^5+1}}$ also diverges. ■

9.3. Exercises 9.3.

EXERCISE 9.3.1. *Determine the following series converges or diverges.*

(a) $\sum_{k=1}^{\infty} \frac{1}{k^k}$.

(b) $\sum_{k=1}^{\infty} \frac{1}{(\ln k)^k}$.

(c) $\sum_{k=1}^{\infty} \frac{k}{2^k}$.

(d) $\sum_{k=1}^{\infty} \frac{k^{100}}{e^k}$.

(e) $\sum_{k=1}^{\infty} \left(\sqrt{k+1} - \sqrt{k}\right)^k$.

(f) $\sum_{k=1}^{\infty} \frac{1}{k!}$.

(g) $\sum_{k=1}^{\infty} \frac{3^k}{k!}$.

(h) $\sum_{k=1}^{\infty} \frac{k^4}{k!}$.

(i) $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdots(2k-1)}{(2k)!}$.

(j) $\sum_{k=1}^{\infty} \frac{3^k\cdot k!}{k^k}$.

Sol.

(a) Since $\lim\limits_{k\to\infty} \sqrt[k]{\frac{1}{k^k}} = \lim\limits_{k\to\infty} \frac{1}{k} = 0 < 1$, by the root test (Theorem 9.3.1), $\sum\limits_{k=1}^{\infty} \frac{1}{k^k}$ converges. ■

(b) Since $\lim\limits_{k\to\infty} \sqrt[k]{\frac{1}{(\ln k)^k}} = \lim\limits_{k\to\infty} \frac{1}{\ln k} = 0 < 1$, by the root test (Theorem 9.3.1), $\sum\limits_{k=1}^{\infty} \frac{1}{(\ln k)^k}$ converges. ■

(c) Since $\lim\limits_{k\to\infty} \sqrt[k]{\frac{k}{2^k}} = \lim\limits_{k\to\infty} \frac{\sqrt[k]{k}}{2} = \frac{1}{2} < 1$, by the root test (Theorem 9.3.1), $\sum\limits_{k=1}^{\infty} \frac{k}{2^k}$ converges. ■

(d) Since $\lim\limits_{k\to\infty} \sqrt[k]{\frac{k^{100}}{e^k}} = \lim\limits_{k\to\infty} \frac{k^{\frac{100}{k}}}{e} = \frac{1}{e} < 1$, by the root test (Theorem 9.3.1), $\sum\limits_{k=1}^{\infty} \frac{k^{100}}{e^k}$ converges. ■

(e) Since

$$\lim_{k\to\infty} \sqrt[k]{\left(\sqrt{k+1}-\sqrt{k}\right)^k} = \lim_{k\to\infty} \left(\sqrt{k+1}-\sqrt{k}\right)$$
$$= \lim_{k\to\infty} \frac{\left(\sqrt{k+1}-\sqrt{k}\right)\left(\sqrt{k+1}+\sqrt{k}\right)}{\sqrt{k+1}+\sqrt{k}}$$
$$= \lim_{k\to\infty} \frac{(k+1)-k}{\sqrt{k+1}+\sqrt{k}} = \lim_{k\to\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}} = 0 < 1,$$

by the root test (Theorem 9.3.1), $\sum\limits_{k=1}^{\infty} \left(\sqrt{k+1}-\sqrt{k}\right)^k$ converges. ■

(f) Since $\lim\limits_{k\to\infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim\limits_{k\to\infty} \frac{1}{k+1} = 0 < 1$, by the ratio test (Theorem 9.3.4), $\sum\limits_{k=1}^{\infty} \frac{1}{k!}$ converges. ■

(g) Since $\lim\limits_{k\to\infty} \frac{\frac{3^{k+1}}{(k+1)!}}{\frac{3^k}{k!}} = \lim\limits_{k\to\infty} \frac{3}{k+1} = 0 < 1$, by the ratio test (Theorem 9.3.4), $\sum\limits_{k=1}^{\infty} \frac{3^k}{k!}$ converges. ■

(h) Since $\lim\limits_{k\to\infty} \frac{\frac{(k+1)^4}{(k+1)!}}{\frac{k^4}{k!}} = \lim\limits_{k\to\infty} \frac{(k+1)^4}{k^4(k+1)} = 0 < 1$. by the ratio test (Theorem 9.3.4), $\sum\limits_{k=1}^{\infty} \frac{k^4}{k!}$ converges. ■

(i) Since

$$\lim_{k\to\infty} \frac{\frac{1\cdot 3\cdot\cdots\cdot(2k-1)\cdot(2k+1)}{(2(k+1))!}}{\frac{1\cdot 3\cdot\cdots\cdot(2k-1)}{(2k)!}} = \lim_{k\to\infty} \frac{2k+1}{(2k+1)(2k+2)}$$
$$= \lim_{k\to\infty} \frac{1}{2k+2} = 0 < 1,$$

by the ratio test (Theorem 9.3.4), $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot\cdots\cdot(2k-1)}{(2k)!}$ converges. ■

(j) First, we have

$$\lim_{k\to\infty} \frac{\frac{3^{k+1}\cdot(k+1)!}{(k+1)^{k+1}}}{\frac{3^k\cdot k!}{k^k}} = \lim_{k\to\infty} \frac{3\cdot(k+1)\cdot k^k}{(k+1)^{k+1}} = \lim_{k\to\infty} \frac{3\cdot k^k}{(k+1)^k} = \lim_{k\to\infty} 3\left(\frac{k}{k+1}\right)^k,$$

Then let $y = \left(\frac{x}{x+1}\right)^x$, $\ln y = x\cdot\ln\left(\frac{x}{x+1}\right)$. By L'Hopital rule, we have

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{x+1}{x}\cdot\frac{1(x+1)-1\cdot x}{(x+1)^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{-x}{x+1} = -1.$$

Therefore, by theorem 8.3.14

$$\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} y = e^{\lim\limits_{x\to\infty} \ln y} = e^{-1} = \frac{1}{e}.$$

Then by the ratio test (Theorem 9.3.4), $\lim\limits_{k\to\infty} 3\left(\frac{k}{k+1}\right)^k = \frac{3}{e} > 1$ implies $\sum_{k=1}^{\infty} \frac{3^k\cdot k!}{k^k}$ diverges. ■

9.4. Exercises 9.4.

EXERCISE 9.4.1. *Do the following series converge absolutely, converge conditionally or diverge?*

(a) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$.

(b) $\sum_{n=1}^{\infty} \frac{1+\cos \pi n}{n!}$.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$.

(e) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$.

(f) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$.

Sol.

(a) First, since $0 \le |\sin n| \le 1,\ \forall\ n \in \mathbb{N}$, we have $0 \le \left|\frac{\sin n}{n^3}\right| \le \frac{1}{n^3}\ \forall\ n \in \mathbb{N}$. THen since $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series, $p = 2$), by the basic comparison test (Theorem 9.2.8), $\sum\limits_{n=1}^{\infty} \left|\frac{\sin n}{n^3}\right|$ converges. Hence $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n^3}$ converges absolutely. ■

(b) First, since

$$0 \le |1 + \cos n\pi| \le |1| + |\cos n\pi| \le 1 + 1 = 2,\ \forall\ n \in \mathbb{N},$$

$0 \le \left|\frac{1+\cos \pi n}{n!}\right| \le \frac{2}{n!},\ \forall\ n \in \mathbb{N}$. Then since $\lim\limits_{n\to\infty} \frac{\frac{2}{(n+1)!}}{\frac{2}{n!}} = \lim\limits_{n\to\infty} \frac{1}{n+1} = 0 < 1$, by the ratio test (Theorem 9.3.4), $\sum\limits_{n=1}^{\infty} \frac{2}{n!}$ converges. As a result, by the basic comparison test (Theorem 9.2.8), $\sum\limits_{n=1}^{\infty} \left|\frac{1+\cos \pi n}{n!}\right|$ also converges. Hence $\sum\limits_{n=1}^{\infty} \left|\frac{1+\cos \pi n}{n!}\right|$ converges absolutely. ■

(c) First, since $\left|\frac{(-1)^n \ln n}{n}\right| = \frac{\ln n}{n},\ \forall\ n \ge 3$, and since by exercise 9.2.1.(a), $\sum\limits_{n=3}^{\infty} \frac{\ln n}{n}$ diverges, $\sum\limits_{n=1}^{\infty} \left|\frac{(-1)^n \ln n}{n}\right|$ also diverges.

Then since $\left\{\frac{\ln n}{n}\right\}_{n=3}^{\infty}$ is positive, decreasing and $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$, by the alternating test (Theorem 9.4.5), $\sum\limits_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$ converges, that is, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ converges. Hence $\sum\limits_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ converges conditionally. ■

(d) First, since

$$0 \le \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \cdots \cdot \frac{n}{n} \le \frac{1}{n} \cdot 1 \cdot \cdots \cdot 1 = \frac{1}{n}$$

and $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} \frac{1}{n} = 0$, by the pinching theorem, $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$. Then since

$$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{1}{\frac{n!}{n^n}} = \infty$$

does not exist, $\lim\limits_{n\to\infty} \frac{n^n}{n!} \ne 0$. Thus by theorem 9.1.6, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$ diverges. ■

(e) First, let $f(x) = \frac{1}{x\ln x}$, then $f(x)$ is continuous, positive and decreasing on $[3,\infty)$. Let $u = \ln x$, $du = \frac{1}{x}dx$, then we have

$$\begin{aligned}\int_3^\infty f(x)dx &= \lim_{b\to\infty}\left(\int_3^b \frac{1}{x\ln x}dx\right) = \lim_{b\to\infty}\left(\int_{\ln 3}^{\ln b}\frac{1}{u}du\right)\\ &= \lim_{b\to\infty}\left(\ln|u|\ \Big|_{\ln 3}^{\ln b}\right) = \lim_{b\to\infty}\left(\ln|\ln b| - \ln|\ln 3|\right) = \infty,\end{aligned}$$

diverges. Hence $\sum\limits_{n=3}^\infty f(n) = \sum\limits_{n=3}^\infty \frac{1}{n\ln n}$ diverges by the integral test (Theorem 9.2.3). Thus $\sum\limits_{n=2}^\infty \frac{1}{n\ln n}$ also diverges.

Then since $\left\{\frac{1}{n\ln n}\right\}_{n=3}^\infty$ positive, decreasing and $\lim\limits_{n\to\infty}\frac{1}{n\ln n} = 0$, by the alternating test (Theorem 9.4.5), $\sum\limits_{n=3}^\infty \frac{(-1)^n}{n\ln n}$ converges, that is, $\sum\limits_{n=2}^\infty \frac{(-1)^n}{n\ln n}$ converges. Hence $\sum\limits_{n=2}^\infty \frac{(-1)^n}{n\ln n}$ converges conditionally. ■

(f) First, let $f(x) = \frac{1}{x(\ln x)^2}$, then $f(x)$ is continuous, positive and decreasing on $[3,\infty)$. Let $u = \ln x$, $du = \frac{1}{x}dx$, then we have

$$\begin{aligned}\int_3^\infty f(x)dx &= \lim_{b\to\infty}\left(\int_3^b \frac{1}{x\left(\ln x\right)^2}dx\right) = \lim_{b\to\infty}\left(\int_{\ln 3}^{\ln b}\frac{1}{u^2}du\right)\\ &= \lim_{b\to\infty}\left(\frac{-1}{u}\Bigg|_{\ln 3}^{\ln b}\right) = \lim_{b\to\infty}\left(\frac{-1}{\ln b} + \frac{1}{\ln 3}\right) = \frac{1}{\ln 3},\end{aligned}$$

converges. Hence $\sum\limits_{n=3}^\infty f(n) = \sum\limits_{n=3}^\infty \frac{1}{n(\ln n)^2}$ converges by the integral test (Theorem 9.2.3). Thus $\sum\limits_{n=2}^\infty \left|\frac{(-1)^n}{n(\ln n)^2}\right| = \frac{1}{2\ln 2} + \sum\limits_{n=3}^\infty \frac{1}{n(\ln n)^2}$ also converges. Hence $\sum\limits_{n=2}^\infty \frac{(-1)^n}{n\ln n}$ converges absolutely. ■

Exercise 9.4.2. *Show that if $\sum\limits_{n=1}^\infty a_n$ converges absolutely, then $\sum\limits_{n=1}^\infty a_n^2$ converges.*

Sol.

Since $\sum\limits_{n=1}^\infty a_n$ converges absolutely, that is, $\sum\limits_{n=1}^\infty |a_n|$ converges, by Theorem 9.1.6, $\lim\limits_{n\to\infty}|a_n| = 0$. Then for $\epsilon = 1 > 0$, $\exists\ N\in\mathbb{N}$ such that if $n \geq N$ then $||a_n| - 0| = |a_n| < 1$. So we have

$$0 \leq a_n^2 = |a_n|\,|a_n| < 1\cdot|a_n| = |a_n|\,,\ \forall n\geq N.$$

Then since $\sum\limits_{n=N}^{\infty} |a_n|$ converges, by the basic comparison test (Theorem 9.2.8), $\sum\limits_{n=N}^{\infty} a_n^2$ converges. Thus $\sum\limits_{n=1}^{\infty} a_n^2$ also converges. ■

EXERCISE 9.4.3. *Give an example that* $\sum\limits_{n=1}^{\infty} a_n$ *converges, but* $\sum\limits_{n=1}^{\infty} a_n^2$ *diverges.*

Sol.

Consider $a_n = \frac{(-1)^n}{\sqrt{n}}$, then $a_n^2 = \frac{1}{n}$. Since $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is positive, decreasing and $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, by the alternating test (Theorem 9.4.5), $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges. However, $\sum\limits_{n=1}^{\infty} a_n^2 = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p = 2$). ■

9.5. Exercises 9.5.

EXERCISE 9.5.1. *Show that the Taylor series of the following functions at* 0 *converge to* $f(x)$ *for all* $x \in \mathbb{R}$.

(a) $f(x) = \sin(2x)$.
(b) $f(x) = e^{-x}$.
(c) $f(x) = \cos x$.

Sol.

(a) Since

$$f'(x) = 2\cos(2x),\ f''(x) = -4\sin(2x),\ f^{(3)}(x) = -8\cos(2x),\ f^{(4)}(x) = 16\sin(2x),$$

$$\vdots$$

$$\begin{aligned} f^{(4m+1)}(x) &= 2^{4m+1}\cos(2x),\ f^{(4m+2)}(x) = -2^{4m+2}\sin(2x), \\ f^{(4m+3)}(x) &= -2^{4m+3}\cos(2x),\ f^{(4m+4)}(x) = 2^{4m+4}\sin(2x), \end{aligned}$$

we have

$$f(x) = 0 + \frac{2}{1!}x + \frac{0}{2!}x^2 - \frac{2^3}{3!}x^3 - \frac{0}{4!}x^4 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x).$$

Then by Lagrange's estimate, since $\left|f^{(n+1)}(t)\right| \le 2^{n+1}$,

$$|R_n(x)| \le \left(\max_{t\in J}\left|f^{(n+1)}(t)\right|\right)\frac{|x|^{n+1}}{(n+1)!} \le \frac{2^{n+1}|x|^{n+1}}{(n+1)!}.$$

Fixed x, let $k \in \mathbb{N}$ such that $k + 1 > 2|x|$. Then since for $n \ge k$,

$$0 \le |R_n(x)| \le \frac{2^{n+1}|x|^{n+1}}{(n+1)!} = \frac{2^k|x|^k}{k!}\cdot\frac{2|x|}{k+1}\cdot\cdots\cdot\frac{2|x|}{n+1} \le \frac{2^k|x|^k}{k!}\cdot\frac{2|x|}{n+1},$$

and since

$$\lim_{n\to\infty}\left(\frac{2^k|x|^k}{k!}\cdot\frac{2|x|}{n+1}\right)=0,$$

by the pinching theorem, $\lim_{n\to\infty}|R_n(x)|=0$, that is, $\lim_{n\to\infty}R_n(x)=0$. Thus we have

$$\sin(2x)=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n+1}}{(2n+1)!}x^{2n+1}. \qquad \blacksquare$$

(b) Since

$$f'(x)=-e^{-x},\ f''(x)=e^{-x},\ \cdots,\ f^{(n)}(x)=(-1)^ne^{-x},$$

we have

$$f(x)=1-\frac{1}{1!}x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\cdots+\frac{(-1)^n}{n!}x^n+R_n(x).$$

Then since if $x>0$, then $\forall\ t\in[0,x]$, $\left|f^{(n+1)}(t)\right|=e^{-t}\le 1$, and if $x<0$, then $\forall\ t\in[x,0]$, $\left|f^{(n+1)}(t)\right|=e^{-t}\le e^{-x}$, by Lagrange's estimate,

$$0\le|R_n(x)|\le\left(\max_{t\in J}\left|f^{(n+1)}(t)\right|\right)\frac{|x|^{n+1}}{(n+1)!}\le\begin{cases}\frac{|x|^{n+1}}{(n+1)!}, & x>0,\\ \frac{e^{-x}|x|^{n+1}}{(n+1)!}, & x<0.\end{cases}$$

Fixed x, let $k\in\mathbb{N}$ such that $k+1>|x|$. Then since for $n\ge k$,

$$\frac{|x|^{n+1}}{(n+1)!}=\frac{|x|^k}{k!}\cdot\frac{|x|}{k+1}\cdot\cdots\cdot\frac{|x|}{n+1}\le\frac{|x|^k}{k!}\cdot\frac{|x|}{n+1},$$

and since $\lim_{n\to\infty}(\frac{|x|^k}{k!}\cdot\frac{|x|}{n+1})=0$, by the pinching theorem,

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}=0=\lim_{n\to\infty}\frac{e^{-x}|x|^{n+1}}{(n+1)!}.$$

So $\lim_{n\to\infty}|R_n(x)|=0$, that is, $\lim_{n\to\infty}R_n(x)=0$. Thus we have

$$e^{-x}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n. \qquad \blacksquare$$

(c) Since

$$f'(x)=-\sin x,\ f''(x)=-\cos x,\ f^{(3)}(x)=\sin x,\ f^{(4)}(x)=\cos x,$$

$$\vdots$$

$$\begin{aligned}f^{(4m+1)}(x)&=-\sin x,\ f^{(4m+2)}(x)=-\cos x,\\ f^{(4m+3)}(x)&=\sin x,\ f^{(4m+4)}(x)=\cos x,\end{aligned}$$

we have

$$f(x) = 1 - \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x).$$

Then by Lagrange's estimate, since $\left|f^{(n+1)}(t)\right| \leq 1$,

$$0 \leq |R_n(x)| \leq \left(\max_{t\in J}\left|f^{(n+1)}(t)\right|\right)\frac{|x|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Then since $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, by the pinching theorem, $\lim\limits_{n\to\infty} |R_n(x)| = 0$, that is, $\lim\limits_{n\to\infty} R_n(x) = 0$. Thus we have

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}. \qquad \blacksquare$$

EXERCISE 9.5.2. *Show that the Taylor series of $f(x) = \ln(1+x)$ at 0 converges to $f(x)$ for $x = 1$. Find $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.*

Sol.
Since

$$f'(x) = \frac{1}{1+x},\ f''(x) = -\frac{1}{(1+x)^2},\ f^{(3)}(x) = \frac{2}{(1+x)^3},$$
$$\vdots$$
$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n},$$

we have

$$f(x) = 0 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 + \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n + R_n(x).$$

Then by Lagrange's estimate, since for $t \in [0,1]$,

$$\left|f^{(n+1)}(t)\right| = \frac{n!}{(1+t)^{n+1}} \leq n!,$$

we have

$$0 \leq |R_n(1)| \leq \left(\max_{t\in[0,1]}\left|f^{(n+1)}(t)\right|\right)\frac{|1|^{n+1}}{(n+1)!} \leq \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Then since $\lim\limits_{n\to\infty} \frac{1}{n+1} = 0$, by the pinching theorem, $\lim\limits_{n\to\infty} |R_n(1)| = 0$, that is, $\lim\limits_{n\to\infty} R_n(1) = 0$. Thus we have

$$\ln 2 = f(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}. \qquad \blacksquare$$

9.6. Exercises 9.6.
Find the intervals of convergence of the following series.

EXERCISE 9.6.1. $\sum\limits_{n=1}^{\infty} \frac{1}{2n^2+n-1} x^n$.

Sol.
First, since

$$\lim_{n\to\infty} \frac{\left|\frac{1}{2(n+1)^2+(n+1)-1} x^{n+1}\right|}{\left|\frac{1}{2n^2+n-1} x^n\right|} = \lim_{n\to\infty} \left|\frac{2n^2+n-1}{2n^2+5n+2} \cdot x\right| = |x|,$$

by the ratio test (Theorem 9.3.4) and Theorem 9.6.3, the radius of convergence is 1.

Then if $x=1$, since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2$) and

$$\lim_{n\to\infty} \frac{\frac{1}{2n^2+n-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{2n^2+n-1} = \frac{1}{2} > 0,$$

by the limit comparison test (Theorem 9.2.11), $\sum\limits_{n=1}^{\infty} \frac{1}{2n^2+n-1}$ converges. Then if $x=-1$, since $\sum\limits_{n=1}^{\infty} \left|\frac{(-1)^n}{2n^2+n-1}\right| = \sum\limits_{n=1}^{\infty} \frac{1}{2n^2+n-1}$ converges, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n^2+n-1}$ converges absolutely, which implies that $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n^2+n-1}$ also converges. So the interval of convergence of $\sum\limits_{n=1}^{\infty} \frac{1}{2n^2+n-1} x^n$ is $[-1,1]$. ■

EXERCISE 9.6.2. $\sum\limits_{n=1}^{\infty} n x^n$.

Sol.
First, since

$$\lim_{n\to\infty} \frac{\left|(n+1)\, x^{n+1}\right|}{|n x^n|} = \lim_{n\to\infty} \left|\frac{n+1}{n} \cdot x\right| = |x|,$$

by the ratio test (Theorem 9.3.4) and Theorem 9.6.3, the radius of convergence is 1.

Then if $x=1$, since $\lim\limits_{n\to\infty} n \neq 0$, by Theorem 9.1.6, $\sum\limits_{n=1}^{\infty} n$ diverges. Then if $x=-1$, since $\lim\limits_{n\to\infty} n \neq 0$ implies $\lim\limits_{n\to\infty} (-1)^n n \neq 0$, by Theorem 9.1.6, $\sum\limits_{n=1}^{\infty} (-1)^n n$ diverges. So the interval of convergence of $\sum\limits_{n=1}^{\infty} n x^n$ is $(-1,1)$. ■

EXERCISE 9.6.3. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n$.

Sol.

First, since

$$\lim_{n\to\infty}\frac{\left|\frac{(-1)^{n+1}}{(n+1)!}x^{n+1}\right|}{\left|\frac{(-1)^n}{n!}x^n\right|}=\lim_{n\to\infty}\left|\frac{1}{n+1}\cdot x\right|=0,$$

by the ratio test (Theorem 9.3.4) and Theorem 9.6.3, the radius of convergence is ∞. So the interval of convergence of $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n$ is $(-\infty,\infty)=\mathbb{R}$. ■

Exercise 9.6.4. $\sum\limits_{n=1}^{\infty}\frac{n}{\ln n}x^n$.

Sol.

First, we have

$$\lim_{n\to\infty}\frac{\left|\frac{n+1}{\ln(n+1)}x^{n+1}\right|}{\left|\frac{n}{\ln n}x^n\right|}=\lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{\ln n}{\ln(n+1)}\cdot x\right|=|x|,$$

where $\lim\limits_{n\to\infty}\frac{n+1}{n}=1$ and

$$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+1}}=\lim_{x\to\infty}\frac{x+1}{x}=1$$

by L'Hopital's rule. So by the ratio test (Theorem 9.3.4) and Theorem 9.6.3, the radius of convergence is 1.

Then if $x=1$, since by L'Hopital's rule,

$$\lim_{x\to\infty}\frac{x}{\ln x}=\lim_{x\to\infty}\frac{1}{\frac{1}{x}}=\lim_{x\to\infty}x=\infty,$$

that is, $\lim\limits_{n\to\infty}\frac{n}{\ln n}=\infty$, by Theorem 9.1.6, $\sum\limits_{n=1}^{\infty}\frac{n}{\ln n}$ diverges. Then if $x=-1$, since $\lim\limits_{n\to\infty}\frac{n}{\ln n}\neq 0$ implies $\lim\limits_{n\to\infty}(-1)^n\frac{n}{\ln n}\neq 0$, by Theorem 9.1.6, $\sum\limits_{n=1}^{\infty}(-1)^n\frac{n}{\ln n}$ diverges. So the interval of convergence of $\sum\limits_{n=1}^{\infty}\frac{n}{\ln n}x^n$ is $(-1,1)$. ■

Exercise 9.6.5. $\sum\limits_{n=1}^{\infty}\frac{3^n}{n}x^n$.

Sol.

First, since

$$\lim_{n\to\infty}\frac{\left|\frac{3^{n+1}}{n+1}x^{n+1}\right|}{\left|\frac{3^n}{n}x^n\right|}=\lim_{n\to\infty}\left|\frac{3n}{n+1}\cdot x\right|=3|x|,$$

by the ratio test (Theorem 9.3.4) and Theorem 9.6.3, the radius of convergence is $\frac{1}{3}$.

Then if $x = \frac{1}{3}$, $\sum\limits_{n=1}^{\infty} \frac{3^n}{n} \cdot \left(\frac{1}{3}\right)^n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p = 1$). Then if $x = \frac{-1}{3}$, $\sum\limits_{n=1}^{\infty} \frac{3^n}{n} \cdot \left(\frac{-1}{3}\right)^n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating test (Theorem 9.4.5). So the interval of convergence of $\sum\limits_{n=1}^{\infty} \frac{3^n}{n} x^n$ is $[\frac{-1}{3}, \frac{1}{3})$. ■

Exercise 9.6.6. $\sum\limits_{n=1}^{\infty} \frac{\ln n}{2^n} x^n$.

Sol.
First, since

$$\lim_{n\to\infty} \frac{\left|\frac{\ln(n+1)}{2^{n+1}} x^{n+1}\right|}{\left|\frac{\ln n}{2^n} x^n\right|} = \lim_{n\to\infty} \left|\frac{\ln(n+1)}{2\ln n} \cdot x\right| = \frac{1}{2}|x|,$$

by the ratio test (Theorem 9.3.4) and Theorem 9.6.3, the radius of convergence is 2.

Then if $x = 2$, since $\lim\limits_{n\to\infty} \ln n \neq 0$, by Theorem 9.1.6, $\sum\limits_{n=1}^{\infty} \frac{\ln n}{2^n} \cdot 2^n = \sum\limits_{n=1}^{\infty} \ln n$ diverges. Then if $x = -2$, since $\lim\limits_{n\to\infty} (-1)^n \ln n \neq 0$, by Theorem 9.1.6, $\sum\limits_{n=1}^{\infty} \frac{\ln n}{2^n} \cdot (-2)^n = \sum\limits_{n=1}^{\infty} (-1)^n \ln n$ diverges. So the interval of convergence of $\sum\limits_{n=1}^{\infty} \frac{\ln n}{2^n} x^n$ is $(-2, 2)$. ■

Exercise 9.6.7. $\sum\limits_{n=1}^{\infty} \frac{(3n)!}{(n!)^2} x^n$.

Sol.
First, since

$$\lim_{n\to\infty} \frac{\left|\frac{(3(n+1))!}{((n+1)!)^2} x^{n+1}\right|}{\left|\frac{(3n)!}{(n!)^2} x^n\right|} = \lim_{n\to\infty} \left|\frac{\frac{(3n+3)!}{(n+1)!\cdot(n+1)!}}{\frac{(3n)!}{n!\cdot n!}} \cdot x\right|$$

$$= \lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+1)} \cdot x\right| = \infty,$$

by the ratio test (Theorem 9.3.4) and Theorem 9.6.3, the radius of convergence is 0. So $\sum\limits_{n=1}^{\infty} \frac{\ln n}{2^n} x^n$ only converges at $x = 0$. ■

10. Chapter 10

10.1. Exercises 10.1.

EXERCISE 10.1.1. $f(x,y) = \sin^2 xy \cos x^2 + y^2$. *Compute* f_x *and* f_y.

Sol.
$f_x = (2\sin xy \cdot y)\cos x^2 + \sin^2 xy \cdot (-\sin x^2 \cdot 2x)$.
$f_y = (2\sin xy \cdot x)\cos x^2 + 2y$. ■

EXERCISE 10.1.2. $f(x,y) = e^{\sin xy} \tan^{-1} x^2 + y^2$. *Compute* f_x *and* f_y.

Sol.
$f_x = (e^{\sin xy} \cdot \cos xy \cdot y)\tan^{-1} x^2 + e^{\sin xy}(\frac{1}{1+(x^2)^2} \cdot 2x)$.
$f_y = (e^{\sin xy} \cdot \cos xy \cdot x)\tan^{-1} x^2 + 2y$. ■

EXERCISE 10.1.3. $f(x,y) = 1 - x^2 - y^2$. *Compute* f_x *and* f_y.

Sol.
$f_x = -2x$. $f_y = -2y$. ■

EXERCISE 10.1.4. $f(x,y) = \int_x^y \sin t^3 dt$. *Compute* f_x *and* f_y.

Sol.
By the fundamental theorem of calculus I, $f_x = -\sin x^3$. $f_y = \sin y^3$. ■

EXERCISE 10.1.5. $f(x,y,z) = e^{x^4 y^2 \sin z^3}$. *Compute* f_x, f_y *and* f_z.

Sol.
$f_x = e^{x^4y^2\sin z^3} \cdot (4x^3)y^2 \sin z^3$.
$f_y = e^{x^4y^2\sin z^3} \cdot x^4(2y)\sin z^3$.
$f_z = e^{x^4y^2\sin z^3} \cdot x^4y^2(\cos z^3 \cdot 3z^2)$. ■

EXERCISE 10.1.6. $f(x,y,z) = f_1(x)f_2(y)f_3(z)$. *Describe* f_x, f_y *and* f_z *in terms of* f_1, f_2 *and* f_3.

Sol.
$f_x = f_1'(x)f_2(y)f_3(z)$.
$f_y = f_1(x)f_2'(y)f_3(z)$.
$f_z = f_1(x)f_2(y)f_3'(z)$. ■

10.2. Exercises 10.2.

Exercise 10.2.1. $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. *Prove that $f(x,y)$ is continuous at 0.*

Sol.
Since $x^2 + y^2 \geq 2\sqrt{x^2y^2} = 2\,|xy|$, we have

$$\frac{-(x^2+y^2)}{\sqrt{x^2+y^2}} \leq \frac{-2\,|xy|}{\sqrt{x^2+y^2}} \leq \frac{2xy}{\sqrt{x^2+y^2}} \leq \frac{2\,|xy|}{\sqrt{x^2+y^2}} \leq \frac{(x^2+y^2)}{\sqrt{x^2+y^2}}.$$

Then since

$$\begin{aligned} 0 &= \lim_{(x,y)\to(0,0)} [-(x^2+y^2)]^{\frac{1}{2}} = \lim_{(x,y)\to(0,0)} \frac{-(x^2+y^2)}{\sqrt{x^2+y^2}} \\ &\leq \lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} \leq \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)}{\sqrt{x^2+y^2}} \\ &= \lim_{(x,y)\to(0,0)} (x^2+y^2)^{\frac{1}{2}} = 0. \end{aligned}$$

by the pinching theorem, $\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$. So $f(x,y)$ is continuous at 0. ■

Exercise 10.2.2. $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$. *Prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.*

Sol.
$\lim_{h\to 0} f(h,0) = \lim_{h\to 0} \frac{h}{\sqrt{h^2+0}} = 1$, but $\lim_{h\to 0} f(0,h) = \lim_{h\to 0} \frac{0}{\sqrt{0+h^2}} = 0$. ■

Exercise 10.2.3. $f(x,y) = \frac{xy^3}{x^2+y^6}$. *Prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.*

Sol.
$\lim_{h\to 0} f(h,0) = \lim_{h\to 0} \frac{0}{h^2+0} = 0$, but $\lim_{h\to 0} f(h^3,h) = \lim_{h\to 0} \frac{h^6}{h^6+h^6} = \frac{1}{2}$. ■

Exercise 10.2.4. $f(x,y) = \frac{3x^2+7y^2}{x+y^2}$. *Prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.*

Sol.
$\lim_{h\to 0} f(h,0) = \lim_{h\to 0} \frac{3h^2}{h} = 0$, but $\lim_{h\to 0} f(0,h) = \lim_{h\to 0} \frac{7h^2}{h^2} = 7$. ■

Exercise 10.2.5. $f(x,y) = \ln(e^x + e^y)$. *Prove that $f_{xx}f_{yy} - f_{xy}^2 = 0$.*

Sol.
$f_x = \frac{e^x}{e^x+e^y}$. $f_y = \frac{e^y}{e^x+e^y}$.
$f_{xx} = \frac{e^x(e^x+e^y)-e^xe^x}{(e^x+e^y)^2} = \frac{e^xe^y}{(e^x+e^y)^2}$.
$f_{yy} = \frac{e^y(e^x+e^y)-e^ye^y}{(e^x+e^y)^2} = \frac{e^xe^y}{(e^x+e^y)^2}$.
$f_{xy} = \frac{-e^xe^y}{(e^x+e^y)^2}$.
$f_{xx}f_{yy} - f_{xy}^2 = \frac{(e^xe^y)^2}{(e^x+e^y)^4} - \frac{(-e^xe^y)^2}{(e^x+e^y)^4} = 0.$ ■

EXERCISE 10.2.6. *$f = f(x, y, z)$, where $\frac{1}{f} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Compute f_x, f_y and f_z.*

Sol.
$(\frac{1}{f})_x = \frac{-f_x}{f^2} \implies f_x = -f^2 \cdot (\frac{1}{f})_x = f^2 \cdot \frac{1}{x^2}$.
$(\frac{1}{f})_y = \frac{-f_y}{f^2} \implies f_y = -f^2 \cdot (\frac{1}{f})_y = f^2 \cdot \frac{1}{y^2}$.
$(\frac{1}{f})_z = \frac{-f_z}{f^2} \implies f_z = -f^2 \cdot (\frac{1}{f})_z = f^2 \cdot \frac{1}{z^2}$. ■

EXERCISE 10.2.7. *Suppoe $f_1'(x)$ and $f_2'(y)$ are continuous with respect to x and y respectively, and let $f(x, y) = f_1(x)f_2(y)$. Prove that $f_{xy} = f_{yx}$.*

Sol.
Since $f_1'(x)$ and $f_2'(y)$ are continuous, $f_x(x, y) = f_1'(x)f_2(y)$ is continuous, and since $f_1'(x)$ and $f_2'(y)$ are continuous, $f_y(x, y) = f_1(x)f_2'(y)$ is continuous. Then since $f_{xy}(x, y) = f_1'(x)f_2'(y)$ and $f_{yx}(x, y) = f_1'(x)f_2'(y)$ are both continuous, by Remark 10.2.7, $f_{xy} = f_{yx}$. ■

11. Chapter 11

11.1. Exercises 11.1.

11.2. Exercises 11.2.

For Problem 1 to 7,
(a) Find $\bigtriangledown f$.
(b) Evaluate $\bigtriangledown f$ at the given point P.
(c) Find the directional derivative of f at P in the direction of the given vector $\vec{u}$.

EXERCISE 11.2.1. $f(x,y,z) = \sqrt{x^2+yz}$, $P = (1,0,0)$, $\vec{u} = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$.

Sol.
(a) $\bigtriangledown f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (\frac{x}{\sqrt{x^2+yz}}, \frac{z}{2\sqrt{x^2+yz}}, \frac{y}{2\sqrt{x^2+yz}})$.
(b) $\bigtriangledown f(P) = (1,0,0)$.
(c) $||\vec{u}|| = 1$, $f_{\vec{u}}\,'(P) = \bigtriangledown f(P) \cdot \vec{u} = (\frac{\sqrt{2}}{2}, 0, 0)$. ∎

EXERCISE 11.2.2. $f(x,y,z) = \sin(x+y+z)$, $P = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$, $\vec{u} = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$.

Sol.
(a) $\bigtriangledown f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (\cos(x+y+z), \cos(x+y+z), \cos(x+y+z))$.
(b) $\bigtriangledown f(P) = (-1,-1,-1)$.
(c) $||\vec{u}|| = 1$, $f_{\vec{u}}\,'(P) = \bigtriangledown f(P) \cdot \vec{u} = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$. ∎

EXERCISE 11.2.3. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $P = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$, $\vec{u} = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

Sol.
(a) $\bigtriangledown f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}})$.
(b) $\bigtriangledown f(P) = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$.
(c) $||\vec{u}|| = 1$, $f_{\vec{u}}\,'(P) = \bigtriangledown f(P) \cdot \vec{u} = (\frac{2}{9}, \frac{2}{9}, -\frac{4}{9})$. ∎

EXERCISE 11.2.4. $f(x,y,z) = \arctan(x^2+y^2+z^2)$, $P = (\frac{\sqrt{\pi}}{3}, \frac{\sqrt{\pi}}{3}, \frac{\sqrt{\pi}}{3})$, $\vec{u} = (0,1,0)$.

Sol.
(a) $\bigtriangledown f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (\frac{2x}{(x^2+y^2+z^2)^2+1}, \frac{2y}{(x^2+y^2+z^2)^2+1}, \frac{2z}{(x^2+y^2+z^2)^2+1})$.
(b) $\bigtriangledown f(P) = (\frac{9\sqrt{\pi}}{\pi^2+1}, \frac{9\sqrt{\pi}}{\pi^2+1}, \frac{9\sqrt{\pi}}{\pi^2+1})$.
(c) $||\vec{u}|| = 1, f_{\vec{u}}\,'(P) = \bigtriangledown f(P) \cdot \vec{u} = (0, \frac{9\sqrt{\pi}}{\pi^2+1}, 0)$. ∎

EXERCISE 11.2.5. $f(x,y,z) = \frac{xyz}{\sqrt{x^2+y^2+z^2}}$, $P = (1,1,1)$, $\vec{u} = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Sol.

(a) $\bigtriangledown f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (\frac{yz(y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{xz(x^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{xy(x^2+y^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}})$.
(b) $\bigtriangledown f(P) = (\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}})$.
(c) $||\vec{u}|| = 1,\ f_{\vec{u}}\,'(P) = \bigtriangledown f(P) \cdot \vec{u} = (\frac{2}{9}, -\frac{2}{9}, \frac{2}{9})$. ∎

EXERCISE 11.2.6. *$f(x, y, z) = e^{x^2+y^2+z^2}$, $P = (1, 2, 3)$, $\vec{u} = (\frac{4}{5}, -\frac{3}{5}, 0)$.*

Sol.
(a) $\bigtriangledown f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (2xe^{x^2+y^2+z^2}, 2ye^{x^2+y^2+z^2}, 2ze^{x^2+y^2+z^2})$.
(b) $\bigtriangledown f(P) = (2e^{14}, 4e^{14}, 6e^{14})$.
(c) $||\vec{u}|| = 1,\ f_{\vec{u}}\,'(P) = \bigtriangledown f(P) \cdot \vec{u} = (\frac{8e^{14}}{5}, -\frac{12e^{14}}{5}, 0)$. ∎

EXERCISE 11.2.7. *$f(x, y) = e^x \cos y + e^y \sin x$, $P = (0, 0)$, $\vec{u} = (-1, 0)$.*

Sol.
(a) $\bigtriangledown f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (e^x \cos y + e^y \cos x, -e^x \sin y + e^y \sin x)$.
(b) $\bigtriangledown f(P) = (2, 0)$.
(c) $||\vec{u}|| = 1,\ f_{\vec{u}}\,'(P) = \bigtriangledown f(P) \cdot \vec{u} = (-2, 0)$. ∎

EXERCISE 11.2.8. *Let $h(x, y) = e^x \cos y$. Find the tangent line of the curve $h(x, y) = 2$ at $P = (\ln 2, 0)$.*

Sol.
First, we have $\bigtriangledown h = (e^x \cos y, -e^y \sin x)$, and $\bigtriangledown h(P) = (2, 0)$.
Then since $h(x, y) = 2$ is a level curve of $z = h(x, y)$, that is, $h(x, y)$ is a constant on the curve, if $\vec{u}$ is a tangent vector of the curve at P, then the direcional derivative $h_{\vec{u}}\,'(P) = 0$. So

$$h'_{\vec{u}}(P) = \bigtriangledown h(P) \cdot \vec{u} = (2, 0) \cdot \vec{u} = 0,$$

that is, the vector $\bigtriangledown h(P) = (2, 0)$ is normal to the tangent line of $h(x, y) = 2$ at P. Therefore, the tangent line is

$$2(x - \ln 2) = 0. \quad ∎$$

EXERCISE 11.2.9. *Let $h(x, y, z) = \sin\sqrt{x^2 + y^2 + z^2}$. Find the tangent plane of the surface $h(x, y, z) = 0$ at $P = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3})$*

Sol.
First, we have

$$\bigtriangledown h = (\frac{x\cos(\sqrt{x^2 + y^2 + z^2})}{\sqrt{x^2 + y^2 + z^2}}, \frac{y\cos(\sqrt{x^2 + y^2 + z^2})}{\sqrt{x^2 + y^2 + z^2}}, \frac{z\cos(\sqrt{x^2 + y^2 + z^2})}{\sqrt{x^2 + y^2 + z^2}})$$

and $\bigtriangledown h(P) = (-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$.

Then since $h(x,y,z)=0$ is a level surface of $w=h(x,y,z)$, that is, $h(x,y,z)$ is a constant on the surface, if $\vec{u}$ is a vector at P lies on the tangent plane, then the direcional derivative $h_{\vec{u}}\,'(P)=0$. So

$$h'_{\vec{u}}(P)=\bigtriangledown h(P)\cdot\vec{u}=(-\frac{2}{3},-\frac{2}{3},-\frac{1}{3})\cdot\vec{u}=0,$$

that is, the vector $\bigtriangledown h(P)=(-\frac{2}{3},-\frac{2}{3},-\frac{1}{3})$ is normal to the tangent plane of $h(x,y,z)=0$ at P. Therefore, the tangent plane is

$$-\frac{2}{3}(x-\frac{2\pi}{3})-\frac{2}{3}(y-\frac{2\pi}{3})-\frac{1}{3}(z-\frac{\pi}{3})=0. \qquad \blacksquare$$

11.3. Exercises 11.3.

EXERCISE 11.3.1. $u=\sin^2 x+\cos^2 y$, $x=e^{s^2+t^2}$, $y=st$. *Find* $\frac{\partial u}{\partial s}$ *and* $\frac{\partial u}{\partial t}$.

Sol.
$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=4s\sin x\cos xe^{s^2+t^2}-2t\cos y\sin y.$
$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=4t\sin x\cos xe^{s^2+t^2}-2s\cos y\sin y.$ $\blacksquare$

EXERCISE 11.3.2. $u=f(x)g(y)$, $x=f_1(s)f_2(t)$, $y=g_1(s)g_2(t)$. *Find* $\frac{\partial u}{\partial s}$ *and* $\frac{\partial u}{\partial t}$.

Sol.
$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=f\,'(x)g(y)f_1\,'(s)f_2(t)+f(x)g'(y)g_1'(s)g_2(t).$
$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=f\,'(x)g(y)f_1(s)f_2\,'(t)+f(x)g'(y)g_1(s)g_2'(t).$ $\blacksquare$

EXERCISE 11.3.3. $u=\arctan xy$, $x=\sin st$, $y=s^2+t^2$. *Find* $\frac{\partial u}{\partial s}$ *and* $\frac{\partial u}{\partial t}$.

Sol.
$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=\frac{ty\cos st}{x^2y^2+1}+\frac{2sx}{x^2y^2+1}.$
$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=\frac{sy\cos st}{x^2y^2+1}+\frac{2tx}{x^2y^2+1}.$ $\blacksquare$

EXERCISE 11.3.4. *Let* $e^{\cos x\sin y}+x^2+y^2=0$. *Find* $\frac{dy}{dx}$ *and* $\frac{dx}{dy}$.

Sol.
Let $u=e^{\cos x\sin y}+x^2+y^2$.

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\sin x\sin ye^{\cos x\sin y}+2x}{\cos x\cos ye^{\cos x\sin y}+2y}.$$

$$\frac{dx}{dy} = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}=\frac{\cos x\cos ye^{\cos x\sin y}+2y}{\sin x\sin ye^{\cos x\sin y}+2x}. \qquad \blacksquare$$

EXERCISE 11.3.5. *Let* $\frac{x-y}{\sqrt{x^2+y^2+1}} = 0$*. Find* $\frac{dy}{dx}$ *and* $\frac{dx}{dy}$*.*

Sol.
Let $u = \frac{x-y}{\sqrt{x^2+y^2+1}}$.

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{\frac{\sqrt{x^2+y^2+1}-(x-y)\frac{2x}{2\sqrt{x^2+y^2+1}}}{x^2+y^2+1}}{\frac{-\sqrt{x^2+y^2+1}-(x-y)\frac{2y}{2\sqrt{x^2+y^2+1}}}{x^2+y^2+1}} = \frac{y^2+xy+1}{x^2+xy+1}.$$

$$\frac{dx}{dy} = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = \frac{x^2+xy+1}{y^2+xy+1}. \qquad \blacksquare$$

EXERCISE 11.3.6. *Let* $\sin x^2 \tan y^2 = 0$*. Find* $\frac{dy}{dx}$ *and* $\frac{dx}{dy}$*.*

Sol.
Let $u = \sin x^2 \tan y^2$.
$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{2x\cos x^2 \tan y^2}{2y \sin x^2 \sec y^2} = -\frac{x}{y}\cot x^2 \sin y^2$.
$\frac{dx}{dy} = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -\frac{y}{x}\tan x^2 \csc y^2$. $\blacksquare$

EXERCISE 11.3.7. *Let* $f(x)g(y) = 0$*. Find* $\frac{dy}{dx}$ *and* $\frac{dx}{dy}$*.*

Sol.
Let $u = f(x)g(y)$.
$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{f'(x)g(y)}{f(x)g'(y)}$.
$\frac{dx}{dy} = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -\frac{f(x)g'(y)}{f'(x)g(y)}$. $\blacksquare$

11.4. Exercises 11.4 and 11.5.

For Problem 1 to 5,
(a) Find all local extreme values and saddle points of f on $\mathbb{R}^2$.
(b) Find the absolute extreme values of f on the indicated domain D.

EXERCISE 11.4.1. $f(x,y) = \sin x \cos y$, $D = \mathbb{R}^2$

Sol.
(a) Since

$$\nabla f(x,y) = (\cos x \cos y, -\sin x \sin y),$$

and since $\begin{cases} \cos x \cos y = 0, \\ -\sin x \sin y = 0, \end{cases}$ if and only if

$$\begin{cases} x = \frac{\pi}{2} + k_1\pi, & k_1 \in Z, \\ y = k_2\pi, & k_2 \in Z, \end{cases} \quad or \begin{cases} x = k_3\pi, & k_3 \in Z, \\ y = \frac{\pi}{2} + k_4\pi, & k_4 \in Z, \end{cases}$$

the critical points are $(\frac{\pi}{2}+k_1\pi, k_2\pi)$, $(k_3\pi, \frac{\pi}{2}+k_4\pi)$, $k_1, k_2, k_3, k_4 \in Z$.

Then since

$$\begin{aligned} A &= f_{xx}(x,y) = -\sin x\cos y, \\ B &= f_{xy}(x,y) = -\cos x\sin y, \\ C &= f_{yy}(x,y) = -\sin x\cos y, \\ D &= AC - B^2 = \sin^2 x\cos^2 y - \cos^2 x\sin^2 y, \end{aligned}$$

we have

	A	D	Result
$(\frac{\pi}{2}+2n\pi, 2m\pi)$	-1	1	local max
$(\frac{\pi}{2}+(2n+1)\pi, 2m\pi)$	1	1	local min
$(\frac{\pi}{2}+2n\pi, (2m+1)\pi)$	1	1	local min
$(\frac{\pi}{2}+(2n+1)\pi, (2m+1)\pi)$	-1	1	local max
$(k_3\pi, \frac{\pi}{2}+k_4\pi)$	0	-1	saddle

(b) Since $-1 \le \sin x \le 1$, $-1 \le \cos y \le 1, \forall\ x, y \in R$,

$$-1 \le f(x) = \sin x\cos y \le 1, \forall\ (x,y) \in D = \mathbb{R}^2.$$

So these local maximum points are also absolute maximum points, and these local minimum points are also absolute minimum points. ■

EXERCISE 11.4.2. $f(x,y) = x^2 + 6xy + 8y^2$, $D = [-10, 0] \times [-1, 8]$

Sol.

(a) Since

$$\nabla f(x,y) = (2x + 6y, 6x + 16y),$$

and since

$$\begin{cases} 2x + 6y = 0, \\ 6x + 16y = 0, \end{cases} \Rightarrow x = 0,\ y = 0,$$

the only critical point is $(0,0)$.

Then since

$$\begin{aligned} A &= f_{xx}(x,y) = 2,\ B = f_{xy}(x,y) = 6,\ C = f_{yy}(x,y) = 16, \\ D &= AC - B^2 = -4 < 0, \end{aligned}$$

$(0,0)$ is a saddle point.

(b)

Line	Equation	Maximum	Minimum
$x=0$	$f(0,y) = 8y^2$	$f(0,8) = 512$	$f(0,0) = 0$
$x=-10$	$f(-10,y) = 100 - 60y + 8y^2$	$f(-10,-1) = 168$	$f(-10, \frac{15}{4}) = -\frac{25}{2}$
$y=-1$	$f(x,-1) = x^2 - 6x + 8$	$f(-10,-1) = 168$	$f(3,-1) = -1$
$y=8$	$f(x,8) = x^2 + 48x + 512$	$f(0,8) = 512$	$f(-10,8) = 132$

So the absolute maximum point is $(0,8)$ and the absolute minimum point is $(-10,\frac{15}{4})$. ■

EXERCISE 11.4.3. $f(x,y)=\arctan(x^2+y^2)$, $D=[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]\times[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]$

Sol.
(a) Since

$$\nabla f(x,y)=(\frac{2x}{(x^2+y^2)^2+1},\frac{2y}{(x^2+y^2)^2+1}),$$

and since

$$\begin{cases}\frac{2x}{(x^2+y^2)^2+1}=0,\\ \frac{2y}{(x^2+y^2)^2+1}=0,\end{cases}\Rightarrow x=0,\ y=0,$$

the only critical point is $(0,0)$.
Then since

$$\begin{aligned}A&=f_{xx}(x,y)=\frac{2((x^2+y^2)^2+1)-8x^2(x^2+y^2)}{((x^2+y^2)^2+1)^2},\\ B&=f_{xy}(x,y)=\frac{-8xy(x^2+y^2)}{((x^2+y^2)^2+1)^2},\\ C&=f_{yy}(x,y)=\frac{2((x^2+y^2)^2+1)-8y^2(x^2+y^2)}{((x^2+y^2)^2+1)^2},\end{aligned}$$

we have

	A	B	C	D	result
$(0,0)$	2	0	2	4	local min

(b) Since $f(0,0)=0$, and since

Line	Equation	Maximum	Minimum
$x=\pm\frac{1}{\sqrt{2}}$	$f(x,y)=\arctan(y^2+\frac{1}{2})$	$f(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})=\frac{\pi}{4}$	$f(\pm\frac{1}{\sqrt{2}},0)=\arctan(\frac{1}{2})$
$y=\pm\frac{1}{\sqrt{2}}$	$f(x,y)=\arctan(x^2+\frac{1}{2})$	$f(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})=\frac{\pi}{4}$	$f(0\pm\frac{1}{\sqrt{2}})=\arctan(\frac{1}{2})$

the absolute maximum point are $(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})$ and the absolute minimum point is $(0,0)$. ■

EXERCISE 11.4.4. $f(x,y)=\sqrt{|1-x^2-y^2|}$, $D=\{x^2+y^2\leq 16\}$

Sol.
(a) For the region $x^2+y^2>1$, $f(x,y)=\sqrt{x^2+y^2-1}$, so

$$\nabla f(x,y)=(\frac{x}{\sqrt{x^2+y^2-1}},\frac{y}{\sqrt{x^2+y^2-1}}).$$

Then since

$$\begin{cases} \frac{x}{\sqrt{x^2+y^2-1}} = 0, \\ \frac{y}{\sqrt{x^2+y^2-1}} = 0, \end{cases} \Rightarrow x = 0,\ y = 0,$$

but $(0,0)$ does not satisfy $x^2 + y^2 > 1$, there are no critical points in the region.

For the region $x^2 + y^2 = 1$, ∇f does not exist $\forall\ (x, y)$ in the region. Hence all the points are critical points. Then since $f(x, y) \geq 0,\ \forall\ (x, y) \in \mathbb{R}^2$, and since $f(x, y) = 0,\ \forall\ (x, y)$ in the region, all these points are local and absolute minimum points.

For the region $x^2 + y^2 < 1$, $f(x, y) = \sqrt{1 - x^2 - y^2}$, so

$$\nabla f(x, y) = \left(-\frac{x}{\sqrt{1 - x^2 - y^2}}, -\frac{y}{\sqrt{1 - x^2 - y^2}}\right).$$

Then since

$$\begin{cases} -\frac{x}{\sqrt{1-x^2-y^2}} = 0, \\ -\frac{y}{\sqrt{1-x^2-y^2}} = 0, \end{cases} \Rightarrow x = 0,\ y = 0,$$

$(0,0)$ is a critical point. Then since

$$\begin{aligned} A &= f_{xx}(x, y) = \frac{-\sqrt{1 - x^2 - y^2} - \frac{x^2}{\sqrt{1-x^2-y^2}}}{1 - x^2 - y^2}, \\ B &= f_{xy}(x, y) = \frac{-\frac{xy}{\sqrt{1-x^2-y^2}}}{1 - x^2 - y^2}, \\ C &= f_{yy}(x, y) = \frac{-\sqrt{1 - x^2 - y^2} - \frac{y^2}{\sqrt{1-x^2-y^2}}}{1 - x^2 - y^2}, \end{aligned}$$

we have

	A	B	C	D	result
$(0,0)$	-1	0	-1	1	local min

(b) Since $f(0,0) = 1$ and since $f(x, y) = \sqrt{15},\ \forall\ (x, y)$ on the circle $x^2 + y^2 = 16$, the absolute maximum points are the points on the circle $x^2 + y^2 = 16$ and the absolute minimum points are the points on the circle $x^2 + y^2 = 1$. ■

EXERCISE 11.4.5. $f(x, y) = |x| + |y|$, $D = [-6, 5] \times [-5, 6]$

Sol.

(a) Since ∇f does not exist on $x = 0$ or $y = 0$, the points on the x-axis or y-axis are critical points. Then since $f(x, y) \geq 0,\ \forall\ (x, y) \in \mathbb{R}^2$, and since $f(x, y) = 0,\ \forall\ (x, y)$ on the x-axis or y-axis, all these points are local and absolute minimum points.

Then since for $xy \neq 0$, $\frac{\partial f}{\partial x} = \pm 1 \neq 0, \frac{\partial f}{\partial y} = \pm 1 \neq 0$, there are no other critical points.

(b)

Line	Equation	Maximum	Minimum
$x=-6$	$f(-6,y)=\lvert y\rvert+6$	$f(-6,6)=12$	$f(-6,0)=6$
$x=5$	$f(5,y)=\lvert y\rvert+5$	$f(5,6)=11$	$f(5,0)=5$
$y=-5$	$f(x,-5)=\lvert x\rvert+5$	$f(-6,-5)=11$	$f(0,-5)=5$
$y=6$	$f(x,6)=\lvert x\rvert+6$	$f(-6,6)=12$	$f(0,6)=6$

So the absolute maximum point is $(-6,6)$ and the absolute minimum point is $(0,0)$. ■

EXERCISE 11.4.6. *Under the condition $x+y+z=1$ and $x,y,z\geq 0$, find the maximum of $x^2+y^2+z^2$ and xyz.*

Sol.
Write $z=1-x-y$, then we have

$$\begin{aligned} x^2+y^2+z^2 &= x^2+y^2+(1-x-y)^2, \\ xyz &= xy(1-x-y), \end{aligned}$$

and the region is bounded by the lines $x=0,\ y=0$, and $1-x-y=0$.

Let $f(x,y)=x^2+y^2+(1-x-y)^2$. Then since

$$\nabla f(x,y)=(2x-2(1-x-y),2y-2(1-x-y)),$$

and since

$$\begin{cases} 2x-2(1-x-y)=0, \\ 2y-2(1-x-y)=0, \end{cases} \Rightarrow x=\frac{1}{3},\ y=\frac{1}{3},$$

$(\frac{1}{3},\frac{1}{3})$ is the only critical point. Then since

$$\begin{aligned} A &= f_{xx}(x,y)=4,\ B=f_{xy}(x,y)=2,\ C=f_{yy}(x,y)=4, \\ D &= AC-B^2=12>0, \end{aligned}$$

$(\frac{1}{3},\frac{1}{3})$ is a local minimum point and $f(\frac{1}{3},\frac{1}{3})=\frac{1}{3}$. Then since

Line	Equation	Maximum	Minimum
$x=0$	$f(x,y)=2(y-\frac{1}{2})^2+\frac{1}{2}$	$f(0,1)=f(0,0)=1$	$f(0,\frac{1}{2})=\frac{1}{2}$
$y=0$	$f(x,y)=2(x-\frac{1}{2})^2+\frac{1}{2}$	$f(1,0)=f(0,0)=1$	$f(\frac{1}{2},0)=\frac{1}{2}$
$y=1-x$	$f(x,y)=2(x-\frac{1}{2})^2+\frac{1}{2}$	$f(0,1)=f(1,0)=1$	$f(\frac{1}{2},\frac{1}{2})=\frac{1}{2}$

So the maximum of $x^2+y^2+z^2$ is 1 at the points $(1,0,0)$,$(0,1,0)$, $(0,0,1)$.

Now let $g(x,y)=xy(1-x-y)=xy-x^2y-xy^2$. Then since

$$\nabla g(x,y)=(y-2xy-y^2,x-x^2-2xy),$$

and since

$$\begin{cases} y-2xy-y^2=0, \\ x-x^2-2xy=0, \end{cases} \Rightarrow x=\frac{1}{3},\ y=\frac{1}{3},\ or\ x=0,\ y=0,$$

$(\frac{1}{3}, \frac{1}{3})$ and $(0,0)$ are the critical points. Then since

$$A = g_{xx}(x,y) = -2y,\ B = g_{xy}(x,y) = 1-2x-2y,\ C = g_{yy}(x,y) = -2x,$$

we have

	A	D	result
$(\frac{1}{3}, \frac{1}{3})$	$-\frac{2}{3}$	$\frac{1}{3}$	local max
$(0,0)$	0	-1	saddle

.

Then since $g(\frac{1}{3}, \frac{1}{3}) = \frac{1}{27}$, and since

Line	Equation	Maximum	Minimum
$x=0$	$g(x,y)=0$	0	0
$y=0$	$g(x,y)=0$	0	0
$y=1-x$	$g(x,y)=0$	0	0

the maximum of xyz is $\frac{1}{27}$ at the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. ■

EXERCISE 11.4.7. *Find the volume of the largest rectangular box in the first octant with 3 faces in the coordinate planes, and one vertex on the plane* $3x+2y+4z=9$.

Sol.

It means that we need to find the maximum of xyz for (x,y,z) satisfies $3x+2y+4z=9$ and $x,y,z \geq 0$. Write $z = \frac{9-3x-2y}{4}$, then we have $xyz = xy(\frac{9-3x-2y}{4})$ and the region is bounded by the lines $x=0$, $y=0$, and $9-3x-2y=0$.

Let $f(x,y) = xy(\frac{9-3x-2y}{4}) = \frac{1}{4}(9xy-3x^2y-2xy^2)$. Then since

$$\nabla f(x,y) = (\frac{1}{4}(9y-6xy-2y^2), \frac{1}{4}(9x-3x^2-4xy)),$$

and since

$$\begin{cases} \frac{1}{4}(9y-6xy-2y^2)=0, \\ \frac{1}{4}(9x-3x^2-4xy)=0, \end{cases} \Rightarrow x=1,\ y=\frac{3}{2},\ \textit{or}\ x=0,\ y=0,$$

$(1, \frac{3}{2})$ and $(0,0)$ are the critical points. Then since

$$\begin{aligned} A &= f_{xx}(x,y) = -\frac{3}{2}y,\ B = f_{xy}(x,y) = \frac{1}{4}(9-6x-4y), \\ C &= f_{yy}(x,y) = -x, \end{aligned}$$

we have

	A	D	result
$(1, \frac{2}{3})$	-1	$\frac{27}{16}$	local max
$(0,0)$	0	$-\frac{81}{16}$	saddle

.

Then since $f(1, \frac{3}{2}) = \frac{9}{8}$, and since

Line	Equation	Maximum	Minimum
$x = 0$	$f(x, y) = 0$	0	0
$y = 0$	$f(x, y) = 0$	0	0
$9 - 3x - 2y$	$f(x, y) = 0$	0	0

the maximum of xyz is $\frac{9}{8}$ at the point $(1, \frac{3}{2}, \frac{3}{4})$. Thus the volume of the largest rectangular box is $\frac{9}{8}$. $\blacksquare$

12. Chapter 12

12.1. Exercises 12.1.

EXERCISE 12.1.1. *Compute* $\int_2^3 \int_4^5 (\frac{x}{y} + \frac{y}{x}) dydx = ?$

Sol.

$$\int_2^3 \int_4^5 (\frac{x}{y} + \frac{y}{x}) dydx = \int_2^3 [(x \ln y + \frac{1}{2x} y^2)\Big|_{y=4}^{y=5}] dx$$
$$= \int_2^3 (x \ln \frac{5}{4} + \frac{9}{2x}) dx = (\frac{x^2}{2} \ln \frac{5}{4} + \frac{9}{2} \ln x)\Big|_2^3 = \frac{9}{2} \ln \frac{5}{2}. \qquad \blacksquare$$

EXERCISE 12.1.2. *Compute* $\int_0^4 \int_1^3 \frac{xy}{\sqrt{x^2+y^2}} dydx = ?$

Sol. Let $u = x^2 + y^2, du = 2ydy.$

$$\int_0^4 \int_1^3 \frac{xy}{\sqrt{x^2+y^2}} dydx = \int_0^4 \int_{x^2+1}^{x^2+9} \frac{x}{2\sqrt{u}} dudx$$
$$= \int_0^4 [(xu^{\frac{1}{2}})\Big|_{y=x^2+1}^{y=x^2+9}] dx = \int_0^4 (x\sqrt{x^2+9} - x\sqrt{x^2+1}) dx.$$

Let $\begin{cases} v = x^2 + 9, dv = 2xdx \\ w = x^2 + 1, dw = 2xdx \end{cases},$

$$\int_0^4 (x\sqrt{x^2+9} - x\sqrt{x^2+1}) dx = \int_9^{25} \frac{1}{2} v^{\frac{1}{2}} dv - \int_1^{17} \frac{1}{2} w^{\frac{1}{2}} dw$$
$$= \frac{1}{3} v^{\frac{3}{2}}\Big|_9^{25} - \frac{1}{3} w^{\frac{3}{2}}\Big|_1^{17} = 33 - \frac{17^{\frac{3}{2}}}{3}. \qquad \blacksquare$$

EXERCISE 12.1.3. *Compute* $\int_0^2 \int_0^2 \sqrt{3x+4y} dydx = ?$

Sol.

$$\int_0^2 \int_0^2 (3x+4y)^{\frac{1}{2}} dydx = \int_0^2 [\frac{1}{6}(3x+4y)^{\frac{3}{2}}\Big|_{y=0}^{y=2}] dx$$
$$= \int_0^2 \frac{1}{6}[(3x+8)^{\frac{3}{2}} - (3x)^{\frac{3}{2}}] dx = \frac{2}{45}[(3x+8)^{\frac{5}{2}}\Big|_0^2 - (3x)^{\frac{5}{2}}\Big|_0^2]$$
$$= \frac{2}{45}(14^{\frac{5}{2}} - 8^{\frac{5}{2}} - 6^{\frac{5}{2}}). \qquad \blacksquare$$

EXERCISE 12.1.4. *Compute* $\int_0^1 \int_x^1 \sin x^2 dxdy = ?$

Sol.

$$\int_0^1 \int_x^1 \sin x^2 dydx = \int_0^1 \int_0^y \sin x^2 dydx$$
$$= \int_0^1 \sin x^2 \Big|_{y=0}^{y=x} dx = \int_0^1 x \sin x^2 dx.$$

Let $u = x^2, du = 2xdx$.

$$\int_0^1 x \sin x^2 dx = \int_0^1 \frac{1}{2} \sin u du = -\frac{1}{2} \cos u \Big|_0^1 = \frac{1}{2} - \frac{1}{2} \cos 1. \qquad \blacksquare$$

EXERCISE 12.1.5. *Compute* $\int_0^1 \int_0^1 xye^{x^2+y^2} dydx$ *= ?*

Sol.

$$\int_0^1 \int_0^1 xye^{x^2+y^2} dydx = \int_0^1 [(\frac{x}{2} e^{x^2+y^2})\Big|_{y=0}^{y=1}]dx$$
$$= \int_0^1 \frac{x}{2} e^{x^2+1} dx - \int_0^1 \frac{x}{2} e^{x^2} dx = \frac{1}{4} e^{x^2+1}\Big|_0^1 - \frac{1}{4} e^{x^2}\Big|_0^1$$
$$= \frac{1}{4}(e^2 - 2e + 1). \qquad \blacksquare$$

EXERCISE 12.1.6. *Compute* $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \sec y \cos(x+y) dydx$ *= ?*

Sol.

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \sec y \cos(x+y) dydx = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \sec y (\cos x \cos y - \sin x \sin y) dydx$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} (\cos x - \sin x \tan y) dydx = \int_0^{\frac{\pi}{2}} \{[y \cos x - \sin x(-\ln|\cos x|)]\Big|_{y=0}^{y=\frac{\pi}{4}}\} dx$$
$$= \int_0^{\frac{\pi}{2}} [\frac{\pi}{4} \cos x + \sin x (\ln \frac{1}{\sqrt{2}})] dx = \frac{\pi}{4} \sin x - \cos x (\ln \frac{1}{\sqrt{2}})\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4} + \ln \frac{1}{\sqrt{2}}. \qquad \blacksquare$$

EXERCISE 12.1.7. *Compute* $\int \int_D y dxdy$ *for*

$$D = \{(x, y) \mid x^2 + y^2 \leq 4,\ x \geq 0,\ y \geq 0\}.$$

Sol.

$$\int\int_D ydxdy = \int_0^2 \int_0^{\sqrt{4-x^2}} ydydx = \int_0^2 \frac{1}{2}y^2\Big|_{y=0}^{y=\sqrt{4-x^2}} dx$$
$$= \int_0^2 \frac{1}{2}(4-x^2)dx = (2x - \frac{1}{6}x^3)\Big|_0^2 = \frac{8}{3}. \qquad \blacksquare$$

12.2. Exercises 12.2.

EXERCISE 12.2.1. *Prove that the determinant* $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$.

Sol.

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$
$$= abc\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = abc[(b-a)(c^2-a^2)-(c-a)(b^2-a^2)]$$
$$= abc(b-a)(c-a)[(c+a)-(b+a)] = abc(a-b)(b-c)(c-a). \qquad \blacksquare$$

EXERCISE 12.2.2. $u = \frac{x}{x+y+z}, v = \frac{y}{x+y+z}, w = \frac{z}{x+y+z}$. *Compute the Jacobian detenminant.*

Sol.

$$J = \frac{1}{(x+y+z)^2}\begin{vmatrix} y+z & -y & -z \\ -x & x+z & -z \\ -x & -y & x+y \end{vmatrix}$$
$$= \frac{1}{(x+y+z)^2}\begin{vmatrix} 0 & -y & -z \\ 0 & x+z & -z \\ 0 & -y & x+y \end{vmatrix} = 0. \qquad \blacksquare$$

EXERCISE 12.2.3. $u = \frac{x}{\sqrt{x^2+y^2+z^2}}, v = \frac{y}{\sqrt{x^2+y^2+z^2}}, w = \frac{z}{\sqrt{x^2+y^2+z^2}}$. *Compute the Jacobian detenminant.*

Sol.

$$\begin{aligned}
J &= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\begin{vmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{vmatrix} \\
&= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}[(y^2+z^2)(x^2+z^2)(x^2+y^2) - 2x^2y^2z^2 \\
&\quad -x^2z^2(x^2+z^2) - y^2z^2(y^2+z^2) - x^2y^2(x^2+y^2)] \\
&= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(0) = 0. \qquad \blacksquare
\end{aligned}$$

EXERCISE 12.2.4. $u = \frac{1}{yz}, v = \frac{1}{xz}, w = \frac{1}{xy}$. *Compute the Jacobian detenminant.*

Sol.

$$\begin{aligned}
J &= \begin{vmatrix} 0 & \frac{-z}{(xz)^2} & \frac{-y}{(xy)^2} \\ \frac{-z}{(yz)^2} & 0 & \frac{-x}{(xy)^2} \\ \frac{-y}{(yz)^2} & \frac{-x}{(xz)^2} & 0 \end{vmatrix} \\
&= \frac{-xyz}{x^4y^4z^4} + \frac{-xyz}{x^4y^4z^4} = \frac{-2}{(xyz)^3}. \qquad \blacksquare
\end{aligned}$$

EXERCISE 12.2.5. $u = \sin(yz), v = \cos(xz), w = \tan(xy)$. *Compute the Jacobian detenminant.*

Sol.

$$\begin{aligned}
J &= \begin{vmatrix} 0 & -\sin(xz)\cdot z & \sec^2(xy)\cdot y \\ \cos(yz)\cdot z & 0 & \sec^2(xy)\cdot x \\ \cos(yz)\cdot y & -\sin(xz)\cdot x & 0 \end{vmatrix} \\
&= -2\sin(xz)\cos(yz)\sec^2(xy)xyz. \qquad \blacksquare
\end{aligned}$$

12.3. Exercises 12.3.

EXERCISE 12.3.1. *Compute* $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xydydx$ *= ?*

Sol.

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xydydx = \int_0^{2\pi}\int_0^3 (r^2\sin\theta\cos\theta)rdrd\theta$$
$$= \int_0^{2\pi}[(\frac{1}{4}r^4\sin\theta\cos\theta)\Big|_{r=0}^{r=3}]d\theta = \int_0^{2\pi}\frac{81}{4}\sin\theta\cos\theta d\theta$$
$$= \int_0^{2\pi}\frac{81}{4}\sin\theta d(\sin\theta) = \frac{81}{8}\sin^2\theta\Big|_0^{2\pi} = 0. \qquad \blacksquare$$

Exercise 12.3.2. *Compute* $\int_0^1\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}}(x^2+y^2)dydx$ *= ?*

Sol.
Since the region is bounded by $y^2 = x - x^2,\ 0 \le x \le 1$, which is inside the circle $(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$, if we let $u = x - \frac{1}{2},\ v = y$, then the region is inside the circle $u^2 + v^2 = 1$ and

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Then let $u = r\cos\theta,\ v = r\sin\theta$, we have

$$\int_0^1\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}}(x^2+y^2)dydx = \int_{-\frac{1}{4}}^{\frac{1}{4}}\int_{-\sqrt{\frac{1}{4}-u^2}}^{\sqrt{\frac{1}{4}-u^2}}((u+\frac{1}{2})^2+v^2)\cdot 1dvdu$$
$$= \int_0^{2\pi}\int_0^{\frac{1}{2}}(r^2 + r\cos\theta + \frac{1}{4})rdrd\theta = \int_0^{2\pi}[(\frac{1}{4}r^4 + \frac{1}{2}r^2\cos\theta + \frac{1}{8}r^2)\Big|_{r=0}^{r=\frac{1}{2}}]d\theta$$
$$= \int_0^{2\pi}(\frac{3}{64} + \frac{1}{8}\cos\theta)d\theta = (\frac{3}{64}\theta - \frac{1}{8}\sin\theta)\Big|_0^{2\pi} = \frac{3\pi}{32}. \qquad \blacksquare$$

Exercise 12.3.3. *Compute* $\int_0^2\int_0^x\sqrt{x^2+y^2}dydx$ *= ?*

Sol.
Since under the polar coordinates, the line $x = 2$ can be written by $r\cos\theta = 2$, that is, $r = 2\sec\theta$, and since the y-axis is $\theta = 0$ and the line $y = x$ is $\theta = \frac{\pi}{4}$, the region bounded the lines $x = 2,\ y = x$ and the y-axis is

$$\left\{(r\cos\theta, r\sin\theta) \mid 0 \le r \le 2\sec\theta,\ 0 \le \theta \le \frac{\pi}{4}\right\}.$$

Then by the solution of Exercise 7.3.8, we have

$$\int_0^2 \int_0^x \sqrt{x^2+y^2}dydx = \int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} r^2 drd\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{3}r^3\Big|_{r=0}^{r=2\sec\theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{8}{3}\sec^3\theta d\theta$$
$$= \frac{8}{3}(\frac{1}{2}\tan x\sec x + \frac{1}{2}\ln|\sec x+\tan x|)\Big|_0^{\frac{\pi}{4}}$$
$$= \frac{8}{3}(\frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left|\sqrt{2}+1\right|). \qquad \blacksquare$$

12.4. Exercises 12.4.

EXERCISE 12.4.1. *Compute* $\iiint\limits_{\Omega} e^{\left(x^2+y^2+z^2\right)^{3/2}} dxdydz$, *where*

$$\Omega = \left\{(x,y,z) \ \mid x^2+y^2+z^2 \le 1\right\}$$

.

Sol.
Since

$$\Omega = \{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \ \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\},$$

we have

$$\iiint\limits_{\Omega} e^{\left(x^2+y^2+z^2\right)^{3/2}} dxdydz = \int_{-1}^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} e^{\left(x^2+y^2+z^2\right)^{3/2}} dxdydz$$
$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{\left(\rho^2\right)^{3/2}} \rho^2 \sin\phi d\rho d\theta d\phi = \int_0^{\pi}\int_0^{2\pi}\left(\int_0^1 e^{\rho^3}\rho^2\sin\phi d\rho\right) d\theta d\phi$$
$$= \int_0^{\pi}\int_0^{2\pi}\left(\frac{1}{3}e^{\rho^3}\sin\phi\Big|_0^1\right) d\theta d\phi = \int_0^{\pi}\left(\int_0^{2\pi}\left(\frac{1}{3}e\sin\phi - \frac{1}{3}\sin\phi\right)d\theta\right)d\phi$$
$$= \int_0^{\pi}\left(\frac{1}{3}\sin\phi\,(e-1)\,\theta\Big|_0^{2\pi}\right)d\phi = \int_0^{\pi}\frac{2\pi}{3}\sin\phi\,(e-1)\,d\phi$$
$$= \frac{-2\pi}{3}\cos\phi\,(e-1)\Big|_0^{\pi} = \frac{4\pi}{3}\cos\phi\,(e-1). \qquad \blacksquare$$

EXERCISE 12.4.2. *Find the volumn of the solid bounded inside the sphere* $x^2+y^2+z^2=1$ *and bounded below by the cone* $z=\sqrt{x^2+y^2}$.

Sol.

Since under the spherical coordinates, the sphere $x^2+y^2+z^2=1$ can be written by $\rho=1$ and the cone $z=\sqrt{x^2+y^2}$ can be written by $\rho\cos\phi=\rho\sin\phi$, that is, $\tan\phi=1$, that is, $\phi=\frac{\pi}{4}$, the region is

$$\left\{(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\ \mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{4}\right\}$$

So the volume is

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 \rho^2\sin\phi d\rho d\theta d\phi &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\frac{1}{3}\rho^3\sin\phi\Big|_0^1 d\theta d\phi\\ =\quad \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\frac{1}{3}\sin\phi d\theta d\phi &= \int_0^{\frac{\pi}{4}}\frac{2\pi}{3}\sin\phi d\phi\\ =\quad -\frac{2\pi}{3}\cos\phi\Big|_0^{\frac{\pi}{4}} &= -\frac{\sqrt{2}\pi}{3}+\frac{2\pi}{3}. \qquad \blacksquare\end{aligned}$$

Exercise 12.4.3. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)^{\frac{1}{2}}\,dzdydx=?$

Sol.

Since the region is $\Omega=\left\{(x,y,z)\ \mid x^2+y^2+(z-2)^2\le 4\right\}$ and under the spherical coordinates,

$$\begin{aligned}4 \quad &= \quad x^2+y^2+(z-2)^2 = x^2+y^2+z^2-4z+4=\rho^2-4\rho\cos\phi+4\\ \Longleftrightarrow \quad & \rho^2=4\rho\cos\phi \Longleftrightarrow \rho=4\cos\phi,\end{aligned}$$

we can consider the region Ω by

$$\left\{(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\ \mid 0\le\rho\le 4\cos\phi,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{2}\right\}.$$

So

$$\begin{aligned}&\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}dzdydx\\ =\quad &\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{4\cos\phi}\left(\rho^2\right)^{\frac{1}{2}}\rho^2\sin\phi d\rho d\theta d\phi = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{4\cos\phi}\rho^3\sin\phi d\rho d\theta d\phi\\ =\quad &\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\frac{1}{4}\rho^4\sin\phi\Big|_{\rho=0}^{\rho=4\cos\phi}d\theta d\phi = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}64\cos^3\phi\sin\phi d\theta d\phi\\ =\quad &\int_0^{\frac{\pi}{2}}128\pi\cos^3\phi\sin\phi d\phi = \int_0^{\frac{\pi}{2}}-128\pi\cos^3\phi d(\cos\phi)\\ =\quad &-32\pi\cos^4\phi\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} = 32\pi. \qquad \blacksquare\end{aligned}$$

12.5. Exercises 12.5.

Exercise 12.5.1. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^3 xdzdydx = ?$

Sol.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^3 xdzdydx = \int_0^{\frac{\pi}{2}} \int_0^1 \int_{\sqrt{x^2+y^2}}^3 r\cos\theta \cdot rdzdrd\theta$$
$$= \int_0^{\frac{\pi}{2}} \int_0^1 zr^2 \cos\theta\Big|_{z=r}^{z=3} drd\theta = \int_0^{\frac{\pi}{2}} \int_0^1 \left(3r^2\cos\theta - r^3\cos\theta\right) drd\theta$$
$$= \int_0^{\frac{\pi}{2}} \left(r^3\cos\theta - \frac{1}{4}r^4\cos\theta\Big|_0^1 \right) d\theta = \int_0^{\frac{\pi}{2}} \left(\cos\theta - \frac{1}{4}\cos\theta\right) d\theta$$
$$= \frac{3}{4}\int_0^{\frac{\pi}{2}} \cos\theta d\theta = \frac{3}{4}\left(\sin\theta|_0^{\frac{\pi}{2}}\right) = \frac{3}{4}. \quad \blacksquare$$

Exercise 12.5.2. $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{9-x^2-y^2}} zdzdydx = ?$

Sol.

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{9-x^2-y^2}} zdzdydx = \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{9-r^2}} zrdzdrd\theta$$
$$= \int_0^{2\pi} \int_0^2 \frac{1}{2}z^2r\Big|_{z=0}^{z=\sqrt{9-r^2}} drd\theta = \int_0^{2\pi} \int_0^2 \frac{9r-r^3}{2} drd\theta$$
$$= \int_0^{2\pi} \left(\frac{9}{4}r^2 - \frac{1}{8}r^4\Big|_0^2\right) d\theta = \int_0^{2\pi} 7d\theta = 7\theta|_0^{2\pi} = 14\pi. \quad \blacksquare$$

Exercise 12.5.3. *Find the volumn of the solid bounded below by $z = x^2 + y^2$ and bounded inside the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 3$.*

Sol.
Since under the cylindrical coordinates, $z = x^2 + y^2$ can be written by $z = r^2$, and $x^2 + y^2 + \frac{z^2}{4} = 3$ can be written by $r^2 + \frac{z^2}{4} = 3$, and since these two graphs intersect

when $\begin{cases} z = 2, \\ r = \sqrt{2}, \end{cases}$ the volume is

$$\begin{aligned}
& \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{12-4r^2}} r dz dr d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} (\sqrt{12-4r^2} - r^2) r dr d\theta \\
= \ & \int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{12-4r^2} - r^3 dr d\theta = \int_0^{2\pi} \frac{-1}{12}(12-4r^2)^{\frac{3}{2}} - \frac{1}{4}r^4 \Big|_0^{\sqrt{2}} d\theta \\
= \ & \int_0^{2\pi} (\frac{-1}{12}(12-8)^{\frac{3}{2}} - \frac{4}{4}) - (\frac{-1}{12}(12)^{\frac{3}{2}}) d\theta \\
= \ & \int_0^{2\pi} (\frac{-8}{12} - 1 + \frac{24\sqrt{3}}{12}) d\theta = 2\pi(2\sqrt{3} - \frac{5}{3}). \qquad \blacksquare
\end{aligned}$$