CS5319 Advanced Discrete Structure

Exam 1 – November 01, 2022 (13:20–15:10)

Answer all six questions. Total marks = 105. Maximum score = 105/100.

1. How many sequence of length k that you can construct, such that the sequence satisfies all the following properties:

   - Numbers in the sequence are distinct, and they are integers chosen from $[1, 100]$.
   - All odd numbers appear at the beginning of the sequence, in increasing order.
   - All even numbers appear at the end of the sequence, in decreasing order.

   Example: When $k = 5$, $\langle 1, 3, 17, 8, 2 \rangle$ is a possible sequence.

   (a) (5%) Express your answer in terms of k, and explain why it is correct.

   (b) Suppose now we have an extra property to be satisfied:

   - There are more odd numbers than even numbers in the sequence.

   How many sequence of length k can we construct?

   (15%) Express your answer in terms of k, and explain why it is correct.

   Hint: Separate the discussion for the cases when k is odd and when k is even.

2. (20%) Simplify the following in terms of k, ℓ, m, n, and show why your answer is correct:

$$\sum_{i=1}^{k-1} \sum_{j=1}^{k-i} \binom{n}{i} \binom{m}{j} \binom{\ell}{k-i-j}$$

   Here, we assume that $\binom{x}{y} = 0$ whenever $y < 0$.

   Hints: Combinatorial argument, inclusion-exclusion principle

3. (20%) Find the coefficient of x^n in the following generating function:

$$\frac{1}{(x^2 - 2)(1 - 3x^2)}.$$

   Hint: Think a bit more before working on this problem

4. Consider the sequence

$$(a_0, \quad a_1, \quad a_2, \quad a_3, \quad \ldots \quad) \quad = \quad (1, \quad 2, \quad 2 \times 7, \quad 2 \times 7 \times 12, \quad \ldots \quad),$$

   where in general $a_r = 2 \times 7 \times \cdots \times (5r - 3)$ for $r \geq 1$.

   (15%) Give the EGF for the sequence (in the simplest form) and show why it is correct.

5. Let n be a positive integer. Let a_n denote the number of ways to partition n into exactly p integers, for some prime p. Let b_n denote the number of ways to partition n, whose largest part is a prime number.

   (15%) Show that $a_n = b_n$.

   Example: Consider $n = 5$. The following are the ways to partition n into a prime number of integers:

   $$\{1,4\}, \quad \{2,3\}, \quad \{1,1,3\}, \quad \{1,2,2\}, \quad \{1,1,1,1,1\}$$

   so that $a_5 = 5$. In contrast, we can partition n so that the largest part is a prime number:

   $$\{5\}, \quad \{2,3\}, \quad \{1,1,3\}, \quad \{1,2,2\}, \quad \{1,1,1,2\}$$

   so that $b_5 = 5$.

6. Given a permutation of $1, 2, \ldots, n$, we can associate it with a *change vector*, which describes the changes between adjacent terms in the permutation by $+$ or $-$, when the value is increased or decreased, respectively.

   Example: When $n = 5$ and the permutation is $\langle 3,4,1,5,2 \rangle$, the change vector would be: $\langle +,-,+,- \rangle$. Here, the leftmost $+$ is due to the increase of value from 3 to 4 in the permutation, and the following $-$ is due to the decrease of value from 4 to 1 in the permutation.

   Example: Note that a different permutation may also share the same change vector. For instance, $\langle 3,5,2,4,1 \rangle$ would have the same change vector as $\langle +,-,+,- \rangle$.

   (15%) For all permutations of $1, 2, \ldots, n$, how many different possible change vectors are there? Express your answer in terms of n, and explain why it is correct.