

排序演算法

合併排序、快速排序……

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1 序

- Bubble Sort

2 原理

- Merge Sort
 - Inversion
- Quicksort
 - Quickselect

3 實務

- `std::sort()`

Section 1

序

- 1 序
 - Bubble Sort
- 2 原理
 - Merge Sort
 - Quicksort
- 3 實務

Introduction to Sorting

Definition (Sort)

Rearrange elements in an array into a sort of order.

- Monotonicity
- Permutation of original array

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Definition (Sort)

Rearrange elements in an array into a sort of order.

- Monotonicity
- Permutation of original array

Reason

- Ranking
- Prerequisite of other algorithms such as binary search, greedy, ...

Naïve Approach: Bubble Sort

```
procedure BUBBLE SORT( $\{a_0, a_1, \dots, a_{n-1}\}$ )  
  for  $i \in [0, n - 1)$  do  
    for  $j \in [0, n - 1 - i)$  do  
      if  $a_j > a_{j+1}$  then  
        SWAP( $a_j, a_{j+1}$ )  
      end if  
    end for  
  end for  
  return  $a$   
end procedure
```

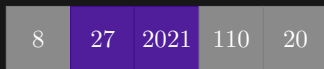
Example

8	27	2021	110	20
---	----	------	-----	----

Example

8	27	2021	110	20
---	----	------	-----	----

Example



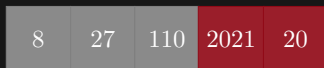
Example



Example

8	27	110	2021	20
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Example



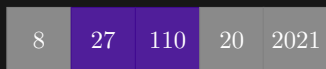
Example

8	27	110	20	2021
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Example

8	27	110	20	2021
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Example



Example

8	27	110	20	2021
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Example

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Example



Example



Example

8	20	27	110	2021
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Section 2

原理

- 1 序
 - Bubble Sort

- 2 原理
 - Merge Sort
 - Inversion
 - Quicksort
 - Quickselect

- 3 實務

分（而）治（之）法 Divide & Conquer

分治三部曲：

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分治三部曲：

Divide Split original problem into several subproblems

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Conquer Recur to each subproblem until it could be easily solved

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Combine Merge the results of subproblems

分（而）治（之）法 Divide & Conquer

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Divide Split original problem into several subproblems

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時間複雜度：Master Theorem

分（而）治（之）法 Divide & Conquer

分治三部曲：

Divide Split original problem into several subproblems

Conquer Recur to each subproblem until it could be easily solved

Combine Merge the results of subproblems

時間複雜度：Master Theorem

$$T(n) = aT\left(\frac{n}{b}\right) + O(n^d) = \begin{cases} O(n^d), & d > \log_b a \\ O(n^d \log n), & d = \log_b a \\ O(n^{\log_b a}), & d < \log_b a \end{cases}$$

合併排序

Divide Split array into *left* and *right* subarrays ($b = 2, O(1)$)

合併排序

Divide Split array into *left* and *right* subarrays ($b = 2, O(1)$)

Conquer Sort two subarrays recursively ($a = 2$)

合併排序

Divide Split array into *left* and *right* subarrays ($b = 2, O(1)$)

Conquer Sort two subarrays recursively ($a = 2$)

Combine Merge two sorted subarrays in $O(n)$

合併

Given two sorted subarrays *left*, *right*, how could we combine them to a sorted array efficiently??

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If **itr** \neq the end of *left* \wedge **jtr** \neq the end of *right*, then we choose the min one and forward the pointer.

合併

Given two sorted subarrays $left, right$, how could we combine them to a sorted array efficiently??

Let itr , jtr point to the begin of $left, right$ respectively.

If $itr \neq \text{the end of } left \wedge jtr \neq \text{the end of } right$, then we choose the min one and forward the pointer.

Else if $itr \neq \text{the end of } left \vee jtr \neq \text{the end of } right$, then we choose the pointer and forward it.

合併

Given two sorted subarrays $left, right$, how could we combine them to a sorted array efficiently??

Let itr , jtr point to the begin of $left, right$ respectively.

If $itr \neq \text{the end of } left \wedge jtr \neq \text{the end of } right$, then we choose the min one and forward the pointer.

Else if $itr \neq \text{the end of } left \vee jtr \neq \text{the end of } right$, then we choose the pointer and forward it.

Repeat until $itr = \text{the end of } left \wedge jtr = \text{the end of } right$.

虛擬碼

```
procedure MERGE SORT(*begin, *end{})
```

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```
  if end - begin = 1 then
```

```
    return
```

```
  end if
```

▷ 0. Recursion boundary

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   $mid \leftarrow \frac{begin+end}{2}$ ,  $left \leftarrow [begin, mid)$ ,  $right \leftarrow [mid, end)$  ▷ 1. Divide
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procedure MERGE SORT(*begin, *end{})
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   $mid \leftarrow \frac{begin+end}{2}$ ,  $left \leftarrow [begin, mid)$ ,  $right \leftarrow [mid, end)$  ▷ 1. Divide
```

```
  MERGE SORT( $left.begin()$ ,  $left.end()$ )
```

▷ 2. Conquer

```
  MERGE SORT( $right.begin()$ ,  $right.end()$ )
```


虛擬碼

```
procedure MERGE SORT(*begin, *end{})  
  if end - begin = 1 then                                ▷ 0. Recursion boundary  
    return  
  end if  
   $mid \leftarrow \frac{begin+end}{2}$ ,  $left \leftarrow [begin, mid)$ ,  $right \leftarrow [mid, end)$  ▷ 1. Divide  
  MERGE SORT( $left.begin()$ ,  $left.end()$ )                    ▷ 2. Conquer  
  MERGE SORT( $right.begin()$ ,  $right.end()$ )  
   $itr \leftarrow left.begin()$ ,  $jtr \leftarrow right.begin()$   
  while  $begin \neq end$  do                                  ▷ 3. Combine  
    if  $itr \neq left.end() \wedge (jtr = right.end() \vee *itr < *jtr)$  then  
       $*begin++ \leftarrow *itr++$   
    else  
       $*begin++ \leftarrow *jtr++$   
    end if  
  end while  
end procedure
```

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8	27	2021	110	20
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Example

8

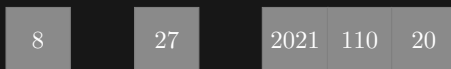
27

2021

110

20

Example



Example

8

27

2021

110

20

Example

8

27

2021

110

20

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8

27

2021

110

20

Example

8

27

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20

110

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8

27

20

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類題演練

- AtCoder Beginner Contest 154 p. C - Distinct or Not
- LeetCode 0004. Median of Two Sorted Arrays (Hard!?)

Distinct or Not

AtCoder Beginner Contest 154 p. C

Problem

Determine if each element in the array is unique.

Distinct or Not

AtCoder Beginner Contest 154 p. C

Problem

Determine if each element in the array is unique.

Idea

Use nested loop??

Distinct or Not

AtCoder Beginner Contest 154 p. C

Problem

Determine if each element in the array is unique.

Idea

Use nested loop??

Algorithm

In **combine** process of MERGE SORT, we could check whether ***itr == *jtr**.

Median of Two Sorted Arrays

LeetCode 0004 (Hard!?)

Definition (Median)

For an array a whose size is n , if n is odd, then its median is $a_{n/2}$; otherwise, it is $\frac{a_{n/2} + a_{n/2-1}}{2}$

Median of Two Sorted Arrays

LeetCode 0004 (Hard!?)

Definition (Median)

For an array a whose size is n , if n is odd, then its median is $a_{n/2}$; otherwise, it is $\frac{a_{n/2} + a_{n/2-1}}{2}$

Algorithm

Obviously, all we have to do is to perform **combine** process of MERGE SORT.

逆序對 (Inversion) 數量

Definition (Inversion)

Given a sequence S . If $i < j \iff S_i > S_j$, then (i, j) or (S_i, S_j) is called a **inversion** of S .

逆序對 (Inversion) 數量

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Idea

$\forall i \in [0, |S|), \forall j \in [i, |S|)$, count whether $i < j \iff S_i > S_j??$

逆序對 (Inversion) 數量

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Idea

$\forall i \in [0, |S|), \forall j \in [i, |S|)$, count whether $i < j \iff S_i > S_j$??

Algorithm

When **combining** two sorted subarrays, if $*itr > *jtr$, then a **inversion** exists, and $*(itr + 1), \dots$ are also greater than $*jtr$ with a smaller index.

類題演練

- AtCoder Beginner Contest 190 p. F - Shift and Inversions
- UVa 10810: Ultra-QuickSort
- UVa 11858: Frosh Week

Shift and Inversions

AtCoder Beginner Contest 190 p. F

- 1 $v = \{0, 1, 2, 3, 4\}$, there are 0 inversions.

Shift and Inversions

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Shift and Inversions

AtCoder Beginner Contest 190 p. F

- ① $v = \{0, 1, 2, 3, 4\}$, there are 0 inversions.
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- ③ $v = \{2, 3, 4, 0, 1\}$, there are 6 inversions:
 $(2, 0), (3, 0), (4, 0), (2, 1), (3, 1), (4, 1)$

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- ④ $v = \{3, 4, 0, 1, 2\}$, there are 6 inversions:
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Shift and Inversions

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- ❺ $v = \{4, 0, 1, 2, 3\}$, there are 4 inversions: $(4, 0), (4, 1), (4, 2), (4, 3)$

Shift and Inversions

AtCoder Beginner Contest 190 p. F

- 1 $v = \{0, 1, 2, 3, 4\}$, there are 0 inversions.
- 2 $v = \{1, 2, 3, 4, 0\}$, there are 4 inversions: $(1, 0), (2, 0), (3, 0), (4, 0)$
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- 4 $v = \{3, 4, 0, 1, 2\}$, there are 6 inversions:
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- 5 $v = \{4, 0, 1, 2, 3\}$, there are 4 inversions: $(4, 0), (4, 1), (4, 2), (4, 3)$
- When removing a from the front of the array, the number of inversion would decrease by a for a is the a -th smallest element in the array.

Shift and Inversions

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 - ❺ $v = \{4, 0, 1, 2, 3\}$, there are 4 inversions: $(4, 0), (4, 1), (4, 2), (4, 3)$
- When removing a from the front of the array, the number of inversion would decrease by a for a is the a -th smallest element in the array.
 - In like manner, when adding a to the back of the array, the number would increase by $N - 1 - a$ because a is less than $N - 1 - a$ elements in the array.

快速排序

Divide Select a *pivot* value and separate the array into *less* partition and *greater* partition ($b = 2, O(n)$)

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Combine Nothing to do, the array have been sorted. ($O(1)$)

原地分割 (In-place Division)

How could we separate the array into *less* partition and *greater* partition??

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Initially, $itr \leftarrow begin$, $pivot \leftarrow end - 1$.

We traverse from *begin* through *pivot* with *jtr*.

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Suppose two slices are $[begin, itr)$ and $[itr + 1, end)$.

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We traverse from *begin* through *pivot* with *jtr*.

If $*jtr < *pivot$, then we put it to $*itr$ and forward *itr*.

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How could we separate the array into *less* partition and *greater* partition??

Suppose two slices are $[begin, itr)$ and $[itr + 1, end)$.

Initially, $itr \leftarrow begin$, $pivot \leftarrow end - 1$.

We traverse from *begin* through *pivot* with *jtr*.

If $*jtr < *pivot$, then we put it to $*itr$ and forward *itr*.

Finally, swap $*itr$ and $*pivot$.

Pivot

The choice of *pivot* plays a significant role when it comes to the efficiency of QUICKSORT.

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Nevertheless, fixed *pivot* may run into troubles because if unfortunately *pivot* is minimum or maximum every time, the time complexity would decline to $O(n^2)$.

Pivot

The choice of *pivot* plays a significant role when it comes to the efficiency of QUICKSORT.

In classical textbook “Introduction to Algorithm”, C., S., L., R. chose the last element to be *pivot*.

Nevertheless, fixed *pivot* may run into troubles because if unfortunately *pivot* is minimum or maximum every time, the time complexity would decline to $O(n^2)$.

With an eye to avoiding this, we can pick *pivot* randomly or use the median of $\{*\textit{begin}, *(\frac{\textit{begin} + \textit{end}}{2}), *(\textit{end} - 1)\}$.

虛擬碼

```
procedure QUICKSORT( $\{ *begin, *end \}$ )
```

虛擬碼

```
procedure QUICKSORT( $\{ *begin, *end \}$ )  
  if  $end - begin \leq 1$  then  
    return  
  end if
```

▷ 0. Recursion boundary

虛擬碼

```
procedure QUICKSORT( $\{*begin, *end\}$ )  
  if  $end - begin \leq 1$  then  
    return  
  end if  
   $itr \leftarrow begin, pivot \leftarrow end - 1$   
  for  $jtr \leftarrow begin; jtr \neq pivot; jtr++$  do  
    if  $*jtr < *pivot$  then  
      SWAP( $*itr++$ ,  $*jtr$ )  
    end if  
  end for  
  SWAP( $*itr$ ,  $*pivot$ )
```

▷ 0. Recursion boundary

▷ 1. Divide

虛擬碼

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procedure QUICKSORT( $\{*begin, *end\}$ )
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  if  $end - begin \leq 1$  then
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```

```
    if  $*jtr < *pivot$  then
```

```
      SWAP( $*itr++$ ,  $*jtr$ )
```

```
    end if
```

```
  end for
```

```
  SWAP( $*itr$ ,  $*pivot$ )
```

```
  QUICKSORT( $begin, itr$ )
```

```
  QUICKSORT( $itr + 1, end$ )
```

▷ 0. Recursion boundary

▷ 1. Divide

▷ 2. Conquer

虛擬碼

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procedure QUICKSORT( $\{ *begin, *end \}$ )
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```

```
    if  $*jtr < *pivot$  then
```

```
      SWAP( $*itr++$ ,  $*jtr$ )
```

```
    end if
```

```
  end for
```

```
  SWAP( $*itr$ ,  $*pivot$ )
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```
  QUICKSORT( $begin, itr$ )
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```
  QUICKSORT( $itr + 1, end$ )
```

```
end procedure
```

▷ 0. Recursion boundary

▷ 1. Divide

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▷ 3. Combine

Example

8	27	2021	110	20
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Example



Example

8

20

27

110

2021

Example



Example

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快速選擇 Quickselect

Problem

Find the k -th element in the array.

快速選擇 Quickselect

Problem

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Idea

Sort the array??

快速選擇 Quickselect

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Algorithm

When **dividing** array into *less* and *greater* parts in QUICKSORT, we could know the size of them. If $k < |less|$, we only need to recur to less part and greater one could be ignored; vice versa.

Note that if we recur to greater part, we should update k to $k - |less| - 1$ because the k -th element in the original array is $k - |less| - 1$ -th element in the greater part.

類題演練

- LeetCode 0215. Kth Largest Element in an Array (Medium)
- AtCoder Beginner Contest 161 p. C - Popular Vote

Kth Largest Element in an Array

LeetCode 0215 (Medium)

非常單純的模板題，我們以此來示範 *pivot* 之選擇。

Always last 40ms, beats 17.10% submissions

Randomly 8ms, faster than 75.44% codes

Section 3

實務

- 1 序
 - Bubble Sort
- 2 原理
 - Merge Sort
 - Quicksort
- 3 實務
 - `std::sort()`

std::merge()

```
void merge_sort(vector<int>::iterator begin,
               vector<int>::iterator end)
{
    // 0. recursion boundary
    if (end - begin == 1)
        return;
    // 1. Divide
    auto mid = begin + ((end - begin) >> 1); // equal to
        'begin+(end-begin)/2'
    vector<int> left(begin, mid), right(mid, end);
    // 2. Conquer
    merge_sort(left.begin(), left.end());
    merge_sort(right.begin(), right.end());
    // 3. Combine
    merge(left.begin(), left.end(), right.begin(),
          right.end(), begin);
}
```

std::partition()

```
int pivot;  
bool cmp_partition(int x) { return x < pivot; }  
void quicksort(vector<int>::iterator begin,  
               vector<int>::iterator end)  
{  
    // 0. recursion boundary  
    if (end - begin <= 1)  
        return;  
    // 1. Divide  
    pivot = *(end - 1);  
    auto itr = partition(begin, end - 1, cmp_partition);  
    swap(*itr, *(end - 1));  
    // 2. Conquer  
    quicksort(begin, itr);  
    quicksort(itr + 1, end);  
    // 3. Combine  
    // Nothing to do  
}
```

std::nth_element()

```
nth_element(v.begin(), v.begin() + 2, v.end());  
cout << "\n2-nd element (0-indexed) is " <<  
      *(v.begin() + 2) << '\n';
```

std::sort()

```
sort(v.begin(), v.end());  
for (const int &i : v)  
    cout << '□' << i;
```

`std::sort()` with `std::string` and other...

```
string s = "hello_world_QWERTY_QAZ_QSC_QQ_wasd";  
cout << '\n' + s << '\n';  
sort(s.begin(), s.end());  
cout << s << '\n';
```

std::sort() using custom comparator

```

bool cmp_sort(int lhs, int rhs) // result be like: {1,
    3, 5, 4, 2, 0}
{
    if (lhs & 1 ^ rhs & 1) // if l%2 != r%2
        return lhs & 1; // return true if l is odd
            otherwise return false
    else
        return lhs & 1 ? lhs < rhs : lhs > rhs; // if l,
            r are both odd then return l<r else return
            l>r
}

// ...

sort(v.begin(), v.end(), cmp_sort);
for (const int &i : v)
    cout << '□' << i;

```