AI353: Introduction to Quantum Computing

Tutorial Week 4 — Solutions

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Q1. Mean and Standard Deviation for an Observable

Claim. For a system in state $|\psi\rangle$ and an observable M with spectral decomposition $M = \sum_m mP_m$ (with projectors P_m), the average outcome is $\langle\psi|M|\psi\rangle$ and the standard deviation is

$$\sigma(M) = \sqrt{\langle\psi|M^2|\psi\rangle - \langle\psi|M|\psi\rangle^2}.$$

Proof (step by step).

1. By the postulates, upon measuring M the outcome m occurs with probability $p_m = \langle\psi|P_m|\psi\rangle$.
2. The average (expectation) is $\mathbb{E}[M] = \sum_m m\, p_m = \sum_m m \langle\psi|P_m|\psi\rangle = \langle\psi|\sum_m mP_m|\psi\rangle = \langle\psi|M|\psi\rangle$.
3. The second moment is $\mathbb{E}[M^2] = \sum_m m^2 p_m = \langle\psi|M^2|\psi\rangle$ (since $M^2 = \sum_m m^2 P_m$).
4. The variance is $\mathrm{Var}(M) = \mathbb{E}[M^2] - \mathbb{E}[M]^2 = \langle\psi|M^2|\psi\rangle - \langle\psi|M|\psi\rangle^2$; the standard deviation is its square root.

Q2. Measuring an Eigenstate

Suppose $|\psi\rangle$ is an eigenstate of M with eigenvalue m. Then, $M|\psi\rangle = m|\psi\rangle$

- $\mathbb{E}[M] = \langle\psi| M |\psi\rangle = \langle\psi| m |\psi\rangle = m$ and $\sigma(M) = \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{m^2 - m^2} = 0$
- Therefore, $\mathbb{E}[M] = m$ and $\sigma(M) = 0$.

Q3. Measure X on $|0\rangle$

Since $X = \sigma_x$ is the Pauli operator with $X^2 = I$, then:

$$\begin{aligned}
\langle X\rangle &= \langle 0|X|0\rangle = 0, \\
\langle X^2\rangle &= \langle 0|I|0\rangle = 1, \\
\sigma(X) &= \sqrt{\langle 0|X^2|0\rangle - \langle X\rangle^2} = \sqrt{1-0} = 1.
\end{aligned}$$

Q4. Eigenvalues and projectors of $\vec{v} \cdot \vec{\sigma}$

Let $\vec{v} = (v_1, v_2, v_3)$ be a real unit vector ($v_1^2 + v_2^2 + v_3^2 = 1$). Using

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we write

$$A := \vec{v} \cdot \vec{\sigma} = v_1\sigma_x + v_2\sigma_y + v_3\sigma_z = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}.$$

Note that A is Hermitian (entries on the diagonal are real and the off-diagonals are conjugates).

Eigenvalues via characteristic polynomial. Compute $\det(A - \lambda I)$:

$$\det \begin{pmatrix} v_3 - \lambda & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - \lambda \end{pmatrix} = (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2).$$

Since $(v_1 - iv_2)(v_1 + iv_2) = v_1^2 + v_2^2$, we get

$$\det(A - \lambda I) = -(v_3^2 - \lambda^2) - (v_1^2 + v_2^2) = \lambda^2 - (v_1^2 + v_2^2 + v_3^2) = \lambda^2 - 1.$$

Thus the eigenvalues are $\boxed{\lambda_\pm = \pm 1}$.

A convenient pair of eigenvectors. Solve $(A - I)\,|u_+\rangle = 0$ and $(A + I)\,|u_-\rangle = 0$. For $\lambda = +1$ we can take

$$|u_+\rangle \propto \begin{pmatrix} v_1 - iv_2 \\ 1 + v_3 \end{pmatrix} \quad \text{(any nonzero scalar multiple works).}$$

For $\lambda = -1$ we can take

$$|u_-\rangle \propto \begin{pmatrix} v_1 - iv_2 \\ -1 + v_3 \end{pmatrix}.$$

(Students can check these by direct substitution; normalize if desired.)

Projectors: two equivalent ways. *(i) From eigenvectors.* If $|v_\pm\rangle$ denote the normalized versions of $|u_\pm\rangle$, then

$$P_\pm = |v_\pm\rangle\langle v_\pm|$$

are the orthogonal projectors onto the ± 1 eigenspaces.

(ii) Direct 2×2 formula. Since A is Hermitian with eigenvalues ± 1, its minimal polynomial is $x^2 - 1$, hence $A^2 = I$. The spectral projectors are then

$$\boxed{P_\pm = \frac{I \pm A}{2} = \frac{1}{2}(I \pm \vec{v} \cdot \vec{\sigma}).}$$

(Students can verify $P_\pm^2 = P_\pm$, $P_+P_- = 0$, and $P_+ + P_- = I$ by direct matrix multiplication.)

Summary. With $A = \left(\begin{smallmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{smallmatrix}\right)$ and $\vec{v}$ unit:

$$\operatorname{spec}(A) = \{+1, -1\}, \qquad P_\pm = \frac{I \pm A}{2} = \frac{I \pm \vec{v} \cdot \vec{\sigma}}{2}.$$

Q5. Probability and Post-Measurement State for Outcome $+1$

The probability for $+1$ is

$$p(+1) = \langle 0|P_+|0\rangle = \frac{1}{2}\Big(1 + \langle 0|\vec{v}\cdot\vec{\sigma}|0\rangle\Big) = \frac{1}{2}(1+v_3),$$

since $\langle 0|\sigma_x|0\rangle = \langle 0|\sigma_y|0\rangle = 0$ and $\langle 0|\sigma_z|0\rangle = 1$. The (normalized) post-measurement state is

$$|\psi_+\rangle = \frac{P_+\,|0\rangle}{\sqrt{p(+1)}} = \frac{(\mathbb{I}+\vec{v}\cdot\vec{\sigma})\,|0\rangle}{\sqrt{2(1+v_3)}} = \frac{(1+v_3)\,|0\rangle + (v_1+iv_2)\,|1\rangle}{\sqrt{2(1+v_3)}},$$

valid when $v_3 \neq -1$ (if $v_3 = -1$, the probability is 0 and this branch never occurs).

Q6. (Optional) No perfect discrimination of non-orthogonal states.

Refer to the Book.

Q7. When $M_m = E_m$, the measurement is projective

Solved in the class. Please refer to your notes.

Q8. Factorization $M_m = U_m\sqrt{E_m}$

Solved in the class. Please refer to your notes.

Q9. Unambiguous discrimination for linearly independent states

Solved in the class. Please refer to your notes.

Q10. Expressing $|\pm\rangle$ in a different basis

Solved in the class. Please refer to your notes.

Q11. Bell basis is orthonormal (and complete)

Define

$$\left|\Phi^\pm\right\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \qquad \left|\Psi^\pm\right\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

Orthonormality. Each has unit norm, and distinct pairs are orthogonal, e.g., $\langle\Phi^+|\Phi^-\rangle = 0$, $\langle\Phi^\pm|\Psi^\pm\rangle = 0$, $\langle\Psi^+|\Psi^-\rangle = 0$ (verify by expansion). **Completeness.** The four projectors sum to identity on $\mathbb{C}^2\otimes\mathbb{C}^2$: $\sum_{\beta\in\{\Phi^\pm,\Psi^\pm\}} |\beta\rangle\langle\beta| = \mathbb{I}_4$. Hence they form an orthonormal basis.

Q12. (Optional) Eve learns nothing from Alice's qubit in superdense coding

Discussed in the class. Please refer to your notes.