AI353: Introduction to Quantum Computing

Tutorial Sheet - Week 3

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Instructions

This tutorial focuses on the topic **Linear Algebra and Postulates of Quantum Mechanics**, as covered in lectures. You are encouraged to work in pairs for solving these problems. Active participation during the tutorial session is rewarded: volunteers who solve and present a problem on the whiteboard will receive points.

Points Distribution:

- Attendance: 15 points
- Active participation: 10 points
- **Total: 25 points**

Q1. What is the Gram–Schmidt procedure? Write down the procedure. (Refer to Nielsen & Chuang, Section 2.1.4)

Q2. (Optional) Prove that the Gram–Schmidt procedure produces an orthonormal basis.

Q3. Discuss the Cauchy–Schwarz inequality. (Refer to Nielsen & Chuang, Box 2.1)

Q4. Prove that $U(t_1, t_2)$ defined as

$$U(t_1, t_2) = \exp\left(-\frac{iH(t_2 - t_1)}{\hbar}\right)$$

is unitary. Here, H is the Hamiltonian as described in Postulate $2'$.

Q5. Use the spectral decomposition to show that $K \equiv -i\log(U)$ is Hermitian for any unitary U, and thus $U = \exp(iK)$ for some Hermitian K.

Q6. Suppose $\{L_l\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement defined by $\{L_l\}$ followed by one defined by $\{M_m\}$ is physically equivalent to a single measurement defined by operators $\{N_{lm}\}$ with $N_{lm} = M_m L_l$.

Q7. Show that the eigenvalues of a projector P are all either 0 or 1.

Q8. Show that a positive operator is necessarily Hermitian.

Q9. Basis change: Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases $\{|v_i\rangle\}$ and $\{|w_i\rangle\}$. Derive the relationship between A' and A''.

Q10. Show that any projector P satisfies the equation $P^2 = P$.

Q11. The Hadamard gate is given by:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Verify that $H^2 = I$ and that H is unitary. Find the eigenvalues and eigenvectors of H.

Q12. (Eigendecomposition of the Pauli matrices) Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y ,and Z.