AI353: Introduction to Quantum Computing

Tutorial Solution - Week 3

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Instructions

This tutorial focuses on the topic **Linear Algebra and Postulates of Quantum Mechanics**, as covered in lectures. You are encouraged to work in pairs for solving these problems. Active participation during the tutorial session is rewarded: volunteers who solve and present a problem on the whiteboard will receive points.

Points Distribution:

- Attendance: 15 points
- Active participation: 10 points
- **Total: 25 points**

Q1. Let $\{|w_1\rangle, \dots, |w_d\rangle\}$ be a basis of a finite-dimensional inner-product space V. The Gram–Schmidt procedure constructs vectors $\{|v_1\rangle, \dots, |v_d\rangle\}$ as follows:

$$|v_1\rangle := \frac{|w_1\rangle}{\| \, |w_1\rangle \, \|},$$

$$|u_{k+1}\rangle := |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle \, |v_i\rangle, \quad 1 \le k \le d-1,$$

$$|v_{k+1}\rangle := \frac{|u_{k+1}\rangle}{\| \, |u_{k+1}\rangle \, \|}.$$

We prove that $\{|v_1\rangle, \dots, |v_d\rangle\}$ is an orthonormal basis for V in the next question.

Q2. Proof:

Let $W_k = \text{span}\{|w_1\rangle, \dots, |w_k\rangle\}$ and $V_k = \text{span}\{|v_1\rangle, \dots, |v_k\rangle\}$.

Base Case ($k = 1$):

$|v_1\rangle$ is normalized by construction, so $\{|v_1\rangle\}$ is orthonormal. Moreover $V_1 = \text{span}\{|v_1\rangle\} = \text{span}\{|w_1\rangle\} = W_1$.

Inductive Step:

Assume for some $k \ge 1$ that $\{|v_1\rangle, \dots, |v_k\rangle\}$ is orthonormal and $V_k = W_k$. We show the claim for $k + 1$.

(a) Orthogonality of $|u_{k+1}\rangle$ For $1 \le j \le k$,

$$\begin{aligned}\langle v_j|u_{k+1}\rangle &= \langle v_j|w_{k+1}\rangle - \sum_{i=1}^{k}\langle v_i|w_{k+1}\rangle\langle v_j|v_i\rangle \\ &= \langle v_j|w_{k+1}\rangle - \langle v_j|w_{k+1}\rangle \\ &= 0,\end{aligned}$$

since $\langle v_j|v_i\rangle = \delta_{ji}$. Hence $|u_{k+1}\rangle \perp V_k$.

(b) Non-zeroness If $|u_{k+1}\rangle = 0$, then

$$|w_{k+1}\rangle = \sum_{i=1}^{k}\langle v_i|w_{k+1}\rangle\,|v_i\rangle \in V_k = W_k,$$

which contradicts the linear independence of $\{|w_i\rangle\}$. Thus $|u_{k+1}\rangle \neq 0$ and $|v_{k+1}\rangle$ is well-defined and normalized.

(c) Extended orthonormality $|v_{k+1}\rangle$ is orthogonal to each $|v_j\rangle$ for $j \le k$ and has norm 1, so $\{|v_1\rangle, \ldots, |v_{k+1}\rangle\}$ is orthonormal.

(d) Equality of spans From the definition,

$$\begin{aligned}|w_{k+1}\rangle &= |u_{k+1}\rangle + \sum_{i=1}^{k}\langle v_i|w_{k+1}\rangle\,|v_i\rangle \\ &\in \operatorname{span}\{|u_{k+1}\rangle, V_k\} = \operatorname{span}\{|v_{k+1}\rangle, V_k\} = V_{k+1}.\end{aligned}$$

Thus $W_{k+1} \subseteq V_{k+1}$. The reverse inclusion is clear since each $|v_i\rangle \in W_i \subseteq W_{k+1}$, hence $V_{k+1} \subseteq W_{k+1}$. Therefore $V_{k+1} = W_{k+1}$.

Conclusion

By induction, $\{|v_1\rangle, \ldots, |v_d\rangle\}$ is orthonormal and spans $W_d = V$. Therefore, the Gram–Schmidt procedure produces an orthonormal basis for V. □

Q3. Refer to Nielsen and Chunag, Box 2.1

Q4. Solved and shown in the lecture.

Q5. You are supposed to use spectral decomposition instead of directly using the identity of the Hermitian of an operator function.
Write $U = \sum_j \lambda_j |j\rangle\langle j|$ such that $\lambda_j \in \mathbb{C}$ be the spectral decomposition of U. If U is unitary, then

$$U^\dagger = \sum_j \lambda_j * |j\rangle\langle j|$$
$$U^\dagger U = \sum_j |\lambda_j|^2 |j\rangle\langle j| = I$$

By completeness relation $\sum_j |j\rangle\langle j| = I$ and therefore,

$$U^\dagger U = \sum_j |\lambda_j|^2 |j\rangle\langle j| = \sum_j |j\rangle\langle j|$$
$$\implies |\lambda_j| = 1 \implies \lambda_j = e^{i\theta} \text{ for some real } \theta.$$

Hence,

$$\begin{aligned} U &= \sum_j e^{i\theta} |j\rangle\langle j| \\ \log U &= \sum_j i\theta |j\rangle\langle j| \\ \implies K = -i \log U &= \sum_j -i^2 \theta |j\rangle\langle j| \\ &= \sum_j \theta |j\rangle\langle j| \implies K^\dagger = K. \end{aligned}$$

Therefore, K is Hermitian and $U = \exp(iK)$.

Q6. Discussed in the lecture. Please turn back your notes.

Q7. As is known that for any projector P,

$$P^2 = P \text{ (Please refer to the proof in Q.10)}$$

Then, let λ be the eigenvalues of P with corresponsing eigenvector $|\nu\rangle$ such that $P|\nu\rangle = \lambda|\nu\rangle$. Because of the above result, we can say the following

$$\begin{aligned} P^2|\nu\rangle &= \lambda P|\nu\rangle \\ &= \lambda^2|\nu\rangle \end{aligned}$$

Also, because, $P^2|\nu\rangle = P|\nu\rangle = \lambda|\nu\rangle$, we have that,

$$\begin{aligned} \lambda^2|\nu\rangle &= P^2|\nu\rangle \\ &= \lambda|\nu\rangle \end{aligned}$$

And therefore, $\lambda = 0$ or 1.

Q8. Let $|\psi$ be a state vector in the Hilbert space V. Then, for any positive operator A, we have $\langle\psi|A|\psi\rangle \in \mathbb{R}_{\geq 0}$. Now, see that the following is true.

$$A = \left(\frac{A + A^\dagger}{2}\right) + i \left(\frac{A - A^\dagger}{2i}\right)$$

Let's call $B = \left(\frac{A+A^\dagger}{2}\right)$ and $C = \left(\frac{A-A^\dagger}{2i}\right)$ and observe that: $B^\dagger = B$ & $C^\dagger = C$. Now, let's look at the following expression involving $\langle\psi|A|\psi\rangle$.

$$\begin{aligned} \langle\psi|A|\psi\rangle &= \langle\psi|B|\psi\rangle + i \cdot \langle\psi|C|\psi\rangle \\ &\in \mathbb{R}_{\geq 0} \end{aligned}$$

Use the fact that for any Hermitian operator B or C, $\langle\psi|B|\psi\rangle$ and $\langle\psi|C|\psi\rangle$ are both real numbers.
Therefore, from the above equation for $\langle\psi|A|\psi\rangle \in \mathbb{R}_{\geq 0}$, it implies that $\langle\psi|C|\psi\rangle = 0$.
Hence, $A = B$, which is a Hermitian operator.

Q9. Discussed in the lecture. Please turn back your notes.

Q10. Any projector P on a subspace $\{|k\rangle\}$ of the inner product space V spanned by the orthonormal basis $\{|i\rangle\}$ is defined as:

$$P = \sum_k |k\rangle\langle k|$$

Note that $\{|k\rangle\}$ are also orthonormal and therefore,

$$\begin{aligned} P^2 &= \left(\sum_k |k\rangle\langle k|\right)\left(\sum_k |k\rangle\langle k|\right) \\ &= \sum_k |k\rangle\langle k| \text{ (because, for } k \neq k',\ \langle k|k'\rangle = 0 \text{)} \\ &= P \end{aligned}$$

$\square$

Q11. The Hadamard gate is

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(a) Verify $H^2 = I$. Compute

$$H^2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

(b) Show H is unitary. Since H is real symmetric, $H^\dagger = H$. Using (a),

$$H^\dagger H = HH = H^2 = I,$$

hence H is unitary.

(c) Eigenvalues and eigenvectors of H. Because $H^2 = I$, the eigenvalues must satisfy $\lambda^2 = 1$, i.e. $\lambda = \pm 1$.
Solve $(H - \lambda I)\, v = 0$.
For $\lambda = 1$:

$$\left(\tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - I\right)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad\Longrightarrow\quad \left(\tfrac{1}{\sqrt{2}} - 1\right)x + \tfrac{1}{\sqrt{2}}\,y = 0.$$

Thus $y = \left(\tfrac{1}{\sqrt{2}}^{-1} - 1\right)x = (\sqrt{2} - 1)\,x$. One normalized eigenvector is

$$v_+ \;=\; \frac{1}{\sqrt{(\,1\,)^2 + (\sqrt{2} - 1)^2}}\begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix} = \frac{1}{\sqrt{4 - 2\sqrt{2}}}\begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix}.$$

For $\lambda = -1$:

$$\left(\tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} + I\right)\begin{bmatrix}x\\ y\end{bmatrix} = 0 \quad\Longrightarrow\quad (\tfrac{1}{\sqrt{2}} + 1)\,x + \tfrac{1}{\sqrt{2}}\,y = 0.$$

Thus $y = -\big(1 + \tfrac{1}{\sqrt{2}}\big)\sqrt{2}\,x = -(1+\sqrt{2})\,x$. One normalized eigenvector is

$$v_- \;=\; \frac{1}{\sqrt{(\,1\,)^2 + (1+\sqrt{2})^2}}\begin{bmatrix}1\\ -(1+\sqrt{2})\end{bmatrix} = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1\\ -(1+\sqrt{2})\end{bmatrix}.$$

Therefore, the spectral decomposition is

$$H \;=\; (+1)\,|v_+\rangle\langle v_+| \;+\; (-1)\,|v_-\rangle\langle v_-|.$$

Q12. The Pauli matrices are

$$X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \qquad Y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \qquad Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Each is Hermitian and unitary, with eigenvalues ± 1.

Matrix Z. Z is already diagonal in the computational basis:

$$Z|0\rangle = +1\,|0\rangle, \qquad Z|1\rangle = -1\,|1\rangle.$$

Thus the eigenvectors are $|0\rangle = \begin{bmatrix}1\\ 0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\ 1\end{bmatrix}$.

Matrix X. Solve $Xv = \lambda v$. The eigenvalues are $\lambda = \pm 1$, with normalized eigenvectors

$$|+\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}, \qquad |-\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}.$$

In this $\{|+\rangle, |-\rangle\}$ basis,

$$X \sim \mathrm{diag}(1, -1).$$

Matrix Y. Solve $Yv = \lambda v$. The eigenvalues are $\lambda = \pm 1$, with normalized eigenvectors

$$v_+ = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}, \qquad v_- = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}.$$

In the $\{v_+, v_-\}$ basis,

$$Y \sim \mathrm{diag}(1, -1).$$

Summary. All three Pauli matrices have eigenvalues ± 1 and are diagonalizable:

$$Z \sim \mathrm{diag}(1,-1), \quad X \sim \mathrm{diag}(1,-1), \quad Y \sim \mathrm{diag}(1,-1),$$

in the bases $\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$, and $\{v_+, v_-\}$ respectively.