AI353 – Introduction to Quantum Computing
Quiz I (Make-up Version) Solutions

Solutions and Explanations

Q1. Bloch Sphere Representation

(a) General form and normalization
Any single-qubit pure state can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad \alpha, \beta \in \mathbb{C}.$$

Normalization condition:

$$|\alpha|^2 + |\beta|^2 = 1.$$

On the Bloch sphere, every normalized qubit state (up to a global phase) can be parametrized as

$$|\psi\rangle = \cos\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\tfrac{\theta}{2}\right)|1\rangle,$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$.

Check normalization:

$$\cos^2\left(\tfrac{\theta}{2}\right) + \sin^2\left(\tfrac{\theta}{2}\right) = 1.$$

(b) Example state

$$|\psi\rangle = \tfrac{1}{\sqrt{3}}|0\rangle + i\sqrt{\tfrac{2}{3}}|1\rangle.$$

Here $\alpha = \frac{1}{\sqrt{3}},\ \beta = i\sqrt{\frac{2}{3}}$.

$$\cos\left(\tfrac{\theta}{2}\right) = \tfrac{1}{\sqrt{3}}, \quad \sin\left(\tfrac{\theta}{2}\right) = \sqrt{\tfrac{2}{3}}.$$

So, $\theta = 2\arccos\left(\frac{1}{\sqrt{3}}\right)$.

Since $\beta = e^{i\phi}\sin(\theta/2)$, comparing with $\beta = i\sqrt{\frac{2}{3}}$ gives $e^{i\phi} = i = e^{i\pi/2}$, so $\phi = \pi/2$.

$$\theta = 2\arccos\left(\tfrac{1}{\sqrt{3}}\right), \quad \phi = \tfrac{\pi}{2}.$$

Q2. Bell State Measurements

Given:

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

(a) Measure the first qubit in computational basis
- Probability of outcome $|0\rangle$: $\Pr(0) = \frac{1}{2}$, post-state: $|00\rangle$.
- Probability of outcome $|1\rangle$: $\Pr(1) = \frac{1}{2}$, post-state: $|11\rangle$.

(b) Measure both qubits in the Hadamard basis $\{|+\rangle, |-\rangle\}$
Recall:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Expressing $|\beta_{00}\rangle$:

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle).$$

Thus with prob. 1/2, both qubits are $|+\rangle$; with prob. 1/2, both are $|-\rangle$. Outcomes are perfectly correlated.

Q3. Teleportation and Basis Transformations

(a) Standard teleportation protocol
- Alice and Bob share an entangled pair $|\beta_{00}\rangle$.
- Alice has an unknown qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.
- Alice applies CNOT (control = message qubit, target = her half of Bell pair).
- She applies Hadamard to the message qubit.
- She measures her two qubits $\to$ classical bits (M_1, M_2).
- Alice sends two classical bits to Bob.
- Bob applies $Z^{M_1}X^{M_2}$.
- Final state at Bob = $|\psi\rangle$.

Two bits are required to distinguish between I, X, Z, XZ.

(b) Omitting Hadamard and top measurement
Alice sends only M_2. Bob applies X^{M_2}. But without M_1, the missing Z^{M_1} correction cannot be applied. Since $|+\rangle$ is not an eigenstate of Z, Bob fails to always recover $|+\rangle$.

(c) Relation of matrix representations
Please check question 9 from Tutorial sheet 3.