AI353: Introduction to Quantum Computing

Tutorial Sheet - Week 4

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Instructions

This tutorial focuses on the topic **Postulates of Quantum Mechanics and its application**, as covered in lectures. You are encouraged to work in pairs for solving these problems. Active participation during the tutorial session is rewarded: volunteers who solve and present a problem on the whiteboard will receive points.

Points Distribution:

- Attendance: 15 points
- Active participation: 10 points
- **Total: 25 points**

Q1. Show that the average value of the measurement is $\langle\psi|M|\psi\rangle$. Also, compute the formula for standard deviation.

Q2. Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M, with corresponding eigenvalue m. What is the average observed value of M, and the standard deviation?

Q3. Suppose we have a qubit in the state $|0\rangle$, and we measure the observable X. What is the average value of X? What is the standard deviation?

Q4. Suppose $\vec{v}$ be any real three dimensional unit vector and we define an observable as follows:

$$\vec{v}\cdot\vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$$

Here $\vec{\sigma}$ is a shorthand for the **vector of Pauli matrices**:

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3),$$

where

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show that $\vec{v} \cdot \vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by

$$P_{\pm} = \frac{I \pm \vec{v} \cdot \vec{\sigma}}{2}.$$

Q5. (Extending the previous question) Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{v} \cdot \vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement if $+1$ is obtained?

Q6. (**Optional**)Show a formal proof establishing that the two non-orthogonal states cannot be reliably distinguished. (Refer to Nielsen & Chuang, Box 2.3)

Q7. Show that any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Q8. Suppose a measurement is described by measurement operators M_m. Show that there exist unitary operators U_m such that $M_m = U_m\sqrt{E_m}$, where E_m is the POVM associated to the measurement.

Q9. Suppose Bob is given a quantum state chosen from a set $|\psi_1\rangle, ..., |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, E_2, ..., E_{m+1}\}$ such that if outcome E_i occurs, $1 \leq i \leq m$, then Bob knows with certainty that he was given the state $|\psi_i\rangle$. (The POVM must be such that $\langle\psi_i|E_i|\psi_i\rangle > 0$ for each i.)

Q10. Express the states $\frac{(|0\rangle+|1\rangle)}{\sqrt{2}}$ and $\frac{(|0\rangle-|1\rangle)}{\sqrt{2}}$ in a basis in which they are *not* the same up to a relative phase shift.

Q11. Verify that the Bell basis forms an orthonormal basis for the two qubit state space.

Q12. (**Optional**)Suppose E is any positive operator acting on Alice's qubit. Show that $\langle\psi|E \otimes I|\psi\rangle$ takes the same value when $|\psi\rangle$ is any of the four Bell states. Suppose some malevolent third party('Eve') intercepts Alice's qubit on the way to Bob in the superdense coding protocol. Can Eve infer anything about which of the four possible bit strings 00, 01, 10, 11 Alice is trying to send? If so, how, or if not, why not?