Adjoint of an Operator Function

For any operator K and an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with real coefficients a_n, we have:

$$(f(K))^\dagger = f(K^\dagger).$$

Proof:

By definition,

$$f(K) = \sum_{n=0}^{\infty} a_n K^n.$$

Taking adjoints term by term,

$$(f(K))^\dagger = \left(\sum_{n=0}^{\infty} a_n K^n\right)^\dagger = \sum_{n=0}^{\infty} a_n (K^n)^\dagger.$$

Since $(K^n)^\dagger = (K^\dagger)^n$, this equals

$$\sum_{n=0}^{\infty} a_n (K^\dagger)^n = f(K^\dagger).$$

Special Case: Exponential

The exponential function has the real-coefficient series

$$\exp(K) = \sum_{n=0}^{\infty} \frac{1}{n!} K^n.$$

Hence,

$$(\exp K)^\dagger = \exp(K^\dagger).$$

Example: If $U = \exp(K)$, then $U^\dagger = \exp(K^\dagger)$. If K is anti-Hermitian ($K^\dagger = -K$), this shows U is unitary.