AI353: Introduction to Quantum Computing

Tutorial Sheet - Week 6

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Instructions

This tutorial focuses on the topic **Quantum Gates**, as covered in lectures. You are encouraged to work in pairs for solving these problems. Active participation during the tutorial session is rewarded: volunteers who solve and present a problem on the whiteboard will receive points. Questions in this set are adapted from Nielsen &Chuang Exercises 4.2 – 4.14.

Points Distribution:

- Attendance: 15 points
- Active participation: 10 points
- **Total: 25 points**

Conventions. We use the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For a real unit vector $\hat{n} = (n_x, n_y, n_z)$ we write $\hat{n} \cdot \boldsymbol{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$ and define the single-qubit rotation about axis $\hat{n}$ by angle θ as

$$R_{\hat{n}}(\theta) = \exp\left(-\mathrm{i}\,\frac{\theta}{2}\,\hat{n} \cdot \boldsymbol{\sigma}\right).$$

Standard x, y, z rotations are $R_x(\theta) = R_{(1,0,0)}(\theta)$, $R_y(\theta) = R_{(0,1,0)}(\theta)$ and $R_z(\theta) = R_{(0,0,1)}(\theta)$. Global phases $e^{\mathrm{i}\alpha}$ are physically irrelevant but kept symbolically in some problems.

Theorem 0.1. *[Z–Y decomposition for a single qubit] Suppose U is a unitary operation on a single qubit. Then there exist real numbers α, β, γ and δ such that*

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta).$$

Theorem 0.2. *If U is a single-qubit unitary, there exist single-qubit unitaries A, B, C such that $ABC = \mathbb{I}$ and $U = e^{\mathrm{i}\alpha} AXBXC$ for some real α.*

Q1. Let $x \in \mathbb{R}$ and let A be a matrix with $A^2 = \mathbb{I}$. Show that

$$\exp(\mathrm{i}xA) = \cos(x)\,\mathbb{I} + \mathrm{i}\sin(x)\,A.$$

Hint: Expand the exponential in a power series and split even/odd powers. Then use this result to verify the identities (for any real θ)

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - \mathrm{i}\sin\left(\frac{\theta}{2}\right)\sigma_x,$$
$$R_y(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - \mathrm{i}\sin\left(\frac{\theta}{2}\right)\sigma_y,$$
$$R_z(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - \mathrm{i}\sin\left(\frac{\theta}{2}\right)\sigma_z.$$

Q2. Show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_z(\pi/4)$ up to an overall phase factor. State explicitly the phase you factor out.

Q3. Express the Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ as a product of rotations using only R_x and R_z gates and a global phase $e^{\mathrm{i}\varphi}$ for some real φ. (Give one valid decomposition and the corresponding φ.)

Q4. Prove that $(\hat{n} \cdot \boldsymbol{\sigma})^2 = \mathbb{I}$ for any real unit vector $\hat{n}$. Verify the closed-form expression

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - \mathrm{i}\sin\left(\frac{\theta}{2}\right)\left(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z\right).$$

Q5. Show that $\sigma_x\sigma_y\sigma_x = -\sigma_y$ and use this to prove $\sigma_x\, R_y(\theta)\, \sigma_x = R_y(-\theta)$.

Q6. Show that any single-qubit unitary can be written as

$$U = e^{\mathrm{i}\alpha}\, R_{\hat{n}}(\theta)$$

for some real numbers α, θ and a real unit vector $\hat{n}$. Provide a concise proof. Then:

(a) Find $(\alpha, \theta, \hat{n})$ that realize the Hadamard gate H.

(b) Find $(\alpha, \theta, \hat{n})$ that realize the phase gate $S = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$.

Q7. Explain why any single-qubit unitary may be written in the following form

$$U = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix}.$$

Q8. Give a decomposition analogous to Theorem 0.1 but using R_x instead of R_z, i.e. express a general unitary as $U = e^{\mathrm{i}\alpha}\, R_x(\beta)\, R_y(\gamma)\, R_x(\delta)$.

Q9. Suppose $\hat{m}$ and $\hat{n}$ are non-parallel real unit vectors in $\mathbb{R}^3$. Use Theorem 0.1 to show that an arbitrary single-qubit unitary U can be written

$$U = e^{\mathrm{i}\alpha}\, R_{\hat{n}}(\beta)\, R_{\hat{m}}(\gamma)\, R_{\hat{n}}(\delta)$$

for suitable $\alpha, \beta, \gamma, \delta$.

Q10. Read Theorem 0.2 and answer the following. Choose explicit A, B, C, α that realize the Hadamard gate H. (You may use $A = R_z(\beta)R_y(\gamma/2)$, $B = R_y(-\gamma/2)R_z\big(-(\delta+\beta)/2\big)$, $C = R_z\big((\delta-\beta)/2\big)$ as a template, with appropriate β, γ, δ for H.)

Q11. Prove the following three useful conjugation identities:

$$H\sigma_x H = \sigma_z, \qquad H\sigma_y H = -\sigma_y, \qquad H\sigma_z H = \sigma_x.$$

Hint: Either use matrix multiplication or write $H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$ and use Pauli anti/commutation rules.

Q12. Use the previous question to show that HTH equals $R_x(\pi/4)$ up to a global phase. State the phase you factor out and verify your claim by a short calculation.

Reminder. When a statement says "up to a global phase", two matrices U, V are considered equivalent if $U = e^{\mathrm{i}\phi}V$ for some real ϕ.