AI353: Introduction to Quantum Computing

Tutorial Solutions - Week 2

Instructor: Yugarshi Shashwat

Solutions

1. The set $\{(1,1),(1,-1)\}$ is linearly independent in $\mathbb{R}^2$. Suppose $a(1,1)+b(1,-1)=(0,0)$. Then we have $(a+b,a-b)=(0,0)$. So $a+b=0$ and $a-b=0$. Adding gives $2a=0 \implies a=0$, and then $b=0$. Thus the set is independent.

2. The 2×2 identity matrix is
$$I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$
Then $I\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$, so the vector remains unchanged.

3. The solution has been explained in the lecture.

4. With $X=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $Y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$, we compute $XY=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$ and $YX=\begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}$. Thus $XY=-YX$.

5. Suppose U is unitary and $U|\psi\rangle=\lambda|\psi\rangle$. Then
$$\|U|\psi\rangle\|^2=\langle\psi|U^\dagger U|\psi\rangle=\langle\psi|\psi\rangle.$$
But also $\|U|\psi\rangle\|^2=|\lambda|^2\langle\psi|\psi\rangle$. Hence $|\lambda|^2=1$, so $|\lambda|=1$ and $\lambda=e^{i\theta}$ for some real θ.

6. The Pauli matrices are:
$$X=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad Y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad Z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$
Each equals its own conjugate transpose, so Hermitian. Further, $X^2=Y^2=Z^2=I$, so they are unitary.

7. For
$$B=\begin{bmatrix}0 & 1+i\\ 1-i & 0\end{bmatrix},$$
we compute $B^\dagger=B$. Hence B is Hermitian.

8. Consider vectors $(1,0,1),(0,1,1),(1,1,2)$. Note $(1,1,2)=(1,0,1)+(0,1,1)$. So only two are independent. A basis is $\{(1,0,1),(0,1,1)\}$. Dimension $=2$.

9. With $|\psi\rangle=(|0\rangle+|1\rangle)/\sqrt{2}$:

$$|\psi\rangle^{\otimes 2}=\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle),$$

$$|\psi\rangle^{\otimes 3}=\frac{1}{2\sqrt{2}}(|000\rangle+|001\rangle+|010\rangle+|011\rangle+|100\rangle+|101\rangle+|110\rangle+|111\rangle).$$

10. Tensor products of Pauli operators:

    (a) $X\otimes Z=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix}.$

    (b) $I\otimes X=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}.$

    (c) $X\otimes I=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}.$

    The tensor product is not commutative: e.g. $X\otimes Z\neq Z\otimes X$.

11. Distributivity of operations over tensor product:

$$(A\otimes B)^*=A^*\otimes B^*,\quad (A\otimes B)^T=A^T\otimes B^T,\quad (A\otimes B)^\dagger=A^\dagger\otimes B^\dagger.$$

    These follow because conjugation, transpose, and adjoint act entrywise.