AI353: Introduction to Quantum Computing

Tutorial Sheet - Week 5

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Instructions

This tutorial focuses on the topic **Basic Elements of Quantum Computation**, as covered in lectures. You are encouraged to work in pairs for solving these problems. Active participation during the tutorial session is rewarded: volunteers who solve and present a problem on the whiteboard will receive points.

Points Distribution:

- Attendance: 15 points
- Active participation: 10 points
- **Total: 25 points**

Q1. Multiple-qubit gates (CNOT):
Give the truth table of the Controlled-NOT (CNOT) gate when applied to the two-qubit computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Q2. Measurement in a different basis:
Suppose a qubit is in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

(a) What are the possible outcomes if it is measured in the computational basis $\{|0\rangle, |1\rangle\}$?

(b) What are the possible outcomes if it is measured in the Hadamard basis $\{|+\rangle, |-\rangle\}$?

Q3. Circuit reasoning:
Consider a two-qubit circuit where the first qubit undergoes a Hadamard gate, followed by a CNOT gate with the first qubit as control and second qubit as target.

(a) What is the output state when the input is $|00\rangle$?

(b) Which special two-qubit state is generated?

Q4. No-cloning theorem:
Explain why a quantum circuit that copies an arbitrary qubit state $|\psi\rangle$ into $|\psi\rangle|\psi\rangle$ cannot exist. Provide a short mathematical justification.

Q5. Bell states:

(a) List the four Bell states.

(b) Design a quantum circuit that produces the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ starting from the initial state $|00\rangle$.

Q6. Teleportation protocol – Step analysis:
Consider the quantum teleportation circuit (Alice has qubits 1 and 2, Bob has qubit 3).

(a) If Alice's qubit 1 is in an unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and qubits 2 and 3 are in the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, write the combined initial state of the three qubits.

(b) Show how Alice's application of CNOT (control = qubit 1, target = qubit 2) and Hadamard on qubit 1 transforms the state.

Q7. Teleportation protocol – Measurement outcomes:
In the teleportation circuit, after Alice measures qubits 1 and 2 in the computational basis, she sends the results to Bob as two classical bits.

(a) Write down the possible post-measurement states of Bob's qubit (qubit 3).

(b) Explain how Bob's application of Pauli gates (I, X, Z, or XZ) restores the original state $|\psi\rangle$.

Q8. Equivalence of circuits:
Show that applying two consecutive Hadamard gates to a qubit is equivalent to the identity operation. Then, using this, prove that the teleportation circuit still works correctly if Bob mistakenly applies an extra Hadamard both before and after his correction step.

Q9. Entanglement via measurement:
Consider two qubits initialized in $|+\rangle|+\rangle$.

(a) Show the effect of measuring them in the Bell basis instead of the computational basis.

(b) Why does this measurement create entanglement between the qubits even if they started as a product state?

Q10. Circuit design challenge – Teleportation variant:
Design a circuit that teleports the state $|\psi\rangle$ from Alice to Bob **without using CNOT** (but still using Hadamard, Pauli gates, and measurements). Give reasoning why it works.