Discussion 10
First Order Logic

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Outline

- **Syntax:** how to write sentences
  - What kind of sentences are well-formed?

- **Semantics**: how to interpret sentences
  - Is this sentence True given this **possible world(model)**?

- **Inference**
  - How to determine entailment?

Syntax

- The same as in Propositional Logic, we have **sentences** in FOL.
- A sentence is evaluated as **True/False** with respect to a **model**.

Here we discuss how a sentence is formed in FOL.

We first see some sentence examples and move to the basic elements.

Sentences Types and Examples

- Atomic Sentences: objects (terms) and predicates
  - UCLAStudent(Mary) (predicate and constant)
  - UCLAStudent(x) (predicate and variable)
  - Married(Mother(Mary), Father(Mary)) (predicate, constant, function)

- Complex Sentences
  - $\text{Under20(Mary)} \wedge \text{UCLAStudent(Mary)}$
  - $\text{Color(Apple)} = \text{Red}$
  - $\text{Sold(John, Car1, Tom)} \Longrightarrow \neg \text{Owns(John, Car1)}$
  - $\forall x\ \text{UCLAStudent}(x) \Rightarrow \text{Person}(x)$
  - $\exists x\ \text{UCLAStudent}(x) \wedge \text{Under20}(x)$

Basic Elements

- **Objects** (a.k.a. **Terms**)
  - Constants
    - e.g., Apple, Pear, Mary, UCLA
  - Variables
    - e.g., x, y, z ← By convention, variables are represented by lowercase letters.
  - Complex terms (having functions)
    - e.g., Mother(Mary), Color(Apple), Color(x)

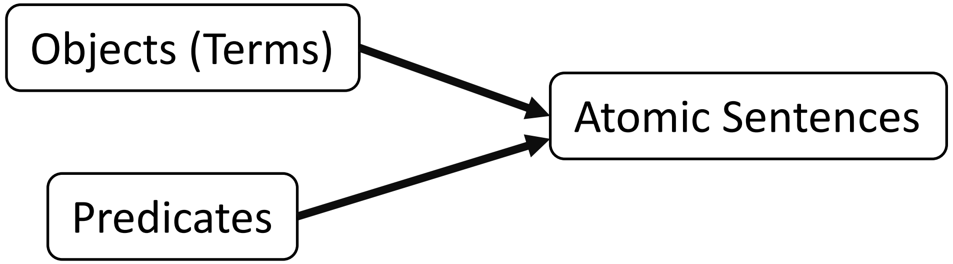
Functions (Return another constant)

A **ground term** is a term without variables
- e.g., Apple, Color(Apple)

Basic Elements

- **Predicates** (Evaluated as <u>True/False</u>)
  - Properties (unary)
    - UCLA_student(Mary)
  - relations (n-ary)
    - Loves(Richard, Dog_of_Richard)
    - Brother(Richard, John)

Atomic Sentences



Atomic Sentences Consist of terms (objects) and predicates

- Examples
  - UCLAStudent(Mary)
  - UCLAStudent(x)
  - Married(Mother(Mary), Father(Mary))

The following is NOT a sentence:

Mary, x, Mother(Mary)

(They are not True or False!)

From Atomic Sentences to Complex Sentences

- Connectives
  - $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ (as in propositional logic)
    - Owns(John, Car1)
    - Sold(John, Car1, Tom) $\Longrightarrow \neg$ Owns(John, Car1)
  - $=, \neq$ (will introduce after quantifiers)
    - Color(Apple)=Red
- Quantifiers
  - $\forall$ for all
  - $\exists$ there exists

Quantifiers

Express properties of <u>entire collections of objects</u>, instead of enumerating the objects by name.

- Universal quantification $\forall$ (For all)
  - Sentence $\forall$x P , where P is any logical expression, says that P is true for every object x
  - $\forall x\ King(x) \Rightarrow Person(x)$
  - Naturally uses $\Rightarrow$

- Existential quantification $\exists$ (There exists)
  - $\exists x\ King(x) \ \land OlderThan30(x)$
  - Naturally uses $\land$
  - $\exists!\, x$ Uniqueness Quantifier
    - $\exists!\, x\ King(x)$: There is exactly one king

Quantifiers - Nesting Quantifiers

- Same type quantifiers: order doesn't matter
  - $\forall x \forall y\ (\mathrm{Paren}(x, y) \land \mathrm{Male}(y) \Rightarrow \mathrm{Son}(y, x))$
  - $\exists\ x \exists y\ (\mathrm{Loves}(x, y) \land \mathrm{Loves}(y, x))$
    - $\exists\ x, y\ (\mathrm{Loves}(x, y) \land \mathrm{Loves}(y, x))$

- Mixed quantifiers: order does matter
  - $\forall\ x \exists y\ (\mathrm{Loves}(x, y))$
    - Everybody has someone they love.
  - $\exists y \forall\ x(\mathrm{Loves}(x, y))$
    - There is someone who is loved by everyone.
  - $\forall\ y \exists x\ (\mathrm{Loves}(x, y))$
    - Everybody has someone who loves them.
  - $\exists x \forall\ y(\mathrm{Loves}(x, y))$
    - There is someone who loves everyone.

Exercise

Are they equivalent? What do they mean?

- $\forall x\ King(x) \Rightarrow Person(x)$
- $\forall x\ King(x) \wedge Person(x)$

- $\exists x\ King(x) \wedge OlderThan30(x)$
- $\exists x\ King(x) \Rightarrow OlderThan30(x)$

Logical Equivalence for Quantifiers

- ∀ and ∃

$$
\begin{aligned}
\forall x \;\; \neg P &\equiv \neg\exists x \;\; P \\
\neg\forall x \;\; P &\equiv \exists x \;\; \neg P \\
\forall x \;\; P &\equiv \neg\exists x \;\; \neg P \\
\exists x \;\; P &\equiv \neg\forall x \;\; \neg P
\end{aligned}
\qquad\qquad
\begin{aligned}
\neg(P \vee Q) &\equiv \neg P \wedge \neg Q \\
\neg(P \wedge Q) &\equiv \neg P \vee \neg Q \\
P \wedge Q &\equiv \neg(\neg P \vee \neg Q) \\
P \vee Q &\equiv \neg(\neg P \wedge \neg Q)\,.
\end{aligned}
$$

$\forall x \; \mathrm{King}(x) \Rightarrow \mathrm{Person}(x)$

$\neg\exists x \;\; \neg \; (\mathrm{King}(x) \Rightarrow \mathrm{Person}(x))$

More about quantifiers

- Variable scope
  - The **scope** of a variable is the sentence to which the quantifier syntactically applies.
    - $\forall \mathbf{x}\ \mathrm{King}(x) \Rightarrow \mathrm{Person}(x)$
    - $\forall \mathbf{x}\ \mathrm{King}(x)\ \lor (\exists\ \mathbf{x}\ \mathrm{Brother}(x, \mathrm{Richard}))$
      - The variable belongs to the **innermost** quantifier that mentions it. Then it will not be subject to any other quantification.
      - Equivalent sentence: $\forall \mathbf{x}$ King(x) $\lor$($\exists$ z Brother(z, Richard))
      - Cause confusion. Not recommended.

  - Not well-formed
    - $\exists x\ P(y)$
      - All variables should be properly introduced!

  - **Ground expression**
    - No variables
    - King(Richard) $\Rightarrow$ Person(Richard)

Logical Connective - Equality

- Equality = (identity relation)
    - Color(Apple)=Red   (True)
    - Color(Apple)=Blue   (False)

Hints for Exercise 1

Are they equivalent? What do they mean?

- $\exists$ x, y Brother(x, Richard) $\wedge$ Bother(y, Richard)
- $\exists$ x, y Brother(x, Richard) $\wedge$ Bother(y, Richard) $\wedge (x \neq y)$

Consider the following cases:

1) Richard only has one brother John
2) Richard has two brothers: John and Tom

Exercise 1

Write FOL sentences:

- Richard has (at least) two brothers

Exercise 1

Write FOL sentences:

- Richard has (at least) two brothers
  - $\exists\ \mathrm{x}, y\ \mathrm{Brother(x, Richard)} \wedge \mathrm{Bother(y, Richard)} \wedge (x \neq y)$

Exercise 2

Translate into FOL:

Everyone has exactly one mother.

- $\mathrm{Mother(y, x)}$ means y is the mother of x

- $\forall \mathrm{x}\ \exists y\ \mathrm{Mother(y, x)}$?
  - Everyone has (at least one) mother.

- $\forall \mathrm{x}\ \exists \mathrm{y}\ \mathrm{Mother(y, x)} \wedge (\forall \mathbf{z}\ \mathbf{Mother(z, x)} \Rightarrow \mathbf{y} = \mathbf{z})$

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Models

In logical system, a sentence is evaluated as True or False with respect to a **model (possible world)**.

- In Propositional Logic, a model is an assignment for this sentence
  - e.g., $f = (\neg A \land B) \leftrightarrow C$
    $w = \{A : 1, B : 1, C : 0\}$
  - If a sentence α is true in model m, we say that model m **satisfies** α
  - $M(\alpha) \coloneqq$ the set of all the models that satisfy α

- What about in First-Order Logic?
  - Much more complex!

Models in FOL

A model in FOL consists of:

- A set of objects
- A set of functions + what values will be returned
- A set of predicates + what values will be returned

Example - Model

Consider:

Objects

$Orange$

$Apple$

Predicates

$IsRed(\cdot)$

$HasVitaminC(\cdot)$

Functions

$OppositeOf(\cdot)$

Example model:

Predicate	Argument	Value
$IsRed$	$Orange$	$False$
$IsRed$	$Apple$	$True$
$HasVitaminC$	$Orange$	$True$
$HasVitaminC$	$Apple$	$True$

Function	Argument	Return
$OppositeOf$	$Orange$	$Apple$
$Opposite$	$Apple$	$Orange$

https://www2.cs.duke.edu/courses/spring15/compsci270/slides/270_6.pdf

Example – A Model

Sentence: $\forall x \ \mathrm{King}(x) \Rightarrow \mathrm{Person}(x)$

Consider the following case:

- Three persons:
Richard (King, 50 years old)
John (Richard's brother, 20 years old)
Elizabeth (Richard's mother)

- A dog:
Gigi (Richard's dog)

Example – A Model (Possible World)

Formalize the setting:

- Objects:
  - Richard, John, Elizabeth, Gigi
- Functions:
  - Age(·)
- Predicates:
  - King(·), Person(·), Brother(·, ·), Mother (·, ·), Dog (·, ·)

A model:

Age(Richard) returns 50, Age(John) returns 20

The following sentences are True and all others are False

- King(Richard), Dog(Gigi, Richard)
- Person(Richard), Person(John), Person(Elizabeth) are True
- Brother(John, Richard), Brother(Richard,John) are True
- Mother(Elizabeth, Richard), Mother(Elizabeth, John) are True

- Three persons:

  Richard (King, 50 years old)

  John (Richard's brother, 20 years old)

  Elizabeth (Richard's mother)

- A dog:

  Gigi (Richard's dog)

Propositionalization

How to evaluate a sentence with quantifiers?

We eliminate the quantifiers and propositionalize it to a propositional logic sentence.

- Given a model, how to determine if $\forall x\ P$ is true?
  - Concatenate all **universal instantiations** by <u>conjunction</u>
    - instantiation: get rid of all variables by replacing them with ground terms
- Given a model, how to determine if $\exists x\ P$ is true?
  - Concatenate all **existential instantiations** by <u>disjunction</u>

Example - Propositionalization

Determine if this is true:
$\forall x \; \mathrm{King}(x) \Rightarrow \mathrm{Person}(x)$

Example - Propositionalization

Determine if this is true:
$\forall x \; \mathrm{King}(x) \Rightarrow \mathrm{Person}(x)$

1. Extend the interpretation:

$x \rightarrow$ Richard

$x \rightarrow$ John

$x \rightarrow$ Elizabeth

$x \rightarrow$ Gigi

Example - Propositionalization

2. Compute the propositional grounding (Do instantiation)

(In this example, we use universal instantiation)

$$\text{King}(\text{Richard}) \Rightarrow \text{Person}(\text{Richard})$$
$$\text{King}(\text{John}) \Rightarrow \text{Person}(\text{John})$$
$$\text{King}(\text{Elizabeth}) \Rightarrow \text{Person}(\text{Elizabeth})$$
$$\text{King}(\text{Gigi}) \Rightarrow \text{Person}(\text{Gigi})$$

Example

What do we do?

2. Compute the propositional grounding

$$
\begin{aligned}
&(\quad King(Richard) \Rightarrow Person(Richard) \quad) \land \\
&(\quad King(John) \Rightarrow Person(John) \quad) \land \\
&(\quad King(Elizabeth) \Rightarrow Person(Elizabeth) \quad) \land \\
&(\quad King(Gigi) \Rightarrow Person(Gigi) \quad)
\end{aligned}
$$

Example

3. See if its' True

($\text{King}(\text{Richard}) \Rightarrow \text{Person}(\text{Richard})$) $\wedge$ True
($\text{King}(\text{John}) \Rightarrow \text{Person}(\text{John})$) $\wedge$ True
($\text{King}(\text{Elizabeth}) \Rightarrow \text{Person}(\text{Elizabeth})$) $\wedge$ True
($\text{King}(\text{Gigi}) \Rightarrow \text{Person}(\text{Gigi})$) True

This sentence is True!