CS 161
Discussion 10
First-order Logic

Outline

- **Syntax**: how to write sentences
  - What kind of sentences are well-formed?
- **Semantics**: how to interpret sentences
  - Is this sentence True given this **possible world(model)**?
- **Inference**
  - How to determine entailment?

Syntax

- The same as in Propositional Logic, we have **sentences** in FOL.
- A sentence is evaluated as **True/False** with respect to a **model**.

Here we discuss how a sentence is formed in FOL.

We first see some sentence examples and move to the basic elements.

Sentences Types and Examples

- Atomic Sentences: objects (terms) and predicates
  - UCLAStudent(Mary) (predicate and constant)
  - UCLAStudent(x) (predicate and variable)
  - Married(Mother(Mary), Father(Mary))
    (predicate, constant, function)

- Complex Sentences
  - Under20(Mary) $\wedge$ UCLAStudent(Mary)
  - Color(Apple)=Red
  - Sold(John, Car1, Tom) $\Longrightarrow \neg$ Owns(John, Car1)
  - $\forall x$ UCLAStudent(x) $\Rightarrow$ Person(x)
  - $\exists x$ UCLAStudent(x) $\wedge$ Under20(x)

Basic Elements

- **Objects** (a.k.a. **Terms**)
  - Constants
    - e.g., Apple, Pear, Mary, UCLA
  - Variables
    - e.g., x, y, z

By convention, variables are represented by lowercase letters.

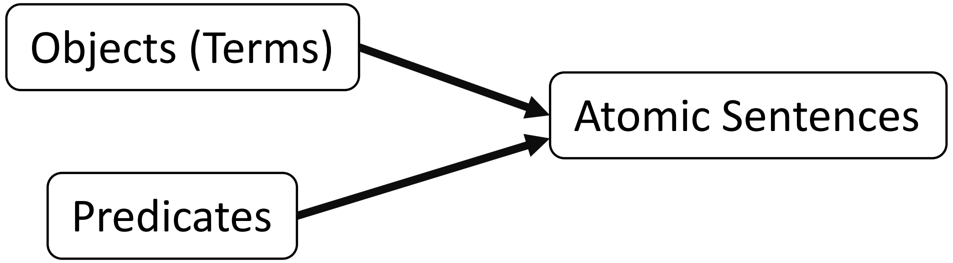
  - Complex terms (having functions)
    - e.g., Mother(Mary), Color(Apple), Color(x)

Functions (Return another constant)

A **ground term** is a term without variables
- e.g., Apple, Color(Apple)

Basic Elements

- **Predicates** (Evaluated as True/False)
  - Properties (unary)
    - UCLA_student(Mary)
  - relations (n-ary)
    - Loves(Richard, Dog_of_Richard)
    - Brother(Richard, John)



Atomic Sentences

Atomic Sentences Consist of terms (objects) and predicates

- Examples
  - UCLAStudent(Mary)
  - UCLAStudent(x)
  - Married(Mother(Mary), Father(Mary))

The following is NOT a sentence:

Mary, x, Mother(Mary)

(They are not True or False!)

From Atomic Sentences to Complex Sentences

- Connectives
  - $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ (as in propositional logic)
    - Owns(John, Car1)
    - Sold(John, Car1, Tom) $\Longrightarrow \neg$ Owns(John, Car1)
  - $=, \neq$ (will introduce after quantifiers)
    - Color(Apple)=Red
- Quantifiers
  - $\forall$ for all
  - $\exists$ there exists

Quantifiers

Express properties of entire collections of objects, instead of enumerating the objects by name.

- Universal quantification $\forall$ (For all)
  - $\forall x \ \mathrm{King}(x) \Rightarrow \mathrm{Person}(x)$
  - Naturally uses $\Rightarrow$
- Existential quantification $\exists$ (There exists)
  - $\exists x \ \mathrm{King}(x) \land \mathrm{OlderThan30}(x)$
  - Naturally uses $\land$

Quantifiers - Nesting Quantifiers

- Same type quantifiers: order doesn't matter
  - $\forall x \forall y\ (\mathrm{Parent}(x, y) \wedge \mathrm{Male}(y) \Rightarrow \mathrm{Son}(y, x))$
  - $\exists\ x \exists y\ (\mathrm{Loves}(x, y) \wedge \mathrm{Loves}(y, x))$
    - $\exists\ x, y\ (\mathrm{Loves}(x, y) \wedge \mathrm{Loves}(y, x))$
- Mixed quantifiers: order does matter
  - $\forall\ x \exists y\ (\mathrm{Loves}(x, y))$
    - Everybody has someone they love.
  - $\exists y \forall\ x(\mathrm{Loves}(x, y))$
    - There is someone who is loved by everyone.
  - $\forall\ y \exists x\ (\mathrm{Loves}(x, y))$
    - Everybody has someone who loves them.
  - $\exists x \forall\ y(\mathrm{Loves}(x, y))$
    - There is someone who loves everyone.

Exercise

Are they equivalent? What do they mean?

- $\forall x \; King(x) \Rightarrow Person(x)$
- $\forall x \; King(x) \land Person(x)$

- $\exists x \; King(x) \land OlderThan30(x)$
- $\exists x \; King(x) \Rightarrow OlderThan30(x)$

Logical Equivalence for Quantifiers

- ∀ and ∃

$$
\begin{aligned}
\forall x\ \ \neg P &\equiv \neg\exists x\ \ P \\
\neg\forall x\ \ P &\equiv \exists x\ \ \neg P \\
\forall x\ \ P &\equiv \neg\exists x\ \ \neg P \\
\exists x\ \ P &\equiv \neg\forall x\ \ \neg P
\end{aligned}
\qquad
\begin{aligned}
\neg(P \lor Q) &\equiv \neg P \land \neg Q \\
\neg(P \land Q) &\equiv \neg P \lor \neg Q \\
P \land Q &\equiv \neg(\neg P \lor \neg Q) \\
P \lor Q &\equiv \neg(\neg P \land \neg Q)\,.
\end{aligned}
$$

$\forall \mathrm{x}\ \ \mathrm{King(x)} \Rightarrow \mathrm{Person(x)}$

$\neg\exists x\ \ \neg\ (\mathrm{King(x)} \Rightarrow \mathrm{Person(x)})$

More about quantifiers

- Variable scope
  - The **scope** of a variable is the sentence to which the quantifier syntactically applies.
    - $\forall \mathbf{x}\ \mathrm{King}(x) \Rightarrow \mathrm{Person}(x)$
    - $\forall \mathbf{x}\ \mathrm{King}(x)\ \vee (\exists\ \mathbf{x}\ \mathrm{Brother}(x, \mathrm{Richard}))$
      - The variable belongs to the **innermost** quantifier that mentions it. Then it will not be subject to any other quantification.
      - Equivalent sentence: $\exists z\ \mathrm{Brother}(z, \mathrm{Richard})$
      - Cause confusion. Not recommended.
  - Not well-formed
    - $\exists x\ P(y)$
      - All variables should be properly introduced!
  - **Ground expression**
    - No variables
    - King(Richard) ⇒ Person(Richard)

Logical Connective - Equality

- Equality = (identity relation)
  - Color(Apple)=Red (True)
  - Color(Apple)=Blue (False)

Exercise 1

Write FOL sentences:

- Richard has (at least) two brothers

Hints for Exercise 1

Are they equivalent? What do they mean?

- $\exists\ \mathrm{x}, y\ \mathrm{Brother(x, Richard)} \land \mathrm{Bother(y, Richard)}$
- $\exists\ \mathrm{x}, y\ \mathrm{Brother(x, Richard)} \land \mathrm{Bother(y, Richard)} \land (x \neq y)$

Consider the following cases:

1) Richard only has one brother John
2) Richard has two brothers: John and Tom

Exercise 1

Write FOL sentences:

- Richard has (at least) two brothers
  - $\exists\ \mathrm{x}, y\ \mathrm{Brother(x, Richard)} \land \mathrm{Bother(y, Richard)} \land (x \neq y)$

Exercise 2

Translate into FOL:

Everyone has exactly one mother.

- $\mathrm{Mother}(\mathrm{y},\mathrm{x})$ means y is the mother of x

- $\forall \mathrm{x}\ \exists y\ \mathrm{Mother}(\mathrm{y},\mathrm{x})$?
  - Everyone has (at least one) mother.

Exercise 2

Translate into FOL:

Everyone has exactly one mother.

- $\text{Mother}(\text{y},\text{x})$ means y is the mother of x

- $\forall \text{x}\ \exists y\ \text{Mother}(\text{y},\text{x})$?
  - Everyone has (at least one) mother.

- $\forall \text{x}\ \exists \text{y}\ \text{Mother}(\text{y},\text{x}) \wedge (\forall \mathbf{z}\ \textbf{Mother}(\mathbf{z},\mathbf{x}) \Rightarrow \mathbf{y} = \mathbf{z})$

at least

What about "Richard has exactly two brothers?"

$\exists x, y,\ Bro(x, Rich) \wedge Bro(y, Rich) \wedge (x \neq y) \wedge (\forall z\ Bro(z, Rich) \Rightarrow ((z=x) \vee (z=y)))$

Exercise 2

Translate into FOL:

Everyone has <u>exactly one</u> mother.

- $\mathrm{Mother}(y, x)$ means y is the mother of x

- $\forall x\ \exists y\ \mathrm{Mother}(y, x)$?
  - Everyone has (at least one) mother.

- $\forall x\ \exists y\ \mathrm{Mother}(y, x) \wedge (\forall \mathbf{z}\ \mathbf{Mother}(\mathbf{z}, \mathbf{x}) \Rightarrow \mathbf{y} = \mathbf{z})$

What about "Richard has exactly two brothers?"

- $\exists\ x, y\ \mathrm{Brother}(x, \mathrm{Richard}) \wedge \mathrm{Bother}(y, \mathrm{Richard}) \wedge (x \neq y) \wedge (\forall z\ Brother(z, Richard) \Rightarrow ((z = x) \vee (z = y)))$

Exercise 3 – Translating English to FOL

- Every gardener likes sunshine

- You can fool some people all the time.
- You can fool all the people some of the time.

Exercise 3 – Translating English to FOL

- Every gardener likes sunshine

- You can fool some people all the time.
    - $\exists x \forall t$
- You can fool all the people some of the time.
    - $\forall x \exists t$

Exercise – Translating English to FOL

- Every gardener likes the sun.
  `(Ax) gardener(x) => likes(x,Sun)`

- You can fool some of the people all of the time.
  `(Ex)(At) (person(x) ^ time(t)) => can-fool(x,t)`

- You can fool all of the people some of the time.
  `(Ax)(Et) (person(x) ^ time(t) => can-fool(x,t)`

- All purple mushrooms are poisonous.
  `(Ax) (mushroom(x) ^ purple(x)) => poisonous(x)`

- No purple mushroom is poisonous.
  `~(Ex) purple(x) ^ mushroom(x) ^ poisonous(x)`
  or, equivalently,
  `(Ax) (mushroom(x) ^ purple(x)) => ~poisonous(x)`

- There are exactly two purple mushrooms.
  `(Ex)(Ey) mushroom(x) ^ purple(x) ^ mushroom(y) ^ purple(y) ^ ~(x=y) ^ (Az) (mushroom(z) ^ purple(z)) => ((x=z) v (y=z))`

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Models

In logical system, a sentence is evaluated as True or False with respect to a **model (possible world)**.

- In Propositional Logic, a model is an assignment for this sentence
  - e.g., $f = (\neg A \wedge B) \leftrightarrow C$
    $w = \{A : 1, B : 1, C : 0\}$
  - If a sentence α is true in model m, we say that model m **satisfies** α
  - $M(\alpha)$:= the set of all the models that satisfy α

- What about in First-Order Logic?
  - Much more complex!

Models in FOL

A model in FOL consists of:

- A set of objects
- A set of functions + what values will be returned
- A set of predicates + what values will be returned

Example - Model

Consider:

Objects

$Orange$

$Apple$

Predicates

$IsRed(\cdot)$

$HasVitaminC(\cdot)$

Functions

$OppositeOf(\cdot)$

Example model:

Predicate	**Argument**	**Value**
$IsRed$	$Orange$	$False$
$IsRed$	$Apple$	$True$
$HasVitaminC$	$Orange$	$True$
$HasVitaminC$	$Apple$	$True$

Function	**Argument**	**Return**
$OppositeOf$	$Orange$	$Apple$
$Opposite$	$Apple$	$Orange$

https://www2.cs.duke.edu/courses/spring15/compsci270/slides/270_6.pdf

Example – A Model

Sentence: $\forall x\ King(x) \Rightarrow Person(x)$

Consider the following case:

○Three persons:
Richard (King, 50 years old)
John (Richard's brother, 20 years old)
Elizabeth (Richard's mother)

○A dog:
Gigi (Richard's dog)

Example – A Model (Possible World)

○Three persons:
Richard (King, 50 years old)
John (Richard's brother, 20 years old)
Elizabeth (Richard's mother)

○A dog:
Gigi (Richard's dog)

Formalize the setting:

- Objects:
  - Richard, John, Elizabeth, Gigi
- Functions:
  - Age(·)
- Predicates:
  - King(·), Person(·), Brother(·, ·), Mother (·, ·), Dog (·, ·)

A model:

Age(Richard) returns 50, Age(John) returns 20

The following sentences are True and all others are False

- King(Richard), Dog(Gigi, Richard)
- Person(Richard), Person(John), Person(Elizabeth) are True
- Brother(John, Richard), Brother(Richard,John) are True
- Mother(Elizabeth, Richard), Mother(Elizabeth, John) are True

Propositionalization

How to evaluate a sentence with quantifiers?

We eliminate the quantifiers and propositionalize it to a propositional logic sentence.

- Given a model, how to determine if $\forall x\ P$ is true?
  - Concatenate all **universal instantiations** by conjunction
    - instantiation: get rid of all variables by replacing them with ground terms
- Given a model, how to determine if $\exists x\ P$ is true?
  - Concatenate all **existential instantiations** by disjunction

Example - Propositionalization

Determine if this is true:

$\forall x \ King(x) \Rightarrow Person(x)$

Example - Propositionalization

Determine if this is true:
$\forall x\ King(x) \Rightarrow Person(x)$

1. Extend the interpretation:

$x \rightarrow$ Richard

$x \rightarrow$ John

$x \rightarrow$ Elizabeth

$x \rightarrow$ Gigi

Example - Propositionalization

Determine if this is true:

$\forall x\ King(x) \Rightarrow Person(x)$

2. Compute the propositional grounding (Do instantiation)

(In this example, we use universal instantiation)

$King(Richard) \Rightarrow Person(Richard)$
$King(John) \Rightarrow Person(John)$
$King(Elizabeth) \Rightarrow Person(Elizabeth)$
$King(Gigi) \Rightarrow Person(Gigi)$

Example

What do we do?

2. Compute the propositional grounding

$$
\begin{aligned}
&(\quad King(Richard) \Rightarrow Person(Richard) \quad) \land \\
&(\quad King(John) \Rightarrow Person(John) \quad) \land \\
&(\quad King(Elizabeth) \Rightarrow Person(Elizabeth) \quad) \land \\
&(\quad King(Gigi) \Rightarrow Person(Gigi) \quad)
\end{aligned}
$$

Example

Determine if this is true:
$\forall x\ King(x) \Rightarrow Person(x)$

3. See if its' True

$($ $King(Richard) \Rightarrow Person(Richard)$ $) \land$ True

$($ $King(John) \Rightarrow Person(John)$ $) \land$ True

$($ $King(Elizabeth) \Rightarrow Person(Elizabeth)$ $) \land$ True

$($ $King(Gigi) \Rightarrow Person(Gigi)$ $)$ True

This sentence is True!

- Three persons:

Richard (King, 50 years old)
John (Richard's brother, 20 years old)
Elizabeth (Richard's mother)

- A dog:

Gigi (Richard's dog)

Exercise 1

$\forall$ $\wedge$

Determine if this is true in the given model:

$\exists x \; King(x) \wedge OlderThan30(x)$

$=$ $\vee$

1. $x \leftarrow R$

$x \leftarrow J$

⋮

True 2. (King(Rich) $\wedge$ OlderTh30(Rich))

$\vee$ (...)

$\vee$ (...)

$\vee$ (...)

○Three persons: Richard (King, 50 years old) John (Richard's brother, 20 years old) Elizabeth (Richard's mother) ○A dog: Gigi (Richard's dog)

True

Exercise 2

Given:

- FOL sentence:
  - $\forall x, y\ (\text{Friend}(x, y) \land \text{LovesBBQ}(x)) \Longrightarrow \text{LovesBBQ}(y)$
- A finite domain {Alice, Bob} for variable x and y

2×2 = 4

4

Compute the propositional grounding for the FOL sentence with the given domain.

x, y ←		
	A	A
	A	B
	B	A
	B	B

Exercise 2

Given:

- FOL sentence:
  - $\forall x, y\ (\mathrm{Friend}(x, y) \land \mathrm{LovesBBQ}(x)) \implies \mathrm{LovesBBQ}(y)$
- A finite domain {Alice, Bob} for variable x and y

Compute the propositional grounding for the FOL sentence with the given domain.

$$(\mathrm{Friend}(\mathrm{Alice}, \mathrm{Alice}) \land \mathrm{LovesBBQ}(\mathrm{Alice})) \implies \mathrm{LovesBBQ}(\mathrm{Alice})$$
$$(\mathrm{Friend}(\mathrm{Alice}, \mathrm{Bob}) \land \mathrm{LovesBBQ}(\mathrm{Alice})) \implies \mathrm{LovesBBQ}(\mathrm{Bob})$$
$$(\mathrm{Friend}(\mathrm{Bob}, \mathrm{Alice}) \land \mathrm{LovesBBQ}(\mathrm{Bob})) \implies \mathrm{LovesBBQ}(\mathrm{Alice})$$
$$(\mathrm{Friend}(\mathrm{Bob}, \mathrm{Bob}) \land \mathrm{LovesBBQ}(\mathrm{Bob})) \implies \mathrm{LovesBBQ}(\mathrm{Bob})$$

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Knowledge Base (KB)

A KB in propositional logic is a set of sentences. Or it can be considered as a single sentence.

A KB in FOL consists of:

- Objects
- Functions
- Predicates
- Sentences that are **asserted to be True**

Example – Knowledge Base

- Objects:
  - Richard, John, Elizabeth, Gigi
- Functions:
  - Age(·)
- Predicates:
  - King(·), Person(·), Brother(·, ·), Mother (·, ·), Dog (·, ·)

Example – Determine Entailment

- Sentences that are asserted to be True:

==

$\forall x\ King(x) \Rightarrow Person(x)$

$\forall x, y\ Person(x) \land Brother(x, y) \Rightarrow Person(y)$

$\forall x, y\ Brother(x, y) \Rightarrow Brother(y, x)$

General Knowledge

$King(Richard)$

$Brother(John, Richard)$

specific problem

==

Entailment

Given the knowledge base

==

$\forall x\ King(\underset{\text{Richard}}{x}) \Rightarrow Person(x)$

$\forall x, y\ Person(\overset{\text{Richard}}{x}) \wedge Brother(x, \overset{\text{John}}{y}) \Rightarrow Person(\overset{\text{John}}{y})$

$\forall x, y\ Brother(x, y) \Rightarrow Brother(y, x)$

$King(Richard)$

$Brother(John, Richard)$

==

We want to know if the following statements are true:

- $Person(Richard)$ ✓
- $Person(John)$ ✓
- $\forall x\ Person(x) \wedge Brother(y, x) \Rightarrow Person(y)$

Example – Determine Entailment

- Δ: (Our KB, all sentences connected by $\wedge$)

$$\forall x \ \text{King}(x) \Rightarrow \text{Person}(x)$$
$$\forall x, y \ \text{Person}(x) \wedge \text{Brother}(x, y) \Rightarrow \text{Person}(y)$$
$$\forall x, y \ \text{Brother}(x, y) \Rightarrow \text{Brother}(y, x)$$
$$\text{King}(\text{Richard})$$
$$\text{Brother}(\text{John}, \text{Richard})$$

- β: $\underline{\text{Person}(\text{John})}$

Does $\Delta \models \beta$?

- We know how to determine entailment by resolution in propositional logic.
- But FOL has quantifiers. What should we do?

Example – Proof by Refutation

- α:

$$(\ \text{Person}(\text{Richard}) \land \text{Brother}(\text{John}, \text{Richard}) \longrightarrow \text{Person}(\text{John})\ ) \land$$
$$(\ \text{Person}(\text{Richard})\ ) \land$$
$$(\ \text{Brother}(\text{John}, \text{Richard})\ )$$

- β: $\text{Person}(\text{John})$

$\alpha \models \beta$

Show that $(\underline{a} \land \underline{\neg\beta})$ is unsatisfiable.

(Unsatisfiable when *resolvents* contain the empty clause)

Thank you!