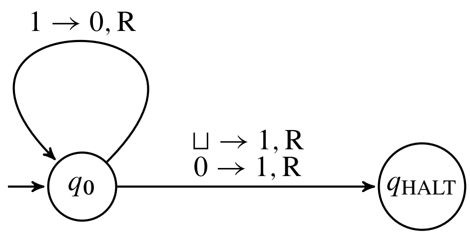
CS 181 FINAL EXAM **NAME** ______________________________

SPRING 2014 **UCLA ID** ______________________________

You have 3 hours to complete this exam. You may assume without proof any statement proved in class.

(2 pts) **1** Give a simple verbal description of the function $f\colon\{0,1\}^* \to \{0,1\}^*$ computed by the Turing machine below.



Solution. The Turing machine interprets its input as an integer n in binary, written left to right starting with the least significant bit, and computes the function $n \mapsto n+1$.

(3 pts) **2** Fermat's conjecture, until recently one of the most famous unproven statements in mathematics, asserts that no positive integers a, b, c can satisfy the equation

$$a^n + b^n = c^n$$

for any integer $n > 2$. Ignoring the fact that the conjecture has now been proved, explain how a solution to the halting problem would allow you to determine the truth or falsity of Fermat's conjecture.

Solution. Consider a Turing machine M that ignores its input and proceeds to enumerate all tuples (a, b, c, n) of positive integers in increasing order by $a + b + c + n$, until it finds a tuple with $n > 2$ and $a^n + b^n = c^n$ at which point it enters the accept state. Clearly $(M, \varepsilon) \notin$ HALT if and only if Fermat's conjecture is true. Thus, a solution to the halting program would allow us to determine the status of Fermat's conjecture.

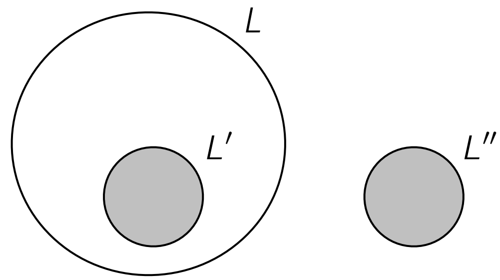
(4 pts) **3** Prove that Turing-recognizable languages are closed under the prefix operation.

Solution. Let L be a Turing-recognizable language, and let M be a Turing machine that recognizes L. Consider the nondeterministic Turing machine M' which on input w, nondeterministically guesses a string $u \in \Sigma^*$, runs M on input wu, and outputs the result (if M halts on wu).

- If $w \in \text{prefix}(L)$, there exists $u \in \Sigma^*$ such that M accepts wu. Therefore, M' accepts w on at least one computational path.
- If $w \notin \text{prefix}(L)$, then M rejects all strings of the form wu, either explicitly or by failing to halt. Therefore, M' does not accept such w on any computational path.

Thus, M' recognizes $\text{prefix}(L)$. This completes the proof because nondeterministic Turing machines recognize only Turing-recognizable languages.

(4 pts) **4** Language L is said to *separate* languages L', L'' if $L' \subseteq L$ and $L'' \subseteq \overline{L}$, as illustrated in the figure below. Prove that any two disjoint languages L', L'' whose complements are Turing-recognizable are separated by some decidable language.



Solution. Let M' and M'' be Turing machines that recognize $\overline{L'}$ and $\overline{L''}$, respectively. On input w, we will run M' and M'' in parallel until one of them accepts w. This will happen eventually because $L(M') \cup L(M'') = \overline{L'} \cup \overline{L''} = \overline{L' \cap L''} = \overline{\varnothing} = \Sigma^*$. If M'' is the first to accept, then $w \notin L''$ and we will accept w; if M' is the first to accept, then $w \notin L'$ and we will reject w. Either way, the described Turing machine always halts, accepts all of L', and rejects all of L'', which means that its language is decidable and separates L' and L''.

5 For each of the languages below, determine whether it is decidable. Prove your answers.

(3 pts) **a.** $\{\langle D_1, D_2\rangle$: the DFAs D_1 and D_2 accept a string in common$\}$

(3 pts) **b.** $\{\langle G, w\rangle$: the CFG G generates a string that starts with $w\}$

(3 pts) **c.** $\{\langle n\rangle$: the decimal expansion of π contains n consecutive 0s$\}$

(3 pts) **d.** $\{\langle M, q\rangle$: the Turing machine M enters the state q on some input$\}$

a. Decidable: create a DFA for $L(D_1) \cap L(D_2)$ using the Cartesian-product construction from class and check whether an accept state is reachable from the start state.

b. Decidable. Construct a PDA for $L(G)$ and a DFA for the regular language $w\Sigma^*$; then combine them into a PDA for $L(G) \cap w\Sigma^*$ using the Cartesian-product construction from class; finally, convert the resulting PDA into a grammar and test whether the grammar generates any strings (using the algorithm from class).

c. Decidable. Let L denote the language in question. If the decimal expansion of π contains arbitrarily long runs of 0s, then every n is included and therefore $L = 0 \cup 1\Sigma^*$; otherwise L is finite. Either way L is regular and hence decidable.

d. Undecidable. Let L denote the language in question. If L were decidable, then we would be able to check whether the language of any given Turing machine M is empty by running L's decider on (M, q_{accept}). This would contradict Rice's theorem, which asserts that every nontrivial property of the language of a Turing machine (such as emptiness) is undecidable.