CS 181 EXAM #2 **NAME** ______________________________

FALL 2016 **UCLA ID** ______________________________

You have 90 minutes to complete this exam. You may state without proof any fact taught in class or assigned as homework.

1 Find a regular expression for each of the following languages over $\Sigma = \{0, 1\}$:

(2 pts) **a.** strings that begin with 00 and do not end with 11;

(2 pts) **b.** strings in which both the number of 0s and the number of 1s are even;

(2 pts) **c.** strings containing no more than one occurrence of 00 (the string 000 has two occurrences).

Solution.

a. $00 \cup 00\Sigma \cup 00\Sigma^*(00 \cup 01 \cup 10)$;

b. $\Big(00 \cup 11 \cup (01 \cup 10)(00 \cup 11)^*(01 \cup 10)\Big)^*$

c. $1^*(01^+)^*(\varepsilon \cup 0)1^*(01^+)^*(\varepsilon \cup 0)$

In all three parts, you could solve the problem in a roundabout manner by constructing an NFA and converting it to a regular expression. But there are easier solutions, as follows. For **a.,** it helps to subdivide into cases and handle each of them separately; think back to my snowdrift metaphor. For **b.,** the problem gets *much* easier if you group the symbols in pairs. For **c.,** it helps to note that the language is precisely LL, where L is the set of all strings that do not contain 00.

(1 pt) **2** Find a string of minimum length *not* in the language $0^*(100^*)^*1^*$.

Solution. 110

(3 pts) **3** Let L be an infinite regular language. Prove that L can be partitioned into two disjoint infinite regular languages.

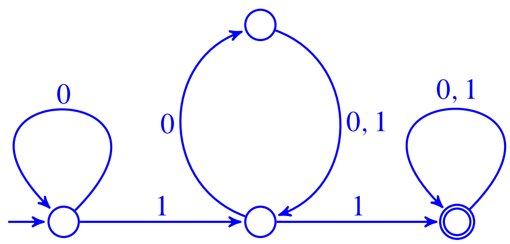
Solution. By the pumping lemma, there is $p \geqslant 0$ such that every string $w \in L$ of length at least p can be written as $w = xyz$, where y is nonempty and $xy^iz \in L$ for all $i = 0, 1, 2, 3, \ldots$.

So, fix an arbitrary string $w \in L$ of length at least p (it exists because L is infinite). Let $w = xyz$ be a decomposition guaranteed by the pumping lemma. Partition $L = A \cup (L \setminus A)$, where $A = \{xy^iz : i = 0, 2, 4, 8, \ldots\}$.

- DISJOINTNESS: A and $L \setminus A$ are disjoint by definition.
- INFINITENESS: $\{xy^iz : i = 0, 2, 4, 8, \ldots\} \subseteq A$ and $\{xy^iz : i = 1, 3, 5, 7, \ldots\} \subseteq L \setminus A$.
- REGULARITY: A is regular because it is given by a regular expression, $x(yy)^*z$, which makes $L \setminus A$ regular as well by the closure properties.

(3 pts) **4** Construct a DFA for $\Sigma^* 1 (\Sigma\Sigma)^* 1 \Sigma^*$ with the smallest possible number of states, where $\Sigma = \{0, 1\}$. Prove that your DFA is the smallest possible.

Solution. The following DFA with four states recognizes $L = \Sigma^* 1 (\Sigma\Sigma)^* 1 \Sigma^*$:



By the Myhill–Nerode theorem, no smaller DFA exists because each of the four strings $\varepsilon, 1, 10, 11$ is in a different equivalence class of $\equiv_L$. Their distinguishing suffixes are as follows:

	ε	1	10	11
ε				
1	**1**			
10	**01**	**01**		
11	$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	

5 True or false? Prove your answer.

(2 pts) **a.** If A, $A \cap B$, and $A \cup B$ are regular languages, then B is regular as well.

(2 pts) **b.** For any nonregular languages $L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots$, the union $\bigcup_{n=1}^{\infty} L_n$ is nonregular.

Solution.

a. True. One can obtain B from the regular languages A, $A \cap B$, and $A \cup B$ using set difference: $B = (A \cup B) \setminus (A \setminus (A \cap B))$. Since regular languages are closed under set difference, B must be regular as well.

b. False. Define $L_n = (\Sigma \cup \varepsilon)^n \cup \{0^m 1^m : m \geqslant 0\}$. We claim that:

- each L_n is nonregular;
- $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$;
- the union $\bigcup_{n=1}^{\infty} L_n$ is the regular language Σ^*.

The second and third claims are obvious. For the first claim, the nonregular language $\{0^m 1^m : m \geqslant 0\}$ is the set difference of L_n and a finite (hence regular!) language. This means that each L_n is nonregular—otherwise, the closure properties would force the regularity of $\{0^m 1^m : m \geqslant 0\}$.

6 For each of the following languages over $\Sigma = \{0, 1\}$, determine whether it is regular, and prove your answer:

(2 pts) **a.** strings that contain exactly twice as many 1s as 0s;

(2 pts) **b.** strings that do not contain a palindrome of length 2016 or shorter as a substring;

(2 pts) **c.** strings that can be made into a palindrome by removing fewer than 2016 symbols;

(2 pts) **d.** odd-length strings in which the middle symbol also occurs elsewhere in the string.

Solution.

In each part, L stands for the language in question.

a. Nonregular. For any positive integers $i \neq j$, we have $0^i\mathbf{1}^{\mathbf{2i}} \in L$ but $0^j\mathbf{1}^{\mathbf{2i}} \notin L$. Therefore, each of the strings $0, 00, 000, \ldots, 0^n, \ldots$ is in a different equivalence class of $\equiv_L$. Since there are infinitely many equivalence classes, L is nonregular by the Myhill–Nerode theorem.

b. Regular. Let F denote the finite (hence regular) language of palindromes of length at most 2016. Then L can be built up from F using the Kleene star, concatenation, and complement operations: $L = \overline{\Sigma^* F \Sigma^*}$. Since regular languages are closed under these operations, L must be regular as well.

c. Nonregular. For any positive integers $i \neq j$, we have $0^{2016i}\mathbf{1}^{\mathbf{2016}}\mathbf{0}^{\mathbf{2016i}} \in L$ but $0^{2016j}\mathbf{1}^{\mathbf{2016}}\mathbf{0}^{\mathbf{2016i}} \notin L$. Therefore, each of the strings $0^{2016}, 0^{4032}, \ldots, 0^{2016n}, \ldots$ is in a different equivalence class of $\equiv_L$. Since there are infinitely many equivalence classes, L is nonregular by the Myhill–Nerode theorem.

d. Nonregular. For any positive integers $i \neq j$, we have $0^{2i}\mathbf{1}\mathbf{0}^{\mathbf{2i}} \notin L$ but $0^{2j}\mathbf{1}\mathbf{0}^{\mathbf{2i}} \in L$. Therefore, each of the strings $0^2, 0^4, \ldots, 0^{2n}, \ldots$ is in a different equivalence class of $\equiv_L$. Since there are infinitely many equivalence classes, L is nonregular by the Myhill–Nerode theorem.