System impulse response

This lecture introduces time-domain analysis of systems, including the impulse response. It also discusses linear time-invariant systems. Topics include:

- Impulse response definition
- Impulse response of LTI systems
- The impulse response as a sufficient characterization of an LTI system
- Impulse response and the convolution integral

Motivation for this lecture

In the last lecture, we defined a system, S. The system equations are used to calculate an output signal given an input signal.

In real life, we often do not have the luxury of knowing exactly what S is, or perhaps we only know it imperfectly. And even if we did know it, it could take on a very complicated form.

The *impulse response* is a characterization of the system that, for linear time-invariant systems, *enables to calculate the output for* **any** *input*. In this manner, it is a full time-domain description of the system.

How do we calculate this convenient quantity? Its name tells us the answer. We input an impulse into the system, and see what comes out at the output.

We will unpack this idea in this lecture.

System response

If we want to know how a system, S, transforms input to output signals, one way to characterize S is by applying an input and measuring the output (or response) of the system.

For example, a type of response is called the *zero-input* response. Here, we input $x(t) = 0$ and measure $y(t) = S(x(t))$.

Question: What is the zero-input response if S is a linear system?

Impulse response

The *impulse response* of a linear system, denoted $h(t, \tau)$ is the output of the system at time t to an impulse at time τ. Denoting the system as H, the impulse response is mathematically:

$$h(t, \tau) = H(\delta(t - \tau))$$

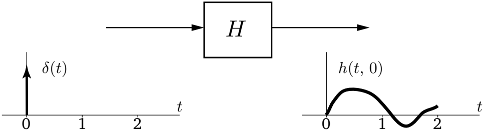
There are important things to be careful of when looking at this equation.

- The t on the left and right hand side of these equations *are not the same*!
- The t on the left hand side is the impulse response at a specific value of time.
- The t on the right hand side varies across all time.
- The output at the specific time t on the left will depend on the input at several times t on the right.
- e.g., the integrator system,

$$h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Impulse response, t

An example of these t's not being the same is shown below. In this example, let $\tau = 0$.



It may be tempting to write:

$$h(1,0) = H(\delta(1))$$

This is wrong.

- On the left, $\delta(1) = 0$. We know if H is linear, then $H(0) = 0$, implying that $h(1,0) = 0$.
- But in general, the impulse response can be non-zero, i.e., $h(1,0) \neq 0$ in the above diagram, if the impulse response produces some non-zero response.

Impulse response for a time-invariant system

If H is time-invariant, then delaying the input by τ should delay the output by τ as well. i.e., if

$$h(t, \tau) \quad = \quad H(\delta(t-\tau))$$

then with time-invariant H, we have that

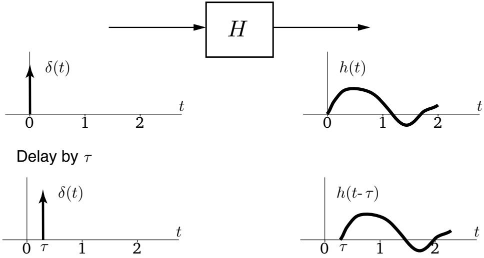
$$\begin{aligned} h(t, 0) &= H(\delta(t)) \\ h(t-\tau, 0) &= H(\delta(t-\tau)) \\ &= h(t, \tau) \end{aligned}$$

Therefore, $h(t, \tau) = h(t-\tau, 0)$ for a time-invariant system. Since the r.h.s. of this equation only depends on $t-\tau$, we omit the second argument, 0, and write that the impulse response as:

$$h(t) = H(\delta(t))$$

Impulse response of a time-invariant system (cont.)

This property of the impulse response for a time-invariant system is drawn below:



A restatement of linearity

Recall that a system, H, is linear if for $y_n = H(x_n)$ where n is a subscript denoting different signals, and a_n are constants, we have that:

$$\sum_n a_n y_n = H\left(\sum_n a_n x_n\right)$$

i.e., it has both homogeneity and superposition. Thus, summation and the system operator can be interchanged.

In particular, this holds over integration (which is summation over infinitesimal intervals). That is, if $y = H(x)$, then:

$$\int_{-\infty}^{\infty} a(\tau)y(t-\tau)d\tau = H\left(\int_{-\infty}^{\infty} a(\tau)x(t-\tau)d\tau\right)$$

Impulse response of a LTI system

A system is called linear time-invariant (LTI) if it is linear and time-invariant. For such a system, we have that its impulse response is given by

$$h(t) = H(\delta(t))$$

For any LTI system, if we know $h(t)$, then we can calculate $H(x(t))$ for any input $x(t)$. Thus, the LTI system is completely characterized by its impulse response.

Let's derive this result. We start by writing $x(t)$ in terms of $\delta(t)$ functions, since our goal is to use the impulse response. We know from our signal models lecture that:

$$x(\tau)\delta(\tau) = x(0)\delta(\tau)$$

Colloquially, $\delta(\tau)$ samples $x(\tau)$ at the impulse location (i.e., 0) and returns x at this sample. Now, if we delay the impulse by t, we have that:

$$x(\tau)\delta(\tau - t) = x(t)\delta(\tau - t)$$

Impulse response of a LTI system (cont.)

Continuing...

$$x(\tau)\delta(\tau - t) = x(t)\delta(\tau - t)$$

At this point, how do we simplify this expression? We can use the fact that the integral of the δ function is 1. Integrating both sides (w.r.t. τ), we get that:

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)\delta(\tau - t)d\tau &= \int_{-\infty}^{\infty} x(t)\delta(\tau - t)d\tau \\ &= x(t) \end{aligned}$$

Note also that this implies that this equation is also true:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

(Why?)

Impulse response of a LTI system (cont.)

Let's now get to showing that the impulse response is a sufficient characterization of an LTI system. First, let's take our expression for $x(t)$ and put it through our system, H.

$$\begin{aligned} y(t) &= H(x(t)) \\ &= H\left(\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right) \end{aligned}$$

Applying the linearity of H, we have that

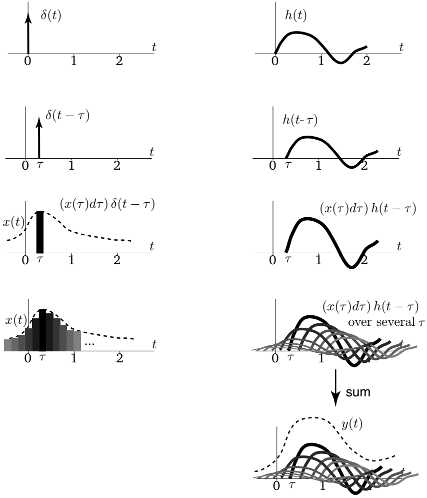
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)H\left(\delta(t-\tau)\right)d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned}$$

This expression may be opaque at the moment. We'll provide some intuition for what it means through illustration, but really, we'll get to the crux of what this expression is in next lecture. This is a form of a signal operation called *convolution*, which is a critical part of signal processing.

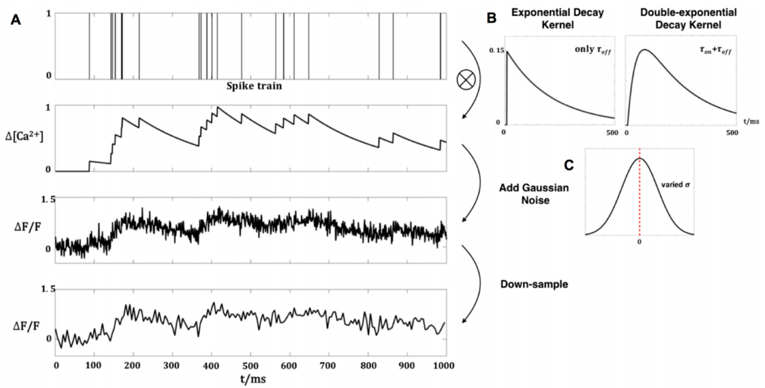
Impulse response of a LTI system (cont.)

Illustration of:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$



Example: calcium imaging



(From Sun, Kao, et al., IEEE NER 2017)