ECE 102, Fall 2020 — **FINAL STATISTICS**

Department of Electrical and Computer Engineering
University of California, Los Angeles
Prof. J.C. Kao
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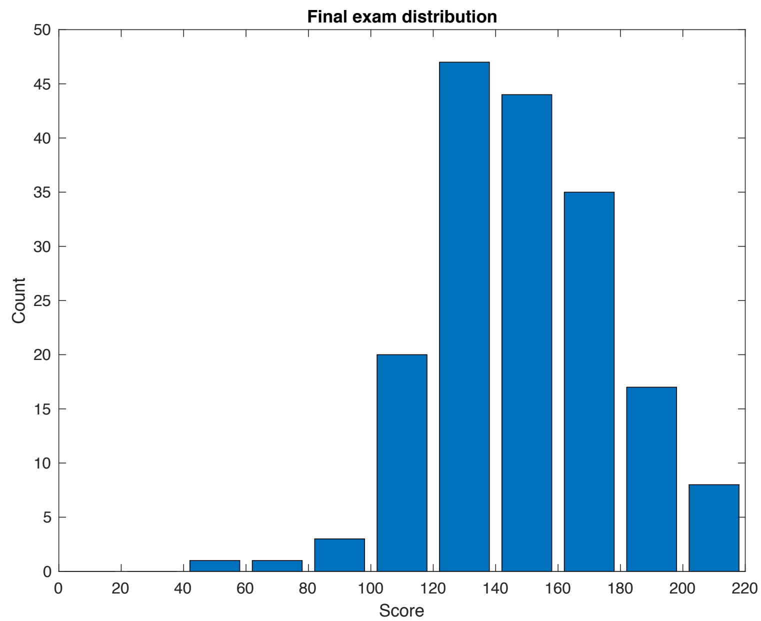

Statistics: Mean: 148.3 (out of 200) or 74.2%, Standard deviation: 29.3 (14.6%), Median: 146 (73%), Maximum score: 217 (100%), Number of exams: 176

Comments:

- If you have grading questions, please submit through Gradescope. Regrades should *only be submitted* if you believe we applied the rubric mistakenly. When we regrade, we re-evaluate the entire question, and so while rare, losing points is possible if we mis-graded your work.
- If you have particular questions about a midterm question and its grading, please see: Q1: Arunabh, Q2: Tonmoy, Q3: Guangyuan, Q4: Arunabh, Bonus: Tonmoy

ECE102, Fall 2020 | **Final Exam**

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UCLA True Bruin academic integrity principles apply.
Open: Notes, Book.
Closed: Internet, except to use Piazza and CCLE.
3:00-6:00pm.
Wednesday, 16 Dec 2020.

State your assumptions and reasoning.
No credit without reasoning.

Name: ____________________

Signature: ____________________

ID#: ____________________

Problem 1	_____	/ 50
Problem 2	_____	/ 50
Problem 3	_____	/ 54
Problem 4	_____	/ 46
BONUS	_____	/ 15 bonus points
Total	_____	/ 200 points + 15 bonus points

1. **Signal and System Basics** (50 points)

   (a) (16 points) Consider an LTI system whose response to the signal $x_1(t)$ in Figure (1a) is the signal $y_1(t)$ illustrated in Figure (1b).

      i. (8 points) Determine and sketch the response of the system to the input $x_2(t)$ shown in Figure (1c).

      ii. (8 points) Determine and sketch the response of the system to the input $x_3(t)$ shown in Figure (1d).

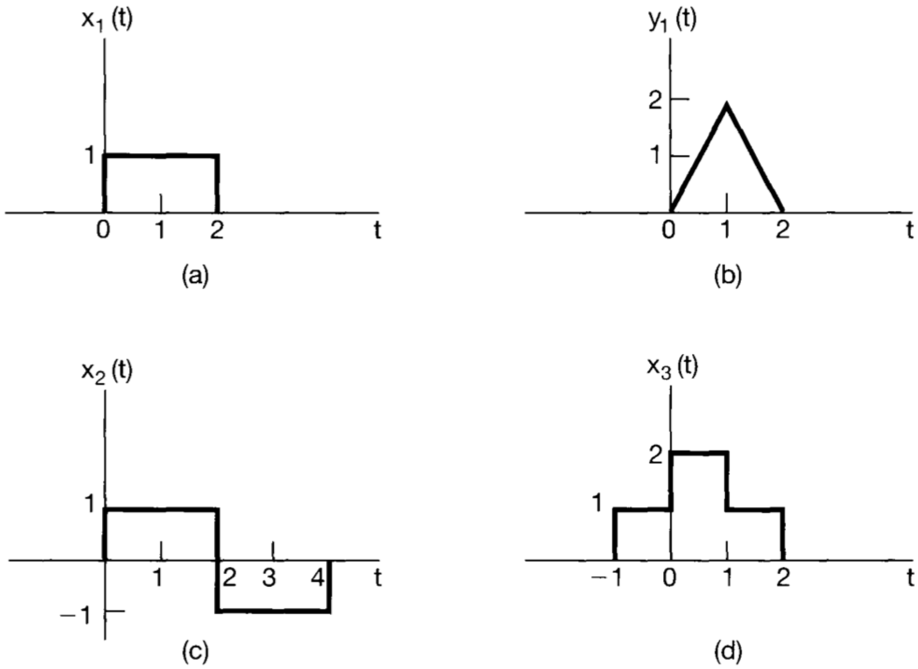


Figure 1: Input-output relationship

Solution:

i. From the input $x_2(t)$, we can see that $x_2(t)$ can be written in terms of $x_1(t)$ as,

$$x_2(t) = x_1(t) - x_1(t-2)$$

As is was said that, the system is an LTI system, the output can be written as:

$$x_2(t) = x_1(t) - x_1(t-2) \rightarrow y_2(t) = y_1(t) - y_1(t-2)$$

The output can be sketched as follows:

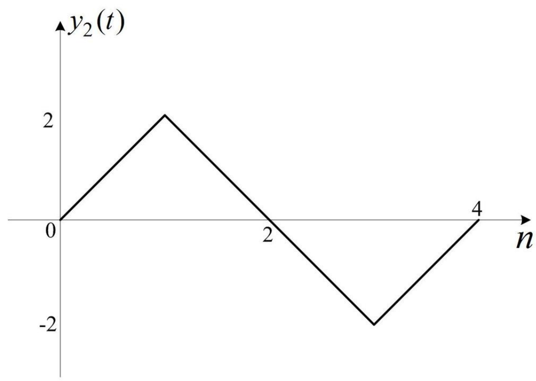


Figure.1

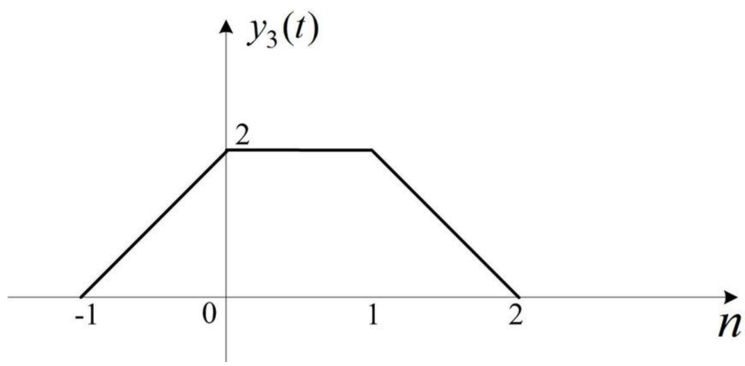
ii. We can see that the input $x_3(t)$ can written as:

$$x_3(t) = x_1(t) + x_1(t+1)$$

As it was said that, the system is linear time-invariant, the output can be written as:

$$x_3(t) = x_1(t) + x_1(t+1) \rightarrow y_3(t) = y_1(t) + y_1(t+1)$$

The signal $y_3(t)$ can be depicted as follows:



(b) (24 points) For each statement below, determine whether it is true or false. You must justify your answer to receive full credit.

i. (8 points) If $T_1(x)$ and $T_2(x)$ are LTI systems, then $T_2(T_1(x))$ must be LTI.

Solution: True. Let system have $T_1(x) = y$ and system have $T_2(y) = z$. As both the systems are linear, their series connection is also linear $T_2(T_1(ax_1 + bx_2)) = T_2(ay_1 + by_2) = az_1 + bz_2$. As both of them are time invariant, their series connection is also time invariant. $T_2(T_1(x(t - t_0))) = T_2(y(t - t_0)) = z(t - t_0)$.

ii. (8 points) If $T_1(x)$ and $T_2(x)$ are nonlinear systems, then $T_2(T_1(x))$ must be nonlinear.

Solution: False. Let $y = T_1(x(t)) = x^2(t)$ and $z = T_2(y(t)) = \sqrt{y(t)}$ be nonlinear systems. The series combination $z(t) = T_2(T_1(x(t))) = T_2(x^2(t)) = x(t)$ is linear.

iii. (8 points) Consider a time-invariant system with input $x(t)$ and output $y(t)$. If $x(t)$ is periodic with period T, then $y(t)$ is also periodic with period T.

Solution: True. Given that $x(t)$ is passed through a time-invariant system to give output $y(t)$:

$$x(t) \rightarrow y(t)$$

Since the system is time-invariant,

$$x(t - t_0) \rightarrow y(t - t_0)$$

If $x(t)$ is periodic with period T, then $x(t - T) = x(t)$, which implies that,

$$x(t - T) = x(t) \implies y(t - T) = y(t)$$

Therefore, we can conclude that $y(t)$ is also periodic.

(c) (10 points) The signal $y(t)$ is generated by convolving a band-limited signal $x_1(t)$ with another band-limited signal $x_2(t)$, that is:

$$y(t) = x_1(t) * x_2(t) \tag{1}$$

where,

$$X_1(j\omega) = 0 \text{ for } |\omega| > 1000\pi$$
$$X_2(j\omega) = 0 \text{ for } |\omega| > 2000\pi$$

Next, $y(t)$ is sampled via an impulse train to obtain

$$y_p(t) = \sum_{n=-\infty}^{+\infty} y(nT)\delta(t - nT)$$

Specify the range of values for the sampling period T which ensures that $y(t)$ is recoverable from $y_p(t)$.

Solution: We know that:

$$Y(j\omega) = X_1(j\omega)X_2(j\omega)$$

Given that $X_1(j\omega) = 0$ for $|\omega| > 1000\pi$ and $X_2(j\omega) = 0$ for $|\omega| > 2000\pi$, hence,

$$Y(j\omega) = 0 \text{ for } |\omega| > 1000\pi$$

Therefore the Nyquist rate would be:

$$\text{Nyquist rate} = 2 \times 1000\pi$$

Therefore the sampling period T can be at most:

$$T = \frac{2\pi}{2000\pi} = 10^{-3}s$$

Therefore in order to recover $y(t)$ from $y_p(t)$ the value of T must be $< 10^{-3}s$.

2. **Fourier transform, Frequency response and Output of LTI systems** (50 points)

(a) (30 points) Consider an LTI system with the impulse response:

$$h(t) = \left(\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}\right) 2j\sin(20t)$$

i. (5 points) Show that

$$\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t} = \frac{15}{\pi}Sa(5t)Sa(15t)$$

where $Sa(t) = \frac{sin(t)}{t}$.

Solution:

$$\begin{aligned}\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t} &= \frac{\sin(5t)}{5t}\frac{\sin(15t)}{\pi t}\\ &= \frac{1}{\pi}Sa(5t)\frac{\sin(15t)}{t}\\ &= \frac{15}{\pi}Sa(5t)\frac{\sin(15t)}{15t}\\ &= \frac{15}{\pi}Sa(5t)Sa(15t)\end{aligned}$$

ii. (15 points) Compute the Fourier transform of

$$\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}$$

You must indicate which properties and pairs you are using to arrive at your answer. You must show all steps to receive credit.

Hint: You may find the following Fourier Transform useful.

$$\frac{B}{2\pi}Sa\left(\frac{Bt}{2}\right) \Longleftrightarrow \text{rect}\left(\frac{\omega}{B}\right)$$

Solution: In the Fourier transform table we have the following pair:

$$\frac{B}{2\pi}Sa(\frac{Bt}{2}) \overset{\mathcal{F}}{\longleftrightarrow} rect(\frac{\omega}{B})$$

Then using the above pair we have

$$\begin{aligned}Sa(5t) &\overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{5}rect(\frac{\omega}{10})\\ Sa(15t) &\overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{15}rect(\frac{\omega}{30})\end{aligned}$$

Now by the multiplication property of Fourier Transform

$$\mathcal{F}\left\{\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}\right\} = \frac{15}{\pi}\times\frac{1}{2\pi}\times\frac{\pi}{15}\text{rect}(\frac{\omega}{30}) * \frac{\pi}{5}\text{rect}(\frac{\omega}{10})$$
$$= \frac{1}{2\pi}\text{rect}(\frac{\omega}{30}) * \frac{\pi}{5}\text{rect}(\frac{\omega}{10})$$

For $0 < \omega < 10$:

$$\frac{1}{2\pi}\text{rect}(\frac{\omega}{30}) * \frac{\pi}{5}\text{rect}(\frac{\omega}{10}) = \int_{\omega-5}^{\omega+5}\frac{1}{10}d\tau$$
$$= 1$$

For $10 < \omega < 20$:

$$\frac{1}{2\pi}rect(\frac{\omega}{30}) * \frac{\pi}{5}rect(\frac{\omega}{10}) = \int_{\omega-5}^{15}\frac{1}{10}d\tau$$
$$= \frac{1}{10}(20-\omega)$$

For $\omega > 20$:

$$\frac{1}{2\pi}rect(\frac{\omega}{30}) * \frac{\pi}{5}rect(\frac{\omega}{10}) = 0$$

Since the convolution of two rectangular pulse is symmetric so,

$$H_1(j\omega) = \begin{cases} 1, & -10 < \omega < 10 \\ \frac{1}{10}(20-\omega), & 10 < \omega < 20 \\ \frac{1}{10}(20+\omega), & -20 < \omega < -10 \\ 0, & |\omega| > 20 \end{cases}$$

where $H_1(j\omega) = \mathcal{F}\left\{\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}\right\}$.

iii. (10 points) Use the properties of Fourier transform to determine the frequency response $H(j\omega) = \mathcal{F}[h(t)]$ and plot it on graph below. You must indicate which property you are using to arrive at your answer. You must show all steps to receive credit. For your convenience, recall that $h(t)$ is:

$$h(t) = \left(\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}\right)2j\sin(20t)$$

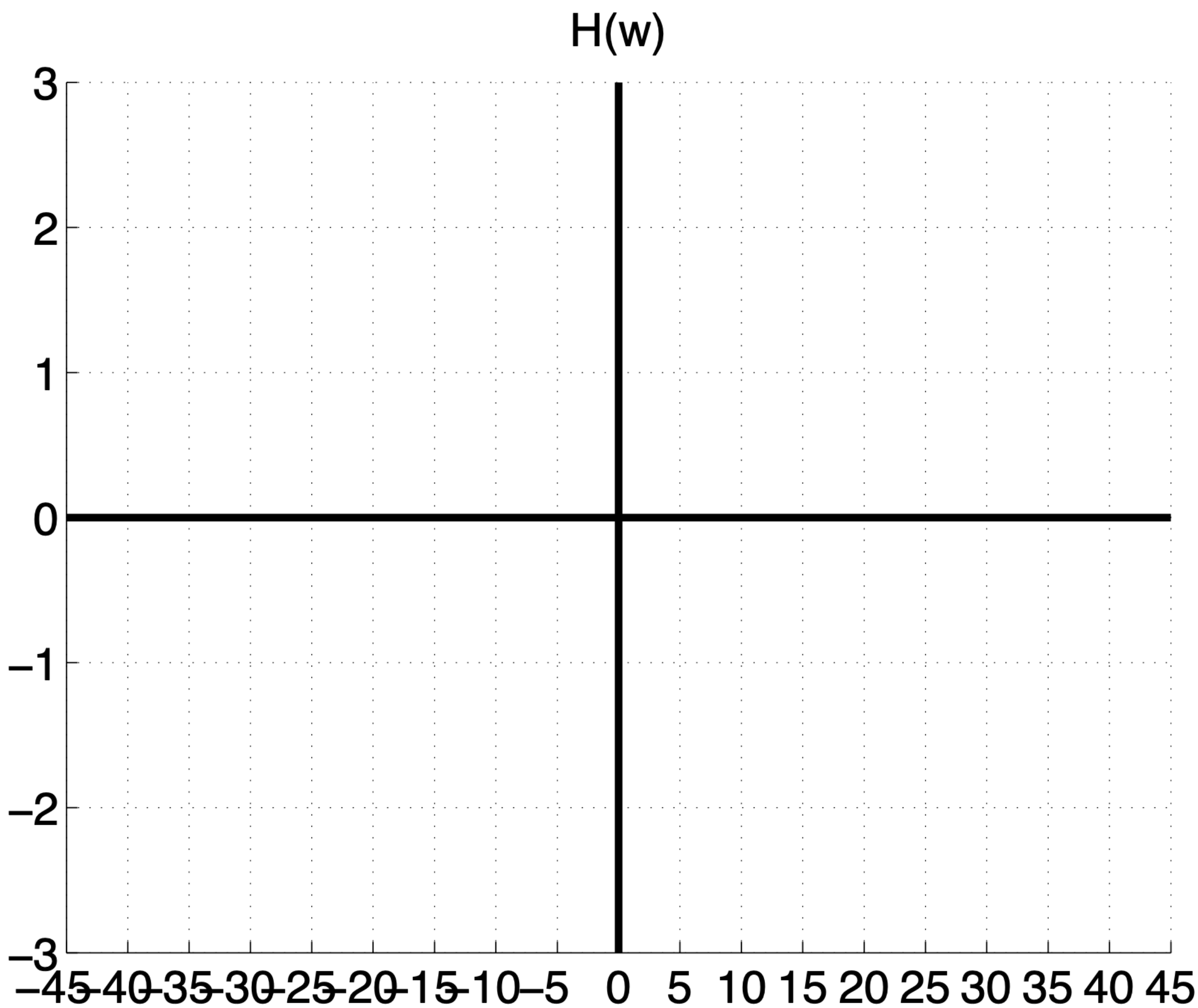


Figure 2: Frequency response of LTI system

Solution: Using the modulation property of Fourier transform we have

$$\begin{aligned} H(j\omega) &= \left\{ \frac{1}{2j} H_1(j(\omega - 20)) - \frac{1}{2j} H_1(j(\omega + 20)) \right\} 2j \\ &= H_1(j(\omega - 20)) - H_1(j(\omega + 20)) \end{aligned}$$

where $H_1(j\omega)$ is defined in (a)(ii). $H(j\omega)$ is plotted below

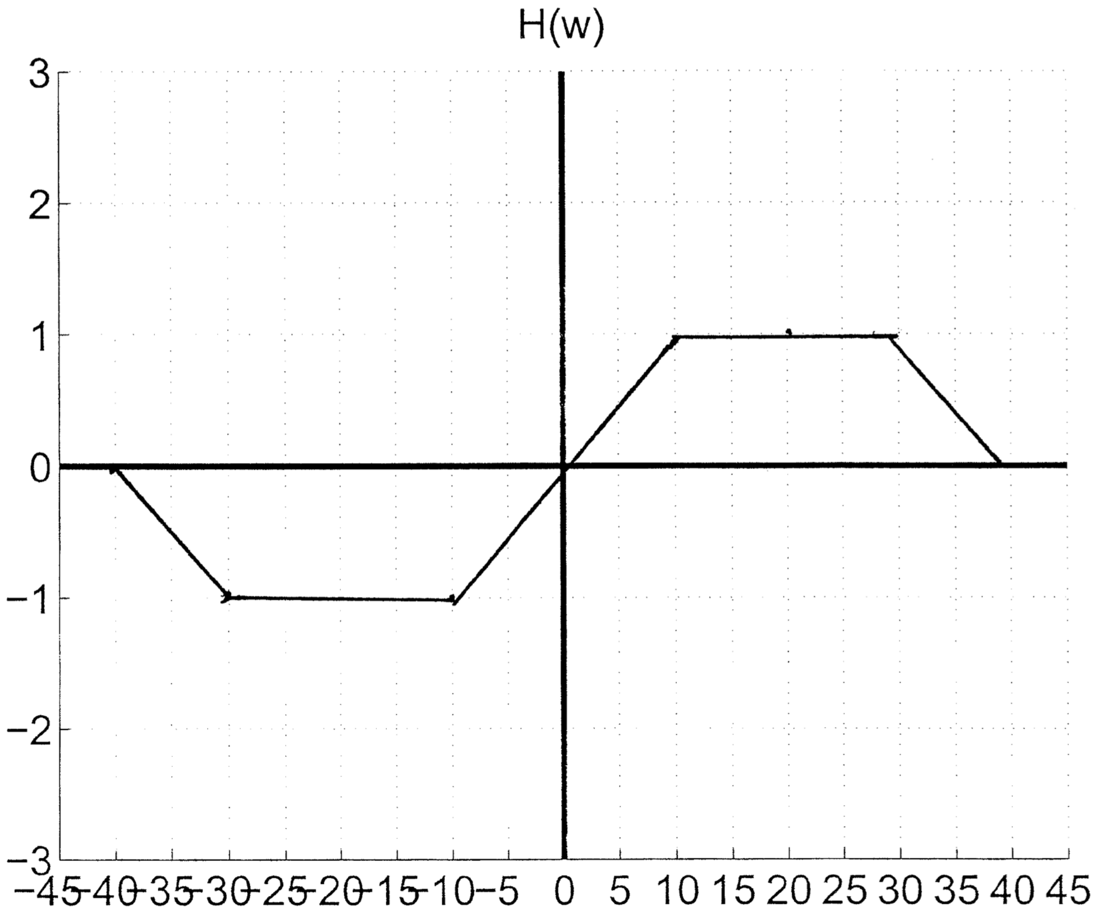


Figure 3: $H(j\omega)$

(b) (20 points) Consider an LTI system with Frequency response shown below:

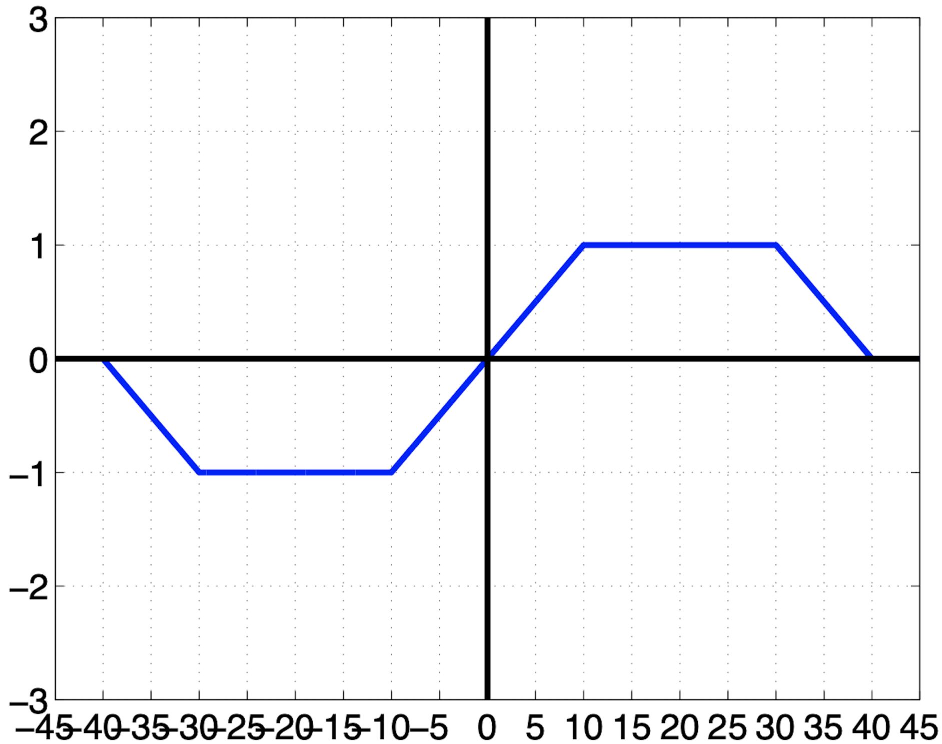


Figure 4: Frequency Response of LTI system

For the LTI system with the frequency response sketched above, determine the output $y(t)$ for the input $x(t)$ given below, which is the Fourier Series expansion for a periodic sawtooth waveform with fundamental frequency $\omega_0 = 5$ rads/sec.

$$x(t) = \sum_{k=-\infty}^{k=\infty} a_k e^{jk5t}$$

where,

$$a_k = \frac{j(-1)^k}{k\pi} \ \ fork \neq 0, \ a_0 = 0$$

Show your work and write your expression for $y(t)$ in the space provided below.

Solution: From the problem statement we observe that $x(t)$ is a sum of weighted complex exponentials. Since complex exponentials are eigenfunctions of LTI systems, so

$$y(t) = \sum_{k=-\infty}^{k=\infty} H(jk5) a_k e^{jk5t}$$

Inspecting the Freequency Response drawn above we know the following:

i. Any frequency $\omega_k = 5k$ for which $|\omega_k| \geq 40$ is rejected

$$H(jk5) = 0, \;\; |k| \geq 8$$

ii. The frequencies $\omega_k = \pm 10, \pm 15, \pm 20, \pm 25, \pm 30$ are passed with $\pm$ unity gain

$$H(jk5) = sgn(k), \;\; 2 \leq |k| \leq 6$$

iii. The frequencies $\omega_k = 5$ and $\omega_k = 35$ are passed with a gain of 0.5

$$H(jk5) = 0.5, \;\; k = 1, 7$$

iv. The frequencies $\omega_k = -5$ and $\omega_k = -35$ are passed with a gain of -0.5

$$H(jk5) = -0.5, \;\; k = -1, -7$$

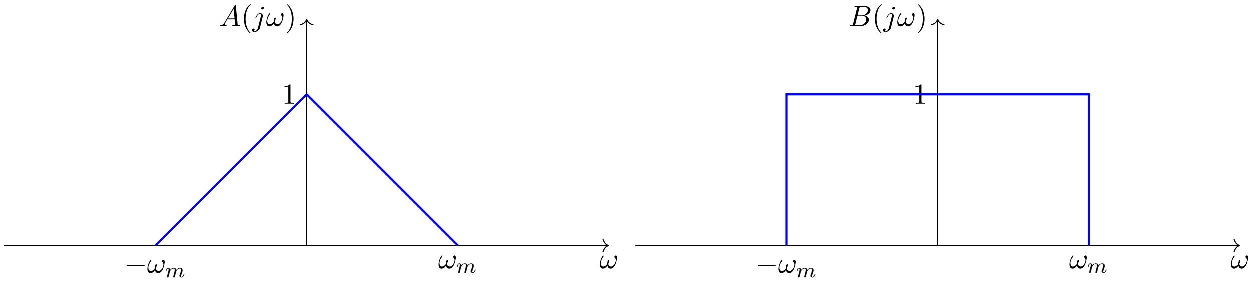
v. DC frequency is rejected

$$H(jk5) = 0, \;\; k = 0$$

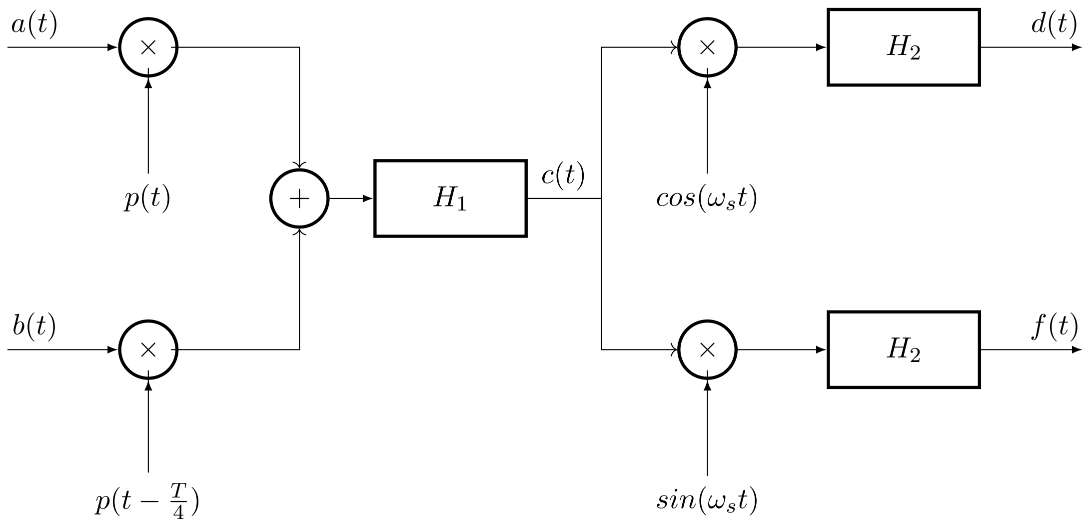
Then using the above information, we have

$$y(t) = -\frac{1}{2}a_{-7}e^{-j35t} + \sum_{\substack{k=-6 \\ k \neq \pm 1 \\ k \neq 0}}^{k=6} sgn(k)a_k e^{jk5t} + \frac{1}{2}a_7 e^{j35t} + \frac{1}{2}a_1 e^{j5t} - \frac{1}{2}a_{-1}e^{-j5t}$$

3. **Modulation and demodulation** (56 points)
It is given that input signals $a(t)$ and $b(t)$ are real and even, with Fourier Transforms shown below.



Consider the system below:



Where we define the following filters with conditions:

$$H_1(j\omega) = \begin{cases} T, & \omega_s - \omega_m \le |\omega| \le \omega_s + \omega_m \\ 0, & \text{otherwise.} \end{cases}$$

$$H_2(j\omega) = \begin{cases} 1, & |\omega| \le \omega_m \\ 0, & \text{otherwise.} \end{cases}$$

$$p(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

$$\frac{2\pi}{T} = \omega_s > 2\omega_m$$

Note: You are not required to consider amplitude scaling when answering this question.

(a) (15 points) In lectures, we found the Fourier Transform of the impulse train using the Fourier Series. Using a similar argument, derive the Fourier Transform of the shifted impulse train, $p(t - \frac{T}{4})$.

Hint: Recall $e^{-j\frac{\pi}{2}} = -j$. *You may need this to simplify your solution.*

Solution:

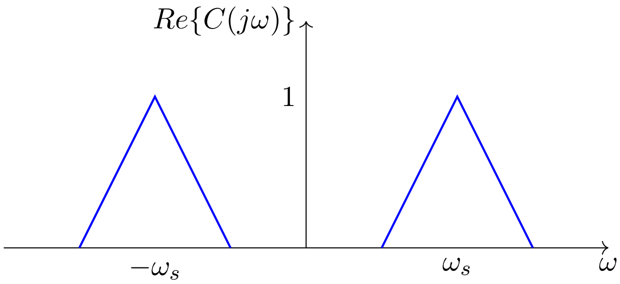
$$\mathcal{F}(\delta_T(t - \frac{T}{4})) = \frac{1}{T}\sum_k \mathcal{F}[e^{j\frac{2\pi}{T}k(t-\frac{T}{4})}] =$$

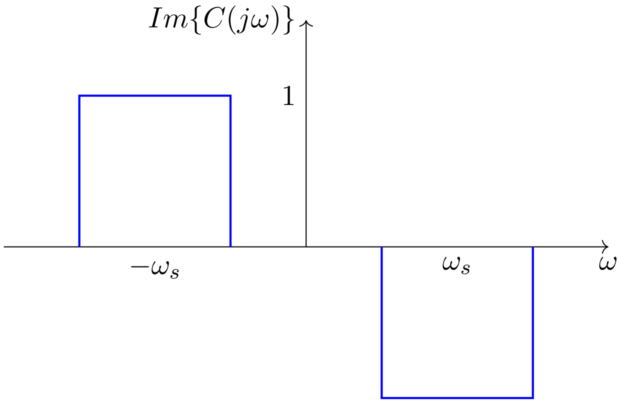
$$\frac{1}{T}\sum_k e^{-j\frac{2\pi k}{4}}\mathcal{F}[e^{j\frac{2\pi}{T}kt}] = \frac{1}{T}\sum_k e^{-j\frac{\pi}{2}k}2\pi\delta(\omega - k\omega_0) = \frac{1}{T}\sum_k (-j)^k 2\pi\delta(\omega - k\omega_0)$$

$$= \omega_0 \sum_k (-j)^k \delta(\omega - k\omega_0)$$

(b) (16 points) Plot the Real and Imaginary components of $C(j\omega)$ (the Fourier Transform of the output signal of the bandpass filter H_1).

Solution:

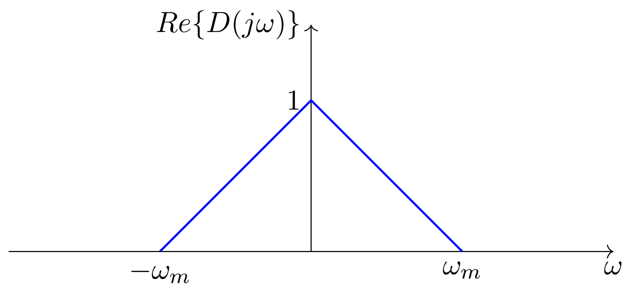


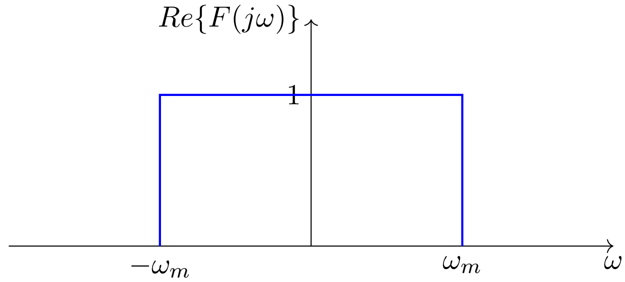


(c) (16 points) Plot the Real and Imaginary Components of $D(j\omega)$ and $F(j\omega)$.

Solution:

Both $D(j\omega)$ and $F(j\omega)$ will be Real as Imaginary part is cancelled.





(d) (9 points) Describe how $d(t)$ relates to $a(t)$ and/or $b(t)$.

Solution: $d(t)$ is scaled version of $a(t)$.

4. **Laplace Transform** (46 points)

(a) (20 points) Find the Laplace transforms of the following signals and determine their region of convergence.

i. $f(t) = e^{-at}\left(\cos(\omega_0 t) - 1\right)u(t)$

ii. $f(t) = \int_{0^-}^{t-1} e^{-ax}\left(\cos(\omega_0 x) - 1\right)\,dx,\;\; t-1 \geq 0$

Solution:

i. We can equivalently write $f(t)$ as follows:

$$f(t) = e^{-at}\cos(\omega_0 t)u(t) - e^{-at}u(t)$$

The Laplace transform of $e^{-at}u(t)$ is:

$$G(s) = \frac{1}{s+a}$$

The Laplace transform of $\cos(\omega_0 t)u(t)$ is:

$$G(s) = \frac{s}{s^2 + \omega_0^2}$$

By using the Laplace transform properties we know that, the laplace transform of $e^{-at}\cos(\omega_0 t)u(t)$ is:

$$G(s) = \frac{s+a}{(s+a)^2 + \omega_0^2}$$

Therefore the laplace transform of the entire function is:

$$G(s) = \frac{s+a}{(s+a)^2 + \omega_0^2} - \frac{1}{s+a}$$

For ROC, we need Re$\{s\} > -a$.

ii. Let $g(t) = \int_{0^-}^{t} e^{-ax} \left(\cos(\omega_0 x) - 1\right) u(x)\, dx, \;\; t \geq 0$.

Thus

$$G(s) = \frac{1}{s}\left(\frac{s+a}{(s+a)^2 + \omega_0^2} - \frac{1}{s+a}\right) = \frac{1}{s}\frac{-\omega_0^2}{(s+a)((s+a)^2 + \omega_0^2)}$$

.

Since, $f(t) = g(t-1)$, we have

$$F(s) = e^{-s}G(s) = \frac{e^{-s}}{s}\frac{-\omega_0^2}{(s+a)((s+a)^2 + \omega_0^2)}$$

For ROC, we need Re$\{s\} > 0$ and Re$\{s\} > -a$. If $a > 0$, the ROC is equivalent to have Re$\{s\} > 0$. If $a < 0$, the ROC is equivalent to have Re$\{s\} > -a$.

(b) (10 points) Find the inverse Laplace transform $f(t)$ for the following $F(s)$. ($f(t)$ is a causal signal.)

$$F(s) = \frac{s^2 + s + 1}{(s+1)(s+2)(s+3)}$$

Solution: We have:

$$F(s) = \frac{r_1}{s+1} + \frac{r_2}{s+2} + \frac{r_3}{s+3}$$

where

$$r_1 = \left.\frac{s^2+s+1}{(s+2)(s+3)}\right|_{s=-1} = \frac{1}{2}$$

$$r_2 = \left.\frac{s^2+s+1}{(s+1)(s+3)}\right|_{s=-2} = -3$$

$$r_3 = \left.\frac{s^2+s+1}{(s+1)(s+2)}\right|_{s=-3} = \frac{7}{2}$$

Therefore,

$$f(t) = \left(\frac{1}{2}e^{-t} - 3e^{-2t} + \frac{7}{2}e^{-3t}\right) u(t)$$

(c) (16 points) Prove the following statements

i. (8 points) If $x(t)$ is an even function, so that $x(t) = x(-t)$, then $X(s) = X(-s)$.

ii. (8 points) If $x(t)$ is an odd function, so that $x(t) = -x(-t)$, then $X(s) = -X(-s)$.

Solution:

i. Given that $x(t) = x(-t)$, calculating $X(s)$:

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} x(-t)e^{-st}dt \\ &= X(-s) \end{aligned}$$

ii. Given that $x(t) = -x(-t)$, calculating $X(s)$:

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} -x(-t)e^{-st}dt \\ &= -\int_{-\infty}^{\infty} x(-t)e^{-st}dt \\ &= -X(-s) \end{aligned}$$

Bonus (15 points)

This problem is about determining the Fourier Transform of the signal $x_a(t) = \tan^{-1}(at)$, working through a sequence of successive steps. Clearly delineate your work and circle your answer for each part below.

1. (10 points) Compute the Fourier transform of

$$x(t) = \tan^{-1}(t)$$

Hint: You might find the following calculus result useful

$$\frac{d}{dt}\tan^{-1}(t) = \frac{1}{1+t^2}$$

Solution: From the Fourier transform table, we know that

$$\mathcal{F}[e^{-a|t|}] = \frac{2a}{\omega^2 + a^2}$$

Then for $a = 1$ we have

$$\mathcal{F}[e^{-|t|}] = \frac{2}{\omega^2 + 1}$$

Hence using duality property we have

$$\mathcal{F}[\frac{2}{t^2+1}] = 2\pi e^{-|\omega|}$$

Using the hint we have

$$\frac{d}{dt}\tan^{-1}(t) = \frac{1}{1+t^2}$$

Taking the Fourier transform of both sides of the equation, we have

$$\begin{aligned} j\omega\mathcal{F}[\tan^{-1}(t)] &= \mathcal{F}[\frac{1}{1+t^2}] \\ \mathcal{F}[\tan^{-1}(t)] &= \frac{1}{j\omega}\pi e^{-|\omega|} \end{aligned}$$

2. (3 points) Does your answer for part (a) have a real part? Explain how your answer is consistent with the symmetry properties of the Fourier Transform.

   Solution: We know that $\tan^{-1}(t)$ is an odd function so the Fourier transform of an odd function is purely imaginary. From (1) we can see that the Fourier transform is purely imaginary, therefore it checks out.

3. (2 points) Now use one of the properties of the Fourier Transform to determine the Fourier Transform for the more general case below, where a is a real-valued positive constant.

$$x_a(t) = \tan^{-1}(at)$$

   Solution: Using the time scaling property of Fourier transform

$$\begin{aligned} \mathcal{F}[x_a(t)] &= \frac{1}{|a|}X(\frac{\omega}{a}) \\ &= \frac{1}{a}\frac{1}{j\frac{\omega}{a}}\pi e^{-|\frac{\omega}{a}|} \end{aligned}$$