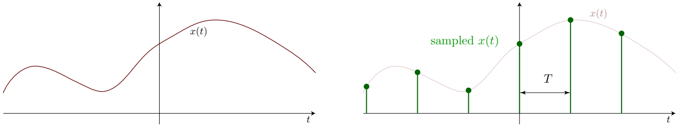
Sampling

This handout covers sampling continuous signals. Topics include:

- Fourier transform of a periodic signal
- Impulse trains
- Sampling via an impulse train
- Sampling theorem
- Aliasing
- Interpolation equation

Motivation

In reality, we could never store a continuous time signal. Instead, as we see in MATLAB, we store the signal's value at various times. This is called sampling, as illustrated below.



A key variable of interest is the sampling frequency, i.e., the time in between our samples, denoted T in the above diagram.

This is related to discrete signals, i.e., $x[n] = x(nT)$.

How to sample a continuous signal?

How do we sample a continuous signal? You may have several intuitions to do so already using the $\delta(t)$ signal and its property that $f(t)\delta(t) = f(0)\delta(t)$.

- We will arrive at sampling by first studying a related problem: the Fourier transform of periodic signals.
- The reason we approach this is that Fourier series are discrete coefficients, c_k, while the Fourier transform is typically some continuous signal. i.e., it seems like there may be a relationship whereby the Fourier series is like a sampled Fourier transform.
- So we ask: what is the relationship between the Fourier series and the Fourier transform?
- To see this, we can begin by identifying the relationship between the Fourier series and the Fourier transform.

Fourier transform of a periodic signal

We cannot directly take the Fourier transform of a periodic signal, since they do not have finite energy. However, we can use a few tricks (like in the Generalized Fourier Transform lecture) to calculate the FT of a periodic signal.

Let $f(t)$ have a Fourier series (with period $T_0 = \omega_0/2\pi$)

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jk\omega_0 t}dt$$

There's a close relationship between the two, as the Fourier series equation looks like the Fourier transform equation but with a $\sum$ instead of an $\int$.

Fourier transform of the Fourier series representation

Let's take the Fourier transform of the Fourier series representation.

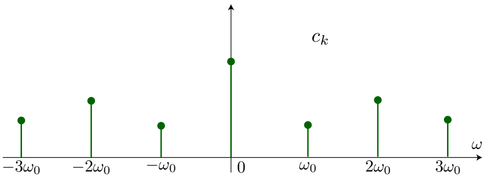
$$
\begin{aligned}
\mathcal{F}[f(t)] &= \mathcal{F}\left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right] \\
&= \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left[e^{jk\omega_0 t}\right] \\
&= \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)
\end{aligned}
$$

This result tells us that the Fourier transform of a periodic signal is simply the coefficients of the Fourier series multiplied by δ signals and scaled by 2π. This is illustrated on the next page.

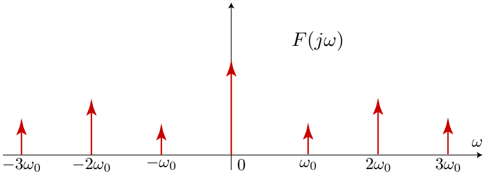
Fourier transform of the Fourier series (cont.)

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \iff \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0)$$

Fourier series



Fourier transform

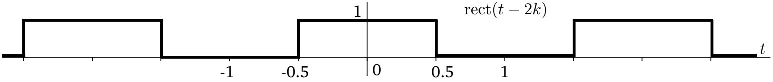


Example: square wave

Consider the square wave below:

$$f(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}(t - 2k)$$

This is illustrated below.



In the Fourier series lecture (slide 8-32), we calculated that the Fourier series of this signal is

$$c_k = \frac{1}{2}\,\mathrm{sinc}(k/2)$$

Therefore, its Fourier transform is

$$\begin{aligned} F(j\omega) &= \sum_{k=-\infty}^{\infty} \pi\,\mathrm{sinc}(k/2)\delta(\omega - k\pi) \\ &= \pi \sum_{k=-\infty}^{\infty} \mathrm{sinc}(\omega/2\pi)\delta(\omega - k\pi) \end{aligned}$$

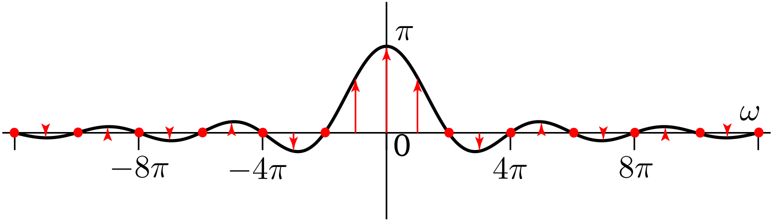
Example: square wave (cont.)

In the last step on the previous page, we used the fact that

$$\begin{aligned} \mathrm{sinc}(\omega/2\pi)\delta(\omega - k\pi) &= \mathrm{sinc}(k\pi/2\pi)\delta(\omega - k\pi) \\ &= \mathrm{sinc}(k/2)\delta(\omega - k\pi) \end{aligned}$$

Hence, the Fourier transform of the square wave is the Fourier transform of a rect multiplied by evenly spaced δ's, i.e.,

$$F(j\omega) = \pi \sum_{k=-\infty}^{\infty} \mathrm{sinc}(\omega/2\pi)\delta(\omega - k\pi)$$

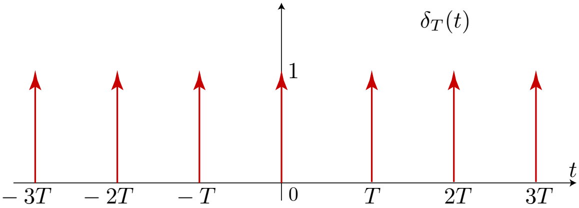


Impulse trains

To simplify notation here, we can define an *impulse train*, which ends up being our sampling function. We let $\delta_T(t)$ be a sequence of unit δ functions spaced by T.

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

This is illustrated below.



With this, we can write the Fourier transform of the square wave as

$$F(j\omega) = \pi \, \mathrm{sinc}(\omega/2\pi)\delta_\pi(\omega)$$

Impulse trains (cont.)

The impulse train may have been your first thought when thinking of how to sample a signal every T.

Indeed, this signal has very important qualities. Let's start off with a simple question: intuitively, what is the Fourier transform of a impulse train?

Let's think through this using our square wave example.

- We know that the Fourier transform of the square wave is a sinc multiplied by $\delta_\pi(\omega)$.
- From the convolution theorem, this means that the inverse Fourier transform (i.e., the square wave) is the inverse Fourier transform of a sinc (i.e., a rect) convolved with the inverse Fourier transform of a impulse train.
- We know that a square wave is simply a rect repeated over and over again, i.e., convolved with a impulse train.
- So intuitively, by duality, the Fourier transform of a impulse train should be a impulse train.

Note, we will sometimes use the term 'delta train' to describe an impulse train.

Fourier transform of an impulse train

Let's check our intuition and compute the Fourier transform of an impulse train. To do so, we'll use our trick of finding the Fourier series of the (periodic) impulse train, and then multiplying by $2\pi\delta(\cdot)$.

The Fourier series coefficients of the impulse train are:

$$\begin{aligned} c_k &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j2\pi kt/T}dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-j2\pi kt/T}dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)dt \\ &= \frac{1}{T} \end{aligned}$$

This is not dependent on k. Therefore, $c_k = 1/T$ for all k. Hence, the Fourier series of $\delta_T(t)$ is

$$\delta_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}$$

Fourier transform of an impulse train (cont.)

With Fourier series

$$\delta_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}$$

we thus have that, with $\omega_0 = 2\pi/T$,

$$\begin{aligned}\mathcal{F}[\delta_T(t)] &= \frac{1}{T}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0) \\ &= \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \\ &= \omega_0 \delta_{\omega_0}(\omega)\end{aligned}$$

This matches our intuition, i.e., the Fourier transform of an impulse train is another impulse train. Hence, we have the following Fourier transform pair:

$$\boxed{\delta_T(t) \iff \omega_0 \delta_{\omega_0}(\omega)}$$

Sampling with an impulse train

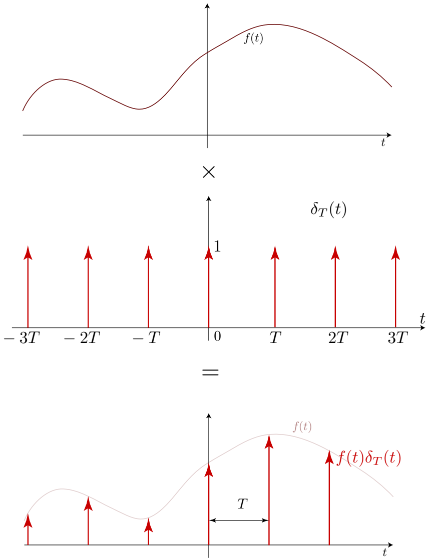
As we saw earlier, one of the things we will use the impulse train for is to sample signals.

Given a signal $f(t)$,

$$
\begin{aligned}
f(t)\delta_T(t) &= f(t) \sum_{k=-\infty}^{\infty} \delta(t-kT) \\
&= \sum_{k=-\infty}^{\infty} f(t)\delta(t-kT) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)
\end{aligned}
$$

and thus $f(t)\delta_T(t)$ samples the signal $f(t)$ at equal intervals kT. This is illustrated on the next slide.

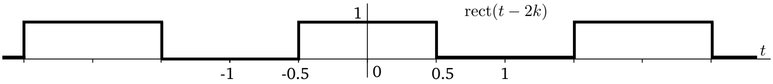
Sampling with an impulse train (cont.)



Square wave, part 2

Let's revisit our square wave example, where

$$f(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}(t - 2k)$$



Another way to represent this square wave is as follows:

$$f(t) = \mathrm{rect}(t) * \delta_2(t)$$

Hence, we can calculate its Fourier transform by using the convolution theorem. Recall that, for $\omega_0 = 2\pi/T$,

$$\mathrm{rect}(t) \iff \mathrm{sinc}(\omega/2\pi)$$

and

$$\delta_T(t) \iff \omega_0 \delta_{\omega_0}(\omega)$$

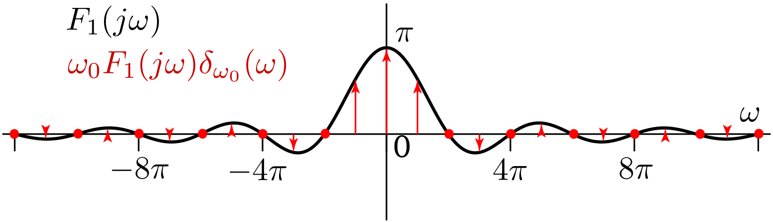
Square wave, part 2 (cont.)

Note that when $T = 2$, then $\omega_0 = \pi$. Then, we have that,

$$\begin{aligned}\mathcal{F}\left[f(t)\right] &= \mathcal{F}\left[\text{rect}(t) * \delta_2(t)\right] \\ &= \mathcal{F}\left[\text{rect}(t)\right]\mathcal{F}\left[\delta_2(t)\right] \\ &= \text{sinc}(\omega/2\pi)\pi\delta_\pi(\omega)\end{aligned}$$

This is exactly the same Fourier transform we calculated earlier using the Fourier series of the square wave.

Another intuition to remember here is that the Fourier transform of a periodic signal is the Fourier transform of one period of the signal (which we can denote f_1), sampled by an impulse train at multiples of ω_0.



Discrete - periodic duality

We can determine the Fourier transform of a signal sampled in the time-domain. Consider

$$\tilde{f}(t) = f(t)\delta_T(t)$$

Its Fourier transform is

$$\begin{aligned}
\tilde{F}(j\omega) &= \mathcal{F}\left[f(t)\delta_T(t)\right] \\
&= \frac{1}{2\pi}\mathcal{F}[f(t)] * \mathcal{F}(\delta_T(t)) \\
&= \frac{1}{2\pi}F(j\omega) * \omega_0\delta_{\omega_0}(\omega) \\
&= \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega)
\end{aligned}$$

Discrete - periodic duality (cont.)

This are merely samples of $F(j\omega)$ repeated every ω_0, since

$$
\begin{aligned}
\tilde{F}(j\omega) &= \frac{1}{T} F(j\omega) * \delta_{\omega_0}(\omega) \\
&= \frac{1}{T} F(j\omega) * \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))
\end{aligned}
$$

This leads us to the realization that:

- A signal that is periodic in time is discrete in spectrum.
- A signal that is discrete in time is periodic in spectrum.

There are important consequences from this result when we consider sampling signals in the time domain.

Sampling theorem motivation

Consider the following problem. We have a signal $f(t)$, and we need to store it. Our experimental set up is able to sample this signal at an interval T. How do we set T so that we can faithfully store $f(t)$? If T is too large, we sample infrequently and may lose information about $f(t)$. If T is too small, we waste memory and resources to store values we don't need.

The sampling theorem uses the results we've derived to tell us the minimum frequency at which we must sample $f(t)$ to not lose information. It is a very important theorem.

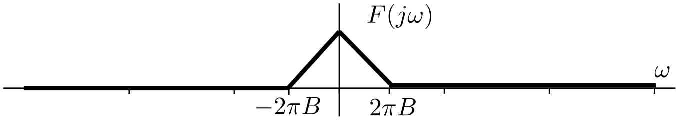
Sampling theorem

If $\tilde{f}(t) = f(t)\delta_T(t)$, then as shown on the previous slides,

$$\tilde{F}(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))$$

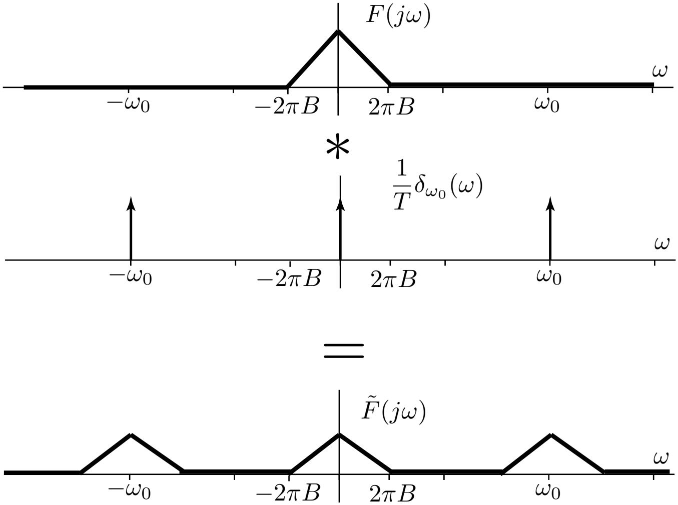
Therefore, the spectrum of $\tilde{f}(t)$ are shifted replicas of the spectrum, $F(j\omega) = \mathcal{F}[f(t)]$ spaced every ω_0 and scaled by $1/T$.

We define the bandwidth of $f(t)$ to be $\pm B$ Hz, e.g.,



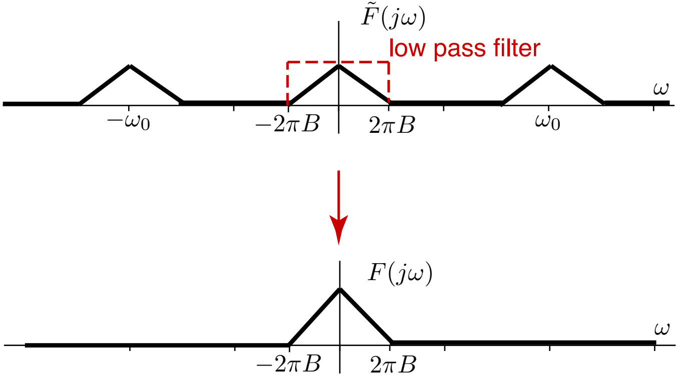
Sampling theorem (cont.)

For a particular choice of ω_0, where $\omega_0 \gg 2\pi B$, we see the spectrum of $\tilde{F}(j\omega)$ looks like:



Sampling theorem (cont.)

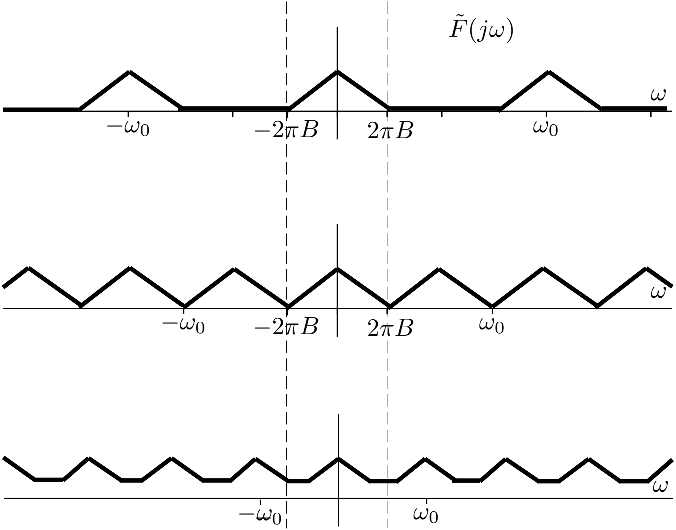
For this choice of ω_0, the original $F(j\omega)$ can be recovered through low pass filtering.



With ideal low pass filtering for the illustrated ω_0, we can *perfectly* recover $f(t)$ after sampling.

Sampling theorem (cont.)

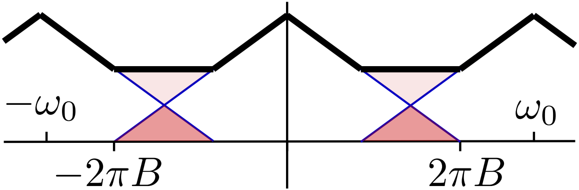
But now, as we increase the time T between samples, which decreases ω_0, the replicas of $F(j\omega)$ get closer and closer together.



Sampling theorem (cont.)

We see that as ω_0 decreases, the bands start to overlap. When the replicas overlap, even with ideal low pass filtering, we cannot recover the original $F(j\omega)$.

This overlap is called *aliasing* because low frequencies of one spectral replica appear (or alias) as high frequencies in the next spectral replica. The vice versa is true as well; high frequencies of one spectral replica alias as low frequencies in an adjacent spectral replica. The alias'd sections are shown in the darker red.



Sampling theorem (cont.)

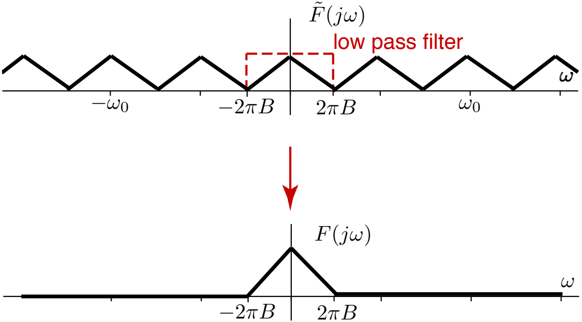
To be able to perfectly recover a signal, we need to sample so as to avoid aliasing. No aliasing happens if $2\pi B < \omega_0/2$. We can simplify this as

$$\begin{aligned} 2B &< \omega_0/2\pi \\ &= \frac{2\pi}{T}\frac{1}{2\pi} \\ &= \frac{1}{T} \end{aligned}$$

Therefore, the signal can only be recovered exactly if the signal bandwidth $2B$ is less than or equal to the sampling rate $1/T$. Hence, we need to sample at intervals less than or equal to $T = 2B$. This sampling rate, $2B$ is called the *Nyquist rate* for $f(t)$, and it is the lowest rate that we can sample $f(t)$ so that it can be perfectly recovered. T is called the *Nyquist interval*.

Recovering the original signal through interpolation

With a sampled signal, $\tilde{f}(t)$, as long as we have sampled at a rate $\geq 2B$, we can perfectly recover the original signal through ideal low pass filtering. Let's formalize how this happens, using the particular instantiation that $T = 1/2B$, i.e., we sample at the Nyquist rate.



Our low pass filter has frequency response

$$H(j\omega) = T\text{rect}\left(\frac{\omega}{4\pi B}\right)$$

Recovering the original signal through interpolation (cont.)

The inverse Fourier transform of $H(j\omega)$ is

$$h(t) = 2BT\,\mathrm{sinc}(2Bt)$$

Since $T = 1/2B$, we can simplify this expression to

$$h(t) = \mathrm{sinc}(2Bt)$$

Therefore, to reconstruct $f(t)$ from $\tilde{f}(t)$, we calculate:

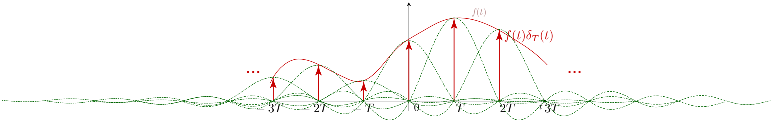
$$\begin{aligned}
\tilde{f}(t) * h(t) &= \left(\sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT) \right) * h(t) \\
&= \sum_{k=-\infty}^{\infty} f(kT)h(t-kT) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2B(t-kT)) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt-k))
\end{aligned}$$

Recovering the original signal through interpolation (cont.)

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula. Intuitively, it does the following:



The sum of the green sinc functions will equal the red function, $f(t)$.

To not mince words, this result, which combines many of the things we've learned thus far, is remarkable. Through this reconstruction, we are able to *perfectly* recover an original signal from samples.

Sampling example: sinusoids

Let's consider sampling a cosine,

$$f(t) = \cos(\omega_0 t)$$

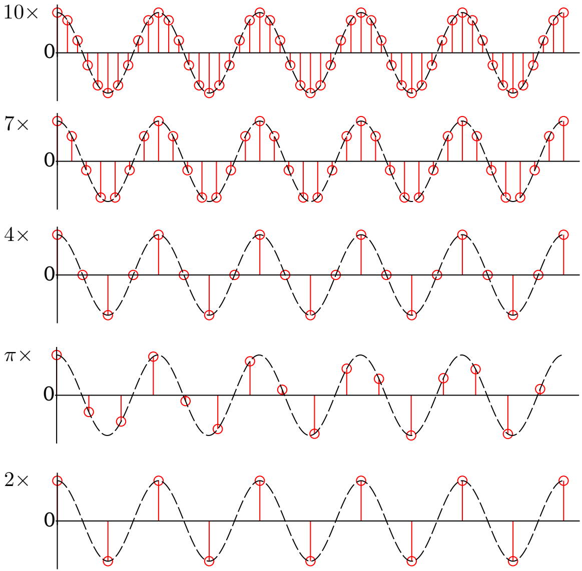
Let's take $\omega_0 = 2\pi$. What is its bandwidth?

What is the Nyquist rate?

How about when $\omega_0 = 1$? What is its bandwidth?

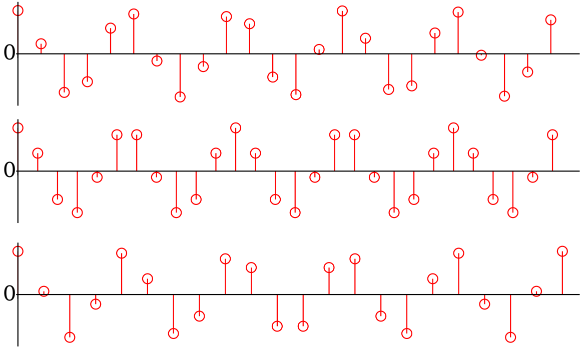
What is the Nyquist rate?

Sampling example: sinusoids (cont.)



Sampling example: sinusoids (cont.)

Now, we're going to sample this same cosine, but not show the dotted cosine. Through the interpolation formula, *all* of these can be reconstructed as $f(t) = \cos(2\pi t)$.

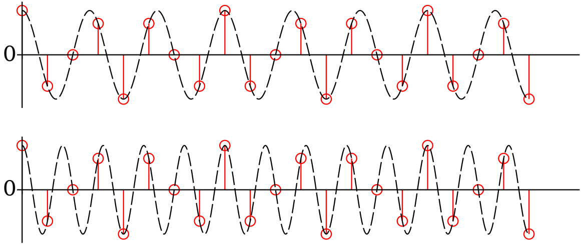


Question for you: how do I know that these are necessarily $\cos(2\pi t)$? Couldn't I draw an arbitrary function that passes through these dots that is not $\cos(2\pi t)$? Why couldn't this be the signal?

Aliasing example

Here's an example of aliasing. Below are two sinusoids. The upper one is at a frequency of $f = 0.75$ Hz and the one below is at $f = 1.25$ Hz.

Sampling both signals at $f_s = 2$ Hz,

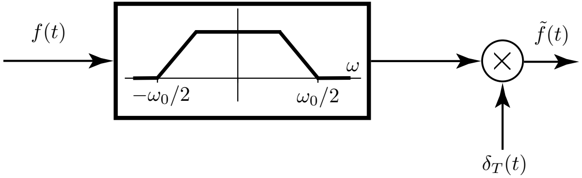


As you can see, the samples are the same for both sinusoids..

How to ameliorate aliasing

If you sample below the Nyquist rate, there will be aliasing. Once aliasing happens, there is no way to eliminate aliasing without having additional information about the signal.

One way we can ameliorate aliasing is to first low pass filter the signal, then sample:



- In reality, our low pass filters are not perfect, and so the bandwidth will be larger than $\omega_0/2$, however, we'll attenuate frequencies outside of range.
- Low pass filtering will distort the signal.
- However, the point is that when sampling, frequencies beyond $\omega_0/2$ would cause artifacts. Low pass filtering ameliorates this.
- It also supresses noise outside of $\omega_0/2$.