Problem 1: complex exponential

a) $z_1 = 16e^{-j\frac{2\pi}{3}}$, $z_2 = 4 - j4\sqrt{3}$

i) Since we want to express $\frac{z_1}{z_2}$ in cartesian form, so we will first express z_1 in cartesian form

using euler's identity,

$$z_1 = 16\cos\left(-\frac{2\pi}{3}\right) + j\,16\sin\left(-\frac{2\pi}{3}\right)$$

$$= 16\cos\left(\frac{2\pi}{3}\right) - j16\sin\left(\frac{2\pi}{3}\right)$$

$$= -8 - j8\sqrt{3}$$

Now,

$$\frac{z_1}{z_2} = \frac{-8 - j8\sqrt{3}}{4 - j4\sqrt{3}}$$

we multiply the numerator and denominator by z_2^*

$$\frac{z_1 z_2^*}{z_2 z_2^*} = \frac{(-8 - j8\sqrt{3})(4 + j4\sqrt{3})}{(4 - j4\sqrt{3})(4 + j4\sqrt{3})}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{-32 - j32\sqrt{3} - j\,32\sqrt{3} - j^2 96}{4^2 + (4\sqrt{3})^2}$$

$$\frac{z_1}{z_2} = \frac{64 - j\,64\sqrt{3}}{16 + 48} = 1 - j\sqrt{3}$$

ii) Since we want to express $\frac{z_1}{z_2}$ in polar form, so we will first express z_2 in polar form

$$|z_2| = \sqrt{64} = 8$$

$$\angle z_2 = \arctan\left(-\frac{4\sqrt{3}}{4}\right)$$

$$= \arctan\left(-\sqrt{3}\right)$$

$$= -\frac{\pi}{3}$$

So, $z_2 = 8e^{-j\pi/3}$

Now,

$$\frac{z_1}{z_2} = \frac{16 e^{-j\frac{2\pi}{3}}}{8 e^{-j\pi/3}}$$

$$= 2 e^{-j\frac{2\pi}{3} + j\frac{\pi}{3}}$$

$$\therefore \frac{z_1}{z_2} = 2 e^{-j\frac{\pi}{3}}$$

Sanity check!

Expressing $2e^{-j\pi/3}$ in cartesian form we get

$$2\cos(\pi/3) - j\,2\sin(\pi/3)$$

$$= 1 - j\sqrt{3}$$

Hence both the answers match.

b) we have,

$$1-e^{j\alpha}$$

$$= e^{j\frac{\alpha}{2}} e^{-j\frac{\alpha}{2}} (1-e^{j\alpha})$$

$$= e^{j\frac{\alpha}{2}} \left(e^{-j\frac{\alpha}{2}} - e^{j\frac{\alpha}{2}}\right)$$

Now we Know the following relation

$$-2j\sin x = e^{-jx} - e^{jx}$$

using the above relation, we have

$$= e^{j\frac{\alpha}{2}} \left(-2j\sin\frac{\alpha}{2}\right)$$

Now from euler's identity we know

$$-j = e^{-j\frac{\pi}{2}}$$

using the above relation, we have

$$= e^{j\frac{\alpha}{2}} \left(2e^{-j\pi/2}\sin\frac{\alpha}{2}\right)$$

$$= 2\sin\frac{\alpha}{2}\, e^{j\left(\frac{\alpha-\pi}{2}\right)}$$

Problem 2: Energy and Power of Signals

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$$

Now,

$$|y(t)| = |e^{-ct}|$$

$$= |e^{-(a+jb)t}|$$

$$= |e^{-at} \cdot e^{-jbt}|$$

$$= |e^{-at}| \cdot |e^{-jbt}|$$

$$= e^{-at}$$

So,

$$E_y = \int_{-\infty}^{\infty} e^{-2at} dt = \infty$$

$$P_y = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |y(t)|^2 dt$$

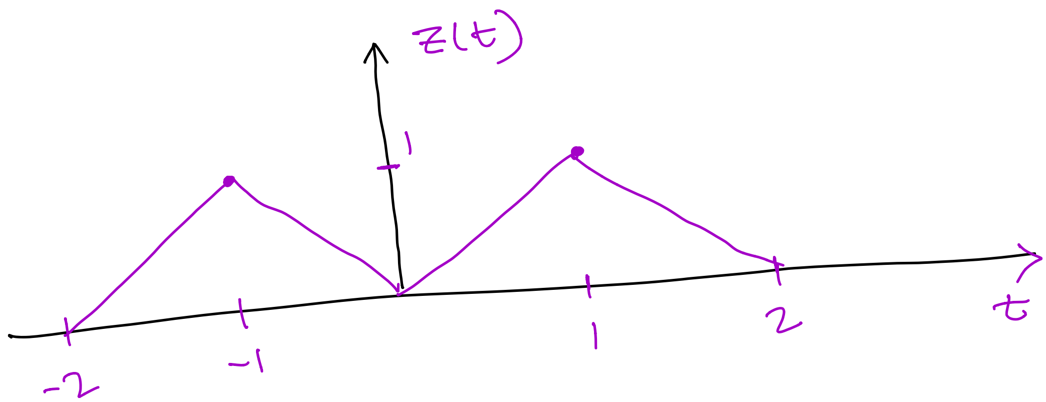
$$P_y = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} e^{-2at} \cdot dt$$

$$P_y = \infty$$

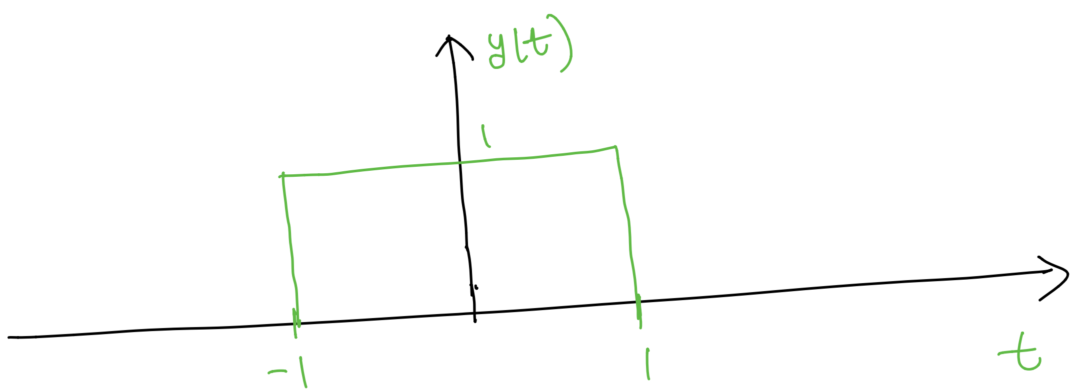
Therefore, e^{-ct} is neither energy nor a power signal for a complex value of c with nonzero real part.

Problem 3: Expression for signals.

We can express $x(t)$ as a combination of scaled and shifted $\Delta(t)$ and $\mathrm{rect}(t)$



$$z(t) = \Delta(t+1) + \Delta(t-1)$$



$$y(t) = \mathrm{rect}(t/2)$$

from the above figures, it is evident that

$$x(t) = z(t) + y(t)$$

$$x(t) = \Delta(t+1) + \Delta(t-1) + \text{rect}(t/2)$$

Problem 4: Elementary Signals

a) i) $y(t) = x(t)\,u(t-1) - x(t)\,u(2t-5)$

$\underline{t < 1:}$

$u(t-1) = 0$, $u(2t-5) = 0$

Hence $y(t) = 0$

$\underline{1 \le t < 5/2:}$

$u(t-1) = 1$, $u(2t-5) = 0$

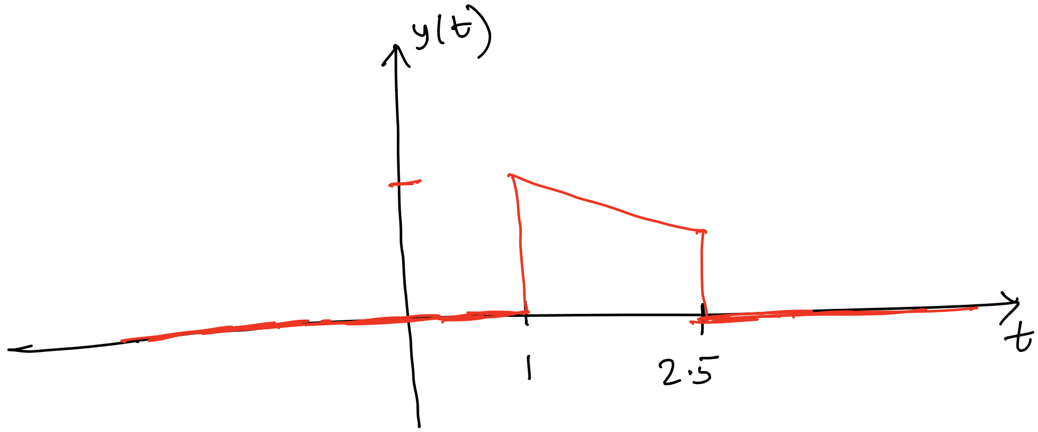
Hence $y(t) = x(t)$

$\underline{t \ge 5/2:}$

$u(t-1) = 1$, $u(2t-5) = 1$

Hence $y(t) = 0$

So, $$y(t) = \begin{cases} 0, & t < 1 \\ x(t), & 1 \le t < 5/2 \\ 0, & t \ge 5/2 \end{cases}$$



ii) $y(t) = \int_{-\infty}^{t} \delta(\tau-2)\, x(\tau)\, d\tau$

Using sifting property of impulse

$$\delta(\tau-2)\, x(\tau) = x(2)\, \delta(\tau-2)$$

Then,

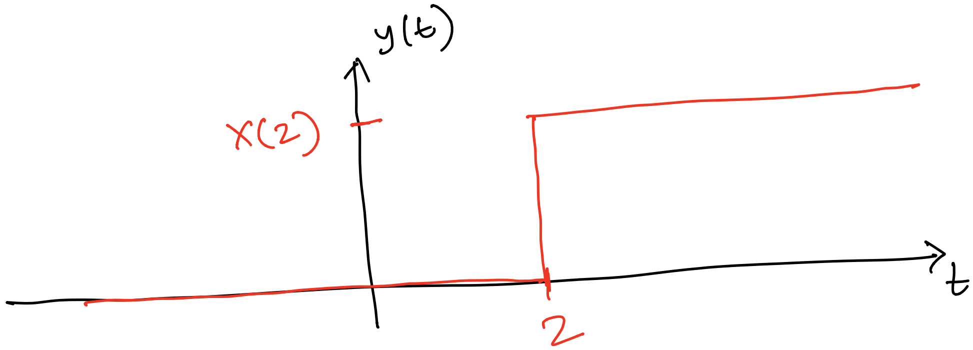
$$y(t) = \int_{-\infty}^{t} x(2)\, \delta(\tau-2)\, d\tau$$

$$= x(2) \int_{-\infty}^{t} \delta(\tau-2)\, d\tau$$

Now, if $t<2$ then the Impulse is not included in the integral and $\int_{-\infty}^{t} \delta(\tau-2)\, d\tau = 0$

If $t \geq 2$ then the impulse is included in the integral and $\int_{-\infty}^{t} \delta(\tau - 2)\, d\tau = 1$

So, $y(t) = \begin{cases} 0, & t < 2 \\ x(2), & t \geq 2 \end{cases}$



b) $$y(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,\delta(t-2)\,d\tau$$

$$\Rightarrow y(t) = \delta(t-2)\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau$$

Using the sifting property

$$f(\tau)\,\delta(t-\tau) = f(t)\,\delta(t-\tau)$$

$$\Rightarrow y(t) = f(t)\,\delta(t-2)\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau$$

$$y(t) = f(t)\,\delta(t-2) = f(2)\,\delta(t-2)$$