Elementary signals

This lecture goes over the major signal models that form the building blocks for the analyses we do later on in this class. We also discuss operations on these signals.

- Review of sinusoidal signals
- Exponential signals
- Complex exponential signals
- Unit step and ramp
- Rectangular pulse
- Delta function

Real sinusoids

We previously discussed the real sinusoid, which we'll recap here for completeness of these notes. A cosine is defined by:

$$
\begin{aligned}
x(t) &= A\cos(\omega t - \theta) \\
&= A\cos(2\pi f t - \theta)
\end{aligned}
$$

with

- A defining the amplitude of the signal (i.e., how large it gets).
- ω defining the *natural* frequency of the signal (in units of radians per second). As ω gets larger, the sinusoid repeats more times in a given time interval.
- The natural frequency is related to the frequency, f, of the signal (in units of Hertz, or s^{-1}) through the relationship: $\omega = 2\pi f$. The frequency, f, is the inverse of the period, i.e.,

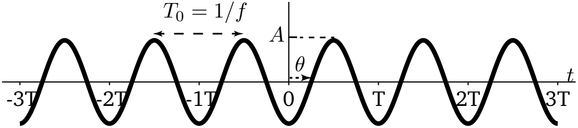
$$T_0 = \frac{1}{f} = \frac{2\pi}{\omega}$$

- θ is the phase of the signal in terms of radians, shifting the sinusoid.

Real sinusoids (cont.)

We illustrate a sinusoid signal below:

$$x(t) = A\cos(\omega t - \theta)$$

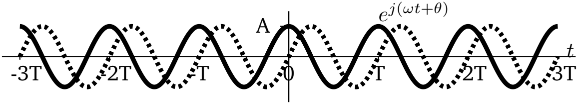


Complex sinusoids

The complex sinusoid is given by:

$$Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + jA\sin(\omega t+\theta)$$

We draw complex signals with dotted lines.



The real part of the complex sinusoid (solid line) is:

$$\Re\left(Ae^{j(\omega t+\theta)}\right) = A\cos(\omega t+\theta)$$

The imaginary part of the complex sinusoid (dotted line) is:

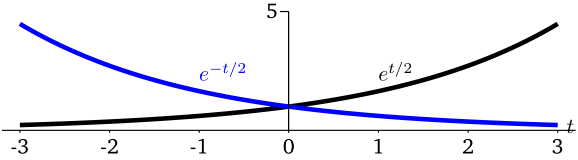
$$\Im\left(Ae^{j(\omega t+\theta)}\right) = A\sin(\omega t+\theta)$$

Exponential

An exponential signal is given by

$$x(t) = e^{\sigma t}$$

- If $\sigma > 0$, this signal grows with increasing t (black signal in plot below). This is called exponential growth.
- If $\sigma < 0$, this signal decays with increasing t (blue signal in plot below). This is called exponential decay.

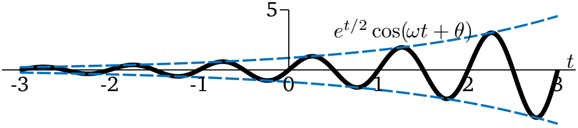


Damped or growing sinusoids

A damped or growing sinusoid is denoted

$$x(t) = e^{\sigma t} \cos(\omega t + \theta)$$

The sinusoid will grow exponentially if $\sigma > 0$ and decay exponentially if $\sigma < 0$.

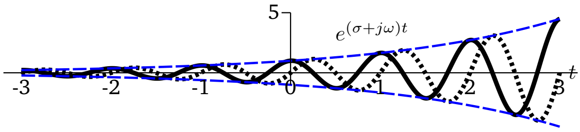


Complex exponential

A complex sinusoid is denoted

$$x(t) = e^{(\sigma + j\omega)t}$$

It is a combination of the complex sinusoid and an exponential. All prior signals are special cases of the complex exponential signal.



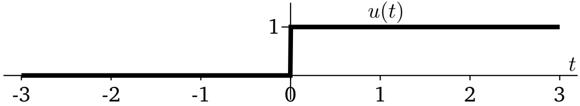
It is helpful to think of σ and $j\omega$ in the complex plane. σ is the x-axis and $j\omega$ is the y-axis. Then complex exponentials in the left complex plane are decreasing signals and those in the right are increasing signals.

Unit (Heavyside) step function

The unit step function, denoted $u(t)$ in this class, is given by

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

It is also called the Heavyside step function. Drawn below:



Unit step function (cont.)

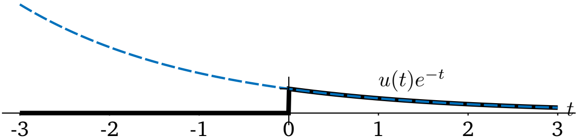
The unit step function is often used to extract part of another signal. For example, say we had a causal decaying exponential, given by

$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

This could be written as

$$x(t) = u(t)e^{-t}$$

This is drawn below. (Original exponential drawn in dotted blue; $x(t)$ in black.)



The unit step function can also be used to create other signals, like a rectangle of length and height 1 beginning at the origin via $x(t) = u(t) - u(t-1)$.

Unit rectangle

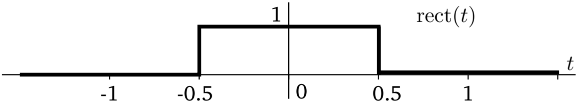
The unit rectangle is defined as:

$$\mathrm{rect}_\Delta(t) = \begin{cases} 1/\Delta, & |t| \leq \Delta/2 \\ 0, & \text{else} \end{cases}$$

The unit rectangle has area 1 and is parametrized by Δ. When we do not specify Δ, e.g., if we were to simply write $\mathrm{rect}(t)$, then we assume $\Delta = 1$. That is,

$$\mathrm{rect}(t) = \begin{cases} 1, & |t| \leq 1/2 \\ 0, & \text{else} \end{cases}$$

This is illustrated below:



The unit rectangle can be used to extract out segments of a signal.

Unit ramp

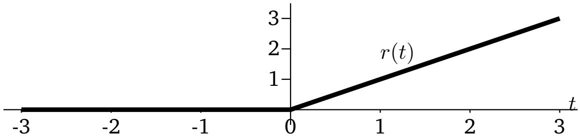
The unit ramp is defined as:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Note that the unit ramp is the integral of the unit step, i.e.

$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

The unit ramp is illustrated below:

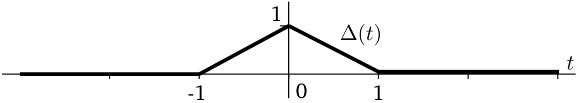


Unit triangle

The unit triangle is defined as:

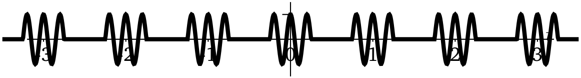
$$\Delta(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{else} \end{cases}$$

The unit triangle is illustrated below:



Using the building blocks

You can create fairly interesting signals using these building blocks. Consider the following example:



In this signal, the bursts have form $A\cos(\omega t)$ and they last for width 0.5 s around each integer (i.e., it bursts from -0.25 to 0.25, 0.75 to 1.25, etc.) Assume it lasts infinitely. How do you use the elementary signals we've discussed so far to write this signal?

Answer:

$$x(t) = \sum_{k=-\infty}^{\infty} A\cos(\omega t)\mathrm{rect}(2(t-k))$$

The impulse or Dirac delta function

A critical function in signals and systems is the Dirac delta function, or impulse, denoted $\delta(t)$. This signal is an idealization, rather than a rigorous mathematical function. Instead, we define it according to its properties.

One may wonder why we use $\delta(t)$ if it isn't a rigorous mathematical function. It turns out that this function is very useful for studying signals and systems, as should be clear as we continue in this class.

(An aside: when working with discrete signals, the impulse function is the Kronecker delta, which is well-defined, i.e., $\delta[n] = 1$ at $n = 0$ and is 0 everywhere else. You'll see this in ECE 113.)

The impulse or Dirac delta function (cont.)

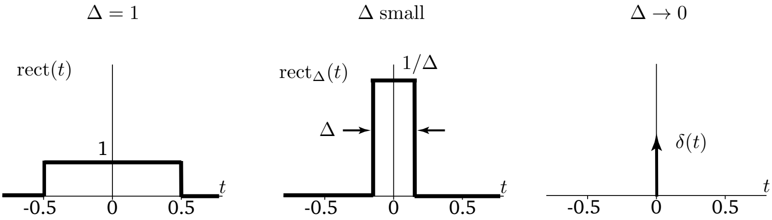
The Dirac delta (or impulse) function, $\delta(t)$, has the following properties:

- It is very large (i.e., approaching infinity) at $t = 0$.
- It is zero for all $t \neq 0$.
- It has area 1.

This function is called an impulse because it is conceptually zero whenever $t \neq 0$, essentially infinite at $t = 0$. An intuitive way to think of $\delta(t)$ is as the limit of $\mathrm{rect}_\Delta(t)$ as $\Delta \to 0$. $\mathrm{rect}_\Delta(t)$ has area 1, and as $\Delta > 0$, its width converges to zero and its amplitude goes to infinity. i.e., $\delta(t)$ behaves like

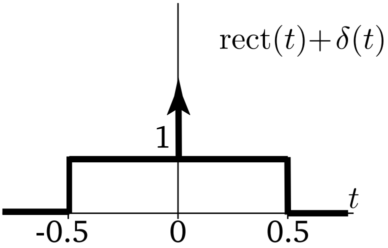
$$\lim_{\Delta \to 0} \mathrm{rect}_\Delta(t)$$

Note, we graph $\delta(t)$ as a solid upward arrow.



Example of a signal with an impulse

Let's say $x(t) = \text{rect}(t) + \delta(t)$. We would plot the signal as follows:



Formal definition of the Dirac delta

We define the Dirac delta, $\delta(t)$, according to the the following property:

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)}$$

as long as f is continuous at $t = 0$.

What is the idea behind this definition?

- $\delta(t)$ acts over a very small interval around $t = 0$.
- Intuitively, $\delta(t)$ makes the integral zero everywhere except in the vicinity of $t = 0$ (since $\delta(t)$ is conceptually zero except at $t = 0$).
- At $t = 0$, where $\delta(t)$ is not zero, it extracts the value of $f(t)|_{t=0}$.

To see this, think again of the approximation of $\delta(t)$ by $\mathrm{rect}_\Delta(t)$. Here, we can write:

$$\begin{aligned} \lim_{\Delta\to 0} \int_{-\infty}^{\infty} f(t)\mathrm{rect}_\Delta(t)dt &= f(0)\int_{-\infty}^{\infty} \mathrm{rect}_\Delta(t)dt \\ &= f(0) \end{aligned}$$

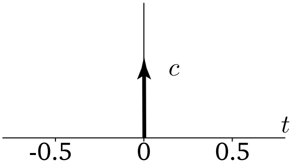
Scaling an impulse

Even though the $\delta(t)$ function is undefined at $t = 0$, we can scale the impulse. Consider the impulse $c\delta(t)$. Then,

$$\int_{-\infty}^{\infty} c\delta(t)f(t)dt = cf(0)$$

if f is continuous at 0.

Here, c is sometimes referred to as the magnitude of the impulse. We graph this by placing a number next to the impulse arrow, as below.



Impulse sampling property

The sampling property of the impulse is that

$$\boxed{f(t)\delta(t) = f(0)\delta(t)}$$

Here, the signal is "sampled" at the location of the impulse (i.e., for $\delta(t)$ at $t = 0$).

The intuition of why this is true is based again on approximating $\delta(t)$ as a rect with vanishingly small width. As we multiply by the infinitesimally small rect, we evaluate $f(t)\mathrm{rect}_\Delta(t)$.

$\mathrm{rect}_\Delta(t)$ is zero everywhere outside $[-\Delta/2, \Delta/2]$, and so as $\Delta \to 0$, this multiplication results in a signal that is zero everywhere except at $t = 0$. Therefore, it extracts out the value of $f(t)$ at the location of the impulse, i.e., $f(0)$, multiplied by $\delta(t)$.

Impulse sifting property

The sifting property of the impulse is that:

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-T)dt = f(T)}$$

To see this, we recognize that the signal $\delta(t-T)$ is an impulse function at $t=T$ using the signal shifting property. By the earlier intuitions, an impulse at $t=T$ will, after integration, extract the value of the function at T, i.e., $f(T)$. This is exactly like how $\delta(t)$, an impulse at 0, extracts out the value of the function at 0, i.e., $f(0)$.

- While we're on shifted impulse functions, you should also convince yourself that the following is true:

$$g(t)\delta(t-T) = g(T)\delta(t-T)$$

- Example: compute $\sin(20\pi t)\delta\left(t-\frac{1}{80}\right)$ and its integral.

Caution on limits of integration

If the impulse is not in the integration bounds, the integral is zero. e.g.,

$$\int_a^b \delta(t)dt = \begin{cases} 1, & \text{if } 0 \in (a,b) \\ 0, & \text{else} \end{cases}$$

If $a = 0$ or $b = 0$, this integral is ambiguous.

- To avoid this ambiguity, we will use $a-$ and $b+$ to denote whether to include the impulse or not. e.g.,

$$\int_{0+}^1 \delta(t)dt = 0$$

  but

$$\int_{0-}^1 \delta(t)dt = 1$$

- We'll do a few examples in class.
- Physical intuition of the delta function?

Integral of Dirac delta

Note that, based on its definition:

$$\begin{aligned}\int_{-\infty}^{t} \delta(\tau) d\tau &= \begin{cases} 1, & t \geq 0 \\ 0, & \text{else} \end{cases} \\ &= u(t)\end{aligned}$$

Hence, the integral of $\delta(\tau)$ up until time t is equal to the unit step signal, $u(t)$. Therefore, we can write

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Using the fundamental theorem of calculus, we also have the following relationship:

$$\delta(t) = \frac{du(t)}{dt}$$

We'll do an example of this integration and differentiation in class.

Additional question: what is

$$\int_{0}^{s} \int_{-\infty}^{t} \delta(\tau) d\tau dt$$

Last caveats of the impulse

Remember, $\delta(t)$ is not a signal or function that is mathematically rigorous; it only makes sense when used as we defined it.

- For the rest of this class, we will manipulate $\delta(t)$ as if it were a function but remember that this isn't mathematically rigorous.
- It is safe to use $\delta(t)$ in integrals consistent with our definition, e.g., expressions like
$$\int_{-\infty}^{\infty} f(t)\delta(t-T)dt$$
make mathematical sense, provided f is continuous at T.
- But some other expressions don't make any sense, like $\delta^2(t)$ or $\delta(t^2)$.