Generalized Fourier transforms

In this lecture, we'll see that naively applying the Fourier transform to some signals of interest doesn't directly work. Instead, we'll use some tricks to take Fourier transforms of important signals. For example, we know from LTI systems that we need to calculate quantities like the impulse and step responses; and so we need to know what the Fourier Transforms of the delta and step.

- FT of delta functions
- FT of a constant
- FT of complex exponentials
- FT of step function
- FT of running integral

Fourier transform of the Dirac delta, δ

Recall the unit impulse, $\delta(t)$, is not a usual function. However, it does make sense inside integrals, and indeed we can take the Fourier transform of it by using the sifting property.

$$\begin{aligned} \mathcal{F}[\delta(t)] &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \\ &= e^0 \\ &= 1 \end{aligned}$$

Therefore,

$$\boxed{\delta(t) \iff 1}$$

Does this make sense intuitively? (Think about our time scaling result; think about frequencies needed to create an impulse; think about convolution theorem.)

Fourier transform of the Dirac delta, δ (cont.)

We can verify that this result is consistent with our prior results. For example, consider the convolution theorem. First, we know that $\delta(t)$ is the identity element of convolution, i.e.,

$$f(t) = f(t) * \delta(t)$$

Therefore, by the convolution theorem,

$$\begin{aligned} \mathcal{F}[f(t) * \delta(t)] &= \mathcal{F}[f(t)]\mathcal{F}[\delta(t)] \\ &= F(j\omega) \cdot 1 \\ &= F(j\omega) \end{aligned}$$

Fourier transform of the shifted delta, $\delta(t-\tau)$

The Fourier transform of the shifted δ is

$$\begin{aligned}\mathcal{F}[\delta(t-\tau)] &= \int_{-\infty}^{\infty} \delta(t-\tau)e^{-j\omega t}dt \\ &= e^{-j\omega\tau}\end{aligned}$$

Therefore,

$$\boxed{\delta(t-\tau) \iff e^{-j\omega\tau}}$$

This matches what we would expect from using the time-shift theorem.

Intuitively, what's going on here? Why is it that if I shift $\delta(t)$ to $\delta(t-\tau)$, I get a complex exponential when the impulse at $t=0$ is just 1?

Another derivation of the shift property for FTs

Since $f(t-\tau) = f(t) * \delta(t-\tau)$, note that we could have derived the time shift property of the Fourier transform via the convolution theorem, i.e.,

$$\begin{aligned} \mathcal{F}[f(t-\tau)] &= \mathcal{F}[\delta(t-\tau) * f(t)] \\ &= \mathcal{F}[\delta(t-\tau)]\mathcal{F}[f(t)] \\ &= e^{-j\omega\tau}F(j\omega) \end{aligned}$$

Fourier transform of a constant signal

What, intuitively, should the Fourier transform of $f(t) = 1$ be?

Let's try to evaluate it using the definition of the Fourier transform.

$$\begin{aligned}\mathcal{F}[1] &= \int_{-\infty}^{\infty} e^{-j\omega t} dt \\ &= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\infty}^{\infty}\end{aligned}$$

We run into a problem here: we cannot evaluate this integral.

Instead, we can use duality. We know that $\delta(t) \iff 1$, and therefore by duality, we have that

$$\boxed{1 \iff 2\pi\delta(\omega)}$$

This confirms our intuition: 1 is a constant, and therefore, its spectrum should only have a DC component (i.e., at $\omega = 0$).

Fourier transform of the complex exponential

What, intuitively, should the Fourier transform of a complex exponential, $f(t) = e^{j\omega_0 t}$ be?

If we try to evaluate this using the definition of the Fourier transform, we'll run into the exact same problem we ran into before. Therefore, we're going to once again use duality.

We know that $\delta(t-\tau) \iff e^{j\omega\tau}$ and therefore,

$$e^{jt\tau} \iff 2\pi\delta(\omega - \tau)$$

We also change τ to ω_0 so that:

$$\boxed{e^{j\omega_0 t} \iff 2\pi\delta(\omega - \omega_0)}$$

Fourier transform of cosine

Now that we know the Fourier transform of the complex exponential, we can derive the Fourier transform of cosine and sine.

Recall that

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_t} + e^{-j\omega_0 t}\right)$$

and therefore,

$$\begin{aligned}
\mathcal{F}[\cos(\omega_0 t)] &= \mathcal{F}\left[\frac{1}{2}\left(e^{-j\omega_0 t} + e^{j\omega_0 t}\right)\right] \\
&= \frac{1}{2}\left(\mathcal{F}\left[e^{j\omega_0 t}\right]\mathcal{F}\left[e^{-j\omega_0 t}\right]\right) \\
&= \frac{1}{2}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right) \\
&= \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)
\end{aligned}$$

Does this make sense intuitively?

$$\boxed{\cos(\omega_0 t) \iff \pi(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))}$$

Fourier transform of sine

Likewise, using the fact that

$$\sin(\omega_0 t) = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

we can derive that

$$\boxed{\sin(\omega_0 t) \iff j\pi(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))}$$

Another view of modulation

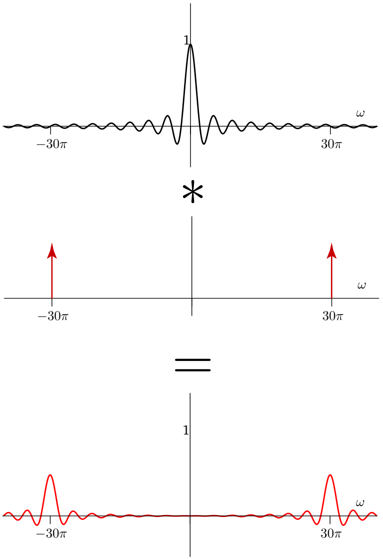
Last lecture, we used the duality of time shifting to derive modulation. We can now view convolution in a more intuitive way: as frequency domain convolution.

If we have a signal $f(t) \iff F(j\omega)$, then

$$
\begin{aligned}
\mathcal{F}[f(t)\cos(\omega_0 t)] &= \frac{1}{2\pi}\left(\mathcal{F}\left[f(t)\right] * \mathcal{F}\left[\cos(\omega_0 t)\right]\right) \\
&= \frac{1}{2\pi}\left(\mathcal{F}\left[f(t)\right] * \left(\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right)\right) \\
&= \frac{1}{2}\left(F(j\omega) * \delta(\omega-\omega_0) + F(j\omega) * \delta(\omega+\omega_0)\right) \\
&= \frac{1}{2}\left(F(j(\omega-\omega_0)) + F(j(\omega+\omega_0))\right)
\end{aligned}
$$

This is diagrammed on the next page.

Another view of modulation (cont.)



Fourier transform of the Heavyside step function

What is the Fourier transform of $u(t)$? If we try to use the Fourier transform formula, we have that

$$\begin{aligned}\mathcal{F}[u(t)] &= \int_{-\infty}^{\infty} u(t)e-j\omega tdt \\ &= \int_{0}^{\infty} e^{-j\omega t}dt \\ &= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{0}^{\infty}\end{aligned}$$

and thus the integral doesn't converge. Instead, we'll use an idea called limiting Fourier transforms.

Limiting Fourier transforms

When the Fourier transform integral doesn't converge, and there's not a "trick" we can use, an alternative approach is to use limiting Fourier transforms.

In this approach, we represent the signal as a limit of a sequence of signals for which the Fourier transforms do exist. i.e., consider $f_n(t)$ which does have a Fourier transform. If

$$f(t) = \lim_{n \to \infty} f_n(t)$$

then we also have that

$$F(j\omega) = \lim_{n \to \infty} F_n(j\omega)$$

if the limit makes sense.

Limiting Fourier transform example

Consider the Fourier transform of $f(t) = \mathrm{sign}(t)$. This signal is defined as

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

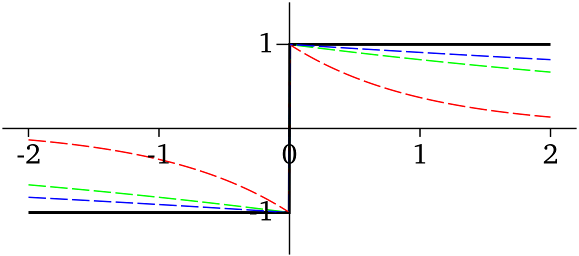
We previously derived the Fourier transform for $e^{-at}u(t)$. We can use this signal to make a limiting approximation to $\mathrm{sign}(t)$ by setting

$$f_a(t) = e^{-at}u(t) - e^{at}u(-t)$$

This approximation is shown on the next slide.

Limiting Fourier transform example (cont.)

Below we show $f_a(t)$ for $a = 1$ (red), $a = 1/5$ (green), and $a = 1/10$ (blue).



As $a \to 0$, then, $f_a(t) \to \text{sign}(t)$.

Limiting Fourier transform example (cont.)

Hence, we can compute the Fourier transform, $F_a(j\omega) = \mathcal{F}[f_a(t)]$, and then compute the Fourier transform of $\text{sign}(t)$ as the limit of $F_a(j\omega)$ as $a \to 0$.

$$\begin{aligned} F_a(j\omega) &= \mathcal{F}[f_a(t)] \\ &= \mathcal{F}\left[e^{-at}u(t) - e^{at}u(-t)\right] \\ &= \mathcal{F}\left[e^{-at}u(t)\right] - \mathcal{F}\left[e^{at}u(-t)\right] \\ &= \frac{1}{a + j\omega} - \frac{1}{a - j\omega} \\ &= \frac{-2j\omega}{a^2 + \omega^2} \end{aligned}$$

When $\omega = 0$, then $F_a(j\omega) = 0$ for any $a \neq 0$. Otherwise, if $\omega \neq 0$, then

$$\begin{aligned} \lim_{a \to 0} F_a(j\omega) &= \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2} \\ &= \frac{-2j\omega}{\omega^2} \\ &= \frac{2}{j\omega} \end{aligned}$$

Limiting Fourier transform example (cont.)

With this, we can state that

$$\boxed{\mathrm{sign}(t) \iff \begin{cases} \frac{2}{j\omega}, & \omega \neq 0 \\ 0, & \omega = 0 \end{cases}}$$

Fourier transform of the step function

The step function can be written in terms of the sign function, i.e.,

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sign}(t)$$

Therefore,

$$\begin{aligned}
\mathcal{F}[u(t)] &= \mathcal{F}\left[\frac{1}{2} + \frac{1}{2}\mathrm{sign}(t)\right] \\
&= \frac{1}{2}2\pi\delta(\omega) + \frac{1}{2}\left(\frac{2}{j\omega}\right) \\
&= \pi\delta(\omega) + \frac{1}{j\omega}
\end{aligned}$$

Note that the second term is zero at $\omega = 0$, and so the spectrum of $u(t)$ is $\pi\delta(\omega)$ at $\omega = 0$. Thus,

$$\boxed{u(t) \iff \pi\delta(\omega) + \frac{1}{j\omega}}$$

Fourier transform of an integral

Now, with the Fourier transform of the step function, it is possible to calculate the Fourier transform of an integral. Recall that we can represent integration as the convolution with the step function, i.e.,

$$\int_{-\infty}^{t} f(\tau)d\tau = (f * u)(t)$$

Therefore,

$$\begin{aligned}\mathcal{F}\left[\int_{-\infty}^{t} f(\tau)d\tau\right] &= \mathcal{F}[f(t)]\mathcal{F}[u(t)] \\ &= F(j\omega)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) \\ &= \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}\end{aligned}$$

Therefore,

$$\boxed{\int_{-\infty}^{t} f(\tau)d\tau \iff \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}}$$