Problem 1: Sampling

From sampling theorem, we know

$$f_s = 2B \text{ Hz}$$

(a) Using the time shift property of FT

$$\mathcal{F}[x(t-1)] = e^{-j\omega} X(j\omega)$$

So,

$$|e^{-j\omega} X(j\omega)|$$

$$= |X(j\omega)|$$

Therefore, $x(t-1)$ will also be a bandlimited signal with bandwidth B Hz.

Hence, Nyquist rate of $x(t-1)$ is also 2B Hz

(b) Using modulation property of FT

$\mathcal{F}[\cos(2\pi Bt)\,x(t)]$

$= \frac{1}{2} X[j(\omega - 2\pi B)] + \frac{1}{2} X[j(\omega + 2\pi B)]$

Therefore, $\cos(2\pi Bt)\,x(t)$ will also be a bandlimited signal with Bandwidth of 2B Hz. Hence, Nyquist rate of $\cos(2\pi Bt)\,x(t)$ is 4B Hz

(c) Using time-scaling property of FT

$\mathcal{F}[x(t) + x(t/2)]$

$= X(j\omega) + 2X(j2\omega)$

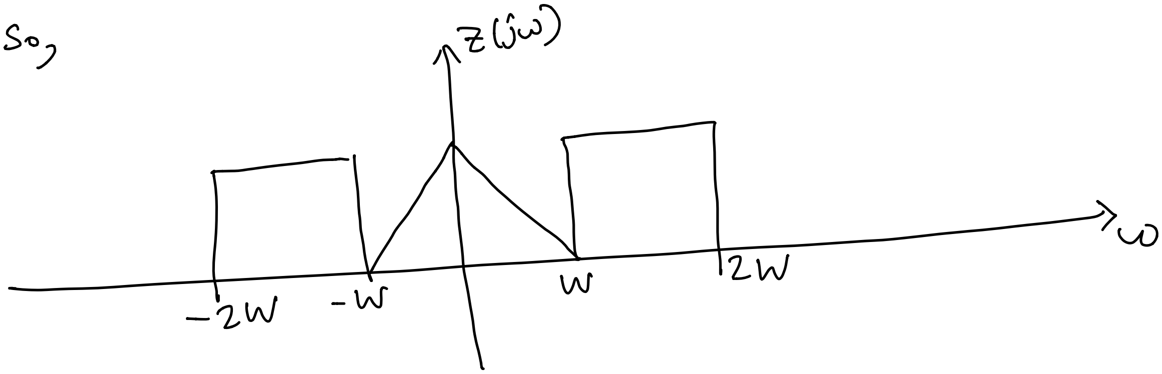
Therefore, $x(t) + x(t/2)$ will also be a bandlimited signal with Bandwidth B Hz. Hence, Nyquist rate of $x(t) + x(t/2)$ is 2B Hz.

Problem 2: Denoising

Let $z(t) = x(t) + n(t)$. Then

$$Z(j\omega) = X(j\omega) + N(j\omega)$$

So,



Now, let $r(t) = z(t)s(t)$. So,

$$r(t) = z(t)\left(\sum_{n=-\infty}^{\infty} \delta(t-nT)\right)$$

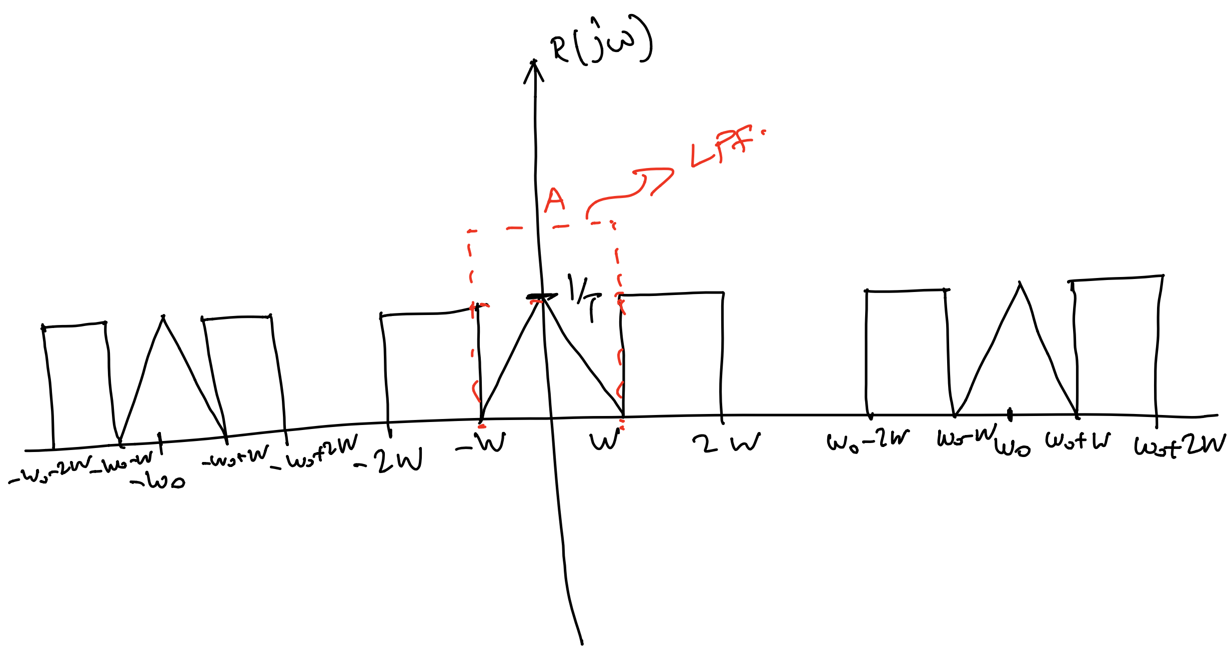
$$= \sum_{n=-\infty}^{\infty} z(nT)\,\delta(t-nT)$$

So, $r(t)$ is a sampled version of $z(t)$ with sampling interval T seconds.

From lecture, we know

$$R(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Z(j(\omega - k\omega_0))$$

where $\omega_0 = \frac{2\pi}{T}$. Therefore $R(j\omega)$ is a periodic extension of $Z(j\omega)$ with repetition every ω_0



From the above sketch, we can recover $x(t)$ if there is no distortion in the baseband and $A = T$. It happens if

$$\omega_0 - 2W > W, \quad -\omega_0 + 2W < -W$$

$$\Rightarrow \quad \omega_0 > 3W$$

$$\Rightarrow \quad \frac{2\pi}{T} > 3W$$

$$\Rightarrow \quad T < \frac{2\pi}{3W}$$

Therefore the maximum value of T for which we can recover $x(t)$ is $\frac{2\pi}{3W}$.

$$\therefore \quad A = T_{max} = \frac{2\pi}{3W}$$

Problem 3: Laplace Transform

$$x(t) = 3e^{2t}u(t) + 4e^{3t}u(t)$$

(a) $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$\Rightarrow X(j\omega) = 3\int_{0}^{\infty} e^{2t} e^{-j\omega t}\, dt + 4\int_{0}^{\infty} e^{3t} e^{-j\omega t}\, dt$$

$$\Rightarrow X(j\omega) = 3\int_{0}^{\infty} e^{(2-j\omega)t}\, dt + 4\int_{0}^{\infty} e^{(3-j\omega)t}\, dt$$

$$\Rightarrow X(j\omega) = 3\left[\frac{1}{2-j\omega} e^{(2-j\omega)t}\Big|_{0}^{\infty}\right] + 4\left[\frac{1}{3-j\omega} e^{(3-j\omega)t}\Big|_{0}^{\infty}\right]$$

Since e^{2t} and e^{3t} are growing exponentials, so the envelope of $e^{(2-j\omega)t}$ and $e^{(3-j\omega)t}$ are growing exponentially and doesn't converge as $t \to \infty$. $\therefore$ FT of $x(t)$ doesn't converge

(b) $\mathcal{F}[x(t)e^{-\sigma t}]$

$$= 3\int_0^\infty e^{(2-j\omega-\sigma)t}\,dt + 4\int_0^\infty e^{(3-j\omega-\sigma)t}\,dt$$

$$= 3\left[\frac{1}{2-j\omega-\sigma} e^{(2-j\omega-\sigma)t}\Big|_0^\infty\right] + 4\left[\frac{1}{3-j\omega-\sigma} e^{(3-j\omega-\sigma)t}\Big|_0^\infty\right]$$

If $e^{(2-\sigma)t}$ and $e^{(3-\sigma)t}$ are decaying exponentials, then the envelope of $e^{(2-j\omega-\sigma)t}$ and $e^{(3-j\omega-\sigma)t}$ are decaying exponentially and converges to zero as $t\to\infty$.

Therefore FT of $x(t)e^{-\sigma t}$ converges if

$$2-\sigma<0 \quad \text{and} \quad 3-\sigma<0$$

$\Rightarrow \quad \sigma>2 \quad \text{and} \quad \sigma>3$

$\Rightarrow \quad \sigma>3$

$\therefore$ FT of $x(t)e^{-\sigma t}$ only converges for $\sigma=3.5$.

ⓒ $X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$

$$= 3\int_{0^-}^{\infty} e^{(2-s)t} dt + 4\int_{0^-}^{\infty} e^{(3-s)t} dt$$

$$= 3\left[\frac{1}{2-s} e^{(2-s)t} \Big|_{0^-}^{\infty} \right] + 4\left[\frac{1}{3-s} e^{(3-s)t} \Big|_{0^-}^{\infty} \right]$$

If $Re\{s\} > 2$ and $Re\{s\} > 3$, then

$e^{(2-s)t}$ and $e^{(3-s)t}$ decays to zero as $t \to \infty$.

Hence $X(s)$ exists for $Re\{s\} > 3$. Then,

$$X(s) = \frac{3}{s-2} + \frac{4}{s-3}$$

$$= \frac{3(s-3) + 4(s-2)}{(s-2)(s-3)}$$

$$X(s) = \frac{7s-17}{(s-2)(s-3)}$$

So, Poles of $X(s)$ are $s=2, s=3$

Zeros of $X(s)$ are $s=17/7$