

Due Friday, 6 Nov 2020, by 11:59pm to Gradescope.

Covers material up to Lecture 8.

100 points total.

This homework covers questions relate to Fourier series and LTI systems.

1. (28 points) **Fourier Series**

(a) (18 points) Find the Fourier series coefficients for each of the following periodic signals:

i. (9 points)  $f(t) = \cos(5\pi t) + \frac{1}{2} \sin(4\pi t)$

**Solution:** We first find the period of  $f(t)$ . The first term  $\cos(5\pi t)$  is periodic with period  $T_1 = \frac{2\pi}{5\pi} = \frac{2}{5}$ . The second term  $\sin(4\pi t)$  is periodic with period  $T_2 = \frac{2\pi}{4\pi} = \frac{1}{2}$ . Since  $\frac{T_1}{T_2} = \frac{4}{5}$ ,  $f(t)$  is then periodic with fundamental period  $T_0 = 5T_1 = 4T_2 = 2$  sec, and fundamental frequency  $\omega_0 = \frac{2\pi}{T_0} = \pi$  rad/s.

Using Euler's identity,  $f(t)$  can be equivalently written as:

$$\begin{aligned} f(t) &= \cos(5\pi t) + \frac{1}{2} \sin(4\pi t) = \frac{1}{2} (e^{j5\pi t} + e^{-j5\pi t}) + \frac{1}{4j} (e^{j4\pi t} - e^{-j4\pi t}) \\ &= \frac{1}{2} e^{j5\pi t} + \frac{1}{2} e^{-j5\pi t} + \frac{-j}{4} e^{j4\pi t} + \frac{j}{4} e^{-j4\pi t} \end{aligned}$$

The fundamental frequency of  $f(t)$  is  $\omega_0 = \pi$ , and since any periodic signal can be written as:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 k t}$$

we deduce for  $f(t)$  the following Fourier series coefficients:

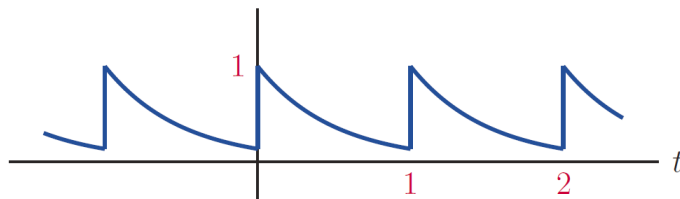
$$c_k = \begin{cases} \frac{-j}{4}, & \text{if } k = 4 \\ \frac{j}{4}, & \text{if } k = -4 \\ \frac{1}{2}, & \text{if } k = -5, 5 \\ 0, & \text{otherwise} \end{cases}$$

ii. (9 points)  $f(t)$  is a periodic signal with period  $T = 1$  s, where one period of the signal is defined as  $e^{-2t}$  for  $0 < t < 1$  s, as shown below.

**Solution:**

Since  $f(t)$  is periodic with period  $T_0 = 1$  s, we can rewrite it as:

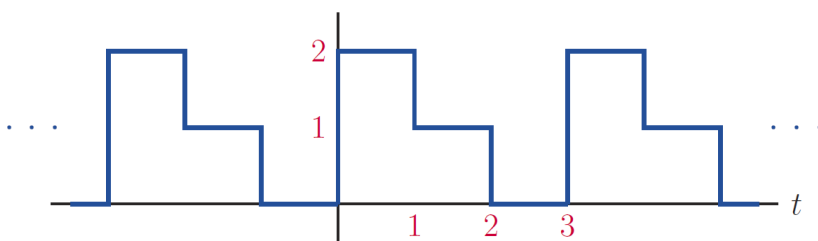
$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$



where  $\omega_0 = \frac{2\pi}{T_0} = 2\pi$  rad/s and the coefficients  $c_k$ 's are as follows:

$$\begin{aligned} c_k &= \frac{1}{T} \int_0^T f(t) e^{-jk\omega_0 t} dt = \int_0^1 e^{-2t} e^{-j2k\pi t} dt \\ &= \frac{1 - e^{-(2+j2\pi k)}}{2 + j2\pi k} = \frac{1 - e^{-2}}{2 + j2\pi k} \end{aligned}$$

iii. (optional) (0 points)  $f(t)$  is the periodic signal shown below:



**Solution:** Since  $f(t)$  is periodic with period  $T_0 = 3$  s, we can rewrite it as:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where  $\omega_0 = \frac{2\pi}{3}$  rad/s and the coefficients  $c_k$ 's are as follows:

$$c_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{3} \left( \int_0^1 2 dt + \int_1^2 1 dt \right) = 1$$

and for  $k \neq 0$ , we have:

$$\begin{aligned} c_k &= \frac{1}{T} \int_0^T f(t) e^{-jk\omega_0 t} dt = \frac{1}{3} \left( \int_0^1 2 e^{-j(2\pi/3)kt} dt + \int_1^2 e^{-j(2\pi/3)kt} dt \right) \\ &= \frac{1}{3} \left( 2 \frac{1 - e^{-j(2\pi/3)k}}{j(2\pi/3)k} + \frac{e^{-j(2\pi/3)k} - e^{-j(4\pi/3)k}}{j(2\pi/3)k} \right) = \frac{2 - e^{-j(2\pi/3)k} - e^{-j(4\pi/3)k}}{j2\pi k} \\ &= \frac{2 - e^{-j(2\pi/3)k} - e^{j(2\pi/3)k}}{j2\pi k} = \frac{2 - 2 \cos\left(\frac{2\pi k}{3}\right)}{j2\pi k} = \frac{1 - \cos\left(\frac{2\pi k}{3}\right)}{j\pi k} \end{aligned}$$

(b) (10 points) Suppose you have two periodic signals  $x(t)$  and  $y(t)$ , of periods  $T_1$  and  $T_2$  respectively. Let  $x_k$  and  $y_k$  be the Fourier series coefficients of  $x(t)$  and  $y(t)$ .

- i. (5 points) If  $T_1 = T_2$ , express the Fourier series coefficients of  $z(t) = x(t) + y(t)$  in terms of  $x_k$  and  $y_k$ .

**Solution:**

If  $T_1 = T_2$ , then  $y(t)$  is also periodic with period  $T_0 = T_1 = T_2$ . If  $\omega_0 = \frac{2\pi}{T_0}$ , then

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t}$$

and

$$y(t) = \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t}$$

Therefore,

$$z(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (x_k + y_k) e^{jk\omega_0 t}$$

Therefore, the Fourier series coefficients of  $z(t)$  are:

$$z_k = x_k + y_k$$

- ii. (5 points) If  $T_1 = \frac{1}{2}T_2$ , express the Fourier series coefficients of  $w(t) = x(t) + y(t)$  in terms of  $x_k$  and  $y_k$ .

**Solution:** First of all,  $w(t)$  is periodic with period  $T_0 = T_1 = \frac{1}{2}T_2$ , and frequency  $\omega_1 = 2\omega_2 = 2\omega_0$ . Let,

$$x(t) = \sum_{m=-\infty}^{\infty} x_m e^{jm\omega_1 t} = \sum_{m=-\infty}^{\infty} x_m e^{2jm\omega_0 t}$$

and

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_2 t} = \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_0 t}$$

Therefore,  $w(t)$  can be written as:

$$w(t) = x(t) + y(t) = \sum_{m=-\infty}^{\infty} x_m e^{j2m\omega_0 t} + \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_0 t}$$

Let  $m' = 2m$ , then

$$\begin{aligned} w(t) &= \sum_{m=-\infty}^{\infty} y_n e^{jn\omega_0 t} + \sum_{\text{even } m'} x_{\frac{m'}{2}} e^{jm'\omega_0 t} \\ &= \sum_{\text{even } n} y_n e^{jn\omega_0 t} + \sum_{\text{odd } n} y_n e^{jn\omega_0 t} + \sum_{\text{even } m'} x_{\frac{m'}{2}} e^{jm'\omega_0 t} \end{aligned}$$

Therefore,

$$w_n = \begin{cases} y_n, & \text{for } n \text{ odd} \\ y_n + x_{\frac{n}{2}}, & \text{for } n \text{ even} \end{cases}$$

2. (20 points) **Fourier series of transformation of signals**

Suppose that  $f(t)$  is a periodic signal with period  $T_0$ , with the following Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Determine the period of each of the following signals, then express its Fourier series in terms of  $c_k$ :

(a) (5 points)  $g(t) = 2f(t)$

**Solution:**

The function  $g(t)$  has the same period of  $f(t)$ . Scaling the signal will affect the Fourier coefficient  $c_k$ :

$$g(t) = 2f(t) = 2 \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} 2c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega_0 t}$$

Therefore  $c'_k = 2c_k$

(b) (5 points)  $g(t) = f(-2t)$

**Solution:**

The period of  $g(t)$  is  $T'_0 = \frac{T_0}{2}$  and its corresponding frequency is  $\omega'_0 = 2\omega_0$

$$g(t) = f(-2t) = \sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 2t} = \sum_{k=-\infty}^{\infty} c_{-k} e^{jk2\omega_0 t} = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega'_0 t}$$

Therefore  $c'_k = c_{-k}$

(c) (5 points)  $g(t) = f(t - t_0)$

**Solution:**

$g(t)$  has the same period of  $f(t)$ .

$$g(t) = f(t - t_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-t_0)} = \sum_{k=-\infty}^{\infty} (c_k e^{-jk\omega_0 t_0}) e^{jk\omega_0 t}$$

Therefore  $c'_k = c_k e^{-jk\omega_0 t_0}$

(d) (5 points)  $g(t) = f(t/a)$ , where  $a$  is a positive real number

**Solution:**

The period of  $g(t)$  is  $T'_0 = aT_0$ , and its corresponding frequency is:  $\omega'_0 = \frac{\omega_0}{a}$ . Therefore,

$$g(t) = f(t/a) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t/a)} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega'_0 t}$$

Therefore, the Fourier series coefficients of  $f(t)$  and  $g(t)$  are the same.

3. (10 points) **Eigenfunctions and LTI systems**

- (a) (5 points) Show that  $f(t) = \cos(\omega_0 t)$  is not an eigenfunction of an LTI system.

**Solution:**

Assume that  $h(t)$  is the impulse response of the system. Then the output  $y(t)$  to input  $f(t) = \cos(\omega_0 t) = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$  is as follows:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{j\omega_0(t-\tau)}h(\tau)d\tau + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j\omega_0(t-\tau)}h(\tau)d\tau \\ &= \frac{1}{2}e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} e^{-j\omega_0 \tau}h(\tau)d\tau}_{=a_1} + \frac{1}{2}e^{-j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} e^{j\omega_0 \tau}h(\tau)d\tau}_{=a_2} \end{aligned}$$

For  $f(t)$  to be an eigenfunction for the system, its corresponding output should be of the form  $af(t)$ , where  $a$  is constant. The output to  $\cos(\omega_0 t)$  is:

$$y(t) = \frac{1}{2}a_1 e^{j\omega_0 t} + \frac{1}{2}a_2 e^{-j\omega_0 t}$$

Since, in general  $a_1 \neq a_2$ , we cannot construct again  $\cos(\omega_0 t)$  in  $y(t)$ . For instance, suppose  $f(t) = \delta(t-4)$ , then  $a_1 = e^{-j4\omega_0}$  and  $a_2 = e^{j4\omega_0}$ . Therefore,

$$y(t) = \frac{1}{2}e^{j\omega_0(t-4)} + \frac{1}{2}e^{-j\omega_0(t-4)} = \cos(\omega_0(t-4))$$

We then see the output is not of the form  $a \cos(\omega_0 t)$ , therefore  $\cos(\omega_0 t)$  is not an eigenfunction for an LTI system. (We will accept a counterexample as correct, since complex exponentials are eigenfunctions of all LTI systems.)

- (b) (5 points) Show that  $f(t) = t$  is not an eigenfunction of an LTI system.

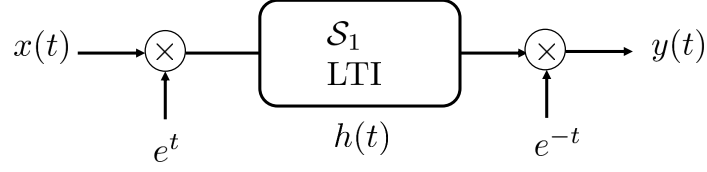
**Solution:**

Assume that  $h(t)$  is the impulse response of the system. Then the output  $y(t)$  to input  $f(t) = t$  is as follows:

$$y(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} (t-\tau)h(\tau)d\tau = t \underbrace{\int_{-\infty}^{\infty} h(\tau)d\tau}_{=a_1} - \underbrace{\int_{-\infty}^{\infty} \tau h(\tau)d\tau}_{=a_2}$$

$y(t)$  is of the form  $a_1 t + a_2$ , therefore the function  $f(t) = t$  is not an eigenfunction of an LTI system.

4. (29 points) **LTI systems**



(a) Consider the following system:

The system takes as input  $x(t)$ , it first multiplies the input with  $e^t$ , then sends it through an LTI system. The output of the LTI system gets multiplied by  $e^{-t}$  to form the output  $y(t)$ .

i. (5 points) Show that we can write  $y(t)$  as follows:

$$y(t) = [(e^t x(t)) * h(t)] e^{-t} \quad (1)$$

**Solution:**

The input  $x(t)$  gets first multiplied by  $e^t$  and forms the intermediate signal:

$$y_1(t) = e^t x(t)$$

Next,  $y_1(t)$  is fed to the LTI system, the output  $y_2(t)$  is then the convolution of  $y_1(t)$  with  $h(t)$ :

$$y_2(t) = y_1(t) * h(t) = (e^t x(t)) * h(t)$$

Finally,  $y_2(t)$  gets finally multiplied by  $e^{-t}$ :

$$y(t) = e^{-t} y_2(t) = [(e^t x(t)) * h(t)] e^{-t}$$

ii. (5 points) Use the definition of convolution to show that (1) can be equivalently written as:

$$y(t) = \int_{-\infty}^{\infty} h'(\tau) x(t - \tau) d\tau \quad (2)$$

where  $h'(\tau)$  is a function to define in terms of  $h(t)$ .

**Solution:**

By applying the definition of convolution, we obtain:

$$\begin{aligned}
 y(t) &= [(e^t x(t)) * h(t)] e^{-t} \\
 &= e^{-t} \int_{-\infty}^{\infty} h(\tau) e^{t-\tau} x(t - \tau) d\tau \\
 &= e^{-t} e^t \int_{-\infty}^{\infty} h(\tau) e^{-\tau} x(t - \tau) d\tau \\
 &= \int_{-\infty}^{\infty} h(\tau) e^{-\tau} x(t - \tau) d\tau \\
 &= \int_{-\infty}^{\infty} h'(\tau) x(t - \tau) d\tau
 \end{aligned}$$

where  $h'(\tau) = h(\tau) e^{-\tau}$ .

- iii. (5 points) Equation (2) represents a description of the equivalent system that maps  $x(t)$  to  $y(t)$ . Show using (2) that the equivalent system is LTI and determine its impulse response  $h_{eq}(t)$  in terms of  $h(t)$ .

**Solution:**

**Linearity:**

Suppose that for inputs  $x_1(t)$  and  $x_2(t)$ , we have respectively the corresponding outputs  $y_1(t)$  and  $y_2(t)$  outputs. Now, let  $x(t) = ax_1(t) + bx_2(t)$ , we then have the following:

Method 1: Using the equation from part b:

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} h'(\tau)x(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} h'(\tau)(ax_1(t-\tau) + bx_2(t-\tau))d\tau \\
 &= \int_{-\infty}^{\infty} (ah'(\tau)x_1(t-\tau) + bh'(\tau)x_2(t-\tau))d\tau \\
 &= \int_{-\infty}^{\infty} ah'(\tau)x_1(t-\tau)d\tau + \int_{-\infty}^{\infty} bh'(\tau)x_2(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} ah'(\tau)x_1(t-\tau)d\tau + \int_{-\infty}^{\infty} bh'(\tau)x_2(t-\tau)d\tau \\
 &= ay_1(t) + by_2(t)
 \end{aligned}$$

Method 2:

$$\begin{aligned}
 y(t) &= [(e^t x(t)) * h(t)]e^{-t} \\
 &= [(e^t (ax_1(t) + bx_2(t))) * h(t)]e^{-t} \\
 &= [(ae^t x_1(t) + be^t x_2(t)) * h(t)]e^{-t} \\
 &= [(ae^t x_1(t)) * h(t) + (be^t x_2(t)) * h(t)]e^{-t} \\
 &= [(ae^t x_1(t)) * h(t)]e^{-t} + [(be^t x_2(t)) * h(t)]e^{-t} \\
 &= ay_1(t) + by_2(t)
 \end{aligned}$$

Therefore system is linear.

**Time invariance:**

Using result from part b, if we delay the input for  $t_0$ :

$$\begin{aligned}
 y_{t_0}(t) &= \int_{-\infty}^{\infty} h'(\tau)x(t-\tau-t_0)d\tau \\
 &= \int_{-\infty}^{\infty} h'(\tau)x(t-t_0-\tau)d\tau \\
 &= y(t-t_0)
 \end{aligned}$$

Therefore system is TI. From part b, we know that  $h'(t) = h(t)e^{-t}$ . Therefore, the impulse response of the equivalent system is:

$$h_{eq}(t) = h(t)e^{-t}$$

- iv. (Optional) (0 points) Suppose that system  $\mathcal{S}_1$  is given by its step response  $s(t) = r(t - 1)$ . Find the impulse response  $h(t)$  of  $\mathcal{S}_1$ . What can you say about the causality and stability of system  $\mathcal{S}_1$ ? What can you say about the causality and stability of the overall equivalent system?

**Solution:**

The impulse response of system  $\mathcal{S}_1$  is:

$$h(t) = \frac{d}{dt}s(t) = u(t - 1)$$

Since  $h(t) = 0$  for  $t < 0$ , the system  $\mathcal{S}_1$  is causal. However, this same system is not stable because

$$\int_{-\infty}^{\infty} |h(t)| dt \rightarrow \infty$$

The equivalent system has the following equivalent impulse response:

$$h_{eq}(t) = e^{-t}u(t - 1)$$

Since  $h_{eq}(t) = 0$  for  $t < 0$ , the system is causal. It is also stable, because:

$$\int_{-\infty}^{\infty} |h_{eq}(t)| dt = \int_{t=1}^{\infty} e^{-t} dt = e^{-1} < \infty$$

- (b) Suppose  $x(t)$  is periodic with period  $T$  and is specified in the interval  $0 < t < T/4$  as shown in figure 1.

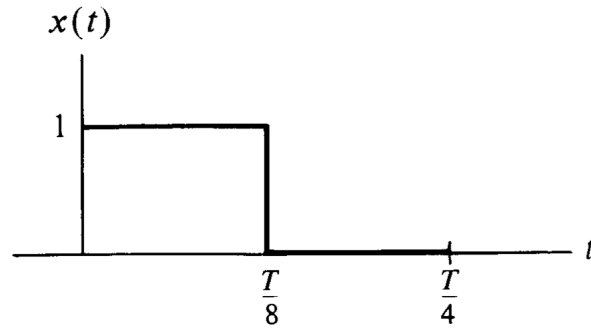


Figure 1:  $x(t)$  in the interval  $0 < t < T/4$

Sketch  $x(t)$  in the interval  $0 < t < T$  if

- i. (7 points) the Fourier series has only odd harmonics and  $x(t)$  is an even function

**Solution:** Since  $x(t)$  is even, we can extend figure 1 as indicated in figure 2. Since  $x(t)$  has only odd harmonics, it must have the property  $x(t - T/2) = -x(t)$ . So we have  $x(t)$  as in figure 3.



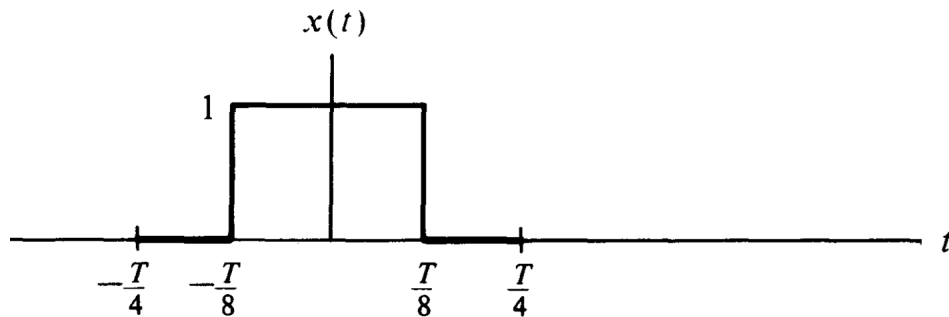


Figure 2:  $x(t)$  (even) in the interval  $-T/4 < t < T/4$

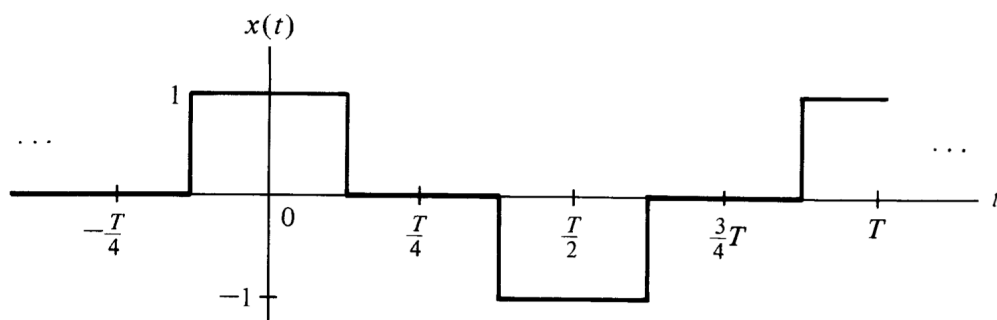


Figure 3:  $x(t)$  (even) in the interval  $-T < t < T$

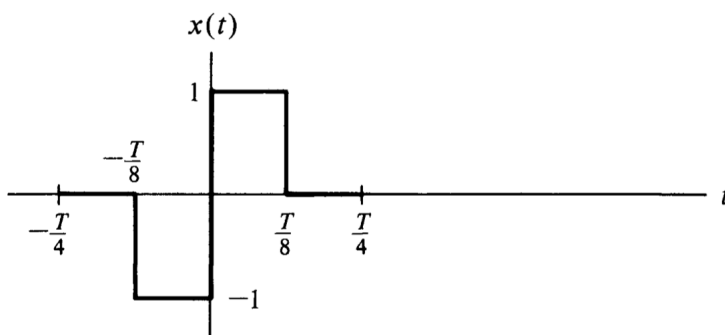


Figure 4:  $x(t)$  (odd) in the interval  $-T/4 < t < T/4$

- ii. (7 points) the Fourier series has only odd harmonics and  $x(t)$  is an odd function

**Solution:** Since  $x(t)$  is odd, for  $-T/4 < t < T/4$  it must be as indicated in figure 4. Since  $x(t)$  has odd harmonics, so  $x(t - T/2) = -x(t)$ . Consequently  $x(t)$  is as shown in figure 5.

5. (13 points) **MATLAB**

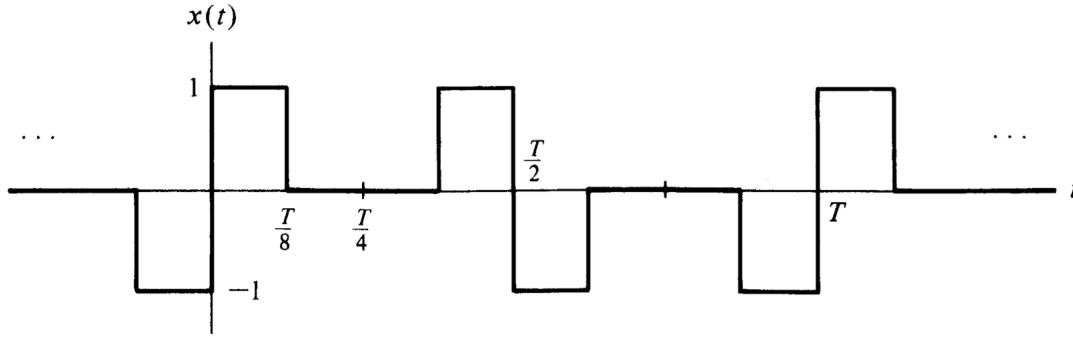


Figure 5:  $x(t)$  (odd) in the interval  $-T < t < T$

(a) (6 points) **Task 1**

Write an m-file that takes a set of Fourier series coefficients, a fundamental frequency, and a vector of output times, and computes the truncated Fourier series evaluated at these times. The declaration and help for the m-file might be:

```
function fn = myfs(Dn,omega0,t)
%
% fn = myfs(Dn,omega0,t)
% % Evaluates the truncated Fourier Series at times t
%
% Dn -- vector of Fourier series coefficients
%
% omega0 -- fundamental frequency
% t -- vector of times for evaluation
%
% fn -- truncated Fourier series evaluated at t
```

The output of the m-file should be

$$f_N(t) = \sum_{n=-N}^N D_n e^{j\omega_0 n t}$$

The length of the vector Dn should be  $2N + 1$ . You will need to calculate  $N$  from the length of Dn.

**Solution:**

```
function fn = myfs(Dn,omega0,t)
% fn = myfs(Dn,omega0,t)
% Evaluates the truncated Fourier Series at times t
% Dn -- vector of Fourier series coefficients
% assumed to run from -N:N, where length(Dn) is 2N+1
% omega0 -- fundamental frequency
% t -- vector of times for evaluation
```

```

% fn -- truncated Fourier series evaluated at t
N = (length(Dn)-1)/2;
fn = zeros(size(t));
for n = -N:N
    D_n = Dn(n+N+1);
    fn = fn + D_n*exp(j*omega0*n*t);
end

```

(b) (7 points) **Task 2**

Verify the output of your routine by checking the Fourier series coefficients for the sawtooth waveform. The sawtooth signal is given by  $f(t) = t \bmod 1$  described in the class notes. Try for  $N = 10$ ,  $N = 50$ . Use the MATLAB subplot command to put multiple plots on a page.

**Solution:**

```

N=10;n1=1:1:N;n2=-N:1:-1;
D1=j./(2*pi*n1);D2=j./(2*pi*n2);D=[D2 0.5 D1];
omega0=2*pi;
t=-2.5:0.001:2.5;
fn = myfs(D,omega0,t);
subplot(3,1,1);
plot(t,fn);
xlabel('t (s)'); ylabel('N=10');
title('Sawtooth Fourier Series Approximation');

```

```

N=50;n1=1:1:N;n2=-N:1:-1;
D1=j./(2*pi*n1);D2=j./(2*pi*n2);D=[D2 0.5 D1];
fn = myfs(D,omega0,t);
subplot(3,1,2);
plot(t,fn);
xlabel('t (s)'); ylabel('N=50');

```

```

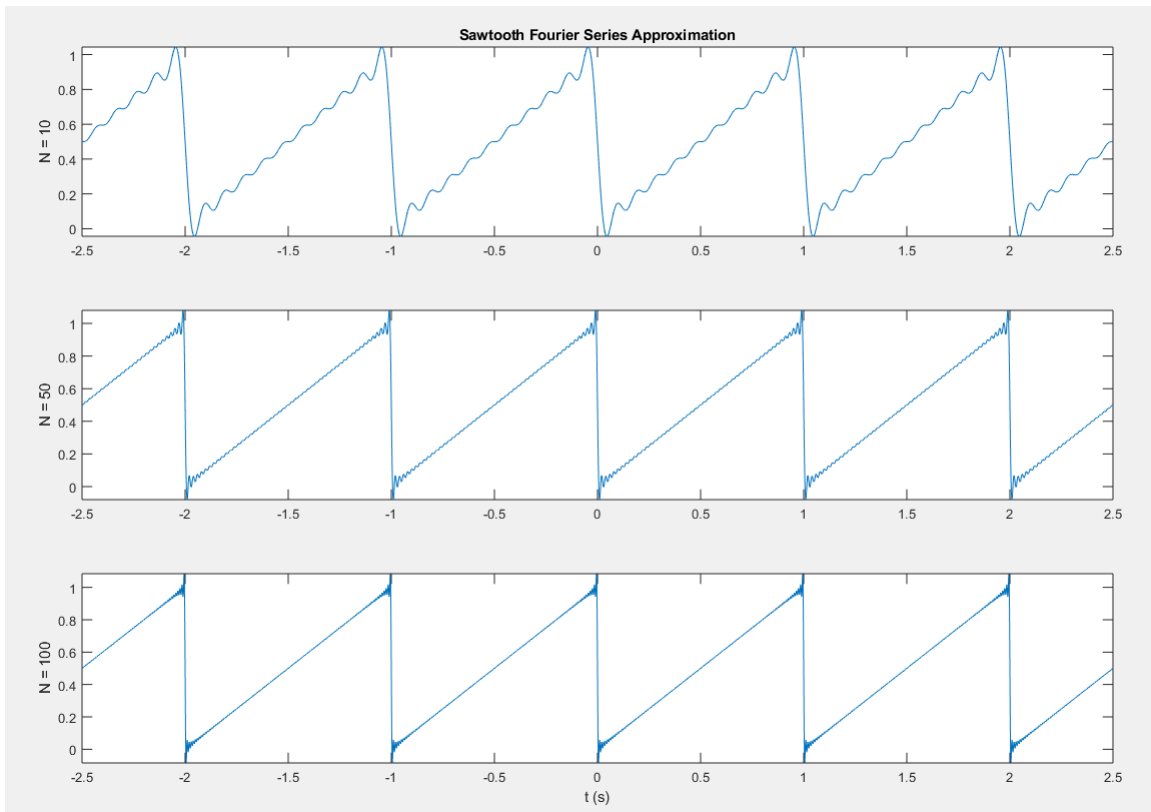
N=100;n1=1:1:N;n2=-N:1:-1;
D1=j./(2*pi*n1);D2=j./(2*pi*n2);D=[D2 0.5 D1];
omega0=2*pi;
t=-2.5:0.001:2.5;
fn = myfs(D,omega0,t);
subplot(3,1,3);
plot(t,fn);
xlabel('t (s)'); ylabel('N=100');

```

(c) (Optional) (0 points) **Task 3**

Repeat the steps of Task 2 for the case of the signal from Problem 1-a-ii.

**Solution:**



```

N=10;n=-N:1:N;D=(1-exp(-2))./(2+2*j*pi*n);
omega0=2*pi;t=-3.5:0.001:3.5;
fn = myfsHs(D,omega0,t);
subplot(3,1,1);
plot(t,fn);
xlabel('t(sec)'); ylabel('N=10');

```

```

N=50;n=-N:1:N;D=(1-exp(-2))./(2+2*j*pi*n);
omega0=2*pi;t=-3.5:0.001:3.5;
fn = myfsHs(D,omega0,t);
subplot(3,1,2);
plot(t,fn);
xlabel('t(sec)'); ylabel('N=50');

```

```

N=100;n=-N:1:N; D=(1-exp(-2))./(2+2*j*pi*n);
omega0=2*pi;t=-3.5:0.001:3.5;fn = myfsHs(D,omega0,t);
subplot(3,1,3);
plot(t,fn);
xlabel('t(sec)'); ylabel('N=100');

```

