Problem 1: Properties of fourier transform

From the convolution theorem, we know

$$Y(j\omega) = X(j\omega)\,H(j\omega)$$

$$\Rightarrow Y(j\omega) = X(j\omega)\left[1+e^{-j\omega}+e^{-3j\omega}\right]$$

$$\Rightarrow Y(j\omega) = X(j\omega) + e^{-j\omega}X(j\omega) + e^{-3j\omega}X(j\omega)$$

By linearity and time-shift properties of fourier transform

$$y(t) = x(t) + x(t-1) + x(t-3)$$

Then,

$$y(0) = x(0) + x(-1) + x(-3)$$

$$= 1 + e^{-1}\cos(A) + e^{-3}\cos(-3A)$$

If $A = \pi\left(n+\frac{1}{2}\right)$, for any integer n, we have $x(-1) = x(-3) = 0$, so $y(0) = x(0) = 1$

Problem 2: Inverse fourier transform

ⓐ Using the definition of inverse fourier transform, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\left[\frac{\pi}{j}\int_{-\infty}^{\infty} 4\delta(\omega-6)\, e^{j\omega t}\, d\omega - \frac{\pi}{j}\int_{-\infty}^{\infty} 4\delta(\omega+6)\, e^{j\omega t}\, d\omega\right]$$

Using the sifting property of impulse,

$$x(t) = \frac{1}{2\pi}\left[\frac{\pi}{j}\cdot 4e^{j6t} - \frac{\pi}{j}\cdot 4e^{-j6t}\right]$$

$$x(t) = \frac{1}{2j}\left[4e^{j6t} - 4e^{-j6t}\right]$$

Using euler's identity,

$$x(t) = 4\sin(6t)$$

(b) $X(j\omega) = \dfrac{12+7j\omega-\omega^2}{(\omega^2-2j\omega-1)(-\omega^2+j\omega-6)}$

As a first step, we factor each of the quadratic polynomials.

Let $j\omega = s$, then

$$X(s) = \frac{12+7s+s^2}{(-s^2-2s-1)(s^2+s-6)}$$

Now let's factor the 3 quadratic polynomials

(1) $s^2+7s+12$

The roots are

$$s_{1,2} = \frac{-7 \pm \sqrt{49-48}}{2}$$

$$s_{1,2} = \frac{-7 \pm 1}{2} = -3, -4.$$

So,

$$s^2+7s+12 = (s+3)(s+4)$$

(2) $-s^2 - 2s - 1$

The roots are,

$$s_{1,2} = \frac{2 \pm \sqrt{4-4}}{-2} = -1, -1$$

So,

$$-s^2 - 2s - 1 = -(s+1)^2$$

(3) $s^2 + s - 6$

The roots are

$$s_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} = 2, -3$$

So,

$$s^2 + s - 6 = (s-2)(s+3)$$

Hence,

$$X(s) = \frac{(s+3)(s+4)}{-(s+1)^2(s-2)(s+3)}$$

$$X(s) = \frac{(s+4)}{-(s+1)^2(s-2)} = \frac{s+4}{(s+1)^2(2-s)}$$

Now, we will use partial fractions to express $X(s)$ in the form below

$$X(s) = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{(2-s)}$$

Now,

$LHS = s+4$

$$RHS = A(2-s) + B(s+1)(2-s) + C(s+1)^2$$

$$= 2A - As + B(2s - s^2 + 2 - s) + C(s^2 + 2s + 1)$$

$$= 2A - As + Bs - Bs^2 + 2B + Cs^2 + 2Cs + C$$

$$= (C-B)s^2 + (B + 2C - A)s + (2A + 2B + C)$$

By equating the coefficients, we have

$$B = C \quad \ldots \text{(i)}$$

$$B + 2C - A = 1$$

$$2A + 2B + C = 4$$

using (i), we have

$$3B - A = 1$$

$$3B + 2A = 4$$

Solving the above system of equations we get,

$$A = 1, \quad B = C = 2/3$$

Now,

$$X(s) = \frac{1}{(s+1)^2} + \frac{2/3}{(s+1)} + \frac{2/3}{(2-s)}$$

Substituting $j\omega = s$, we have

$$X(j\omega) = \frac{1}{(1+j\omega)^2} + \frac{2/3}{(1+j\omega)} + \frac{2/3}{(2-j\omega)}$$

By lookup table,

$$\frac{1}{1+j\omega} \longleftrightarrow e^{-t}u(t)$$

$$\frac{1}{2-j\omega} \longleftrightarrow e^{2t}u(-t)$$

$$\frac{1}{(1+j\omega)^2} \longleftrightarrow te^{-t}u(t)$$

Hence by linearity of IFT, we have

$$x(t) = \frac{2}{3}e^{-t}u(t) + \frac{2}{3}e^{2t}u(-t) + te^{-t}u(t)$$

Problem 3: Diff. eq^n description of LTI systems

(a) From the FT table,

$$H(j\omega) = \frac{1}{3+j\omega}$$

$$Y(j\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega}$$

From convolution theorem,

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

$$\Rightarrow X(j\omega) = \frac{\frac{1}{3+j\omega} - \frac{1}{4+j\omega}}{\frac{1}{3+j\omega}}$$

$$\Rightarrow X(j\omega) = \frac{4+j\omega-3-j\omega}{(3+j\omega)(4+j\omega)} \times (3+j\omega)$$

$$\Rightarrow X(j\omega) = \frac{1}{4+j\omega}$$

Using the FT table,

$$x(t) = e^{-4t}u(t)$$

(b) from the FT table,

$$X(j\omega) = \frac{1}{1+j\omega}$$

$$Y(j\omega) = \frac{1}{2+j\omega}$$

Using the convolution theorem,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$\Rightarrow H(j\omega) = \frac{1+j\omega}{2+j\omega}$$

(i) From the FT table,

$$X_1(j\omega) = \frac{1}{\frac{1}{2}+j\omega}$$

Using the convolution theorem,

$$Y_1(j\omega) = X_1(j\omega)\, H(j\omega)$$

$$\Rightarrow Y_1(j\omega) = \frac{1+j\omega}{\left(\frac{1}{2}+j\omega\right)(2+j\omega)}$$

Let $s = j\omega$, then,

$$Y_1(s) = \frac{1+s}{\left(\frac{1}{2}+s\right)(2+s)}$$

Using partial fraction expansion,

$$Y_1(s) = \frac{A}{\frac{1}{2}+s} + \frac{B}{2+s}$$

$LHS = 1+s$

$RHS = A(2+s) + B\left(\frac{1}{2}+s\right)$

$= 2A + As + \frac{B}{2} + Bs$

$= (A+B)s + \left(2A + \frac{B}{2}\right)$

Equating coefficients,

$A+B = 1$

$2A + \frac{B}{2} = 1$

Solving the above system of equations, we get $A = 1/3$, $B = 2/3$.

So, $Y_1(j\omega) = \frac{1/3}{1/2 + j\omega} + \frac{2/3}{2+j\omega}$

$\therefore\ y_1(t) = \frac{1}{3} e^{-1/2 t} u(t) + 2/3\, e^{-2t} u(t)$

(ii) we know,

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{1+j\omega}{2+j\omega}$$

$$\Rightarrow \quad 2Y(j\omega) + j\omega Y(j\omega) = X(j\omega) + j\omega X(j\omega)$$

Taking the IFT of both sides,

$$2y(t) + \frac{dy(t)}{dt} = x(t) + \frac{dx(t)}{dt}$$