Continuous time Fourier series

This lecture discusses Fourier series, en route to the Fourier transform.

- Motivation
- Complex exponentials as eigenfunctions of LTI systems
- Amplitude and phase spectrum
- Derivation of Fourier series
- Example: square wave
- Fourier convergence
- Fourier properties and symmetries
- Parseval's theorem
- Example: sawtooth signal

Overview

The bottom line of this handout: Fourier series allow us to express any periodic signal as a sum of sinusoids, or more specifically, complex exponentials.

Why would we ever want to do this?

Motivation

We know that for a linear system, if $x(t) = x_1(t) + x_2(t) + \cdots + x_n(t)$, then

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= h(t) * (x_1(t) + x_2(t) + \cdots + x_n(t)) \\ &= h(t) * x_1(t) + h(t) * x_2(t) + \cdots + h(t) * x_n(t) \end{aligned}$$

What this tells us is that if we can decompose our signal, $x(t)$, into its components, we can calculate the system output by considering each of these components in isolation. After this, we sum the outputs of these components together.

Fourier's idea, in 1807, was to decompose $x(t)$ using one of the most basic signals we know of: sines and cosines.

Fourier figured out that, indeed, (almost) every signal is decomposable into sines and cosines. As in the first lecture, this, I believe, is one of the great secrets of the Universe.

We'll see that convolution with sine waves has certain pros that make studying LTI systems *a lot* easier.

Motivation (cont.)

You may have previously heard of the Fourier transform. The Fourier series lays the foundation for deriving and understanding the Fourier transform. With the Fourier transform, we will be able to:

- Decompose signals into their “fundamental” or “primitive” components (sines and cosines).
- Reveal structure in signals and systems.
- In general, do LTI operations in a much simpler way.

The bottom line

These are the main mathematical results of this lecture, written here for convenience.

If $f(t)$ is a well-behaved periodic signal with period T_0, then $f(t)$ can be written as a Fourier series

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

where $\omega_0 = \frac{2\pi}{T_0}$ and

$$\boxed{c_k = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} f(t) e^{-jk\omega_0 t} dt}$$

for all integers k. The c_k are called the *Fourier coefficients* of $f(t)$.

Here, $f(t)$ is the *weighted average* of complex exponentials (which are simply complex sines and cosines).

Why complex exponentials instead of sines and cosines?

Another representation of the Fourier series that is equivalent to the previous page is:

$$f(t) = \sum_{k=0}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=0}^{\infty} b_k \sin(k\omega_0 t)$$

To arrive at the form in the previous slide, we make use of the fact that:

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \& \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

The comple exponential is a succinct representation, incorporating both the sines and cosines. They also have an important property, discussed on the next slide.

Eigenfunctions

$x(t)$ is an *eigenfunction* of a system if, when inputting $x(t)$ to the system, the output is simply a scaled version of $x(t)$, i.e., $y(t) = ax(t)$ where a is a constant (called an eigenvalue). Note that a may be a complex constant.

Consider an LTI system with impulse response $h(t)$. If the input is a complex exponential, i.e., e^{st} where $s = \sigma + j\omega$, then

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\ &= H(s)e^{st} \end{aligned}$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$ is a (complex) constant that is not time dependent (note that the τ is integrated out).

Bottom line: I put e^{st} into the system, and I got out ae^{st}, i.e., a scaled version of my input exponential, with $a = H(s)$.

Eigenfunctions of LTI systems

This shows that the complex exponential is an eigenfunction of an LTI system.

- If I input a complex exponential into the LTI system, I get the same complex exponential out scaled by $H(s)$.

From here, we can see how Fourier series might help:

- First, I decompose my signal, $x(t)$, into the sum of complex exponentials.
- After this, I put each complex exponential into my LTI system. Since the system is LTI, each complex exponential comes out scaled by $H(s)$.
- Then, I can add my scaled complex exponentials at the output to get the system output.
- Conveniently, since the output is a sum of (scaled) complex exponentials, it is also a Fourier series.

Let's formalize this intuition we've stated here and get to the math.

Complex exponentials in LTI systems

Our Fourier series representation is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

From two slides ago, we know that if we input a complex sinusoid, $e^{j\omega t}$ (where $\omega = k\omega_0$ for some k), we have that:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau \\ &= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \\ &= H(j\omega) e^{j\omega t} \end{aligned}$$

where, again, $H(j\omega)$ has no dependence on t, and so its a (complex) constant.

Bottom line intuition again: if I put the complex exponential $e^{j\omega t}$ into my system, and the output is the same complex exponential $e^{j\omega t}$ scaled by a constant $H(j\omega)$.

Complex exponentials in LTI systems (cont.)

Now, what happens when we input $x(t)$ instead of a single complex sinusoid? Again, our input is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Therefore, our output is:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-\tau)}\right)d\tau \\ &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-jk\omega_0\tau}d\tau \\ &= \sum_{k=-\infty}^{\infty} H(jk\omega_0)c_k e^{jk\omega_0 t} \end{aligned}$$

Complex exponentials in LTI systems (cont.)

This matches exactly the intuition we laid out three slides ago. We took our input, $x(t)$, and decomposed it into a sum of complex exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

At the output, we received back our sum of complex exponentials, each of them scaled by a constant $H(j\omega)$ where $\omega = k\omega_0$, i.e., we found that

$$y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) c_k e^{jk\omega_0 t}$$

By doing so, we gave our system an input, represented as a Fourier series, and received the Fourier series of $y(t)$, i.e., the output.

Further, the calculation of $y(t)$ is pretty simple. All I do is take the Fourier series for $x(t)$, and if I can calculate $H(jk\omega_0)$ then I immediately know the Fourier series representation of $y(t)$.

This is one of the major insights of the class

You need to be sure you understand this intuition. This is one of the key insights of this class and lays the groundwork for Electrical Engineering, modern communications, analog circuits, etc.

Since it's so important, we'll state it one more time.

For LTI systems, a simple way to analyze them is to:

- Decompose $x(t)$, the input, into its Fourier series, i.e., a sum of complex exponentials. This represents its decomposition into "fundamental" components, which are sinusoids at different frequencies $k\omega_0$.
- Because the complex exponential is an eigenfunction of an LTI system, if I pass in a complex exponential into my system, I get the same complex exponential at the output, scaled by $H(jk\omega_0)$.
- Since LTI systems are distributive, if I pass in a sum of these complex exponentials into my system, I get back an output, $y(t)$, that is a sum of scaled complex exponentials (where the scale term is $H(jk\omega_0)$).
- This term, $H(jk\omega_0)$, is called the *transfer function*.

Motivation for Fourier series

With this motivation, we now need to know how to actually calculate Fourier series, i.e., how do I find the c_k so that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

And further, when is this possible? The rest of this lecture will cover this.

Fourier series of a cosine (cont.)

Let's start simple. Consider the sinusoid:

$$f(t) = A\cos(\omega_0 t + \theta)$$

Its Fourier series is:

$$\begin{aligned} f(t) &= A\cos(\omega_0 t + \theta) \\ &= A\left(\frac{e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}}{2}\right) \\ &= \frac{Ae^{j\theta}}{2}e^{j\omega_0 t} + \frac{Ae^{-j\theta}}{2}e^{-j\omega_0 t} \end{aligned}$$

This signal is characterized fully by its:

- Magnitude, $A > 0$
- Phase, θ
- Frequency, $\omega_0 > 0$

Spectrum of cosine

We graphically represent such functions through a *spectrum*. In the spectrum, we plot frequency on the x-axis. There are two spectrum we'll plot here:

- The amplitude spectrum, which is the constant multiplying the complex exponential. (Sometimes this is squared to get the "power spectrum.")
- The phase spectrum, which is the phase at each frequency.

From the prior slide, we have that the Fourier series of a cosine is:

$$f(t) = \frac{A}{2}e^{j\omega_0 t + j\theta} + \frac{A}{2}e^{-j\omega_0 t - j\theta}$$

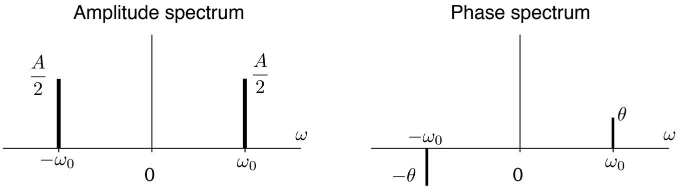
In the spectrum, we have two frequencies we are concerned about:

- The first complex exponential is at frequency ω_0. Its amplitude is $A/2$. Its phase is $+\theta$.
- The second complex exponential is at frequency $-\omega_0$. Its amplitude is $A/2$. Its phase is $-\theta$.

Spectrum of cosine (cont.)

Hence, below is the amplitude and phase spectrum for

$$f(t) = A\cos(\omega_0 t + \theta)$$



Hence, notice we have two ways to represent this signal, $f(t)$.

- Its time domain representation, which is $A\cos(\omega_0 t + \theta)$. We plot this as an oscillating cosine through time.
- Its frequency domain representation, which is $\frac{A}{2}e^{j\omega_0 t + j\theta} + \frac{A}{2}e^{-j\omega_0 t - j\theta}$. We plot this via the spectrum above.

Here, we described the signal in two ways, and both representations are exactly equal to each other.

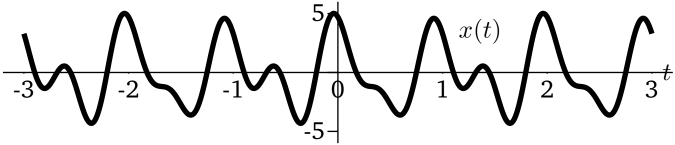
Using spectrum to find structure

For the cosine example, it doesn't look like we made our lives easier by representing it as a spectrum. But for any more complex signal, it can.

Consider the signal

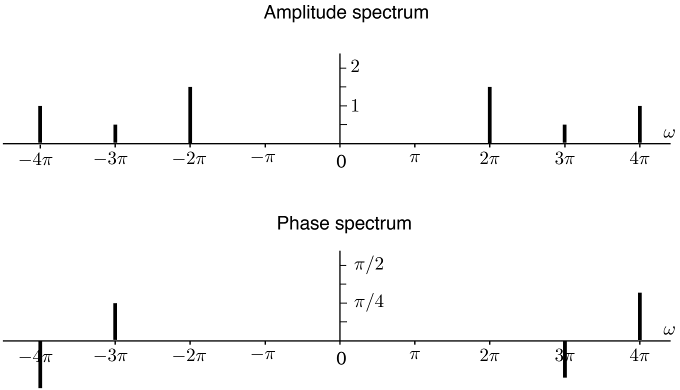
$$x(t) = 3\cos(2\pi t) + \cos(3\pi t - \pi/4) + 2\cos(4\pi t + \pi/3)$$

This signal is very simple. However, if I gave you a plot of its time domain representation, it'd be hard to recover exactly what $x(t)$ is.



Using spectrum to find structure (cont.)

However, if we plotted the spectrum of this signal, it would look like the following:



From the spectrum, we can read off that this signal is composed of sinusoids at three different frequencies (2π, 3π, and 4π) with amplitudes given by the amplitude spctrum and phases given by the phase spectrum.

A music example

Imagine I played a C-major chord, which comprises three unique notes: C, E, and G. What would its spectrum look like?

When does a signal have a Fourier series representation?

One thing we note about the Fourier series is that it is periodic, and therefore the Fourier series can only be used to represent periodic functions or time-limited functions. (We'll address this later when we get to the Fourier transform, which can be used on non-periodic signals.)

To see the Fourier series is periodic, note that for $T_0 = 2\pi/\omega_0$, we have that:

$$\begin{aligned} f(t+T_0) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t+2\pi/\omega_0)} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{j2\pi k} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ &= f(t) \end{aligned}$$

When does a signal have a Fourier series representation? (cont.)

We can approximate $f(t)$ with a Fourier series when:

- The function is periodic, with period $T_0 = 2\pi/\omega_0$, or
- The function is time-limited, defined only on an interval of length $T_0 = 2\pi/\omega_0$.
- The function has integral square error approaching zero (this is unclear now; it is explained later when we discuss Parseval's theorem).

Deriving Fourier series

How do we find the c_k?

Our derivation is as follows:

- First, we *assume* that the signal $f(t)$ can be written as a sum of complex exponentials that are scaled by coefficients c_k.
- Given this assumption, we find if there are c_k such that we can represent $f(t)$ in this way.

We know that we'll be able to find these c_k, since we know that Fourier series exist, but imagine if we were Fourier back in the day. This approach would give an interesting result either way. Consider the outcomes:

- If we can find these c_k, then we've discovered and found out how to represent any periodic function as a sum of complex exponentials.
- If we can't find these c_k, then it means that we can't represent any periodic function as a sum of complex exponentials.

A preliminary result on integrating complex exponentials

Before proceeding, we're going to introduce a handy trick that will simplify our derivation. Let $T_0 = 2\pi/\omega_0$. Consider the complex exponential

$$e^{jk\omega_0 t}$$

in our Fourier series.

- When $k = 0$, then this complex exponential is equal to 1.
- When $k \neq 0$, then this complex exponential is equal to

$$\cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

A preliminary result on integrating complex exponentials (cont.)

If I integrate this expression over a period, I get the following:

$$\begin{aligned}\int_{t_0}^{t_0+T_0} e^{jk\omega_0 t} dt &= \int_{t_0}^{t_0+T_0} e^{j\frac{2\pi k}{T_0}t} dt \\ &= \int_{t_0}^{t_0+T_0} \cos\left(\frac{2\pi k}{T_0}t\right) dt + j\int_{t_0}^{t_0+T_0} \sin\left(\frac{2\pi k}{T_0}t\right) dt\end{aligned}$$

Now, if we integrate a sine or cosine over one period, its integral is zero. Therefore, this integral evaluates to T_0 if $k = 0$ and evaluates to 0 otherwise, i.e.

$$\int_{t_0}^{t_0+T_0} e^{jk\omega_0 t} dt = \begin{cases} T_0, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

This is a handy trick: integrating a complex exponential over its period is zero except when its exponent is 0.

Deriving Fourier series

Let's begin with the derivation then.

Define $\omega_0 \triangleq 2\pi/T_0$, and assume that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

To use our preliminary result, what we'll do is multiply both sides by a complex exponential,

$$e^{-jn\omega_0 t}$$

and then integrate over one period, T_0.

Deriving Fourier series (cont.)

$$\begin{aligned}\int_{t_0}^{t_0+T_0} f(t)e^{-jn\omega_0 t}dt &= \int_{t_0}^{t_0+T_0}\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)e^{-jn\omega_0 t}dt \\ &= \sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_0} e^{jk\omega_0 t}e^{-jn\omega_0 t}dt \\ &= \sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_0} e^{j(k-n)\omega_0 t}dt\end{aligned}$$

Next, we use the result from two slides ago that:

$$\int_{t_0}^{t_0+T_0} e^{j(k-n)\omega_0 t}dt = \begin{cases} T_0, & k=n \\ 0, & k\neq n\end{cases}$$

Deriving Fourier series (cont.)

Returning to our derivation,

$$\begin{aligned}\int_{t_0}^{t_0+T_0} f(t)e^{-jn\omega_0 t}dt &= \sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_0} e^{j(k-n)\omega_0 t}dt \\ &= c_n T_0\end{aligned}$$

Thus, we have that (after replacing n's with k's):

$$\boxed{c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-jk\omega_0 t}dt}$$

These are the Fourier coefficients (!) and demonstrate that indeed, a periodic signal (or one defined over a length T_0) can be written as a sum of complex exponentials.

Fourier series convergence?

After our proof, can we definitely say that for

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-jk\omega_0 t}dt$$

that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

at every point in time, t?

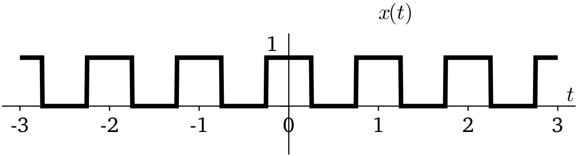
The answer is no. If you notice our derivation, we only showed that this holds when $f(t)$ or its Fourier series representation are in an integral, i.e.,

$$\int_{t_0}^{t_0+T_0} f(t)e^{-jn\omega_0 t}dt \quad = \quad \int_{t_0}^{t_0+T_0}\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)e^{-jn\omega_0 t}dt$$

What this means is that our Fourier series formula really only holds in the sense of the integral average over this period. But it need not be the case that our Fourier series formula holds at exactly every single t. We'll see this in an example.

Example: square wave

Consider the square wave below.



Let's calculate the Fourier series for this. First, we note that we only need to look at one period, so let's look at from $t = -0.5$ to 0.5, where the square wave starts at 0 and transitions to 1 at $t = -0.25$ and from 1 back to 0 at $t = 0.25$. Let's define

$$s(t) = \begin{cases} 0, & -0.5 \leq t < -0.25 \text{ and } 0.25 \leq t < 0.5 \\ 1, & -0.25 \leq t < 0.25 \end{cases}$$

Example: square wave (cont.)

When we calculate the Fourier series, we should worry about two cases: when $k = 0$ and $k \neq 0$. (Usually, if we just solve for when $k \neq 0$, we'll get an expression that is undefined for $k = 0$. This is why we do both.)

For our square wave, we have that $T_0 = 1$. When $k = 0$,

$$
\begin{aligned}
c_k &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t} dt \\
c_0 &= \frac{1}{T_0} \int_{-0.5}^{0.5} s(t) e^{0} dt \\
&= \frac{1}{T_0} \int_{-0.25}^{0.25} 1 dt \\
&= \frac{1}{2T_0}
\end{aligned}
$$

Example: square wave (cont.)

When $k \neq 0$,

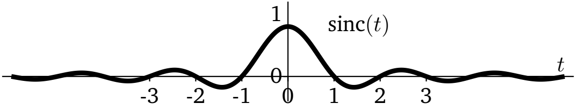
$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-jk\omega_0 t}dt \\
&= \int_{-0.5}^{0.5} s(t)e^{-jk2\pi t}dt \\
&= \int_{-0.25}^{0.25} e^{-jk2\pi t}dt \\
&= \left.\frac{e^{-jk2\pi t}}{-jk2\pi}\right|_{-0.25}^{0.25} \\
&= \left.\frac{\cos(-k2\pi t) + j\sin(-k2\pi t)}{-jk2\pi}\right|_{-0.25}^{0.25} \\
&= \left.\frac{\cos(k2\pi t) - j\sin(k2\pi t)}{-jk2\pi}\right|_{-0.25}^{0.25} \\
&= \frac{\cos(k\pi/2) - \cos(-k\pi/2) - j\left(\sin(k\pi/2) - \sin(-k\pi/2)\right)}{-jk2\pi} \\
&= \frac{\sin(k\pi/2)}{\pi k}
\end{aligned}
$$

Example: square wave (cont.)

The term $\frac{\sin(\pi t)}{\pi t}$ occurs so frequently in signal processing that it has its own name:

$$\boxed{\operatorname{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}}$$

The sinc function looks like:



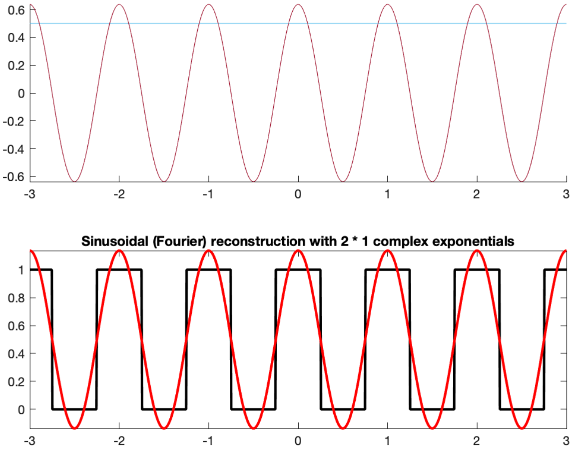
(Note that some define $\operatorname{sinc}(t) = \sin(t)/t$. We do NOT use this definition.)
Thus, we have that

$$c_k = \frac{1}{2}\operatorname{sinc}(k/2)$$

Note that $\operatorname{sinc}(0) = 1$.

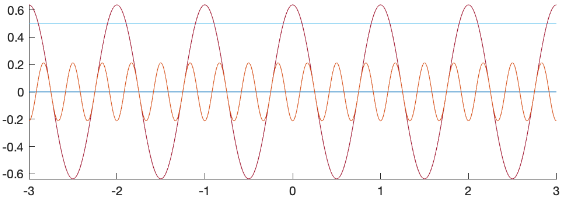
Example: square wave (cont.)

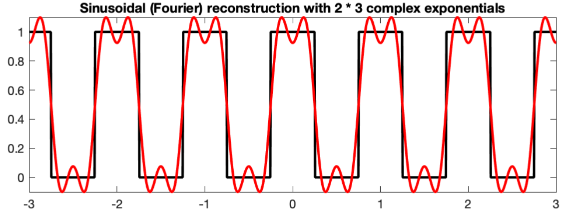
Let's now plot Fourier series fits for our square wave with one complex exponential frequency.



Example: square wave (cont.)

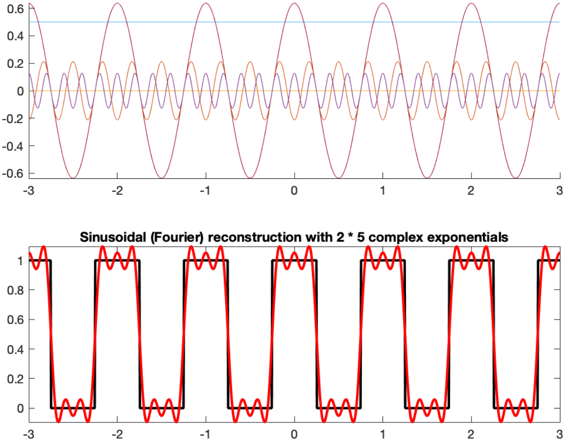
Fourier series with 3 complex exponential frequencies...





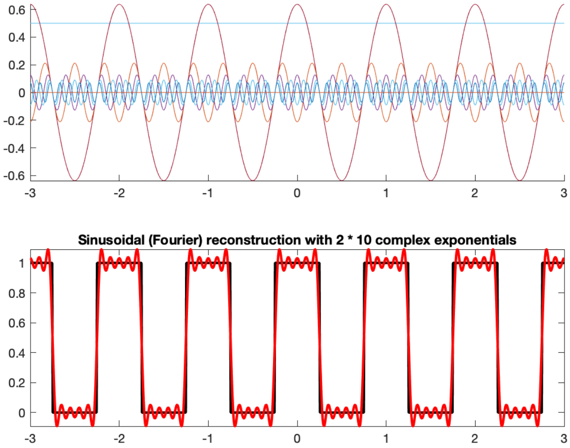
Example: square wave (cont.)

Fourier series with 5 complex exponential frequencies...



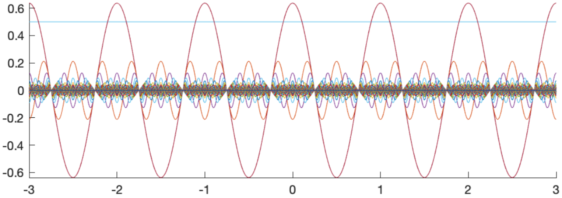
Example: square wave (cont.)

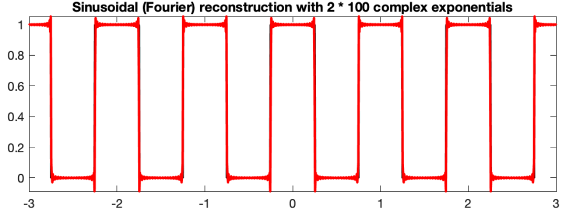
Fourier series with 10 complex exponential frequencies...



Example: square wave (cont.)

Fourier series with 100 complex exponential frequencies...





Fourier series does not equal $f(t)$ everywhere

As we can see, the Fourier series does not equal $f(t)$ everywhere. However, it does a very reasonable job at fitting the square wave. There is work (we won't cover) on things we observe, like the "ringing" (called Gibbs effect) of the Fourier series at discontinuities. Interestingly, increasing the number of terms compresses the ringing but does not reduce its amplitude. You can see it's still present with $k = 100$ complex exponential frequencies in our square wave example.

There are *Dirichlet conditions* that describe when f can be approximated by a Fourier series. We won't talk about these in depth in class, but essentially, the signal should be "smooth" and "well-behaved" for a Fourier series approximation to be good.

Fourier series properties

There are interesting symmetries and properties of the Fourier series that are worth expanding upon.

- c_0 **is the average of the signal**. Note that for $k = 0$, we have that

$$\boxed{c_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t)dt}$$

  Thus, c_0 is exactly the time-averaged mean of the signal and corresponds to a constant value (i.e., it has no sinusoidal component). For this reason, it is sometimes called the "DC component." DC stands for direct current in circuits, and refers to non-alternating (sinusoidal) currents. The DC component is the average value taken on by a signal.

- **Complex representation**. In general, the c_k may be complex, and so they can be expressed in their real / imaginary form or in magnitude / phase form. i.e.,

$$\begin{aligned} c_k &= \Re(c_k) + j\Im(c_k) \\ &= |c_k|\, e^{j\angle c_k} \end{aligned}$$

Fourier symmetry

We can apply Euler's formula to re-write the Fourier coefficients, and reveal some symmetries:

$$\begin{aligned}
c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-j\frac{2\pi k t}{T_0}}\,dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right]dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi k}{T_0}t\right)dt - \frac{j}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi k}{T_0}t\right)dt
\end{aligned}$$

If $f(t)$ is real, then so are:

$$\begin{aligned}
\Re(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi k}{T_0}t\right)dt \\
\Im(c_k) &= -\frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi k}{T_0}t\right)dt
\end{aligned}$$

Fourier symmetry (cont.)

Therefore, for $f(t)$ real, and using the fact that $\cos(k)$ is even and $\sin(k)$ is odd, we have the following symmetries:

$$\begin{aligned}
\Re(c_k) &= \Re(c_{-k}) \\
\Im(c_k) &= -\Im(c_{-k}) \\
c_k^* &= c_{-k} \\
|c_k| &= |c_{-k}| \\
\angle c_k &= -\angle c_{-k}
\end{aligned}$$

Fourier symmetry (cont.)

- If $f(t)$ is even, then $x(t) = x(-t)$ and therefore $c_k = c_{-k}$. You can see this by realizing that kt only appears in the complex exponential, and therefore negating t has the same effect as negating k.

$$f(t) \text{ even } \implies c_k = c_{-k}$$

- If $f(t)$ is odd, then $x(t) = -x(-t)$ and therefore $c_k = -c_{-k}$. This holds for the same reason as for the even case.

$$f(t) \text{ odd } \implies c_k = -c_{-k}$$

- Combining facts, we have that if $f(t)$ is even and real, then $c_k = c_{-k}$ and $c_{-k} = c_k^*$, and so $c_k = c_k^*$. This means that the c_k must be real.

$$f(t) \text{ even and real } \implies c_k \text{ real}$$

- If $f(t)$ is odd and real, then $c_k = -c_{-k}$ and because $c_{-k} = c_k^*$, then $c_k = -c_k^*$. This means the c_k must be imaginary.

$$f(t) \text{ odd and real } \implies c_k \text{ imaginary}$$

Truncated Fourier series

With

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

we can also define the truncated Fourier series, which only uses K components as

$$\hat{f}_K(t) = \sum_{k=-K}^{K} c_k e^{jk\omega_0 t}$$

We now define a quantity, the integral square error, as

$$\mathcal{E}_K = \int_{t_0}^{t_0+T_0} \left| \hat{f}_K(t) - f(t) \right|^2 dt$$

We say that the truncated Fourier series converges if $\mathcal{E}_K \to 0$ as $K \to \infty$ and if the truncated Fourier series converges, then $f(t)$ has a Fourier series representation.

(This is the same intuition we talked about earlier in our Fourier derivation, i.e., that when we approximate $f(t)$ with a Fourier series, then $f(t)$ and its Fourier series are equal in their integral average over a period.)

Parseval's Theorem

We now expand the integral square error to derive Parseval's theorem. This theorem is useful in a few ways:

- It relates signal power in the time and frequency domains.
- It tells us when a function, $f(t)$, has a Fourier series.
- It allows us to do some difficult integrals.

Expanding the integral square error:

$$\begin{aligned}
\mathcal{E}_K &= \int_{t_0}^{t_0+T_0} \left| \sum_{k=-K}^{K} c_k e^{jk\omega_0 t} - f(t) \right|^2 dt \\
&= \int_{t_0}^{t_0+T_0} \left(\sum_{k=-K}^{K} c_k e^{jk\omega_0 t} - f(t) \right) \left(\sum_{n=-K}^{K} c_n e^{jk\omega_0 t} - f(t) \right)^* dt \\
&= \int_{t_0}^{t_0+T_0} \left(\sum_{k=-K}^{K} c_k e^{jk\omega_0 t} - f(t) \right) \left(\sum_{n=-K}^{K} c_n^* e^{-jn\omega_0 t} - f^*(t) \right) dt
\end{aligned}$$

Parseval's Theorem (cont.)

Continuing...

$$
\begin{aligned}
\mathcal{E}_k &= \int_{t_0}^{t_0+T_0} \left[\left(\sum_{k=-K}^{K} c_k e^{jk\omega_0 t} \right) \left(\sum_{n=-K}^{K} c_n^* e^{-jn\omega_0 t} \right) + f(t) * f^*(t) \right. \\
&\quad \left. -f(t) \left(\sum_{n=-K}^{K} c_n^* e^{-jn\omega_0 t} \right) - \left(\sum_{k=-K}^{K} c_k e^{jk\omega_0 t} \right) f^*(t) \right] dt \\
&= \sum_{k=-K}^{K} \sum_{n=-K}^{K} c_k c_n^* \int_{t_0}^{t_0+T_0} e^{jk\omega_0 t} e^{-jn\omega_0 t} + \int_{t_0}^{t_0+T_0} |f(t)|^2 \, dt \\
&\quad - \sum_{n=-K}^{K} c_n^* \int_{t_0}^{t_0+T_0} f(t) e^{-jn\omega_0 t} dt - \sum_{k=-K}^{K} c_k \int_{t_0}^{t_0+T_0} f^*(t) e^{jk\omega_0 t} dt \\
&= \sum_{k=-K}^{K} |c_k|^2 T_0 + \int_{t_0}^{t_0+T_0} |f(t)|^2 \, dt - \sum_{n=-K}^{K} T_0 c_n^* c_n - \sum_{k=-K}^{K} T_0 c_k c_k^* \\
&= T_0 \sum_{k=-K}^{K} |c_k|^2 + \int_{t_0}^{t_0+T_0} |f(t)|^2 \, dt - T_0 \sum_{n=-K}^{K} |c_n|^2 - T_0 \sum_{k=-K}^{K} |c_k|^2
\end{aligned}
$$

Parseval's Theorem (cont.)

Continuing...

$$\mathcal{E}_k = -T_0 \sum_{k=-K}^{K} |c_k|^2 + \int_{t_0}^{t_0+T_0} |f(t)|^2 \, dt$$

Note that in this proof, we used the fact that if

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t}$$

then

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} f^*(t) e^{-jk\omega_0 t} = c_{-k}^*$$

Try to show this yourselves; we'll also prove this when we get to Fourier transforms.

Now, if the integral square error goes to zero as $K \to \infty$, then $f(t)$ has a Fourier transform. Then, we find that

$$\boxed{\lim_{K\to\infty} \sum_{k=-K}^{K} |c_k|^2 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |f(t)|^2 \, dt}$$

Parseval's Theorem (cont.)

$$\boxed{\lim_{K\to\infty}\sum_{k=-K}^{K}|c_k|^2 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0}|f(t)|^2\,dt}$$

A few things:

- If this relationship holds, the integral square error goes to zero. Similarly, when the integral square error goes to zero, this relationship holds. Thus, this relationship holds *if and only if* $f(t)$ has a Fourier series.
- In words, this relationship means the *power* of the signal in the time domain is the same as the sum of the powers of its frequency components.
- It also gives us a way to calculate some annoying integrals, i.e., by simply summing Fourier series coefficients.

Parseval's Theorem integral

Example of using Parseval's Theorem to calculate integrals.

Consider calculating

$$\frac{1}{T}\int_0^T \sin^6(3\pi t)dt$$

where T is the period of $\sin^3(3\pi t)$. We recognize that if $f(t) = \sin^3(3\pi t)$ then this integral is $\frac{1}{T}\int_0^T |f(t)|^2 \, dt$ and we can use Parseval's theorem.

The Fourier coefficients for $\sin^3(3\pi t)$ are

$$c_{\pm 1} = \pm\frac{3}{8j} \qquad c_{\pm 3} = \pm\frac{1}{8j}$$

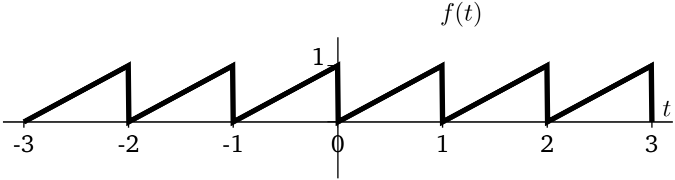
and therefore,

$$\begin{aligned} \int_0^T \sin^6 3\pi t dt &= 2\left(\frac{3}{8}\right)^2 + 2\left(\frac{1}{8}\right)^2 \\ &= 20/64 \end{aligned}$$

Example: sawtooth signal

We'll finish this lecture by doing a few more examples to give more familiarity with the Fourier series.

The sawtooth signal is given by $f(t) = t \mod 1$. It is plotted below:



This signal has a period of $T_0 = 1$. Now, when $k = 0$,

$$\begin{aligned} c_0 &= \int_0^1 te^0 dt \\ &= \left.\frac{t^2}{2}\right|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

Example: sawtooth signal (cont.)

Now, when $k \neq 0$,

$$c_k = \int_0^1 te^{-jk\omega_0 t} dt$$

We use integration by parts, i.e., $\int u dv = uv - \int v du$. We let

$$u = t \qquad dv = e^{-jk\omega_0 t} dt$$

so that

$$du = dt \qquad v = \frac{e^{-jk\omega_0 t}}{-jk\omega_0}$$

This gives that, when $k \neq 0$,

$$\begin{aligned} c_k &= \left. \frac{te^{-jk\omega_0 t}}{-jk\omega_0} \right|_0^1 - \frac{1}{-jk\omega_0} \int_0^1 e^{-jk\omega_0 t} dt \\ &= \frac{je^{-jk\omega_0}}{k\omega_0} + \frac{1}{(k\omega_0)^2} \left. e^{-jk\omega_0 t} \right|_0^1 \\ &= \frac{je^{-jk\omega_0}}{k\omega_0} + \frac{e^{-jk\omega_0} - 1}{(k\omega_0)^2} \end{aligned}$$

Example: sawtooth signal (cont.)

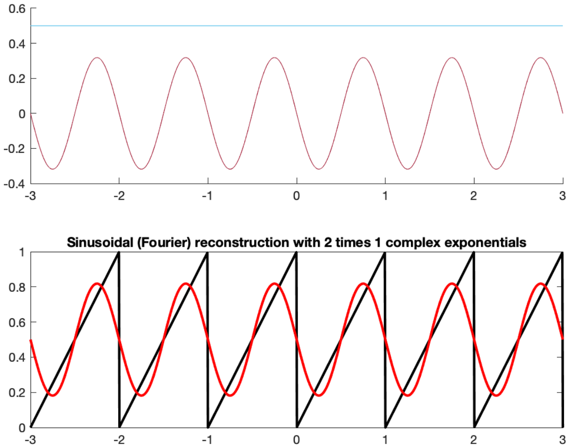
Now, since $T_0 = 1$, we have that $\omega_0 = 2\pi$. This means $c_k = j/(2\pi k)$ for $k \neq 0$. Therefore, the Fourier series of the sawtooth is:

$$t \mod 1 = \frac{1}{2} + \sum_{k \neq 0} \frac{j}{2\pi k} e^{jk2\pi t}$$

We show some illustrations below.

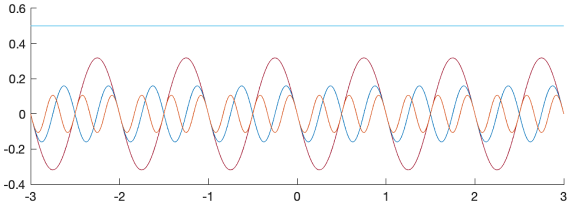
Example: sawtooth signal (cont.)

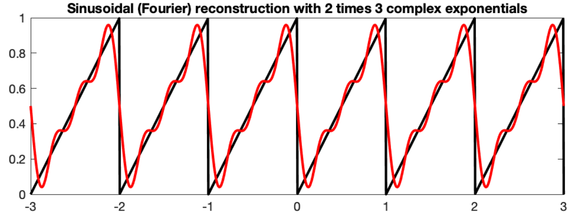
Let's now plot Fourier series fits for our sawtooth signal with one complex exponential frequency.



Example: sawtooth signal (cont.)

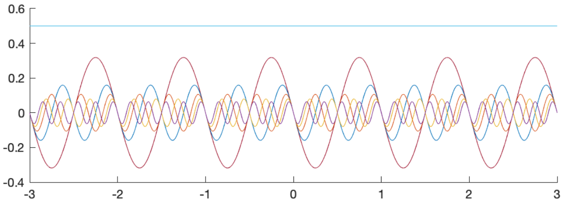
Fourier series with 3 complex exponential frequencies...

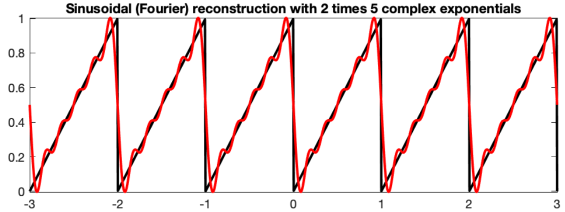




Example: sawtooth signal (cont.)

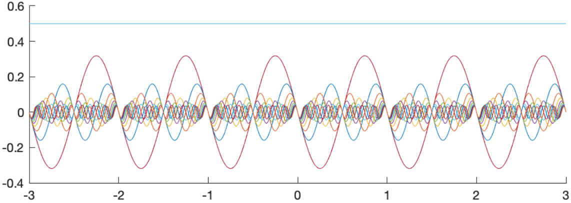
Fourier series with 5 complex exponential frequencies...

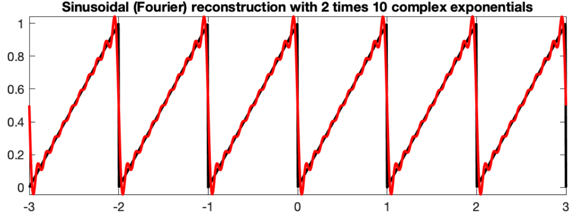




Example: sawtooth signal (cont.)

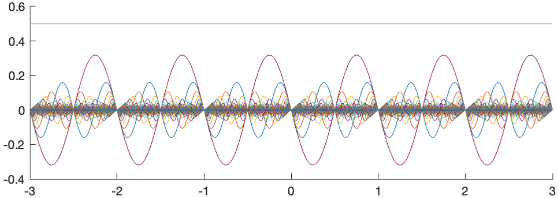
Fourier series with 10 complex exponential frequencies...

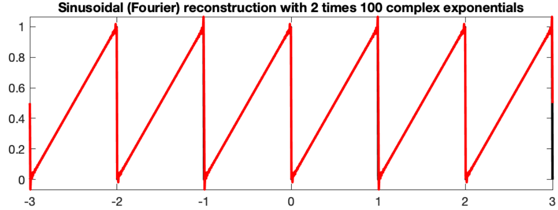




Example: sawtooth signal (cont.)

Fourier series with 100 complex exponential frequencies...





Example: ramp and rect

In this lecture, we derived the Fourier series for a sawtooth signal and a square wave.

Note that these are also the Fourier series for a ramp signal (truncated to the interval $[0, 1)$) and the rect function, respectively.