ECE102, Fall 2020 **Homework #2**

Signals & Systems Prof. J.C. Kao

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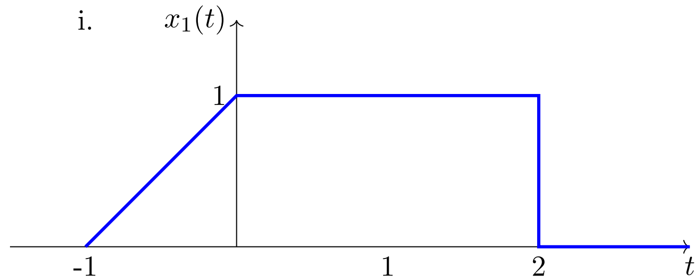
Due Friday, 23 Oct 2020, by 11:59pm to Gradescope.
Covers material up to Lecture 4.
100 points total.

1. (22 points) **Elementary signals.**

   (a) (9 points) Consider the signal $x(t)$ shown below.

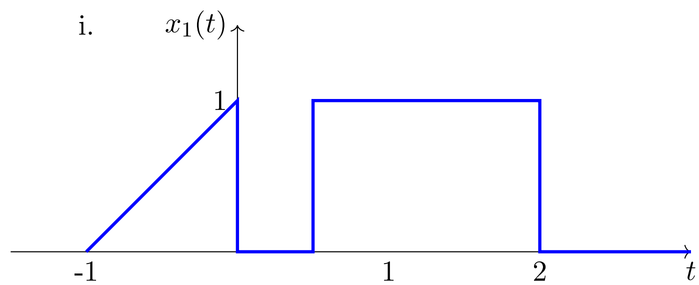


Sketch the following:

i. $y(t) = x(t)\,(1 - u(t) + u(2t-1))$

Solution: If we look at $u(t) - u(2t-1)$, we'll see that it is equal to 1 between $t = 0$ and $t = 0.5$, and equals 0 elsewhere. If we consider $1 - (u(t) - u(2t-1))$, it is equal to 1 when $t < 0$ and $t > 0.5$. Therefore, its multiplication with $x(t)$ will be:

$$y(t) = \begin{cases} 0 & 0 \le t < 0.5 \\ x(t) & else \end{cases}$$

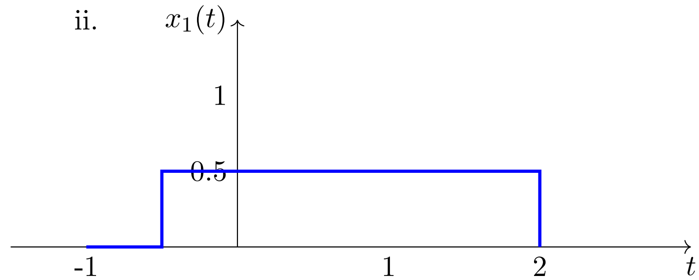


ii. $y(t) = \int_{-\infty}^{t} \delta(\tau + 0.5)x(\tau)d\tau$

Solution: Recall the sifting property:

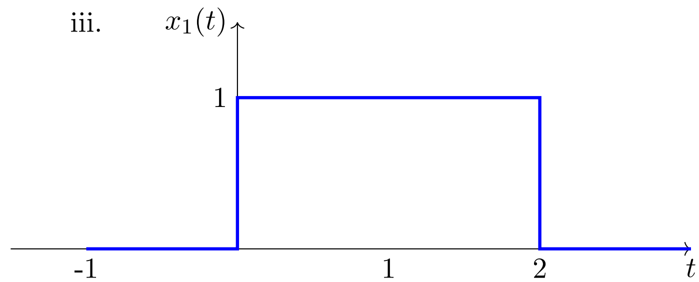
$$\int_{-\infty}^{t} x(\tau)\delta(\tau - T)d\tau = x(T)\int_{-\infty}^{t} \delta(\tau)d\tau \tag{1}$$

Thus, using the sifting property, we get $y(t) = x(-0.5)\int_{-\infty}^{t} \delta(\tau)d\tau$.



iii. $y(t) = x(t) - r(t+1) + r(t) + u(t)$

Solution:



(b) (9 points) Evaluate these integrals:

i. $\int_{-\infty}^{\infty} f(t+1)\delta(t+1)dt$

Solution:

Using the sifting property, we first have: $f(t+1)\delta(t+1) = f(0)\delta(t+1)$. Therefore,

$$\int_{-\infty}^{\infty} f(t+1)\delta(t+1)dt = f(0)\int_{-\infty}^{\infty} \delta(t+1)dt = f(0).$$

ii. $\int_{t}^{\infty} e^{-2\tau}u(\tau-1)d\tau$

Solution:

To evaluate this integral, we have to consider two cases; the first one is when $t \geq 1$ and the second one is when $t < 1$. This is because $u(\tau-1)$ is one when $\tau \geq 1$ and zero otherwise. Thus, if $t \geq 1$, then:

$$\int_{t}^{\infty} e^{-2\tau}u(\tau-1)d\tau = \int_{t}^{\infty} e^{-2\tau}d\tau = \frac{e^{-2\tau}}{-2}\bigg|_{t}^{\infty} = \frac{e^{-2t}}{2}$$

If $t < 1$, then:

$$\int_{t}^{\infty} e^{-2\tau}u(\tau-1)d\tau = \int_{1}^{\infty} e^{-2\tau}d\tau = \frac{e^{-2\tau}}{-2}\bigg|_{t}^{\infty} = \frac{e^{-2}}{2}$$

.

iii. $\int_{0^-}^{\infty} f(t)(\delta(t-1)+\delta(t+1)+\delta(t))dt$

Solution:

The integral can be decomposed as follows:

$$\int_{0^-}^{\infty} f(t)\delta(t-1)dt + \int_{0^-}^{\infty} f(t)\delta(t+1)dt + \int_{0^-}^{\infty} f(t)\delta(t)dt$$

Using the sifting property for the first integral, we have:

$$\int_{0}^{\infty} f(t)\delta(t-1)dt + \int_{0^-}^{\infty} f(t)\delta(t)dt = f(1) + f(0)$$

The second integral is zero, because $\delta(t+1)$ is centred at $t = -1$ and the limits of the integration do not include $t = -1$. Therefore, $\int_{0^-}^{\infty} f(t)(\delta(t-1)+\delta(t+1)+\delta(t))dt = f(0) + f(1)$

(c) (4 points) Let b be a positive constant. Show the following property for the delta function:

$$\delta(bt) = \frac{1}{b}\delta(t)$$

Hint: Solve this problem by defining:

$$\delta(t) = \lim_{\Delta \to 0} \mathrm{rect}_{\Delta}(t)$$

.

Solution:

We know that $\delta(t)$ is defined as follows:

$$\delta(t) = \lim_{\Delta \to 0} \mathrm{rect}_\Delta(t)$$

Therefore,

$$\delta(bt) = \lim_{\Delta \to 0} \mathrm{rect}_\Delta(bt)$$

The rectangle $\mathrm{rect}_\Delta(bt)$ is shown in Fig. 2. Let $\Delta' = \Delta/b$, then the same rectangle can be written as:

$$\mathrm{rect}_\Delta(bt) = \frac{1}{b}\mathrm{rect}_{\Delta'}(t)$$

Therefore,

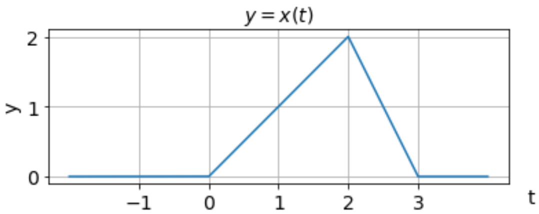
$$\delta(bt) = \lim_{\Delta \to 0} \mathrm{rect}_\Delta(bt) = \lim_{\Delta' \to 0} \frac{1}{b}\mathrm{rect}_{\Delta'}(t) = \frac{1}{b}\delta(t)$$

Note: we can extend this argument to $b < 0$. In general for any $b \neq 0$, we have:

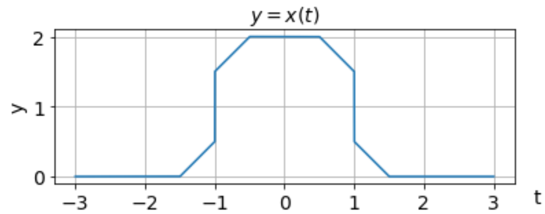
$$\delta(bt) = \frac{1}{|b|}\delta(t)$$

2. (23 points) **Expression for signals.**

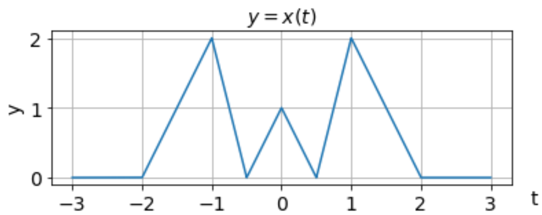
(a) (15 points) Write the following signals as a combination (sums or products) of unit triangles $\Delta(t)$ and unit rectangles $\text{rect}(t)$.



a(i)



a(ii)



a(iii)

Solution:

i. Fig. a)We can see this signal as the sum of two shifted unit triangles, where the first one is scaled by 3/2, i.e., $x(t) = \Delta(t-1) + 2\Delta(t-2)$.

ii. Fig. b) We can express this signal as the sum of two triangles one shifted to the left and the second to the right. Now the parts of these two triangles that are lifted by one for $-1 \leq t \leq 1$ can be obtained by adding a rectangle function. Therefore, $x(t) = \Delta(t-0.5) + \Delta(t+0.5) + \text{rect}(t/2)$.

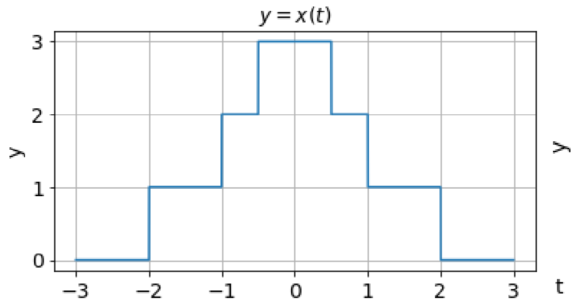
iii. Fig. c) One way to represent this signal is to find an expression for each triangle in terms of the unit triangle. The central triangle can be expressed as: $\Delta(2t)$. The triangle that is on the right side can be expressed as the sum of two triangles that are time-scaled and shifted: $2\Delta(2(t-1)) + \Delta(2(t-3/2))$. Similarly, the part that is on the left can be expressed as follows: $2\Delta(2(t+1)) + \Delta(2(t+3/2))$. Therefore, $x(t) = 2\Delta(2(t+1)) + \Delta(2(t+3/2)) + \Delta(2t) + 2\Delta(2(t-1)) + \Delta(2(t-3/2))$

(b) (8 points) Express each of the signals shown below as sums of scaled and time shifted unit-step functions.

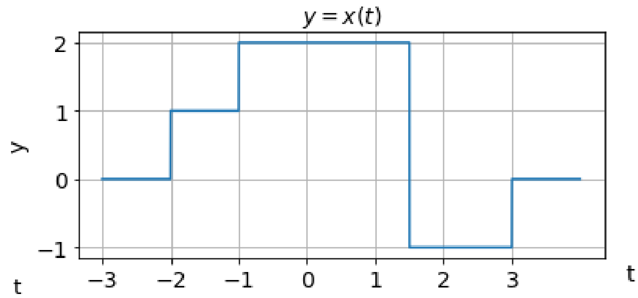
Solution:

i. $x_a(t) = u(t+2) + u(t+1) + u(t+0.5) - u(t-0.5) - u(t-1) - u(t-2)$

ii. $x_b(t) = u(t+2) + u(t+1) - 3u(t-1.5) + u(t-3)$



b(i)



b(ii)

3. (30 points) **System properties.**

(a) (20 points) A system with input $x(t)$ and output $y(t)$ can be time-invariant, causal or stable. Determine which of these properties hold for each of the following systems. Explain your answer.

i. $y(t) = |x(t)| + x(2t)$

Solution:
Linearity: We can check homogeneity. If we scale the input by a, the output is:

$$y_a(t) = |(ax(t))| + (ax(t)) = |a|\,|x(t)| + ax(t) \neq a\,(|x(t)| + x(t))$$

The system is then non-linear.
Time-invariance: If we delay the input by τ, i.e., $x_\tau(t) = x(t-\tau)$, the output is:

$$y_\tau(t) = |x_\tau(t)| + x_\tau(2t) = |x(t-\tau)| + x(2t-\tau)$$

On the other hand,

$$y(t-\tau) = |x(t-\tau)| + x(2(t-\tau))$$

Since $y(t-\tau) \neq y_\tau(t)$, the system is time-variant.
Causality: Since the output can depend on future values of the input, the system is not causal. For instance, the output at $t = 2$ depends on $x(4)$.
Stability: If $|x(t)| \leq B_x$ for any t, then

$$|y(t)| = ||x(t)| + x(2t)| \leq |x(t)| + |x(2t)| \leq 2B_x$$

The output is also bounded, the system is then stable.

ii. $y(t) = \int_{t-T}^{t+T} x(\lambda)d\lambda$, where T is positive and constant.

Solution:
Linearity: We will check here homogeneity and superposition:

$$\begin{aligned}\int_{t-T}^{t+T} (ax_1(\lambda) + bx_2(\lambda))\, d\lambda &= \int_{t-T}^{t+T} ax_1(\lambda)d\lambda + \int_{t-T}^{t+T} bx_2(\lambda)d\lambda \\ &= a\left(\int_{t-T}^{t+T} x_1(\lambda)d\lambda\right) + b\left(\int_{t-T}^{t+T} x_2(\lambda)d\lambda\right) \\ &= ay_1(t) + by_2(t)\end{aligned}$$

The system is then linear.
Time-invariance: If we delay the input by τ, i.e., $x_\tau(t) = x(t-\tau)$, the output is:

$$y_\tau(t) = \int_{t-T}^{t+T} x_\tau(\lambda)d\lambda = \int_{t-T}^{t+T} x(\lambda-\tau)d\lambda$$

Let $\lambda' = \lambda - \tau$, then

$$y_\tau(t) = \int_{t-T-\tau}^{t+T-\tau} x(\lambda')d\lambda' = \int_{(t-\tau)-T}^{(t-\tau)+T} x(\lambda')d\lambda'$$

which is equal to $y(t-\tau)$. The system is then time-invariant.
Causality: The system is integrating values of $x(t)$ from $t-T$ to $t+T$. The output depends on future values of $x(t)$, therefore it is not causal.
Stability: If $|x(t)| \leq B_x$ for any t, then

$$|y(t)| = \left| \int_{t-T}^{t+T} x(\lambda)d\lambda \right| \leq \int_{t-T}^{t+T} |x(\lambda)|\, d\lambda \leq \int_{t-T}^{t+T} B_x d\lambda = 2TB_x$$

The output is also bounded, the system is then stable.

iii. $y(t) = (t+1)\int_{-\infty}^{t} x(\lambda)d\lambda$

Solution:
Linearity: We will check here homogeneity and superposition:

$$\begin{aligned}(t+1)\int_{-\infty}^{t}(ax_1(\lambda)+bx_2(\lambda))\,d\lambda &= (t+1)\int_{-\infty}^{t} ax_1(\lambda)d\lambda + (t+1)\int_{-\infty}^{t} bx_2(\lambda)d\lambda \\ &= a\left((t+1)\int_{-\infty}^{t} x_1(\lambda)d\lambda\right) + b\left((t+1)\int_{-\infty}^{t} x_2(\lambda)d\lambda\right) \\ &= ay_1(t) + by_2(t)\end{aligned}$$

The system is then linear.
Time-invariance: If we delay the input by τ, i.e., $x_\tau(t) = x(t-\tau)$, the output is:

$$y_\tau(t) = (t+1)\int_{-\infty}^{t} x_\tau(\lambda)d\lambda = (t+1)\int_{-\infty}^{t} x(\lambda-\tau)d\lambda$$

Let $\lambda' = \lambda - \tau$, then

$$y_\tau(t) = (t+1)\int_{-\infty}^{t-\tau} x(\lambda')d\lambda'$$

On the other hand,

$$y(t-\tau) = (t-\tau+1)\int_{-\infty}^{t-\tau} x(\lambda)d\lambda$$

Therefore $y(t-\tau) \neq y_\tau(t)$. The system is then time variant.
Causality: The system is integrating values of $x(t)$ up to time t. The output does not depend on future values of $x(t)$, the system is then causal.

Stability: Even if $x(t)$ is absolutely bounded, the integral:

$$\int_{-\infty}^{t} x(\lambda)d\lambda$$

cannot in general be bounded, the system is unstable. For instance, suppose $x(t) = 1$, then $\int_{-\infty}^{t} 1d\lambda \to \infty$. Another example, suppose $x(t) = u(t)$, then

$$y(t) = (t+1)\int_{-\infty}^{t} u(\lambda)d\lambda = (t+1)\int_{0}^{t} 1d\lambda = (t+1)t$$

$(t+1)t$ cannot be bounded as $t \to \infty$, because $(t+1)t \to \infty$ as $t \to \infty$.

iv. $y(t) = 1 + x(t)\cos(\omega t)$

Solution:
Time-invariance: If we delay the input by τ: $x_\tau(t) = x(t-\tau)$, the output is:

$$y_\tau(t) = 1 + x_\tau(t)\cos(\omega t) = 1 + x(t-\tau)\cos(\omega t)$$

On the other hand,

$$y(t-\tau) = 1 + x(t-\tau)\cos(\omega(t-\tau))$$

Since $y(t-\tau) \neq y_\tau(t)$. The system is then time-variant.
Causality: Since the output does not depend on any future values of the input, the system is causal.
Stability: If $|x(t)| \leq B_x$ for any t, then

$$|y(t)| = |1 + x(t)\cos(\omega t)| \leq 1 + |x(t)| \leq 1 + B_x$$

The output is also bounded, the system is then stable.

v. $y(t) = \frac{1}{1+x^2(t)}$

Solution:
Time-invariance: If we delay the input by τ, i.e., $x_\tau(t) = x(t-\tau)$, the output is:

$$y(t) = \frac{1}{1+x_\tau^2(t)} = \frac{1}{1+x^2(t-\tau)}$$

On the other hand,

$$y(t-\tau) = \frac{1}{1+x^2(t-\tau)}$$

Therefore $y(t-\tau) = y_\tau(t)$. The system is then time invariant.
Causality: The output depends on present value of the input. The system is then causal.
Stability: We have the denominator:

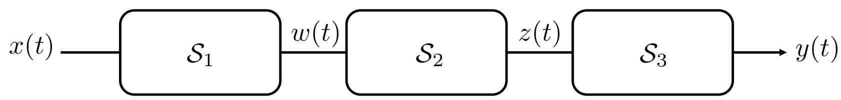
$$1 + x^2(t) \geq 1 \implies \frac{1}{1+x^2(t)} \leq 1$$

for any t. This implies that $y(t) \leq 1$. Moreover $y(t) > 0$, therefore for any t, we always have $|y(t)| \leq 1$. The system is always stable.

(b) (6 points) Consider the following three systems:

$$\mathcal{S}_1 : w(t) = x(3t)$$
$$\mathcal{S}_2 : z(t) = \int_{-\infty}^{t} w(\tau)d\tau$$
$$\mathcal{S}_3 : y(t) = \mathcal{S}_3(z(t))$$

The three systems are connected in series as illustrated here:



Choose the third system $\mathcal{S}_3$, such that overall system is equivalent to the following system:

$$y(t) = \int_{-\infty}^{t-4} x(\tau)d\tau$$

Solution: We first express $z(t)$ in terms of $x(t)$:

$$z(t) = \int_{-\infty}^{t} w(\tau)d\tau = \int_{-\infty}^{t} x(3\tau)d\tau$$

Let $\tau' = 3\tau$, then $d\tau' = 3d\tau$ and $\tau \le t \implies \tau' = 3\tau \le 3t$,

$$z(t) = \frac{1}{3}\int_{-\infty}^{3t} x(\tau')d\tau'$$

To obtain the required $y(t)$ from $z(t)$, we need first to do a time-scaling by $\frac{1}{3}$ for $z(t)$. This step gives us:

$$z(\frac{t}{3}) = \frac{1}{3}\int_{-\infty}^{t} x(\tau')d\tau'$$

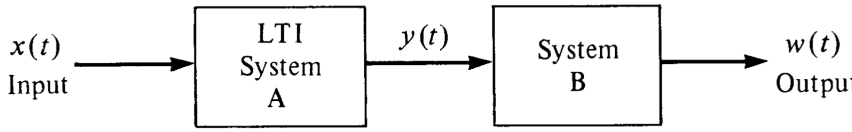
The second step is to do a right shift by 4:

$$z(\frac{1}{3}(t-4)) = \frac{1}{3}\int_{-\infty}^{t-4} x(\tau')d\tau'$$

Therefore, the third system is as follows:

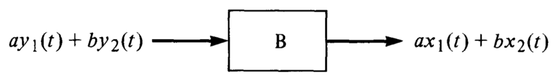
$$y(t) = 3z(\frac{1}{3}(t-4))$$

(c) (4 points) Consider the cascade of two systems shown below. System B is the inverse of system A.



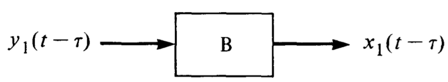
i. Suppose an input $x_1(t)$ produces $y_1(t)$ as System A output and an input $x_2(t)$ produces $y_2(t)$ as System A output. What is $w(t)$ if the input is such that $y(t)$, the output of System A, is $ay_1(t) + by_2(t)$ with a,b constants? *Hint: An inverse system cascaded with the original system is the identity system.*

Solution: If $y(t) = ay_1(t) + by_2(t)$, we know that since System A is linear, $x(t) = ax_1(t) + bx_2(t)$. Since the cascaded system is an identity system, the output $w(t) = ax_1(t) + bx_2(t)$.



ii. Suppose an input $x_1(t)$ produces $y_1(t)$ as System A output. What is $w(t)$ if $x(t)$ is such that $y(t) = y_1(t - \tau)$?

Solution: If $y(t) = y_1(t-\tau)$, then since System A is time-invariant, $x(t) = x_1(t-\tau)$ and also $w(t) = x_1(t - \tau)$.



iii. Is System B an LTI system? Justify your answer.

Solution: From the solution to parts i. and ii. we see that System B is linear and time-invariant.

4. (10 points) **Power and energy of complex signals**

(a) (5 points) Is $x(t) = Ae^{j\omega t} + Be^{-j\omega t}$ a power or energy signal? A and B are both real numbers, not necessarily equal. If it is an energy signal, compute its energy. If it is a power signal, compute its power. (*Hint: Use the fact that the square magnitude of a complex number v is: $|v|^2 = v^*v$, where v^* is the complex conjugate of the complex number v.*)

Solution:

$x(t)$ is a periodic signal, therefore it is not an energy signal (its energy goes to infinity). It is a power signal. To calculate its power, we compute first the magnitude of $x(t)$:

$$\begin{aligned}|x(t)|^2 = x(t)x(t)^* &= (Ae^{j\omega t} + Be^{-j\omega t})(Ae^{-j\omega t} + Be^{j\omega t}) \\ &= A^2 + ABe^{j2\omega t} + ABe^{-j2\omega t} + B^2 \\ &= A^2 + B^2 + 2AB\cos(2\omega t)\end{aligned}$$

Therefore, the power of $x(t)$:

$$\begin{aligned}P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(A^2 + B^2 + 2AB\cos(2\omega t)\right) dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \left(2TA^2 + 2TB^2 + AB\frac{\sin(2\omega t)}{\omega}\Big|_{-T}^{T}\right) \\ &= \lim_{T\to\infty} \frac{1}{2T} \left(2TA^2 + 2TB^2 + 2AB\frac{\sin(2\omega T)}{\omega}\right) \\ &= A^2 + B^2\end{aligned}$$

(b) (5 points) Is $x(t) = e^{-(1+j\omega)t}u(t-1)$ an energy signal or power signal? Again, if it is an energy signal, compute its energy. If it is a power signal, compute its power.

Solution:

The magnitude of $x(t)$ is given by:

$$|x(t)| = e^{-t}u(t-1)$$

Therefore, its energy is:

$$E = \int_1^\infty e^{-2t}dt = \frac{e^{-2t}}{-2}\Big|_{t=1}^{\infty} = \frac{e^{-2}}{2}$$

Therefore, it is an energy signal. Its power is then 0.

5. (15 points) **MATLAB**

(a) (5 points) **Task 1**

A complex sinusoid is denoted:

$$y(t) = e^{(\sigma+j\omega)t}$$

First compute a vector representing time from 0 to 10 seconds in about 500 steps (You can use `linspace`). Use this vector to compute a complex sinusoid with a period of 2 seconds, and a decay rate that reduces the signal level at 10 seconds to half its original value. What σ and ω did you choose? If your complex exponential is y, plot:

```
>> plot(y);
```

What is MATLAB doing here?

Solution:

We want the period to be of 2 seconds, this implies that $\omega = \pi$. We also want a decay rate that reduces the signal level at 10 seconds to half its original value, this implies:

$$e^{10\sigma} = \frac{1}{2} \implies \sigma = -\ln(2)/10$$

```
t=linspace(0,10,500); sigma=-log(2)/10; omega=pi;
y=exp((sigma+j*omega)*t); plot(y);
```

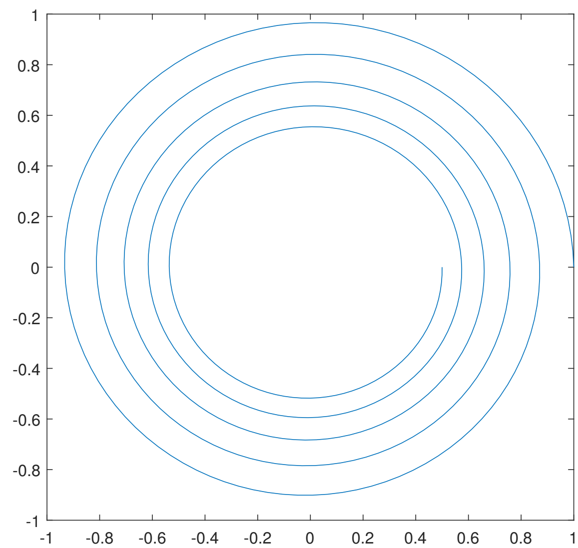


Figure 1: Task 1

When the MATLAB function `plot(y)` takes one argument `y` that is complex, it plots the imaginary part of `y` versus the real part of `y`.

(b) (5 points) **Task 2**

Use the `real()` and `imag()` MATLAB functions to extract the real and imaginary parts of the complex exponential, and plot them as a function of time (plot them separately, you can use `subplot` for this task). This should look more reasonable. Label your axes, and check that your signal has the required period and decay rate.

Solution:

```
subplot(2,1,1);
plot(t,real(y)); xlabel('t(sec)'); ylabel('Real pat of y(t)');
subplot(2,1,2);
plot(t,imag(y)); xlabel('t(sec)'); ylabel('Imaginary pat of y(t)');
```

(c) (5 points) **Task 3**

Use the `abs()` and `angle()` functions to plot the magnitude and phase angle of the complex exponential (plot them in the same figure). Scale the `angle()` plot by dividing

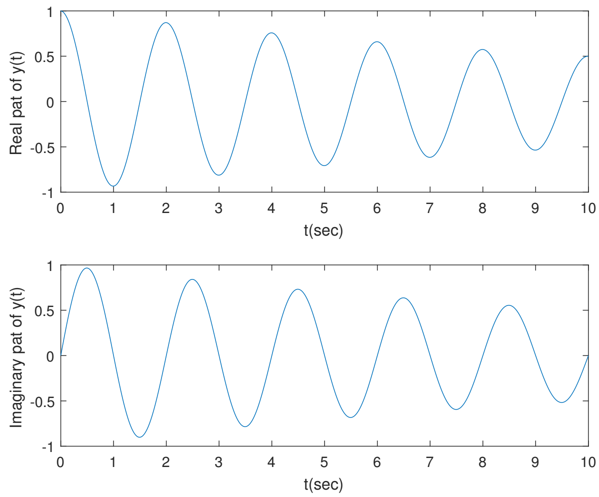


Figure 2: Task 2

it by `2*pi` so that it fits well on the same plot as the `abs()` plot (i.e. plot the angle in cycles, instead of radians, the function `angle(x)` returns the angle in radians).

Solution:

```
plot(t,abs(y),'g',t,angle(y)/(2*pi),'r');
xlabel('t(sec)'); ylabel('Magnitude and phase of y(t)'); grid on;
```

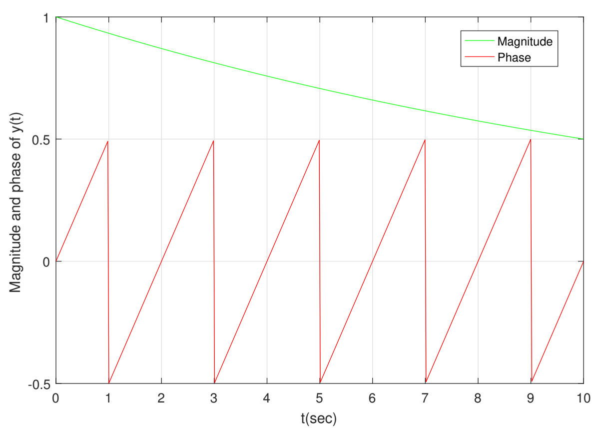


Figure 3: Task 3