Fourier transform properties

This lecture discusses properties of the Fourier transform.

- Linearity
- Time scaling
- Complex conjugate
- Duality
- Time shifting
- Modulation (dual of time shift)
- Derivative
- Parseval's theorem
- $\star\star\star$ **Convolution Theorem** $\star\star\star$
- Frequency domain convolution

Fourier transform operator

In this lecture, we'll use the Fourier transform as something that operates on signals. To denote the operation of taking the Fourier transform, we use $\mathcal{F}(\cdot)$ or $\mathcal{F}[\cdot]$. That is, if

$$f(t) \iff F(j\omega)$$

we may alternately write this as

$$F(j\omega) = \mathcal{F}[f(t)]$$

Likewise, the operator $\mathcal{F}^{-1}$ refers to the inverse Fourier transform. Therefore,

$$\mathcal{F}^{-1}[F(j\omega)] = f(t)$$

This also means that

$$\mathcal{F}^{-1}[\mathcal{F}[f(t)]] = f(t)$$

at all points of continuity in $f(t)$.

Linearity of the Fourier transform

The Fourier transform is linear.

For two signals, $f_1(t)$ and $f_2(t)$, and two complex numbers a and b,

$$\boxed{\mathcal{F}[af_1(t) + bf_2(t)] = a\mathcal{F}[f_1(t)] + b\mathcal{F}[f_2(t)]}$$

Another way to write this is

$$af_1(t) + bf_2(t) \iff aF_1(j\omega) + bF_2(j\omega)$$

where $F_1(j\omega) = \mathcal{F}[f_1(t)]$ and $F_2(j\omega) = \mathcal{F}[f_2(t)]$.

To show this, note

$$\begin{aligned}
\mathcal{F}(af_1(t) + bf_2(t)) &= \int_{-\infty}^{\infty} (af_1(t) + bf_2(t))\, e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} af_1(t)e^{-j\omega t} dt + \int_{-\infty}^{\infty} bf_2(t)e^{-j\omega t} dt \\
&= a\mathcal{F}[f_1(t)] + b\mathcal{F}[f_2(t)]
\end{aligned}$$

Linearity of the Fourier transform (cont.)

This extends to finite combinations, i.e.,

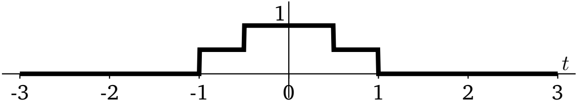
$$\mathcal{F}\left[\sum_{k=1}^{K} a_k f_k(t)\right] = \sum_{k=1}^{K} a_k \mathcal{F}\left[f_k(t)\right]$$

Linearity example

Consider the signal:

$$f(t) = \begin{cases} \frac{1}{2}, & \frac{1}{2} \leq |t| \leq 1 \\ 1, & |t| \leq \frac{1}{2} \end{cases}$$

This signal steps up and then steps down, as shown below.



What is its Fourier transform?

Linearity example (cont.)

We so far know the Fourier transforms of rect and a causal exponential. However, we see we can reconstruct this signal from rects and use the linearity principle. Notice that

$$f(t) = \frac{1}{2}\mathrm{rect}(t/2) + \frac{1}{2}\mathrm{rect}(t)$$

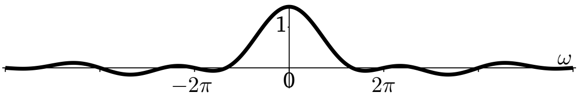
We know that

$$\mathrm{rect}(t/T) \iff T\,\mathrm{sinc}(\omega T/2\pi)$$

and therefore

$$\begin{aligned} F(j\omega) &= \frac{1}{2}2\,\mathrm{sinc}(2\omega/2\pi) + \frac{1}{2}\,\mathrm{sinc}(\omega/2\pi) \\ &= \mathrm{sinc}(\omega/\pi) + \frac{1}{2}\,\mathrm{sinc}(\omega/2\pi) \end{aligned}$$

This is shown below:



Fourier transform of a time scaled signal

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(j\frac{\omega}{a}\right)}$$

Note, for real a:

- If $a > 1$, $f(t)$ contracts, but its Fourier transform expands.
- If $0 < a < 1$, then $f(t)$ expands, but its Fourier transform contracts.
- Thus, stretching a signal in time compresses its Fourier transform, and compacting the signal expands its Fourier transform.
- Does this make intuitive sense?

Fourier transform of a time scaled signal (cont.)

To show this, let's consider $a > 0$. (The proof is essentially the same for $a < 0$, which you can do on your own.) We will use a variable change, $\tau = at$, which means that $d\tau = adt$.

$$\begin{aligned}
\mathcal{F}(f(at)) &= \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau/a}\frac{1}{a}d\tau \\
&= \frac{1}{a}\int_{-\infty}^{\infty} f(\tau)e^{-j(\omega/a)\tau}d\tau \\
&= \frac{1}{a}F\left(j\frac{\omega}{a}\right)
\end{aligned}$$

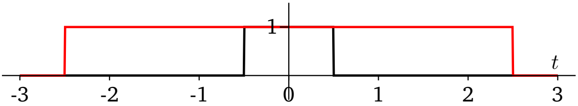
Example: Knowing that

$$\mathrm{rect}(t/T) \iff T\,\mathrm{sinc}(\omega T/2\pi)$$

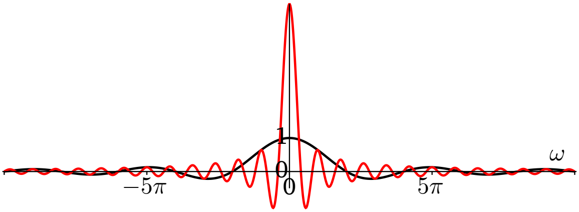
we can then determine that the Fourier transform of $\mathrm{rect}(t)$ is $\mathrm{sinc}(\omega/2\pi)$

Bandwidth example

Let's nail down our intuition for the time-scaling theorem. Consider two rect pulses, $\mathrm{rect}(t)$ and $\mathrm{rect}(t/5)$.



These are their Fourier transforms.



The fatter rect has a narrower spectrum. The width of the spectrum is called bandwidth. So a shorter pulse has a larger bandwidth. What does this mean intuitively?

Fourier transform of a time-reversed signal

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(-t)] = F(-j\omega)}$$

To show this, apply the time-scaling result with $a = -1$.

Time reversal example

Find the Fourier transform of $f(t) = e^{-a|t|}$ (for $a > 0$) without doing integration.

We know that

$$e^{-at}u(t) \iff \frac{1}{a + j\omega}$$

Our goal is to make $f(t)$ out of these basic signals. In particular, note that:

$$\begin{aligned} f(t) &= e^{-a|t|} \\ &= e^{-at}u(t) + e^{at}u(-t) \end{aligned}$$

At this point, we've done a good amount of the work in solving this problem. All that's left to do is determine $\mathcal{F}[e^{at}u(-t)]$ and then apply linearity.

Time reversal example (cont.)

Since

$$\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+j\omega}$$

then by the time-reversal theorem we have that

$$\mathcal{F}[e^{at}u(-t)] = \frac{1}{a-j\omega}$$

Thus, for $a < 0$,

$$\begin{aligned} \mathcal{F}\left[e^{a|t|}\right] &= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} \\ &= \frac{2a}{a^2-(j\omega)^2} \\ &= \frac{2a}{a^2+\omega^2} \end{aligned}$$

Fourier transform of complex conjugate

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{f^*(t) \iff F^*(-j\omega)}$$

Let's show this.

$$\begin{aligned}
\mathcal{F}[f^*(t)] &= \int_{-\infty}^{\infty} f^*(t)e^{-j\omega t}dt \\
&= \left(\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt\right)^* \\
&= \left(\int_{-\infty}^{\infty} f(t)e^{-(-j\omega)t}dt\right)^* \\
&= (F(-j\omega))^* \\
&= F^*(-j\omega)
\end{aligned}$$

Duality of the Fourier transform

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{F(t) \iff 2\pi f(-j\omega)}$$

This expression may be opaque at first. What this is saying is that if I take a Fourier transform pair, I can essentially find the dual pair by replacing all the ω's with t's in $F(j\omega)$ and all the t's with $-\omega$'s in $f(t)$. After scaling by 2π, this results in another Fourier transform pair.

Essentially, every Fourier transform pair we derive (we'll do quite a few) really gives us two Fourier transform pairs.

Duality of the Fourier transform (cont.)

To show this, recognize that as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

then

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega) e^{-j\omega t} d\omega$$

Now, the r.h.s. of this equation is the Fourier transform of $F(j\omega)$ with the roles of ω and t reversed. Hence, $2\pi f(-t)$ is the Fourier transform of $F(j\omega)$ (!) and after we swap the ω and the t's, we arrive at the duality result.

Duality examples

- Since $\text{rect}(t) \iff \text{sinc}(\omega/2\pi)$, then

$$\begin{aligned} \text{sinc}(t/2\pi) &\iff 2\pi\text{rect}(-\omega) \\ &= 2\pi\text{rect}(\omega) \end{aligned}$$

  Thus, we have that $\text{sinc}(t/2\pi) \iff 2\pi\text{rect}(\omega)$.

- Since

$$e^{-at}u(t) \iff \frac{1}{a+j\omega}$$

  then

$$\frac{1}{a+jt} \iff 2\pi e^{a\omega}u(-\omega)$$

- Exercise: find $f(t)$ such that its Fourier transform is $e^{-|\omega|}$.

Fourier transform of a time shifted signal

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}F(j\omega)}$$

To show this, we'll use a change of variables $u = t - \tau$ as follows:

$$\begin{aligned}
\mathcal{F}[f(t-\tau)] &= \int_{-\infty}^{\infty} f(t-\tau)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} f(u)e^{-j\omega(u+\tau)}du \\
&= e^{-j\omega\tau}\int_{-\infty}^{\infty} f(u)e^{-j\omega u}du \\
&= e^{-j\omega\tau}F(j\omega)
\end{aligned}$$

Time shift example

Consider the signal

$$f(t) = \mathrm{rect}\left(\frac{t - T/2}{T}\right)$$

i.e., this is a pulse that happens between 0 and T.

Then,

$$F(j\omega) = e^{-j\omega T/2} T \,\mathrm{sinc}(\omega T / 2\pi)$$

Fourier transform of a modulated signal

A major component of communications has to do with *modulation*. For example, AM and FM radio are amplitude modulation and frequency modulation respectively. These involve multiplying $f(t)$, the signal you wish to transmit, with a complex exponential at a carrier frequency, ω_0. This frequency, ω_0, is the frequency you dial in your car to get AM / FM radio.

There are three modulations we could do. If $\mathcal{F}[f(t)] = F(j\omega)$, then

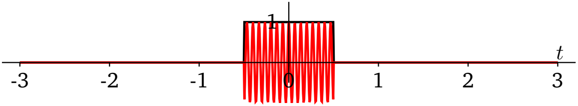
$$\begin{aligned}
\mathcal{F}[f(t)e^{j\omega_0 t}] &= F(j(\omega - \omega_0)) \\
\mathcal{F}[f(t)\cos(\omega_0 t)] &= \frac{1}{2}\left(F(j(\omega - \omega_0)) + F(j(\omega + \omega_0))\right) \\
\mathcal{F}[f(t)\sin(\omega_0 t)] &= \frac{1}{2j}\left(F(j(\omega - \omega_0)) - F(j(\omega + \omega_0))\right)
\end{aligned}$$

Typically, modulation is done through multiplication by $\cos(\omega_0 t)$. Modulation is dual to the time shift Fourier transform.

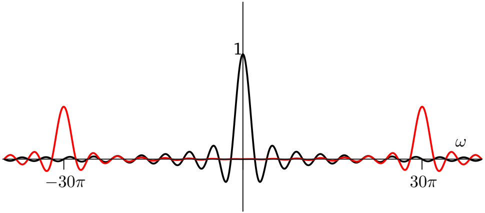
What modulation intuitively does is take $F(j\omega)$ and create replicas at $\pm\omega_0$.

Fourier transform of a modulated signal (cont.)

Below, we show what modulation does. We take a signal (here a rect) and multiply it by a cosine with $\omega_0 = 30\pi$. This is denoted in red in the plot below.



The spectrum takes the FT of our signal (i.e., a sinc) and creates replicas at $\pm 30\pi$.



From here, you can gain some intuition for why different radio stations use different frequencies. They're given these frequencies to transmit whatever signals they like; each radio station occupies a different part of the spectrum!

Fourier transform of a modulated signal (cont.)

To prove the modulation result, note that if $\mathcal{F}[f(t)] = F(j\omega)$ then

$$\begin{aligned} \mathcal{F}[f(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt \\ &= F(j(\omega-\omega_0)) \end{aligned}$$

To get the cosine and sine results, we note that e.g., for cosine,

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

From here, we can use linearity to compute the Fourier transform.

Fourier transform of the derivative

If $\mathcal{F}[f(t)] = F(j\omega)$, and $f(t)$ is differentiable everywhere w.r.t. t with its derivative denoted

$$f'(t) = \frac{df(t)}{dt}$$

then

$$\boxed{\mathcal{F}[f'(t)] = j\omega F(j\omega)}$$

If $f^{(n)}(t)$ denotes the nth derivative of $f(t)$, then

$$\boxed{\mathcal{F}[f^{(n)}(t)] = (j\omega)^n F(j\omega)}$$

Fourier transform of the derivative (cont.)

To show this, note:

$$\begin{aligned} f'(t) &= \frac{d}{dt} f(t) \\ &= \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\frac{d}{dt}e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)j\omega e^{j\omega t}d\omega \end{aligned}$$

Thus, applying the inversion formula, it must be that $\mathcal{F}[f'(t)] = j\omega F(j\omega)$

(Note, you can change differentiation and integration when f and df/dt are continuous across all t; beyond the scope of this class and you won't be responsible for this.)

Derivative dual:

$$(-jt)f(t) \iff F'(j\omega)$$

Parseval's Theorem

Recall that the energy of a signal, $f(t)$, is given by

$$\mathcal{E}_f = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

Parseval's theorem states that the energy of the signal and its Fourier transform are equal up to a scaling factor of 2π, i.e.,

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega}$$

Parseval's Theorem (cont.)

To see this,

$$\begin{aligned}
\mathcal{E}_f &= \int_{-\infty}^{\infty} |f(t)|^2 \, dt \\
&= \int_{-\infty}^{\infty} f(t) f^*(t) dt \\
&= \int_{-\infty}^{\infty} f(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \right)^* dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) F(j\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega
\end{aligned}$$

Parseval's Theorem (cont.)

Like in Fourier series, Parseval's theorem can be used to make integrals a lot easier. Say you were asked to calculate:

$$\int_{-\infty}^{\infty} \text{sinc}^2(t)dt$$

Since $\text{sinc}(t) \iff \text{rect}(\omega/2\pi)$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} \text{sinc}^2(t)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{rect}^2(\omega/2\pi)d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \\
&= 1
\end{aligned}$$

$\star\star\star$ The Convolution Theorem $\star\star\star$

It's probably worth taking up an entire slide here just to say:

This is one of the most important theorems of the class, and is a key reason why a lot of our technology works. (!) This theorem enables us to do convolution, and thus any LTI operation, straightforwardly. With it, we no longer have to do the impulse response integral we saw earlier.

⋆⋆⋆ The Convolution Theorem ⋆⋆⋆

If $f_1(t)$ and $f_2(t)$ are two signals with Fourier transforms $F_1(j\omega)$ and $F_2(j\omega)$, respectively, then

$$\boxed{\mathcal{F}[(f_1 * f_2)(t)] = F_1(j\omega)F_2(j\omega)}$$

Stated simply: **convolution in the time domain is multiplication in the frequency domain.**

(And multiplication is easy.)

This theorem is so practical that even software like `MATLAB` doesn't compute convolution integrals; it calculates the Fourier Transform of the signals, multiplies them in the frequency domain, and then takes the inverse Fourier transform.

The Convolution Theorem

To show this,

$$
\begin{aligned}
\mathcal{F}[(f_1 * f_2)(t)] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \right) e^{-j\omega t} dt \\
&\overset{(a)}{=} \int_{-\infty}^{\infty} f_1(\tau) \int_{-\infty}^{\infty} \left(f_2(t-\tau) e^{-j\omega t} dt \right) d\tau \\
&\overset{(b)}{=} \int_{-\infty}^{\infty} f_1(\tau) \left(e^{-j\omega\tau} F_2(j\omega) \right) d\tau \\
&= F_2(j\omega) \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} d\tau \\
&= F_2(j\omega) F_1(j\omega)
\end{aligned}
$$

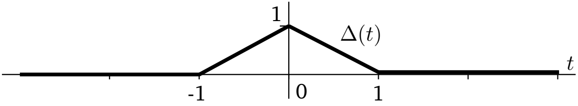
where at (a) we interchanged the order of integration and (b) we used the time shift property of the Fourier transform.

Convolution theorem example

What is the Fourier transform of the unit triangle,

$$\Delta(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Recall:



One could certainly directly solve for the Fourier transform by doing the integral. But instead, we can get the Fourier transform using the convolution theorem.

Since $\Delta(t) = \mathrm{rect}(t) * \mathrm{rect}(t)$, and $\mathrm{rect}(t) \iff \mathrm{sinc}(\omega/2\pi)$, then

$$\Delta(t) \iff \mathrm{sinc}^2(\omega/2\pi)$$

Frequency domain convolution

The frequency domain convolution theorem is that for $f_1(t) \iff F_1(j\omega)$ and $f_2(t) \iff F_2(j\omega)$, then

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\nu)F_2(j(\omega-\nu))d\nu}$$

We typically write this as:

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}(F_1 * F_2)(j\omega)}$$

but note that the convolution is w.r.t. ω, not $j\omega$.

This means that multiplication in the time domain is convolution in the frequency domain. This proof is very similar to the time domain proof.

Frequency domain convolution example

To find the Fourier transform of $\mathrm{sinc}^2(t)$, we first note that

$$\mathrm{sinc}(t) \iff \mathrm{rect}(\omega/2\pi)$$

and therefore

$$\begin{aligned}\mathcal{F}[\,\mathrm{sinc}^2(t)] &= \frac{1}{2\pi}\left[\mathrm{rect}(\omega/2\pi) * \mathrm{rect}(\omega/2\pi)\right] \\ &= \Delta(\omega/2\pi)\end{aligned}$$