Continuous time Fourier transform

This lecture discusses the Fourier transform.

- Motivation
- Intuition for the Fourier transform
- Fourier transform derivation
- Inverse Fourier transform
- Fourier transform symmetry

Motivation

Last lecture, we learned about the Fourier series, which can model (almost) any **periodic** or **time-limited** function as a sum of complex exponentials.

But the Fourier series is limited because it requires the signals be **periodic** or **time-limited**.

The Fourier transform allows us to calculate the spectrum of aperiodic signals.

Intuition of going from Fourier series to Fourier transform

Extending Fourier series to the Fourier transform is fairly intuitive.

The idea is the following.

- We can calculate the Fourier series of a periodic or time-limited signal, over some interval of length T_0.
- A signal that is not periodic can be viewed as a periodic signal, where T_0 is infinite. As T_0 is infinite, it never repeats.
- But the point is that we can replace our Fourier series calculation as, instead of being over a finite period, T_0, being over all time, from $t = -\infty$ to ∞.

Intuition (cont.)

Mathematically, we can calculate the Fourier series of $f(t)$ over the interval $[-T/2, T/2)$ via:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

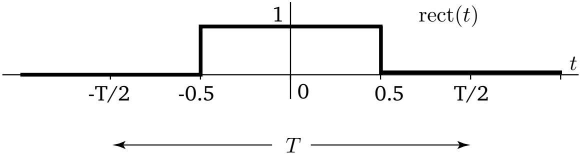
$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0} dt$$

where $\omega_0 = 2\pi/T$.

In the Fourier transform, we're now going to let $T \to \infty$.

The Fourier series of rect as $T \to \infty$

Consider $x(t) = \mathrm{rect}(t)$. Further, let's define a period T over which, if we made a periodic extension of the rect, it would repeat every T. (Again, we're going to set $T \to \infty$ eventually so that it doesn't repeat.)



The Fourier series of this signal is related to the one we did last lecture, but we'll do it again for the sake of completeness:

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{rect}(t) e^{-jk\frac{2\pi}{T}t} dt \\
&= \frac{1}{T}\int_{-1/2}^{1/2} e^{-jk\frac{2\pi}{T}t} dt \\
&= \frac{1}{T} \left. \frac{Te^{-jk\frac{2\pi}{T}t}}{-jk2\pi} \right|_{-1/2}^{1/2}
\end{aligned}
$$

The Fourier series of rect as $T \to \infty$

Continuing ...

$$
\begin{aligned}
c_k &= \left. \frac{e^{-jk\frac{2\pi}{T}t}}{-jk2\pi} \right|_{-1/2}^{1/2} \\
&= \frac{-j\sin(\pi k/T) - j\sin(\pi k/T)}{-j2\pi k} \\
&= \frac{\sin(\pi k/T)}{\pi k} \\
&= \frac{1}{T}\frac{\sin(\pi k/T)}{\pi k/T} \\
&= \frac{1}{T}\operatorname{sinc}\left(\frac{k}{T}\right)
\end{aligned}
$$

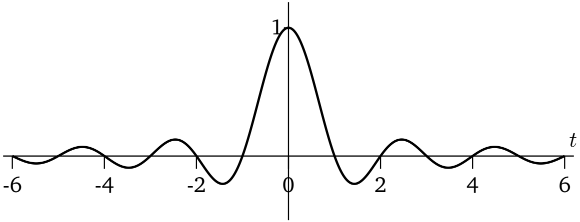
Therefore, the Fourier series of $\mathrm{rect}(t)$ with a periodic extension every T is:

$$
\frac{1}{T}\sum_{k=-\infty} \operatorname{sinc}(k/T)e^{jk\omega_0 t}
$$

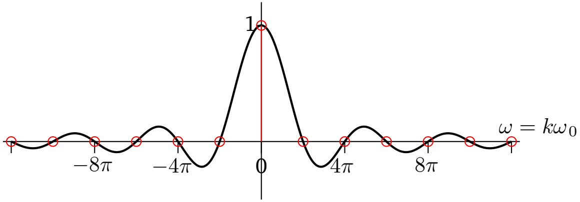
for $\omega_0 = 2\pi/T$.

The Fourier series of rect as $T \to \infty$

Let's now look at what the coefficients look like for varying values of T. First, for a refresher, let's recall what the sinc function looks like.

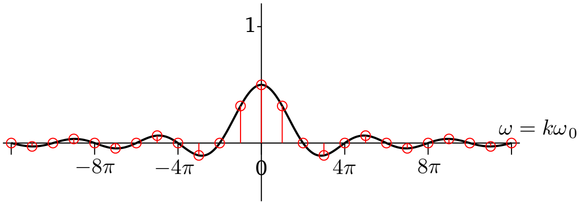


Now, let's set $T = 1$ ($\omega_0 = 2\pi$) and calculate each of the Fourier coefficients.

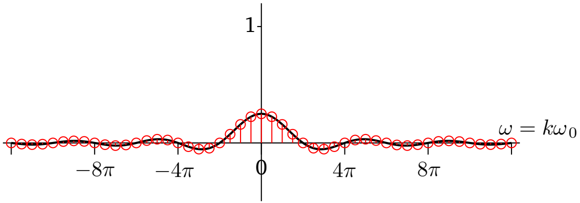


The Fourier series of rect as $T \to \infty$

Set $T = 2$ ($\omega_0 = \pi$) and calculate each of the Fourier coefficients.



Set $T = 4$ ($\omega_0 = \pi/2$) and calculate each of the Fourier coefficients.



What if we replace $k\omega_0$ with a continuous variable, ω, as $T \to \infty$?

The Fourier series of rect as $T \to \infty$

The trend we see is that as we set T larger, we more densely sample the sinc function. This gives us reason to believe that the spectrum of the rect signal, when $T \to \infty$, is a sinc function.

Let's formalize this intuition with math.

Arriving at the Fourier transform

The Fourier series of $f(t)$ an an interval $[-T/2, T/2)$ is given by:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t} dt$$

We define the truncated Fourier transform as:

$$F_T(j\omega) = \int_{-T/2}^{T/2} f(t) e^{-j\omega t} dt$$

so that

$$c_k = \frac{1}{T} F_T(jk\omega_0)$$

Then,

$$f_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(jk\omega_0) e^{jk\omega_0 t}$$

Remember, we're going to replace $k\omega_0$ with ω.

Arriving at the Fourier transform (cont.)

Now, let's set $T \to \infty$. If we do this, then $\omega_0 = 2\pi/T$ will approach 0. So suppose instead that we define a continuous variable,

$$\omega = \frac{2\pi k}{T}$$

which means that k increases with T, so that $\omega = k\omega_0$ is fixed.

The Fourier transform is the limit of the truncated Fourier transform.

$$\begin{aligned} F(j\omega) &= \lim_{T\to\infty} F_T(j\omega) \\ &= \lim_{T\to\infty} \int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \end{aligned}$$

This is the Fourier transform, which takes you from the time domain, $f(t)$, to the frequency domain, $F(j\omega)$.

Inverting the Fourier transform

How do we go from $F(j\omega)$ back to $f(t)$? Let's begin by writing the Fourier series.

$$\begin{aligned} f(t) &= \lim_{T\to\infty} f_T(t) \\ &= \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(jk\omega_0) e^{jk\omega_0 t} \end{aligned}$$

Now as $T \to \infty$, what we see is that this approaches an integral. This is an infinite sum, where the integration "widths" are the infinitesimal $1/T$ and the "heights" are $F_T(jk\omega_0)e^{jk\omega_0 t}$. To make this more clear, we denote $\Delta\omega = 2\pi/T$, and note that $\omega = k\Delta\omega$. Then, this sum becomes

$$\begin{aligned} f(t) &= \lim_{\Delta\omega\to 0} \sum_{k=-\infty}^{\infty} F_T(jk\Delta\omega) e^{jk\Delta\omega t} \frac{\Delta\omega}{2\pi} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \end{aligned}$$

This is the inverse Fourier transform, which takes you from the frequency domain, $F(j\omega)$ to the time domain, $f(t)$.

The Fourier transform

The Fourier transform is:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$$

The inverse Fourier transform is:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega}$$

A few notes:

- Like in Fourier series, the inversion formula for $f(t)$ is accurate when $f(t)$ is continuous, but produces the midpoint when $f(t)$ has jumps.
- These two are almost identical in form, except for the sign of the complex exponential and the factor $1/2\pi$.
- Check your intuition when you look at these formulas: to go from the time domain to frequency domain (Fourier transform) you should integrate away time (giving a function of frequency). Likewise, to go from the frequency domain to the time domain (inverse Fourier transform) you should integrate away frequency (giving a function of time).

The Fourier transform (cont.)

- From $F(j\omega)$, we can determine $f(t)$ and vice versa (if it's well-behaved; e.g., at discontinuities, the Fourier transform will return the midpoint).
- Not every function has a Fourier transform. For example, a sufficient condition for a Fourier transform is that it should have finite energy. Note,

$$\begin{aligned} |F(j\omega)| &= \left| \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right| \\ &\leq \int_{-\infty}^{\infty} \left| f(t) e^{-j\omega t} \right| dt \\ &= \int_{-\infty}^{\infty} |f(t)| \, dt \\ &< \infty \end{aligned}$$

- The above is a sufficient (but not necessary) requirement for the existence of the Fourier transform.
- There is much rigorous theory that has to do with Fourier transform existence, how to deal with discontinuities, etc., that we will not touch upon in this course.

Fourier transform example: rect

Let's find the Fourier transform of $f(t) = \mathrm{rect}(t/T)$ and see how it relates to the Fourier coefficients we derived earlier.

$$
\begin{aligned}
F(j\omega) &= \int_{-\infty}^{\infty} \mathrm{rect}(t/T) e^{-j\omega t} dt \\
&= \int_{-T/2}^{T/2} e^{-j\omega t} dt \\
&= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2} \\
&= \frac{1}{-j\omega} \left(e^{-j\omega T/2} - e^{j\omega T/2} \right) \\
&= \frac{1}{-j\omega} (-2j \sin(\omega T/2)) \\
&= \frac{2\sin(\omega T/2)}{\omega} \\
&= \frac{T \sin(\pi(\omega T/2\pi))}{\pi(\omega T/2\pi)} \\
&= T\,\mathrm{sinc}(\omega T/2\pi)
\end{aligned}
$$

Fourier transform example: rect (cont.)

Note here, we went through some extra algebra to get things into $\mathrm{sinc}(\cdot)$ form. This is out of convenience. Thus, we have that

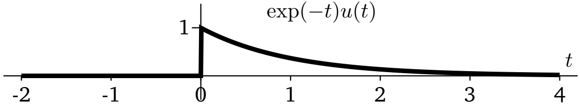
$$\mathrm{rect}(t/T) \iff T\,\mathrm{sinc}(\omega T/2\pi)$$

Fourier transform example: causal exponential

Let's find the Fourier transform of

$$\begin{aligned} f(t) &= \begin{cases} e^{-at}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \\ &= e^{-at}u(t) \end{aligned}$$

for $a > 0$.



Fourier transform example: causal exponential (cont.)

Its Fourier transform is

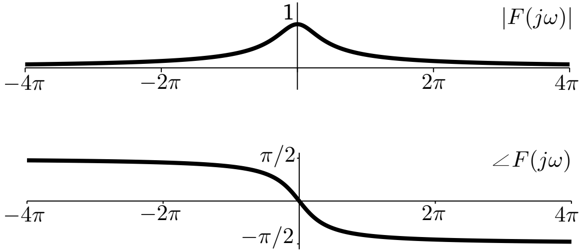
$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-(a+j\omega)t}dt \\ &= \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty} \\ &= \frac{1}{a+j\omega} \end{aligned}$$

Thus,

$$e^{-at}u(t) \iff \frac{1}{a+j\omega}$$

Fourier transform example: causal exponential

Below is the spectrum of the causal exponential for $a = 1$.



Does the amplitude spectrum make intuitive sense?

Fourier transform example: causal exponential

- The inverse Fourier transform of $F(j\omega)$ equals $f(t)$ everywhere except possibly at the origin, where the inverse FT will evaluate to the midpoint of the discontinuity.
- Notice that the magnitude spectrum is symmetric, i.e.,

$$|F(-j\omega)| = |F(j\omega)|$$

- Notice that the phase spectrum is antisymmetric, i.e.,

$$\angle F(-j\omega) = -\angle F(j\omega)$$

- This hints at symmetry properties of the FT.
- We will derive some of these properties and compute several more FTs by the end of our next few lectures in Fourier transforms.
- A goal is to build up a Fourier Transform lookup table, so we don't have to do so many integrals.

Fourier transform symmetries

Checking symmetry can be helpful for a quick intuitive check on your results; it can also be used for computational shortcuts and signal reconstruction.

Recall that every function, $f(t)$, can be decomposed into its even and odd components,

$$f(t) = f_e(t) + f_o(t)$$

Further, you should convince yourself that if $e_1(t)$ and $e_2(t)$ are even, and $o_1(t)$ and $o_2(t)$ are odd, then:

- $e_1(t) \pm e_2(t)$ is even.
- $o_1(t) \pm o_2(t)$ is odd.
- $e_1(t)e_2(t)$ is even.
- $o_1(t)o_2(t)$ is even.
- $e_1(t)o_2(t)$ is odd.

Even + odd Fourier transforms

Assume $f(t) = f_e(t) + f_o(t)$. Then,

$$\begin{aligned}
F(j\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \left(f_e(t) + f_o(t)\right)\left(\cos(\omega t) - j\sin(\omega t)\right)dt \\
&= \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f_e(t)\sin(\omega t)dt \\
&\qquad + \int_{-\infty}^{\infty} f_0(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)dt \\
&\overset{(a)}{=} \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)dt \\
&= F_e(j\omega) + F_o(j\omega)
\end{aligned}$$

where we used in step (a) the fact that the integral of an odd signal from $-\infty$ to ∞ is zero.

Even + odd Fourier transforms (cont.)

From here we see that the Fourier transform also has even and odd components; $F_e(j\omega)$ is the cosine transform of $f_e(t)$ and $F_o(j\omega)$ is $-j$ times the sine transform of $f_o(t)$. These functions are even in ω.

The following also are consequences:

- If $f(t)$ is even in t, then $F(j\omega)$ is even in ω.
- If $f(t)$ is odd in t, then $F(j\omega)$ is odd in ω.

Fourier transform of real functions

We have that

$$F(j\omega) = \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)dt$$

If $f(t)$ is real, then $f_e(t)$ and $f_o(t)$ are real. Therefore, $F_e(j\omega)$ is real and $F_o(j\omega)$ is imaginary.

Hence, $f(t)$ real implies that for $f(t) \iff F(j\omega)$,

$$\begin{aligned} \Re(F(j\omega)) &= \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)dt \\ \Im(F(j\omega)) &= -\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)dt \end{aligned}$$

with its real part being even in ω and its imaginary part being odd in ω.

Fourier transform of real functions (cont.)

Thus, we have that:

- If $f(t)$ is real and even in t, then $F(j\omega)$ is real and even in ω.
- If $f(t)$ is real and odd in t, then $F(j\omega)$ is imaginary and odd in ω.

Continuing, for real $f(t)$, we have that:

- $\Re(F(j\omega)) = \Re(F(-j\omega))$
- $\Im(F(j\omega)) = -\Im(F(-j\omega))$

And hence, for real $f(t)$,

$$\begin{aligned} F(-j\omega) &= \Re(F(-j\omega)) + j\Im(F(-j\omega)) \\ &= \Re(F(j\omega)) - j\Im(F(j\omega)) \\ &= F^*(j\omega) \end{aligned}$$

where $F^*(j\omega)$ is the complex conjugate of $F(j\omega)$. A Fourier transform with this property is called *Hermitian*.

Fourier transform of imaginary functions

We have that

$$F(j\omega) = \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)dt$$

If $f(t)$ is imaginary, then $f_e(t)$ and $f_o(t)$ are imaginary. Therefore, $F_e(j\omega)$ is imaginary and $F_o(j\omega)$ is real.

Hence, $f(t)$ imaginary implies that for $f(t) \iff F(j\omega)$,

$$\begin{aligned} \Re(F(j\omega)) &= -j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)dt \\ \Im(F(j\omega)) &= \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)dt \end{aligned}$$

with its real part being odd in ω and its imaginary part being even in ω.

Fourier transform of imaginary functions (cont.)

Thus, we have that:

- If $f(t)$ is imaginary and even in t, then $F(j\omega)$ is imaginary and even in ω.
- If $f(t)$ is imaginary and odd in t, then $F(j\omega)$ is real and odd in ω.

Continuing, for imaginary $f(t)$, we have that:

- $\Re(F(j\omega)) = -\Re(F(-j\omega))$
- $\Im(F(j\omega)) = \Im(F(-j\omega))$

And hence, for imaginary $f(t)$,

$$\begin{aligned} F(-j\omega) &= \Re(F(-j\omega)) + j\Im(F(-j\omega)) \\ &= -\Re(F(j\omega)) + j\Im(F(j\omega)) \\ &= -\left(\Re(F(j\omega)) - j\Im(F(j\omega))\right) \\ &= -F^*(j\omega) \end{aligned}$$

A Fourier transform with this property is called *anti-Hermitian*.

Summary of symmetries

- For any $f(t)$, whether it be real, imaginary or complex:
  - $f(t)$ even $\rightarrow F(j\omega)$ even.
  - $f(t)$ odd $\rightarrow F(j\omega)$ odd.
- A *real* signal has a Hermitian Fourier transform:

$$F(-j\omega) = F^*(j\omega)$$

- An *imaginary* signal has an anti-Hermitian Fourier transform:

$$F(-j\omega) = -F^*(j\omega)$$