ECE102, Fall 2020 **Homework #3**

Signals & Systems Prof. J.C. Kao

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Due Friday, 30 Oct 2020, by 11:59pm to Gradescope.
Covers material up to Lecture 6.
100 points total.

1. (20 points) **Linear systems**

   Determine whether each of the following systems is linear or not. Explain your answer.

   (a) $y(t) = sin(t)x(t)$

   Solution:
   Let input $x_1(t)$ and $x_2(t)$ have outputs $y_1(t)$ and $y_2(t)$, respectively:

   $$\begin{aligned} y_1(t) &= sin(t)x_1(t) \\ y_2(t) &= sin(t)x_2(t) \end{aligned}$$

   For an input $x(t) = a_1x_1(t) + a_2x_2(t)$, we get the output:

   $$\begin{aligned} y(t) &= a_1 sin(t)x_1(t) + a_2 sin(t)x_2(t) \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

   So, the system is linear. We could also check homogeneity and additivity separately if we wanted.

   (b) $y(t) = \frac{d}{dt}(\frac{1}{3}x(t)^3)$

   Solution: We can simplify the equation as:

   $$y(t) = (x(t))^2 \frac{d}{dt}x(t)$$

   We'll first check for homogeneity. Let $y_a(t)$ be the output for the input $ax(t)$. Then:

   $$\begin{aligned} y_a(t) &= (ax(t))^2 \frac{d}{dt}(ax(t)) \\ &= a^3x(t)\frac{d}{dt}x(t) = a^2y(t) \neq ay(t) \end{aligned}$$

   Then the system is not linear. To be thorough, we'll check additivity as well. Let $y_{12}(t)$ be the output for the input $x_1(t) + x_2(t)$. Then:

   $$\begin{aligned} y_{12}(t) &= (x_1(t) + x_2(t))^2 \frac{d}{dt}(x_1(t) + x_2(t)) \\ &= (x_1(t) + x_2(t))^2 * (\frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t)) \\ &\neq y_1(t) + y_2(t) \end{aligned}$$

(c) $y(t) = e^{x(t)}$

Solution: Like above, we'll define $x(t) = a_1x_1(t) + a_2x_2(t)$. Then, we'll get the output:

$$\begin{aligned} y(t) &= e^{a_1x_1(t)+a_2x_2(t)} \\ &= e^{a_1x_1(t)}e^{a_2x_2(t)} \\ &\neq a_1e^{x_1(t)} + a_2e^{x_2(t)} \\ &= a_1y_1 + a_2y_2 \end{aligned}$$

The system is not linear.

(d) $y(t) = x(t) + u(t+1)$

Solution:
Let's check homogeneity first:

$$\begin{aligned} y_a(t) &= ax(t) + u(t+1) \\ &\neq ay(t) \end{aligned}$$

The system is non-linear.

For additivity:

$$\begin{aligned} y_{12}(t) &= (x_1(t) + x_2(t)) + u(t+1) \\ &\neq y_1(t) + y_2(t) \end{aligned}$$

2. (13 points) **LTI systems**

(a) The input $x(t)$ and the corresponding output $y(t)$ of a linear time-invariant (LTI) system are

$$x(t) = u(t) - u(t-1) \qquad \longrightarrow y(t) = r(t) - 2r(t-1) + r(t-2)$$

where $r(t)$ is the ramp signal defined in lecture. Determine the outputs $y_i(t), i = 1, 2, 3$ corresponding to the following inputs

i. (2 points) $x_1(t) = u(t) - u(t-1) - u(t-2) + u(t-3)$

Solution: We can express $x_1(t)$ in terms of $x(t)$

$$x_1(t) = x(t) - x(t-2)$$

Then, using the linearity and time-invariance property of the system, we have

$$y_1(t) = y(t) - y(t-2)$$

ii. (2 points) $x_2(t) = u(t+1) - 2u(t) + u(t-1)$

Solution: We can express $x_2(t)$ in terms of $x(t)$

$$x_2(t) = x(t+1) - x(t)$$

Then, using the linearity and time-invariance property of the system, we have

$$y_2(t) = y(t+1) - y(t)$$

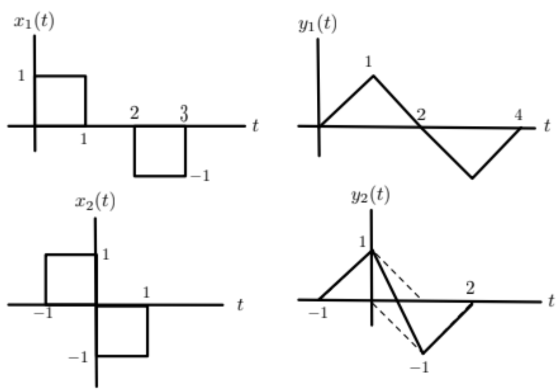


Figure 1: $y_1(t)$ and $y_2(t)$

iii. (3 points) $x_3(t) = \delta(t) - \delta(t-1)$

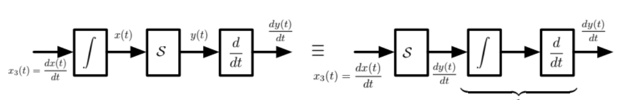
Solution: We can express $x_3(t)$ in terms of $x(t)$

$$x_3(t) = \frac{dx(t)}{dt}$$

So,

$$y_3(t) = \frac{dy(t)}{dt} = u(t) - 2u(t-1) + u(t-2)$$

Considering that the output of $x(t)$ is $y(t)$, i.e., $y(t) = S[x(t)]$, and that the integrator and the differentiator are LTI systems, the figure below shows how to visualize the result in this problem by considering that you can change the order of the cascading of LTI systems.



(b) (6 points) Assume we have a linear system with the following input-output pairs:

- the output is $y_1(t) = cos(t)u(t)$ when the input is $x_1(t) = u(t)$;
- the output is $y_2(t) = cos(t)(u(t+1) - u(t))$ when the input is $x_2(t) = \text{rect}(t + \frac{1}{2})$.

Is the system time-invariant?

Solution:
The signal $x_2(t)$ can be written as: $x_2(t) = u(t+1) - u(t)$. Let,

$$x_3(t) = x_1(t) + x_2(t) = u(t) + u(t+1) - u(t) = u(t+1)$$

We see that $x_3(t) = x_1(t+1)$. Let us now use the properties of linear system to get the output $y_3(t)$ to input $x_3(t)$, we then compare $y_3(t)$ to $y_1(t+1)$. Since, $x_3(t) = x_1(t) + x_2(t)$, the output is then

$$y_3(t) = y_1(t) - y_2(t) = cos(t)u(t) + cos(t)(u(t+1) - u(t)) = cos(t)u(t+1)$$

On the other hand,

$$y_1(t+1) = cos(t+1)u(t+1)$$

Since $y_3(t) \neq y_1(t+1)$, the system is not time-invariant.

3. (38 points) **Convolution**

(a) (10 points) For each pair of the signals given below, compute their convolution using the flip-and-drag technique. Please provide a piecewise formula for $y(t)$.

i. $f(t) = \delta(t+1) + 2\delta(t-2)$, $\qquad g(t) = e^{-t}u(t)$

Solution:

$$\begin{aligned} y(t) = f(t) \star g(t) &= (\delta(t+1) + 2\delta(t-2)) \star e^{-t}u(t) \\ &= \delta(t+1) \star e^{-t}u(t) + 2\delta(t-2) \star e^{-t}u(t) \\ &= e^{-t-1}u(t+1) + 2e^{-t+2}u(t-2) \end{aligned}$$

Note that if you flip the impulse signal, you get the same impulse. Therefore, when you do a flip and drag operation when convolving impulses with the other signal, you are essentially shifting the other signal.

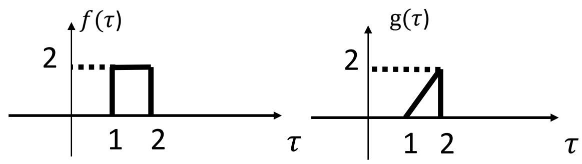
ii. $f(t) = 2\ \text{rect}(t - \frac{3}{2})$, $\qquad g(t) = 2\ r(t-1)\text{rect}(t - \frac{3}{2})$

Solution: Let,

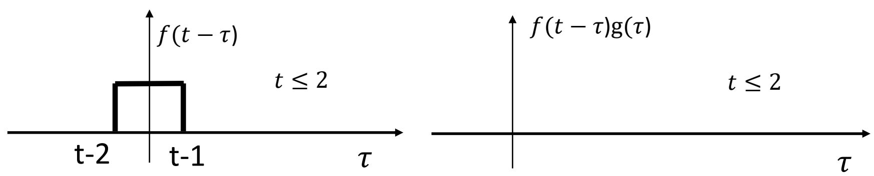
$$y(t) = f(t) \star g(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau$$

We will flip and drag the rect function. As we can see in the figure below, for $t \leq 2$, there is no overlap between the two plots, therefore $y(t) = 0$ for for $t \leq 2$. For $2 < t \leq 3$, the rect function starts to overlap with the triangle, the convolution integral in this case is equal to the overlapped area. For $3 < t \leq 4$, the rect function starts to go out from the triangle, the convolution integral is also equal to the overlapped area. For $t > 4$, there is no overlap between the two plots, then $y(t) = 0$ for $t > 4$. Therefore, $y(t)$ is given as:

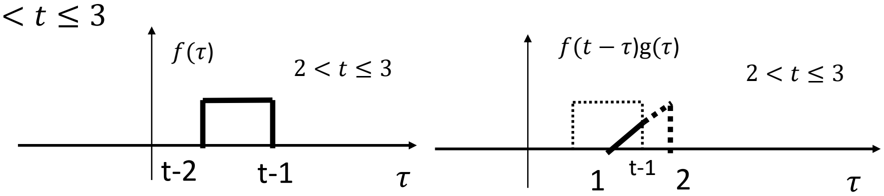
$$y(t) = \begin{cases} 0, & t \leq 2 \\ \int_1^{t-1} 4(\tau - 1)\, d\tau = 2(t-2)^2, & 2 < t \leq 3 \\ \int_{t-2}^{2} 4(\tau - 1)\, d\tau, = -2(t-4)(t-2) & 3 < t \leq 4 \\ 0, & t > 4 \end{cases}$$



Case 1: $t \leq 2$



Case 2: $2 < t \leq 3$



(b) (10 points) For each of the following, find a function $h(t)$ such that $y(t) = x(t) * h(t)$.

i. $y(t) = \int_{t-T}^{t} x(\tau) d\tau$

Solution:
We can think about $h(t)$ as the impulse response of the give LTI system, therefore

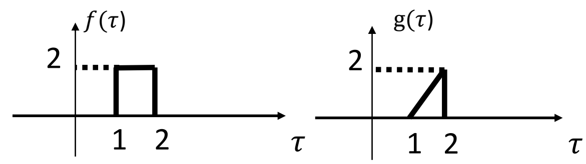
$$h(t) = \int_{t-T}^{t} \delta(\tau) d\tau = u(t) - u(t-T)$$

ii. $y(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$

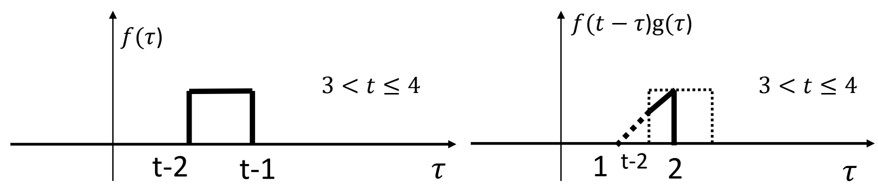
Solution:
Also by applying the sifting property,

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

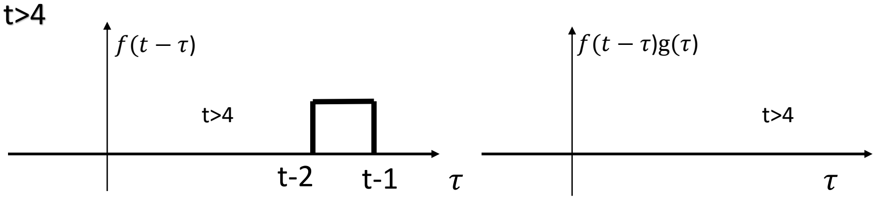
Note: this last operation creates a periodic extension of $x(t)$ where the period is T_s.



Case 3: $3 < t \leq 4$



Case 4: t>4



(c) (10 points) Use the properties of convolution to simplify the following expressions:

i. $[\delta(t-3)+\delta(t+2)] * \left[e^{3t}u(-t)+\delta(t+2)+2\right]$

Solution:
We apply the sifting property:

$$\begin{aligned} e^{3(t-3)}u(-t+3)+\delta(t-1)+2+e^{3(t+2)}u(-t-2)+\delta(t+4)+2 &= \\ e^{3(t-3)}u(-t+3)+\delta(t-1)+e^{3(t+2)}u(-t-2)+\delta(t+4)+4 \end{aligned}$$

ii. $\frac{d}{dt}\left[(u(t)-u(t-1)) * u(t-2)\right]$, *Hint: Show first that* $u(t) * u(t) = r(t)$ *where* $r(t)$ *is the ramp function.*

Solution:
We first show that $u(t) \star u(t) = r(t)$:

$$u(t) \star u(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \left(\int_0^t 1 d\tau\right) u(t) = tu(t) = r(t)$$

Therefore, using the properties of convolutions,

$$(u(t)-u(t-1)) \star u(t-2) = r(t-2) - r(t-3)$$

Thus,

$$\frac{d}{dt}\left(r(t-2)-r(t-3)\right) = u(t-2)-u(t-3)$$

(d) (8 points) Explain whether each of the following statements is true or false.

i. If $x(t)$ and $h(t)$ are both odd functions, and $y(t) = x(t) * h(t)$, then $y(t)$ is an even function.

Solution:
True: We will show this statement by applying the definition of convolution.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Let $\tau' = -\tau$, then

$$y(t) = -\int_{\infty}^{-\infty} x(-\tau')h(t+\tau')d\tau' = \int_{-\infty}^{\infty} x(-\tau')h(t+\tau')d\tau'$$

Since $x(t)$ and $h(t)$ are both odd functions, we have:

$$y(t) = \int_{-\infty}^{\infty} (-x(\tau'))(-h(-t-\tau'))d\tau' = \int_{-\infty}^{\infty} x(\tau')h(-t-\tau')d\tau' = y(-t)$$

Therefore, $y(t)$ is even.

ii. If $y(t) = x(t) * h(t)$, then $y(2t) = h(2t) * x(2t)$.

Solution:
False: Consider the following counter example; let $x(t) = \delta(t)$ and $h(t) = u(t)$, then $x(2t) = \delta(2t) = \frac{1}{2}\delta(t)$ and $h(2t) = u(2t) = u(t)$. Therefore, we have: $y(t) = x(t) \star h(t) = u(t)$, so $y(2t) = u(2t) = u(t)$. On the other hand, $x(2t) \star h(2t) = \frac{1}{2}u(t)$. Thus, $y(2t) \neq h(2t) \star x(2t)$.
What is true instead is that $y(2t) = 2\left(h(2t) \star x(2t)\right)$. This can be shown using the definition of convolution:

$$x(2t) \star h(2t) = \int_{-\infty}^{\infty} x(2\tau)h(2t-2\tau)d\tau$$

Let $\tau' = 2\tau$, then

$$x(2t) \star h(2t) = \frac{1}{2}\int_{-\infty}^{\infty} x(\tau')h(2t-\tau')d\tau' = \frac{1}{2}y(2t)$$

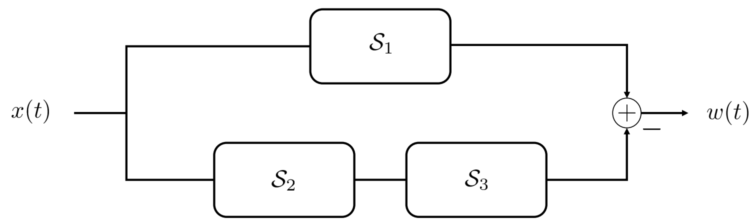
4. (12 points) **Impulse response and LTI systems**

Consider the following three LTI systems:

- The first system $\mathcal{S}_1$ is given by its input-output relationship: $y(t) = \int_{-\infty}^{t-t_0} x(\tau-2)d\tau$;
- The second system $\mathcal{S}_2$ is given by its impulse response: $h_2(t) = u(t+2)$;
- The third system $\mathcal{S}_3$ is given by its impulse response: $h_3(t) = \delta(t-4)$.

(a) (4 points) Compute the impulse responses $h_1(t)$ of system $\mathcal{S}_1$.
Solution: For system $\mathcal{S}_1$, the impulse response is given by:

$$h_1(t) = \int_{-\infty}^{t-t_0} \delta(\tau-2)d\tau = u(t-t_0-2)$$



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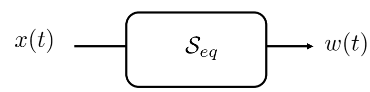
(b) (4 points) The three systems are interconnected as shown below.
Determine the impulse response $h_{eq}(t)$ of the equivalent system.



Solution:

$$h_{eq}(t) = h_1(t) - (h_2(t) \star h_3(t)) = u(t - t_0 - 2) - (u(t + 2) \star \delta(t - 4)) = u(t - t_0 - 2) - u(t - 2)$$

(c) (4 points) Determine the response of the overall system to the input $x(t) = 0.5 * \delta(t - 2) + \delta(t - 3)$.

Solution: The response is: $y(t) = x(t) \star h_{eq}(t) = 0.5u(t - t_0 - 4) - 0.5u(t - 4 - u(t - t_0 - 5) - u(t - 5)$

5. (17 points) **MATLAB**

To complete the following MATLAB tasks, we will provide you with a MATLAB function (`nconv()`), which numerically evaluates the convolution of two continuous-time functions. Make sure to download it from CCLE and save it in your working directory in order to use it.

The function syntax is as follows:

```
[y, ty] = nconv(x,tx,h,th)
```

where the inputs are:

`x` : input signal vector
`tx`: times over which `x` is defined
`h` : impulse response vector
`th`: times over which `h` is defined

and the outputs are:

`y` : output signal vector
`ty`: times over which `y` is defined.

The function is implemented with the MATLAB's `conv()` function. You are encouraged to look at the implementation of the function provided (the explanations are included as comments in the code).

(a) (5 points) **Task 1**

Using the `nconv()` function, perform the convolution of two unit rect functions: rect(t) $*$ rect(t). Plot and label the result.

Solution:

Code:
```
t=[-0.5:0.001:0.5];x=ones(1,length(t));[y, ty] = nconv(x,t,x,t);
plot(ty,y); grid on;
xlabel('t(sec)'); ylabel('Convolution of two rect');
```

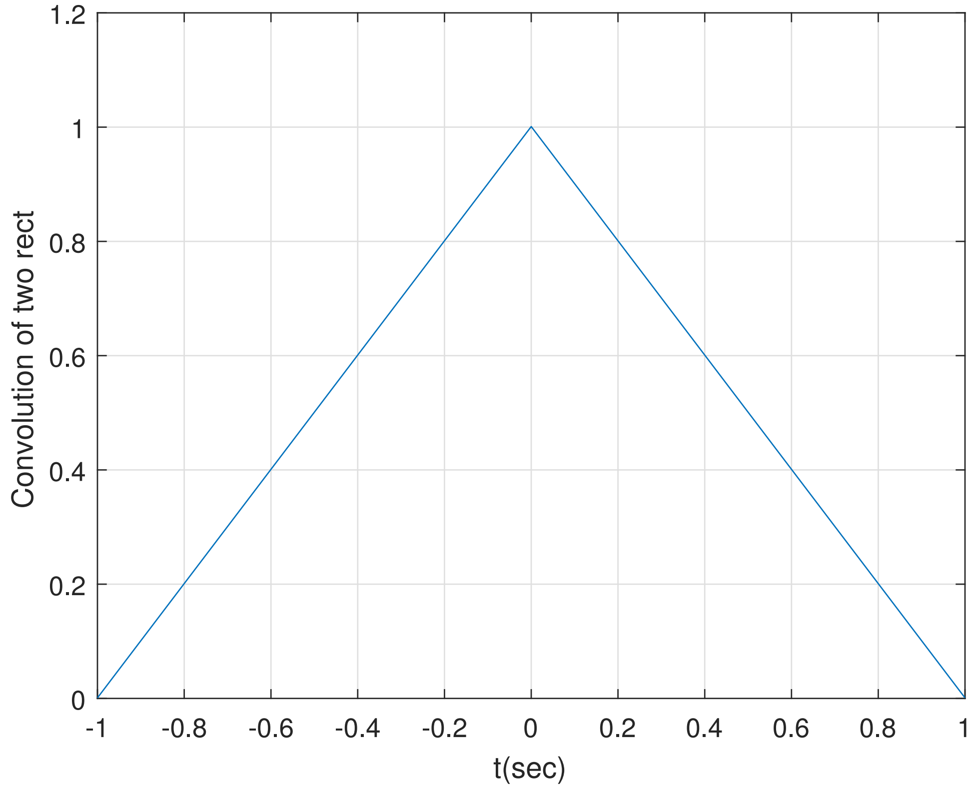


Figure 2: Task 1

(b) (5 points) **Task 2**

Using the result of task 2 and the same MATLAB function, calculate $y(t) = \text{rect(t)} * \text{rect(t)} * \text{rect(t)}$. Plot and label the result.

Solution: Code: we used as input to nconv the same output we obtained for task 2.

```
 [y, ty] = nconv(x,t,y,ty);
plot(ty,y); grid on; xlabel('t(sec)'); ylabel('Convolution of a rect and a triangle');
```

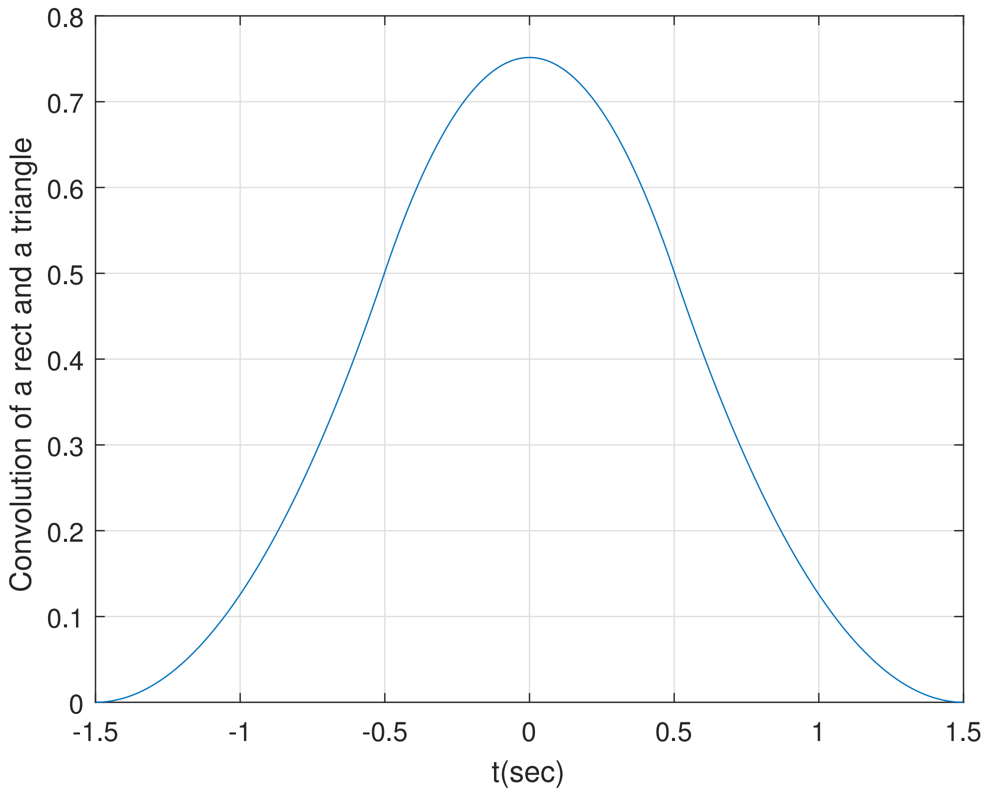


Figure 3: Task 2

(c) (7 pointss) **Task 3**

Now, what happens if we consider $\text{rect(t)} * \text{rect(t)} * \cdots * \text{rect(t)} = \text{rect}^{(N)}(t)$? Using for loop, calculate the result of convolving N rect(t) functions together. Plot and label the result (use $N = 100$).

Solution: Code:

```
 t=[-0.5:0.001:0.5]; x=ones(1,length(t)); [y, ty] = nconv(x,t,x,t);
N=100-2;
for n=1:N
[y, ty] = nconv(x,t,y,ty);
end
plot(ty,y); grid on;
xlabel('t(sec)'); ylabel('Convolution of N rect');
```

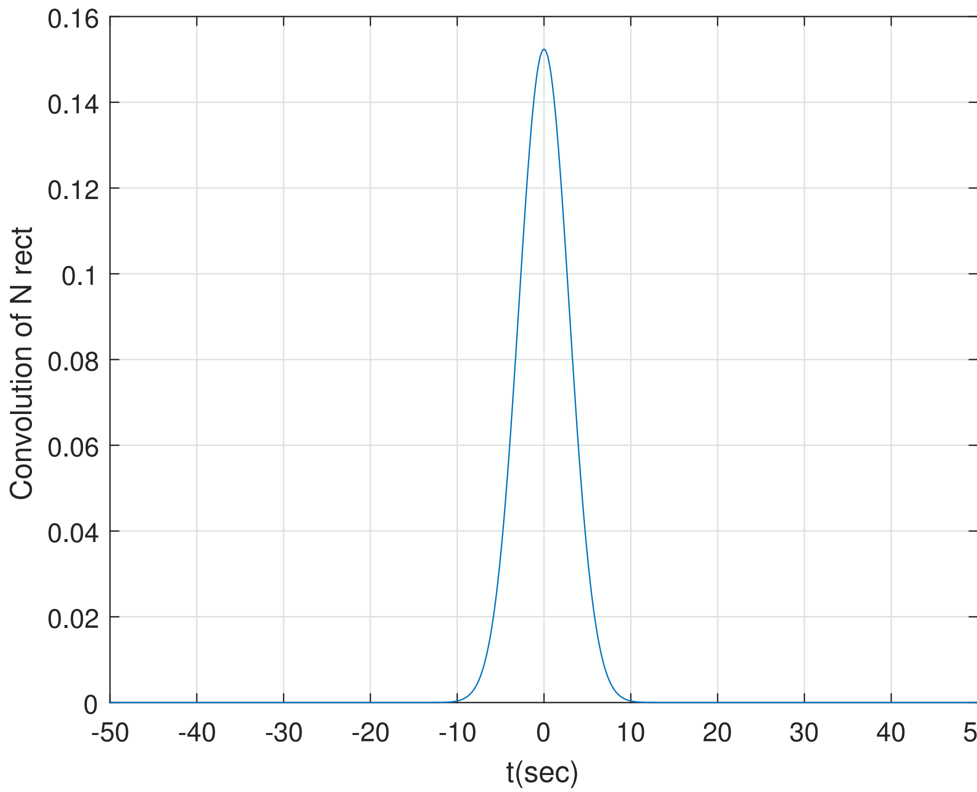


Figure 4: Task 3

Side note in case you have taken any probability course before: Convolution is an operator that is also useful in statistics. We use it to compute the pdf (probability density function) of the sum of N independent random variables. So if we have $Y = X_1 + X_2 + X_3$, the pdf of Y is the convolution of the pdfs of X_1, X_2 and X_3. In task 4, we are computing the pdf of the sum of N uniform random variables (the pdf of a uniform random variable is a rect function), by convolving N times the rect function. The resulting curve will have a bell-shape. This is related to a theorem in statistics called 'The Central Limit Theorem'.