Signal operations and properties

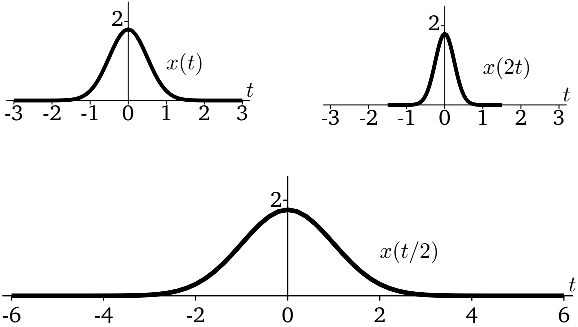
This lecture overviews several mathematical operations and properties that will provide a foundation for the rest of the class. It jumps between various topics as we need to know all of these before moving on.

- Time scaling, reversal and shifting.
- Even and odd signals
- Periodicity
- Review of sinusoids and complex numbers
- Causality
- Energy and power signals
- Euler's formula

Time scaling

A signal $x(t)$ can be compressed or expanded in time by multiplying the time variable by a positive constant, a, to arrive at $x(at)$.

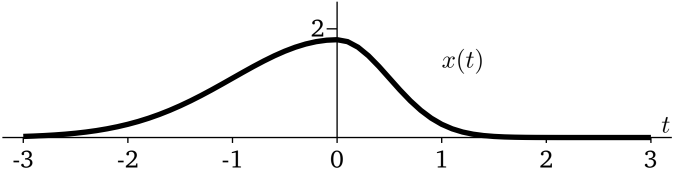
- If $a > 1$ then the signal is compressed in time.
- If $0 < a < 1$ then the signal is expanded in time.

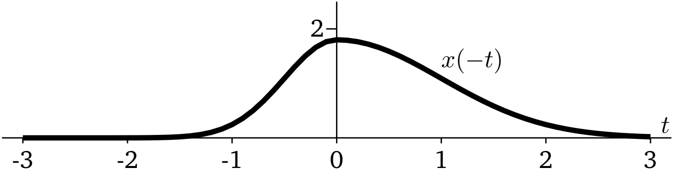


As you work on examples of this, it is sometimes helpful to plug in values of t to make sure you have compressed / expanded the values correctly.

Time reversal

A signal $x(t)$ can be reversed by replacing t with $-t$. The reversed signal is $x(-t)$.



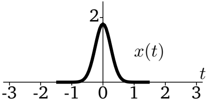


Time shift

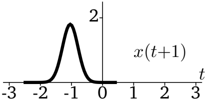
A signal $x(t)$ can be shifted in time by some amount $t_1 > 0$.

- The signal $x(t - t_1)$ is delayed in time by t_1.
- The signal $x(t + t_1)$ is advanced in time by t_1.

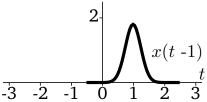
Original



Advanced



Delayed

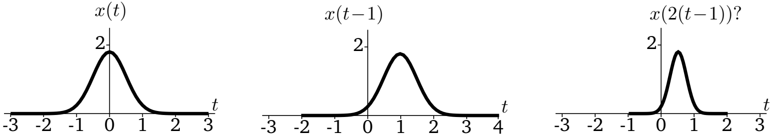


As you work on time shift examples, it may be helpful to consider when $t - t_1 = 0$.

Combining operations

Take care when combining time scaling, shifting, and reversal. For example, consider $x(2(t-1))$.

Let's try the following. First, let's do the operation in the parentheses and shift by 1; after that we compress by 2.

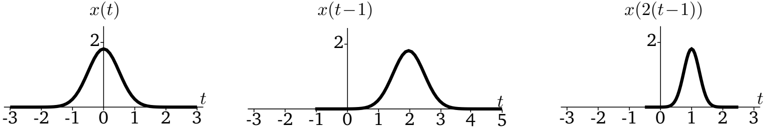


Is this correct? (Hint: for what t is $2(t-1)=0$?)

Combining operations (cont)

The following slide was wrong. Instead, we should recognize that $x(2(t-1)) = x(2t-2)$.

In this manner, we shift by 2 and then scale by 2.

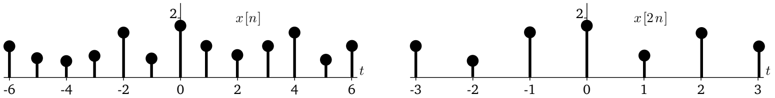


This correct achieves the peak at $t = 1$. Be careful to check where the signal should start and end.

Time operations for discrete signals

The concepts of time scaling, reversal, and shift apply to discrete-time signals as well. While the concepts of time reversal and shift are straightforward to generalize to discrete signals, it is worth touching briefly on time scaling for discrete signals.

Time compression:

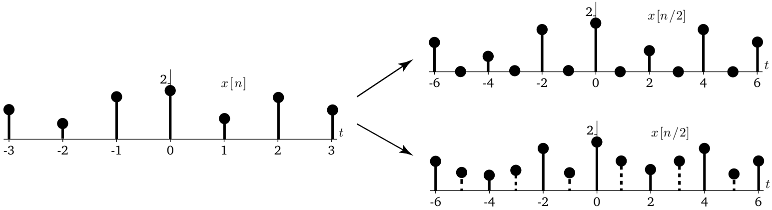


Note the time-axis values.

Time operations for discrete signals (cont.)

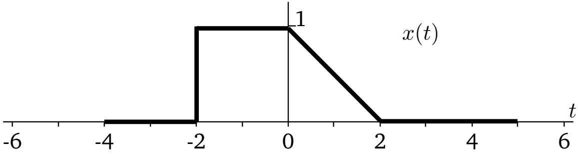
How about time expansion? We are adding samples here, and so it appears that we must synthesize some intermediate samples.

This raises another question: how do we fill in values? This could be done in a few ways; e.g., the intermediate values can be set to zero (top arrow path), or can be *interpolated* (bottom arrow path). It is more common to interpolate the values if you can.



Examples

Consider $x(t)$ below.



Plot:

- $x(-t/2)$
- $x(2(t+2))$
- $0.5x(-t+1)$

Even and odd signals

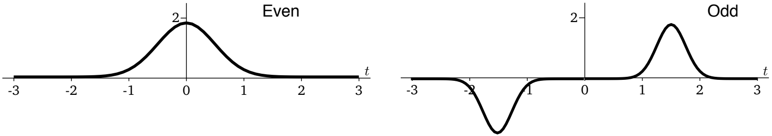
- An *even* signal is symmetric about $t = 0$, i.e.,

$$x(t) = x(-t)$$

- An *odd* signal is antisymmetric about the origin, i.e.,

$$x(t) = -x(-t)$$

- An example:



Even and odd decomposition

Any signal can be decomposed into even and odd parts. We define $x_e(t)$ and $x_o(t)$ to be the even and odd component of $x(t)$, respectively. To show that any signal, $x(t)$ can be decomposed in this way, we need to show:

$$x(t) = x_e(t) + x_o(t)$$

- The even component must have the property that $x(t) = x(-t)$. This can be achieved by defining:

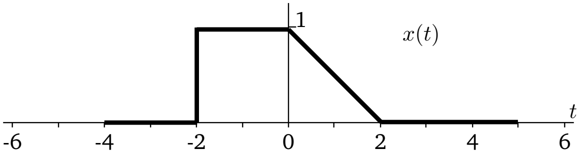
  $$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

  It can be verified that this signal is even.
- Likewise, the odd component, having the property that $x(t) = -x(-t)$, can be achieved via:

  $$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

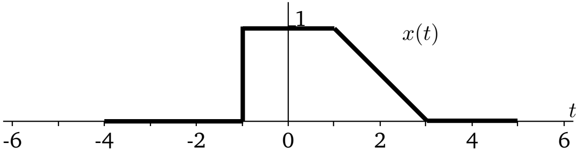
Example

What are the even and odd parts of the following signal?



Example

How about the even and odd components of the signal on the prior page shifted by 1?



Periodic signals

The concept of periodic signals is very important in this class. Colloquially, these are signals that repeat after a given interval, T_0.

The formal definition of a periodic signal is as follows:

A continuous time signal is periodic if and only if there exists a $T_0 > 0$ such that

$$x(t + T_0) = x(t)$$

for all t. T_0 is called the period of $x(t)$.

Periodic signals (cont.)

A few notes about the definition of a periodic signal.

- If a signal is periodic, i.e., $x(t + T_0) = x(t)$ is true for T_0, it is also true for $2T_0$, $3T_0$, etc., i.e.,

$$x(t + nT_0) = x(t)$$

  for $n = \pm 1, \pm 2, \ldots$.
- Thus, we call the smallest T_0 the *fundamental* period of the periodic signal.
- A signal that is not periodic is called *aperiodic*. In this class, I'll use "not periodic" and "aperiodic" interchangeably.
- Sines and cosines are very common periodic signals, which we will discuss in more detail in this class.

Sinusoids

The most basic signal in this class is the sine or cosine wave. We'll use them *extensively* so it's worth reviewing their properties. By the end of this class, you'll be proficient at manipulating sinusoids.

A cosine is defined by:

$$\begin{aligned} x(t) &= A\cos(\omega t - \theta) \\ &= A\cos(2\pi f t - \theta) \end{aligned}$$

with

- A defining the amplitude of the signal (i.e., how large it gets).
- ω defining the *natural* frequency of the signal (in units of radians per second). As ω gets larger, the sinusoid repeats more times in a given time interval.
- The natural frequency is related to the frequency, f, of the signal (in units of Hertz, or s^{-1}) through the relationship: $\omega = 2\pi f$. The frequency, f, is the inverse of the period, i.e.,

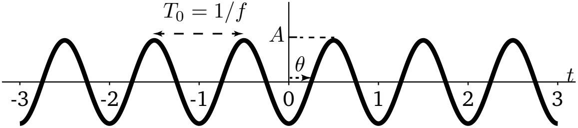
$$T_0 = \frac{1}{f} = \frac{2\pi}{\omega}$$

- θ is the phase of the signal in terms of radians, shifting the sinusoid.

Sinusoids

We illustrate a sinusoid signal below:

$$x(t) = A\cos(\omega t - \theta)$$

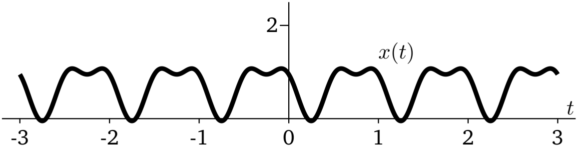


Some additional properties that you should be familiar with from trigonometry:

- $\sin(\theta) = \cos(\theta - \pi/2)$.
- Are either $\cos(\theta)$ or $\sin(\theta)$ even or odd?
- $\frac{d}{dt}\sin(\theta) = \cos(\theta)$ and $\frac{d}{dt}\cos\theta = -\sin(\theta)$.
- $\sin^2(\theta) + \cos^2(\theta) = 1$.

Periodic example

We have a signal below that is not symmetric around $t = 0$. Is it periodic, and if so, what is its period?



Periodic example 2

Consider two periodic signals,

$$x_1(t) = \sin\left(\frac{\pi}{8}t\right)$$

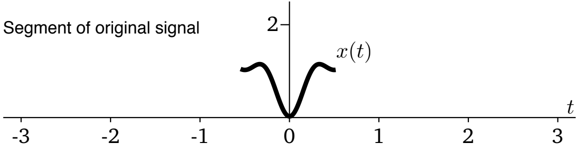
and

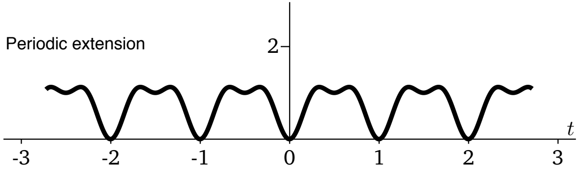
$$x_2(t) = 3\cos\left(\frac{\pi}{2}t\right)$$

- What is the period of $x_1(t)$?
- What is the period of $x_2(t)$?
- Is their sum periodic? If so, what is its period?
- Find the period of the signal: $x(t) = \cos(2\pi\frac{1}{2}t) \cdot \sin(2\pi 5t)$.
- In general, are the sum of two periodic signals always periodic?

Periodic extension

In this class, we will sometimes be interested in taking an aperiodic signal and making its periodic extension. What this mean is that we take some interval on this signal of length T_0 and repeat it, as illustrated below:





Causality

Another signal property that will come up later on is the concept of causality. It is important to define causality because in real life, we typically can only work with signals in the present and past (i.e., we don't know the future). This will become more clear when we discuss convolution and filtering.

- Causal signals are non-zero only for $t \geq 0$. Colloquially, the signal starts at $t = 0$ or later.
- Noncausal signals are non-zero for some $t < 0$. Colloquially, the signal starts before $t = 0$.
- Anticausal signals are non-zero only for $t \leq 0$. Colloquially, these signals only exist before $t = 0$ and typically are interpreted as running backwards in time.

Signal energy and power

We know from circuits that the voltage signal, $v(t)$, and current signal, $i(t)$, across a resistor, R is related through:

$$v(t) = i(t)R$$

and further, that the signal's instantaneous power at time t is:

$$p(t) = v(t)i(t)$$

If we consider that $R = 1\ \Omega$, then we see that

$$\begin{aligned} p(t) &= v^2(t) \\ &= i^2(t) \end{aligned}$$

This is to show that power is proportional to the signal quantity squared; if resistance was not 1, it would only scale power by a constant. In most cases, this is not *actual* power in that most signals aren't applied to a $1\ \Omega$ resistor.

Signal energy and power (cont.)

Signal power has units of Watts (Joules per time). Hence, to get the total energy of a signal, $x(t)$, across all time, we integrate the power.

$$\boxed{E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt}$$

(We incorporate the absolute value, $|\cdot|$, in case $x(t)$ is a complex signal, reviewed in the next slides.) Like signal power, signal energy is usually not an *actual* energy.

We can also calculate the *average power* of the signal by calculating:

$$\boxed{P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}$$

Can we simplify this expression to obtain the power of a periodic signal?

Energy and power signals

Signals are classified by whether they have finite energy or power.

- An *energy signal* has finite energy, i.e., $0 < E_x < \infty$. (What is the signal power of an energy signal?)
- A *power signal* has finite power, i.e., $0 < P_x < \infty$. (What is the signal energy of a power signal?)

Can a signal be both a power and energy signal?

We introduce this concept as energy and power signals are treated differently later on in the class. We'll do a few examples in class to classify signals as energy or power signals.

Complex sinusoids

So far all signals we've presented are real-valued. But signals can also be complex.

- A complex signal is one that takes the form:

$$z(t) = x(t) + jy(t)$$

  where $x(t)$ and $y(t)$ are real-valued signals and $j = \sqrt{-1}$.
- Do complex signals practically arise?

A review of complex numbers

Because complex numbers play a large role in this class, we'll briefly review them.

- A complex number is formed from two real numbers, x and y, via:

  $$z = x + jy$$

  with $j = \sqrt{-1}$. Hence, a complex number is simply an ordered pair of real numbers, (x, y).
- $x = \Re(z)$ is called the *real* part of z. (In this class we will also write $x = \text{Re}(z)$.)
- $y = \Im(z)$ is called the *imaginary* part of z. (In this class we will also write $y = \text{Im}(z)$.)

- An aside: why do EE's use j as the imaginary number, while mathematicians and scientists commonly use i?

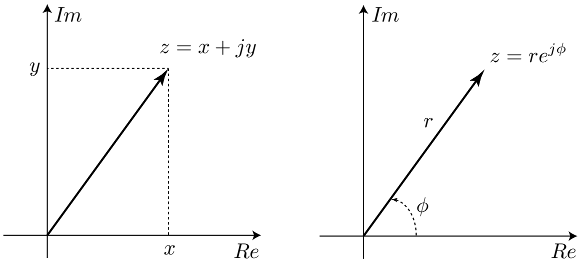
Polar representation of complex numbers

The same complex number can be written in polar form,

$$\begin{aligned} z &= x + jy \\ &= re^{j\phi} \end{aligned}$$

where

- r is the *modulus* or *magnitude* of z.
- ϕ is the *angle* or *phase* of z.
- $e^{j\phi} = \cos(\phi) + j\sin(\phi)$. We will sometimes write this as $\exp(j\phi)$. (More on this below.)



Cartesian vs polar coordinates

$$\begin{aligned} z &= x + jy \\ &= re^{j\phi} \end{aligned}$$

Here, the same intuitions from Cartesian and polar coordinates hold.

- $x = r\cos(\phi)$
- $y = r\sin(\phi)$
- $r = \sqrt{x^2 + y^2}$
- $\phi = \arctan y/x$

Some complex relations

Here are a few relations.

- **Complex conjugate**. If $z = x + jy$, then z^*, the complex conjugate of z, is
$$z^* = x - jy$$
- **Modulus and complex conjugate**. The following relation holds:
$$|z|^2 = z^* z = z z^*$$
This is because
$$\begin{aligned} zz^* &= (x + jy)(x - jy) \\ &= x^2 + y^2 \\ &= r^2 \end{aligned}$$
where $r = \sqrt{x^2 + y^2}$ as on the last slide.
- **Inverse of j**. Since $j^2 = -1$, we have that $-j = \frac{1}{j}$.

Euler's identity

Relating terms in our Cartesian and polar coordinate representation of complex numbers, we arrive at Euler's formula:

$$\begin{aligned} z &= x + jy \\ &= re^{j\phi} \end{aligned}$$

This tells us that, for $r = 1$,

$$\boxed{e^{j\phi} = \cos(\phi) + j\sin(\phi)}$$

Aside: this leads to one of the most elegant equations in mathematics:

$$e^{i\pi} + 1 = 0$$

With five terms, it incorporates Euler's constant (e), pi (π), the imaginary number (i), the multiplicative identity (1) and the additive identity (0).

Familiarity with complex numbers

You should be very familiar knowing how to add, multiply, and divide complex numbers, and be able to go between Cartesian and polar representations easily. Here are some identities that will facilitate this:

- $e^z = e^{x+jy} = e^x e^{jy} = e^x(\cos(y) + j\sin(y))$.
- $z(t) = Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + jA\sin(\omega t+\theta)$.
- The inverse formulas:

$$\begin{aligned} \cos(\theta) &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin(\theta) &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{aligned}$$

Complex examples

We'll do these examples in class to refresh your familiarity with complex numbers.

- Prove that $(\cos(\theta) + j\sin(\theta))^n = \cos(n\theta) + j\sin(n\theta)$.
- Derive the law of the cosines:

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$