Laplace Transform

This lecture introduces the Laplace Transform and its properties. Topics include:

- s spectrum and region of convergence
- Bilateral Laplace transform
- Unilateral Laplace transform
- Relationship between Fourier and Laplace transforms
- Laplace transforms of e^{at}, $u(t)$, t^n, $\delta(t)$, and $\cos(\omega t)$
- Laplace transform properties
- Examples
- Solving differential equations

Motivation

The Fourier transform is powerful, but it doesn't exist for some signals and systems. In several applications, including image processing, communications, and circuit design, its sufficient for analysis.

However, some systems are unstable, or are power signals where the Fourier transform can not be straightforwardly generalized. Some examples of this are signals that grow with time, like (ideally) your bank account, or the S&P 500.

How do we analyze these systems in a similar framework to what Fourier analysis enables us to do?

Motivating example

Let

$$f(t) = e^{at}u(t)$$

When $a > 1$, this signal does not have a Fourier transform.

One approach to arrive at a Fourier transform is to define a new function

$$g(t) = f(t)e^{-\sigma t}$$

If $\sigma > a$, then $g(t)$ is a decreasing exponential, which has a Fourier transform.

If we have changed the function, how does this help?

Motivating example (cont.)

The function $g(t) = f(t)e^{-\sigma t}$ has a Fourier transform for σ sufficiently large. The Fourier transform of $g(t)$ comprises how to sum spectral components $e^{j\omega t}$,

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}d\omega$$

The intuition here is that because $f(t) = g(t)e^{\sigma t}$, $f(t)$ has spectral components

$$e^{\sigma t}e^{j\omega t} = e^{(\sigma + j\omega)t}$$

Hence, the Laplace transform gives us a spectrum of $f(t)$ in terms of a complex exponential with both real and imaginary components (where as the Fourier transform was only with imaginary components).

When does the s-spectrum exist?

For what values of σ does this work? In the case where $f(t) = e^{at}u(t)$, this is clear, i.e., $\sigma > a$.

In general, there is some σ_0 for which

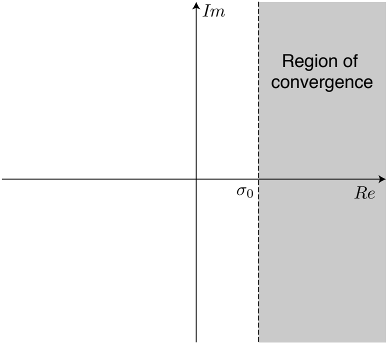
$$f(t)e^{-\sigma_0 t}$$

goes to zero. If it does, then this $f(t)e^{-\sigma_0 t}$ is an energy signal, and its spectrum will exist.

The portion of the complex plane where $\sigma > \sigma_0$ is called the "region of convergence."

Region of convergence

The region of convergence is illustrated below:



Bilateral Laplace transform

The Laplace transform incorporates the real exponential. With $s = \sigma + j\omega$, as before,

- $j\omega$ is related to the oscillatory component of the complex exponential
- σ is related to the decay or growth of the complex exponential

Then, the **bilateral** Laplace transform is:

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt}$$

To invert the bilateral Laplace transform, we calculate:

$$\boxed{f(t) = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds}$$

for $c > \sigma_0$.

We won't use the bilateral Laplace transform, but it's worth mentioning this for completeness.

Laplace transform notation

Our notation for the Laplace transform is very similar to our prior notation. We denote

$$\begin{aligned} F(s) &= \mathcal{L}[f(t)] \\ f(t) &= \mathcal{L}^{-1}[F(s)] \end{aligned}$$

We will also denote this:

$$f(t) \iff F(s)$$

The unilateral Laplace transform

Usually, we are interested in analyzing causal signals. In this case, we can simplify the bilateral Laplace transform. A causal signal can be written as $f(t)u(t)$, and its Laplace transform is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)u(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} f(t)e^{st}dt \end{aligned}$$

When we write 0^-, this indicates that impulses at the origin are included (e.g., $\delta(t)$ would have a contribution to this integral).

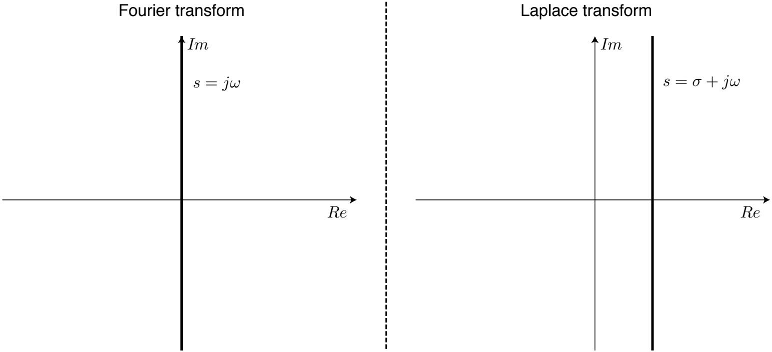
The Laplace transform is (essentially) unique. From now on, we'll use $\mathcal{L}[f(t)]$ to denote the unilateral Laplace transform of $f(t)$.

Relationship between the Fourier and Laplace transforms

The Fourier transform is a special case of the Laplace transform, i.e.,

$$\boxed{F(j\omega) = F(s)|_{s=j\omega}}$$

The Fourier transform is evaluated at $s = j\omega$ and the Laplace transform is evaluated at a particular $s = \sigma + j\omega$.



Relationship between the Fourier and Laplace transforms (cont.)

You may imagine that for signals where we know the Fourier transform, the Laplace transform merely replaces $j\omega$ with s. This is sometimes the case. Let's consider $f(t) = e^{-at}u(t)$.

$$\begin{aligned} F(s) &= \int_0^\infty e^{-at}e^{-st}dt \\ &= \int_0^\infty e^{-(a+s)t}dt \\ &= -\frac{1}{a+s}e^{-(a+s)t}\bigg|_0^\infty \\ &= \frac{1}{a+s} \end{aligned}$$

as long as $e^{-(a+s)t} \to 0$ as $t \to \infty$. When does this happen?

Relationship between the Fourier and Laplace transforms (cont.)

If $e^{-(a+s)t}$ goes to zero, then so does $\left|e^{-(a+s)t}\right|$.

$$\begin{aligned}\left|e^{-(a+s)t}\right| &= \left|e^{-(a+\sigma+j\omega)t}\right| \\ &= \left|e^{-j\omega t}\right|\left|e^{-(\sigma+a)t}\right| \\ &= \left|e^{-(\sigma+a)t}\right| \\ &= e^{-(\sigma+a)t}\end{aligned}$$

since $\left|e^{-j\omega t}\right| = 1$. This means that the region of convergence is for $\sigma > -a$, or equivalently $\Re(s) > -a$.

Relationship between the Fourier and Laplace transforms (cont.)

Hence, we have that

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

and we know prior, for $a > 0$,

$$\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+j\omega}$$

Here, the Laplace transform is the Fourier transform with $j\omega$ replaced with s.

A key thing to note here is that the region of convergence,

$$\sigma > -a$$

includes the $j\omega$ axis. It turns out that when the region of convergence includes the $j\omega$ axis, then we can replace $j\omega$ with s.

Relationship between the Fourier and Laplace transforms (cont.)

A key thing to note is that with

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

holds for all a, positive or negative, as long as $\sigma > -a$.

This means that, for $a > 0$,

$$\mathcal{L}[e^{at}u(t)] = \frac{1}{s-a}$$

Of course, this signal does not have a Fourier transform.

Relationship between the Fourier and Laplace transforms (cont.)

What if the region of convergence does not include the $j\omega$ axis? This is true of some of the signals where we had to extend the Fourier transform. Consider the unit step. Its Laplace transform is:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} dt \\ &= -\frac{1}{s} e^{-st}\Big|_0^\infty \\ &= \frac{1}{s} \end{aligned}$$

as long as e^{-st} goes to 0 as $t \to \infty$. This means the region of convergence is $\Re(s) > 0$. The Fourier transform is evaluated along $\Re(s) = 0$ which is not in the region of convergence.

Recall the Fourier transform of the unit step is:

$$\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

This resembles the Laplace transform with $s = j\omega$, but there is an additional $\pi\delta(\omega)$ term.

Relationship between the Fourier and Laplace transforms (cont.)

We will see this tends to be the case for some of our generalized Fourier transforms. For example, consider the Laplace transform of

$$\begin{aligned} f(t) &= \cos(\omega t) \\ &= \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right] \end{aligned}$$

Then,

$$\begin{aligned} F(s) &= \int_0^\infty \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right] e^{-st} dt \\ &= \frac{1}{2}\int_0^\infty e^{(-s+j\omega)t} + e^{(-s-j\omega)t} dt \\ &= \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) \\ &= \frac{s}{s^2+\omega^2} \end{aligned}$$

The region of convergence is for when $e^{(-s\pm j\omega)t} \to 0$ as $t \to \infty$ and thus is $\Re(s) > 0$. Like the unit step, the Laplace and Fourier transforms disagree, as the Laplace region of convergence does not include the $j\omega$ axis.

Laplace transform of powers of t

Laplace transforms, given all we've learned thus far, should be fairly straightforward to evaluate. We'll go over a few examples here. Let

$$f(t) = t^n$$

for $n \geq 1$. Then,

$$F(s) = \int_0^\infty t^n e^{-st} dt$$

We integrate by parts, setting $u(t) = t^n$ and $v'(t) = e^{-st}$. This means that $u'(t) = nt^{n-1}$ and $v = -\frac{1}{s}e^{-st}$. Then,

$$\begin{aligned} F(s) &= -\frac{t^n e^{-st}}{s}\bigg|_0^\infty + \int_0^\infty \frac{nt^{n-1}}{s} e^{-st} dt \\ &= 0 + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt \\ &= \frac{n}{s}\mathcal{L}[t^{n-1}] \end{aligned}$$

with region of convergence $\Re(s) > 0$.

Laplace transform of powers of t (cont.)

This provides a recurrence relation:

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}]$$

Because the Laplace transform of t^0, or $u(t)$ (recall this is all causal) is $1/s$, we have that:

$$\boxed{\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}}$$

When $n \geq 1$, this signal does not have a Fourier transform.

Laplace transform of impulse

Let

$$f(t) = \delta(t)$$

Then,

$$\begin{aligned} F(s) &= \int_0^\infty \delta(t) e^{-st} dt \\ &= e^{-s \cdot 0} \\ &= 1 \end{aligned}$$

Thus,

$$\boxed{\mathcal{L}[\delta(t)] = 1}$$

Pattern for integration and differentiation?

Notice the following trends:

$$\begin{aligned} \delta(t) &\iff 1 \\ u(t) &\iff \frac{1}{s} \\ tu(t) &\iff \frac{1}{s^2} \\ \frac{1}{2}t^2u(t) &\iff \frac{1}{s^3} \\ \frac{1}{6}t^3u(t) &\iff \frac{1}{s^4} \\ &\vdots \end{aligned}$$

We see a clear pattern: differentiating a signal is equivalent to multiplying the Laplace transform by s while integrating is equivalent to multiplying the Laplace transform by $1/s$.

This is reminiscent of the Fourier transform derivative property (where the Fourier transform of the derivative of a signal is the Fourier transform of the signal multiplied by $j\omega$).

Laplace transform properties

As you might expect, the Laplace and Fourier transforms share many properties. For this reason, we won't go into detailed derivations unless they differ.

- **Linearity**. Given

$$f_1(t) \iff F_1(s) \qquad \& \qquad f_2(t) \iff F_2(s)$$

then

$$\boxed{\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)}$$

indicating that homogeneity and superposition hold.

- **Time scaling**. If $f(t) \iff F(s)$ and a is a positive scalar, then

$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)}$$

Note, we only consider $a > 0$ since if $a < 0$ and $f(t)$ is a causal signal, then $f(at)$ would be anticausal. The unilateral Laplace transform is for causal signals only.

Laplace transform properties (cont.)

- **Time shift**. Let $f(t) \iff F(s)$. If we delay a signal by T, i.e., $f(t-T)$, then we proceed with the understanding that:
  - $T > 0$, since if $T < 0$ the signal would be noncausal.
  - For delays $T > 0$, the signal $f(t-T)$ is zero in the interval from 0 to T.

  Then,

$$\begin{aligned}
\mathcal{L}[f(t-T)] &= \mathcal{L}[f(t-T)u(t-T)] \\
&= \int_0^\infty f(t-T)u(t-T)e^{-st}dt \\
&= \int_T^\infty f(t-T)e^{-st}dt \\
&= \int_0^\infty f(\tau)e^{-s(\tau+T)}d\tau \\
&= e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}d\tau \\
&= e^{-sT}F(s)
\end{aligned}$$

  Hence,

$$\boxed{\mathcal{L}[f(t-T)] = e^{-sT}F(s)}$$

Laplace transform properties (cont.)

- **Frequency shift**. If $f(t) \iff F(s)$, then for some complex frequency $s_0 = \sigma_0 + j\omega_0$, we have that

$$\boxed{\mathcal{L}\left[f(t)e^{s_0 t}\right] = F(s - s_0)}$$

- **Convolution**. If

$$f_1(t) \iff F_1(s) \qquad \& \qquad f_2(t) \iff F_2(s)$$

  then

$$\boxed{\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)}$$

  Hence, just like the Fourier transform, convolution in the time domain is multiplication in the frequency domain.

Laplace transform properties (cont.)

- **Integration**. Using the convolution property:

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] &= \mathcal{L}[f(t) * u(t)] \\ &= \mathcal{L}[f(t)]\mathcal{L}[u(t)] \\ &= \frac{1}{s}F(s)\end{aligned}$$

Hence,

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)}$$

Laplace transform properties (cont.)

- **Derivative**. Let $f'(t)$ be the derivative of $f(t)$ w.r.t. time. Then,

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}dt$$

Integrating by parts, we set $u = e^{-st}$ and $v' = f'(t)$. Then, $u' = -se^{-st}$ and $v = f(t)$. Hence,

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty f'(t)e^{-st}dt \\ &= f(t)e^{-st}\Big|_0^\infty + \int_0^\infty se^{-st}f(t)dt \\ &= -f(0) + sF(s) \end{aligned}$$

if $e^{-st} \to 0$ as $t \to \infty$. Hence,

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$$

Laplace transform properties (cont.)

The following table summarizes how differentiation and integration are related for the Laplace transform.

$g(t)$	$G(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

Laplace transform properties (cont.)

- **Multiplication by** t. If $f(t) \iff F(s)$, then we can differentiate both sides to see that:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t) dt \\ F'(s) &= \int_0^\infty (-t) e^{-st} f(t) dt \\ &= \mathcal{L}[-t f(t)] \end{aligned}$$

Hence,

$$\boxed{\mathcal{L}[t f(t)] = -F'(s)}$$

Example: rectangular pulse

Consider a rectangular pulse:

$$f(t) = \begin{cases} 1 & \text{if } a \le t < b \\ 0 & \text{otherwise} \end{cases}$$

What is its Laplace transform?

We can write $f(t) = u(t-a) - u(t-b)$. Therefore,

$$\begin{aligned} F(s) &= \mathcal{L}[u(t-a) - u(t-b)] \\ &= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} \\ &= \frac{e^{-as} - e^{-bs}}{s} \end{aligned}$$

Example: unit step and ramp functions

We were able to calculate the Fourier transform of the unit step; however, this required generalizing the Fourier transform. How can we find the Laplace transforms of the unit step and unit ramp function using the integral properties of the Laplace transform?

If $f(t) = \delta(t)$, then $F(s) = 1$. Then,

$$\mathcal{L}[u(t)] = \frac{1}{s}\mathcal{L}[\delta(t)] = \frac{1}{s} \cdot 1 = \frac{1}{s}$$

Similarly,

$$\mathcal{L}[r(t)] = \frac{1}{s}\mathcal{L}[u(t)] = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

Example: exponential

Let $f(t) = e^t$. What is its Laplace transform?

We could do this straightforwardly by plugging into the Laplace transform integral, i.e.,

$$\begin{aligned}\mathcal{L}[e^t] &= \int_0^\infty e^t e^{-st} dt \\ &= \int_0^\infty e^{t(1-s)} dt \\ &= \frac{1}{1-s}\ e^{t(1-s)}\Big|_0^\infty \\ &= \frac{1}{s-1}\end{aligned}$$

for $\Re(s) > 1$.

Example: exponential (cont.)

Another way to determine the Laplace transform of $f(t) = e^t$ is to recognize that $f(t) = f'(t)$. Then,

$$\begin{aligned} \mathcal{L}[f'(t)] &= s\mathcal{L}[f(t)] - f(0) \\ &= s\mathcal{L}[f'(t)] - e^0 \end{aligned}$$

Solving for $\mathcal{L}[f'(t)] = \mathcal{L}[e^t]$, we have that

$$\begin{aligned} \mathcal{L}[e^t] &= \frac{-e^0}{1-s} \\ &= \frac{1}{s-1} \end{aligned}$$

agreeing with the prior result.

Example: Laplace transform of sine

We know that

$$\cos(\omega t) \iff \frac{s}{s^2+\omega^2}$$

and since

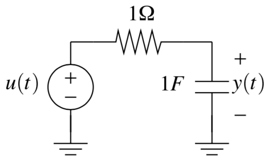
$$\sin(\omega t) = -\frac{1}{\omega}\frac{d}{dt}\cos(\omega t)$$

we have that

$$\begin{aligned}
\mathcal{L}[\sin(\omega t)] &= \mathcal{L}\left[-\frac{1}{\omega}\frac{d}{dt}\cos(\omega t)\right] \\
&= -\frac{1}{\omega}\left(s\mathcal{L}[\cos(\omega t)] - \cos(0)\right) \\
&= -\frac{1}{\omega}\left(s\frac{s}{s^2+\omega^2} - 1\right) \\
&= -\frac{1}{\omega}\left(\frac{s^2-(s^2+\omega^2)}{s^2+\omega^2}\right) \\
&= \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

Solving differential equations

Consider the RC circuit below.



In frequency response, we talked about how this is a low pass filter.

Because of the derivative and integral rules for the Laplace transform, differential equations are convenient to solve via the Laplace transform. We derived that the equation for this circuit is (plugging in $R = 1\Omega$ and $C = 1F$) that:

$$y'(t) + y(t) = u(t)$$

Taking the Laplace transform of both sides, we get that:

$$sY(s) + Y(s) = 1/s$$

where we also used the fact that $y(0) = 0$.

Solving differential equations (cont.)

Now, we can solve for $Y(s)$, i.e.,

$$\begin{aligned} Y(s) &= \frac{1/s}{s+1} \\ &= \frac{1}{s(s+1)} \end{aligned}$$

To invert $Y(s)$, we write it in a form where we can invert the Laplace transforms by inspection. We've seen this technique (partial fractions) once before, and we'll expand upon it more in the next lecture notes. For now, check that

$$Y(s) = \frac{1}{s} - \frac{1}{s+1}$$

Therefore,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] \\ &= 1 - e^{-t} \end{aligned}$$

Key take home point

With the Laplace Transform, *differential equations are turned into algebraic equations.*