ECE102, Fall 2020 **Homework #1**

Signals & Systems Prof. J.C. Kao

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Due Friday, 16 Oct 2020, by 11:59pm to Gradescope.
Covers material up to Lecture 3.
100 points total.

1. (10 points) **Even and odd parts.**

   (a) (3 points) Show that the product of two odd signals is even.
   Solution: Let $x_1(t)$ and $x_2(t)$ be two odd signals. Then their product $y(t)$, given by

   $$\begin{aligned} y(t) &= x_1(t)x_2(t) \\ &= x_1(-t)x_2(-t) \\ &= y(-t) \end{aligned}$$

   is even.

   (b) (3 points) Show that the product of an even signal and an odd signal is odd.
   Solution: Let $x_1(t)$ and $x_2(t)$ be an even and odd signal, respectively. Then their product $y(t)$, given by

   $$\begin{aligned} y(t) &= x_1(t)x_2(t) \\ &= -x_1(-t)x_2(-t) \\ &= -y(-t) \end{aligned}$$

   is odd.

   (c) (4 points) Use the properties derived in the previous parts to find the even and odd component of:

   $$x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$$

   Solution: Using the properties derived in previous parts we have the following classification:

   - $t\cos(t)$ is **odd** because t is an odd signal and $\cos(t)$ is an even signal (Property in (b))
   - $t^2\sin(t)$ is **odd** because t^2 is an even signal and $\sin(t)$ is an odd signal (Property in (b))
   - $\sin(t)\cos(t)$ is **odd** because $\cos(t)$ is an even signal and $\sin(t)$ is an odd signal (Property in (b))
   - $t^3\sin(t)\cos(t)$ is **even** because $\sin(t)\cos(t)$ is odd and t^3 is an odd signal (Property in (a))

Since adding two odd signals results in an odd signal, so the odd component of $x(t)$ is

$$x_o(t) = t\cos(t) + t^2\sin(t)$$

Hence, the even component of $x(t)$ is

$$x_e(t) = 1 + t^3\sin(t)\cos(t)$$

2. (15 points) **Time scaling and shifting.**

   (a) (10 points) For $x(t)$ indicated in the figure below, sketch the following:

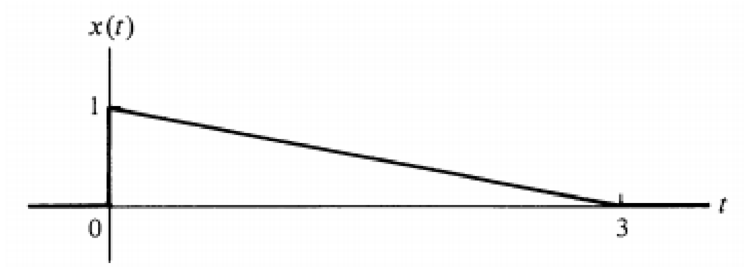


Figure 1: $x(t)$

i. $x(2t+2)$
Solution: We can obtain $x(2t+2)$ from $x(t)$ by first shifting $x(t)$ by 2 units to the left and then compressing $x(t+2)$ by a factor of 2.

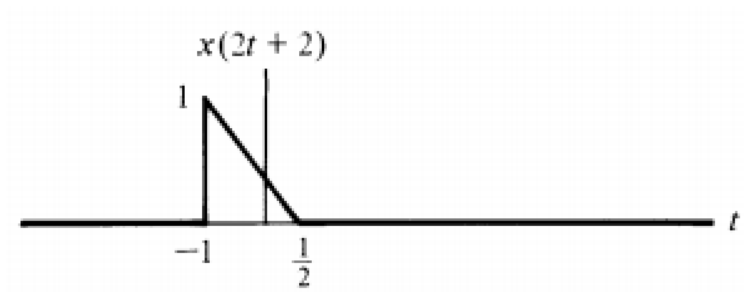


Figure 2: $x(2t+2)$

ii. $x(1-3t)$
Solution: We can obtain $x(1-3t)$ from $x(t)$ by first reflecting $x(t)$ on the vertical axis, then compressing $x(-t)$ by a factor of 3 and finally shifting $x(-3t)$ by $\frac{1}{3}$ units to the right.

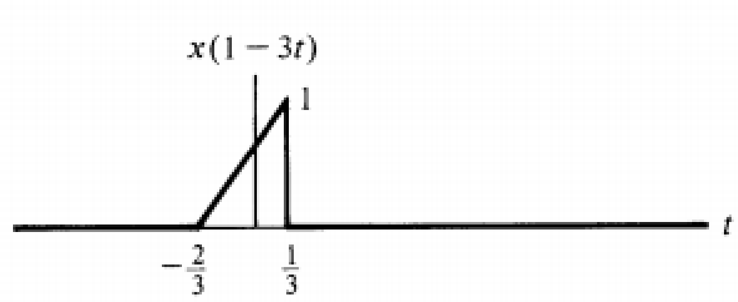
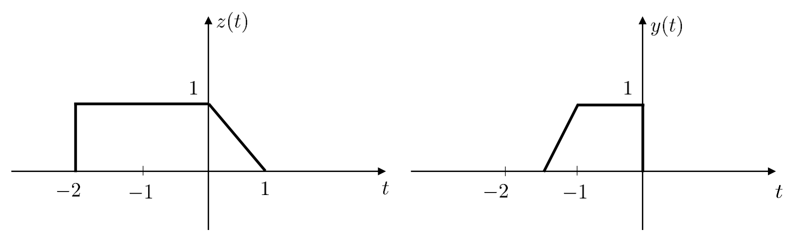


Figure 3: $x(1-3t)$

(b) (5 points) The figure below shows two signals: $z(t)$ and $y(t)$. Please express $y(t)$ in terms of $z(t)$?



Solution: We can obtain $y(t)$ from $z(t)$ by first reflecting $z(t)$ on the vertical axis, then shifting $z(-t)$ by 2 units to the left and finally compressing $z(-t-2)$ by a factor of 2. Therefore,

$$y(t) = z(-2t-2)$$

3. (22 points) **Periodic signals.**

(a) (12 points) For each of the following signals, determine whether it is periodic or not. If the signal is periodic, determine the fundamental period and frequency.

i. $x(t) = \cos(\sqrt{2}\pi t)$

ii. $x(t) = \sin^2(3\pi t + 3)$

iii. $x(t) = e^{-t}\cos(\sqrt{2}\pi t)$

iv.

$$x(t) = \begin{cases} \cos(t) & t < 0 \\ \sin(t) & t \geq 0 \end{cases} \tag{1}$$

Solution:

i. This signal is periodic. The fundamental frequency is $2\pi f = \sqrt{2}\pi \implies f = \frac{1}{\sqrt{2}}$.

ii. $\sin^2(3\pi t + 3) = \frac{1}{2} - \frac{1}{2}\cos(6\pi t + 6)$. For periodicity we don't need to worry about constants. So we can ignore the phase and the constant. We just need to find the period for $\cos(6\pi t)$. The fundamental frequency is $2\pi f = 6\pi \implies f = 3$.

iii. This signal is not periodic. e^{-t} continuously decreases in magnitude. So even though $\cos(\sqrt{2}\pi t)$ is periodic, $e^{-t}\cos(\sqrt{2}\pi t)$ is not periodic.

iv. For $t > 0$, the signal is periodic because the signal is just $\cos(t)$. For $t < 0$, also the signal is periodic because it is just $\sin(t)$. However the entire signal is not periodic because at $t = 0$, the signal is discontinuous. The signal does not repeat at $t = 0$.

(b) (5 points) Assume that the signal $x(t)$ is periodic with period T_0, and that $x(t)$ is odd (i.e. $x(t) = -x(-t)$). What is the value of $x(T_0)$?

Solution: It is given that $x(t)$ is odd. This implies $x(t) = -x(-t)$. For $t = 0$, it means $x(0) = -x(0) \implies x(0) = 0$. Also given, signal is periodic so $x(t) = x(t + T_0) \implies x(0) = x(T_0) \implies x(T_0) = 0$.

(c) (5 points) If $x(t)$ is periodic, are the even and odd components of $x(t)$ also periodic?

Solution: Given that $x(t)$ is periodic, so $x(t) = x(t+T_0)$. The even component of $x(t)$ is $x_e(t) = \frac{x(t)+x(-t)}{2}$. $x_e(t+T_0) = \frac{x(t+T_0)+x(-t-T_0)}{2} \implies \frac{x(t)+x(-t)}{2} = x_e(t)$. So $x_e(t)$ is also periodic. We know $x_o(t) = \frac{x(t)-x(-t)}{2}$. So $x_o(t+T_0) = \frac{x(t+T_0)-x(-t-T_0)}{2} \implies \frac{x(t)-x(-t)}{2}$. So $x_o(t)$ is also periodic.

4. (21 points) **Energy and power signals.**

(a) (15 points) Determine whether the following signals are energy or power signals. If the signal is an energy signal, determine its energy. If the signal is a power signal, determine its power.

i. $x(t) = e^{-|t|}$

ii. $x(t) = \begin{cases} \frac{1}{\sqrt{t}}, & \text{if } t \geq 1 \\ 0, & \text{otherwise} \end{cases}$

iii. $x(t) = \begin{cases} 1 + e^{-t}, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Solutions:

i. $x(t) = e^{-|t|}$

The energy is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} \left|e^{-|t|}\right|^2 dt = \int_{-\infty}^{\infty} e^{-2|t|}dt = \int_{0}^{\infty} e^{-2t}dt + \int_{-\infty}^{0} e^{2t}dt$$
$$= 2\int_{0}^{\infty} e^{-2t}dt = -e^{-2t}\big|_{t=0}^{\infty} = 1$$

Therefore it's a energy signal. Its power is then 0.

ii. $x(t) = \begin{cases} \frac{1}{\sqrt{t}}, & \text{if } t \geq 1 \\ 0, & \text{otherwise} \end{cases}$

The energy of the signal is:

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \lim_{T\to\infty} \int_{-T}^{1} 0dt + \int_{1}^{T} \frac{1}{t}dt = \lim_{T\to\infty} ln(T) - ln(1) = \infty$$

The energy of this signal is infinite. Let's calculate the power:

$$E = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{1} 0dt + \frac{1}{2T} \int_{1}^{T} \frac{1}{t} dt = \lim_{T\to\infty} \frac{ln(T)}{2T} = 0$$

The power of the signal is zero. Therefore this is neither a energy signal nor a power signal.

iii. $x(t) = \begin{cases} 1 + e^{-t}, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

The energy of the signal is:

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T\to\infty} \int_{0}^{T} (1 + e^{-t})^2 dt = \lim_{T\to\infty} \int_{0}^{T} 1 + e^{-2t} + 2e^{-t} dt = \infty$$

The energy of the signal is thus infinite. Let's find the power of the signal.

$$E = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} (1 + e^{-t})^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} 1 + e^{-2t} + 2e^{-t} dt$$
$$= \frac{1}{2} + 0 + 0$$

The power of the signal is $\frac{1}{2}$. Therefore it is a power signal.

(b) (6 points) Show the following two properties:

- If $x(t)$ is an even signal and $y(t)$ is an odd signal, then $x(t)y(t)$ is an odd signal;
- If $z(t)$ is an odd signal, then for any $\tau > 0$ we have:

$$\int_{-\tau}^{\tau} z(t) dt = 0$$

Use these two properties to show that the energy of $x(t)$ is the sum of the energy of its even component $x_e(t)$ and the energy of its odd component $x_o(t)$, i.e.,

$$E_x = E_{x_e} + E_{x_o}$$

Assume $x(t)$ is a real signal.

Solutions:

First property: $x(-t)y(-t) = x(t)(-y(t)) = -x(t)y(t)$, therefore it's odd.

Second property:

$$\int_{-\tau}^{\tau} z(t) dt = \int_{-\tau}^{0} z(t) dt + \int_{0}^{\tau} z(t) dt$$

We apply to the first integral the following variable change: $t = -\lambda$.

$$\int_{-\tau}^{\tau} z(t) dt = -\int_{\tau}^{0} z(-\lambda) d\lambda + \int_{0}^{\tau} z(t) dt$$

We then change the order of the limits of the first integral:

$$\int_{-\tau}^{\tau} z(t)dt = \int_{0}^{\tau} z(-\lambda)d\lambda + \int_{0}^{\tau} z(t)dt$$

Since $z(t)$ is an odd signal, we then have $z(-\lambda) = -z(\lambda)$. Thus,

$$\int_{-\tau}^{\tau} z(t)dt = -\int_{0}^{\tau} z(\lambda)d\lambda + \int_{0}^{\tau} z(t)dt = 0$$

The energy of signal $x(t)$ is given by:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x_e(t) + x_o(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} \left(x_e^2(t) + x_o^2(t) + 2x_e(t)x_o(t)\right) dt \\ &= \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt = E_e + E_o \end{aligned}$$

This is because $2x_e(t)x_o(t)$ is odd, therefore its integral is zero (according to the second property).

5. (16 points) **Euler's identity and complex numbers.**

   (a) (8 points) Use Euler's formula to prove the following identities:

      i. $\cos^2(\theta) + \sin^2(\theta) = 1$

      ii. $\cos(\theta + \psi) = \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)$

   (b) (8 points) $x(t) = (5 + \sqrt{2}j)e^{j(t+2)}$ and $y(t) = 1/(2 - j)$.

      i. Compute the real and imaginary parts of $x(t)$ and $y(t)$.

      ii. Compute the magnitude and phase of $x(t)$ and $y(t)$.

Solutions:

(a) i. $e^{j\theta} = cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = cos(\theta) - j\sin(\theta)$.
Thus, $(cos(\theta) + j\sin(\theta))(cos(\theta) - j\sin(\theta)) = \cos^2(\theta) + \sin^2(\theta) = e^{j\theta} \times e^{-j\theta} = 1$

ii. $\cos(\theta) = (e^{j\theta} + e^{-j\theta})/2$
$\sin(\theta) = (e^{j\theta} - e^{-j\theta})/2j$
$\cos(\theta) \times \cos(\psi) = (e^{j(\theta+\psi)} + e^{-j(\theta+\psi)} + e^{j(\theta-\psi)} + e^{j(\psi-\theta)})/4$
$\sin(\theta) \times \sin(\psi) = (-e^{j(\theta+\psi)} - e^{-j(\theta+\psi)} + e^{j(\theta-\psi)} + e^{j(\psi-\theta)})/(-4)$
$Thus, \cos(\theta) \times \cos(\psi) + \sin(\theta) \times \sin(\psi) = (e^{j(\theta+\psi)} + e^{-j(\theta+\psi)})/2 = \cos(\theta + \psi)$

(b) i. $x(t) = (5 + \sqrt{2}j)e^{j(t+2)} = (5 + \sqrt{2}j)(\cos(t + 2) + j\sin(t + 2)) = 5\cos(t + 2) - \sqrt{2}\sin(t + 2) + j(\sqrt{2}\cos(t + 2) + 5\sin(t + 2))$.
Therefore, the real part is: $5\cos(t + 2) - \sqrt{2}\sin(t + 2)$. The imaginary part is:$\sqrt{2}\cos(t + 2) + 5\sin(t + 2)$

$y(t) = 1/(2 - j) = \frac{2+j}{(2-j)(2+j)} = 2/5 + 1/5j$,

ii. magnitude of $x(t)$ is $\sqrt{5^2 + (\sqrt{2})^2} = 3\sqrt{3}$. phase of $x(t)$ is $\arctan(\sqrt{2}/5) + (t+2)$.

magnitude of $y(t)$ is $\sqrt{\frac{2}{5}^2 + \frac{1}{5}^2} = \frac{\sqrt{5}}{5}$. phase of $y(t)$ is $\arctan(\frac{1}{2})$.

6. (16 points) **MATLAB tasks**
For this question, please include all relevant code in text format. For plots, please include axis labels and preferably include a grid.

(a) (5 points) **Task 1**
Plot the waveform

$$x(t) = e^{-t}\cos(2\pi t)$$

for $-10 \leq t \leq 10$, with a step size of 0.2.

Solutions:
The code is:

```
t=-10:0.2:10;
x=exp(-t).*cos(2*pi*t);
plot(t,x);
grid on;
title('Plot of x(t)=e^{-t}cos(2\pit)'); xlabel('t(sec)');ylabel('x(t)');
```

The code generates the plot shown in Fig. 1.

(b) (5 points) **Task 2**

Create a function `relu(t)` that implements the function from Question 1. You will need to create a file called "relu.m" containing:

```
function out = relu(t)
out = 0; %replace this line with the appropriate implementation of the
%relu function.
end
```

Then plot the function for $-5 \leq t \leq 5$, with a step size of 0.1.

Solutions:

In file relu.m:

```
function out = relu(t)
out = max(0, t);
end
```

Then run:

```
t = -5:0.1:5;
plot(t, relu(t));
xlabel('t');
ylabel('relu(t)')
grid;
```

The code generates the plot shown in Fig. 2.

(c) (6 points) **Task 3**

Create functions `even(t, f)` and `odd(t, f)` that take inputs time `t` and function (handle) `f` that compute the respective even and odd parts of `f(t)` at points t.

For example, the square of a function could be implemented in a file square.m as:

```
function out = square(t, f)
out = f(t).^2;
end
```

and run as:

```
t = -10:0.5:10;
y = square(t, @relu);
```

where `@relu` is called a function handle of the function `relu`, and is necessary for passing a function as input to another function.

Running `plot(t, y); grid;` yields the result:

For this question, plot the even and odd components of $relu(t)$ for $-5 \leq t \leq 5$, with a step size of 0.1 using the functions `even(t, f)` and `odd(t, f)`. Feel free to also define and play around with arbitrary functions to look at their even and odd components.

Solutions:
In file even.m:
```
function out = even(t, f)
out = 0.5*f(t) + 0.5*f(-t);
end
```

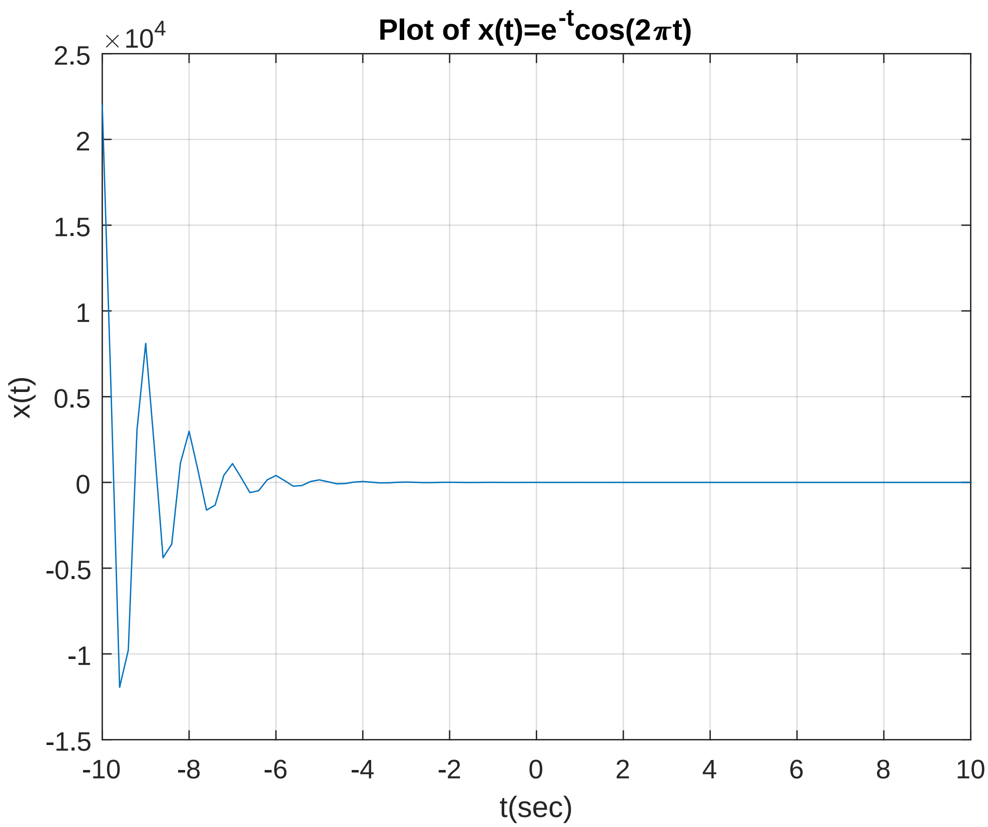
In file odd.m:
```
function out = odd(t, f)
out = 0.5*f(t) - 0.5*f(-t);
end
```

Command line code:
```
t = -5:0.1:5;
figure;
plot(t, even(t, @relu));
xlabel('t');
ylabel('even part of relu(t)');

t = -5:0.1:5;
figure;
plot(t, odd(t, @relu));
xlabel('t');
ylabel('odd part of relu(t)');
```

This code generates the plots shown in Fig. 3 and 4.

(Note: you can also have a function even_func:
```
function outfunc = even_func(f)
outfunc = @(t)0.5*f(t) + 0.5*f(-t);
end
```
which you can run like so:
```
ef = even_func(@relu)
plot(t, ef(t))
```
which returns the even component of f(t) and you can similarly construct an odd function which return the odd component of f(t)).


Plot of x(t)=e^-t cos(2πt)
×10^4
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
x(t)
-10
-8
-6
-4
-2
0
2
4
6
8
10
t(sec)

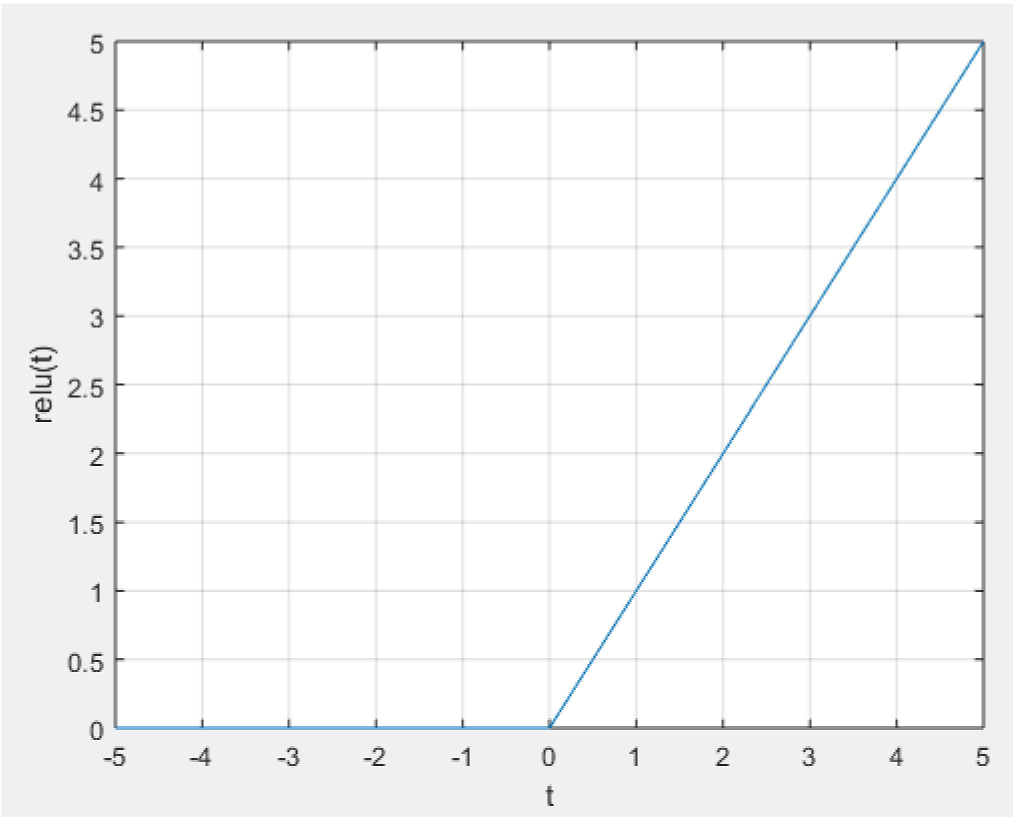
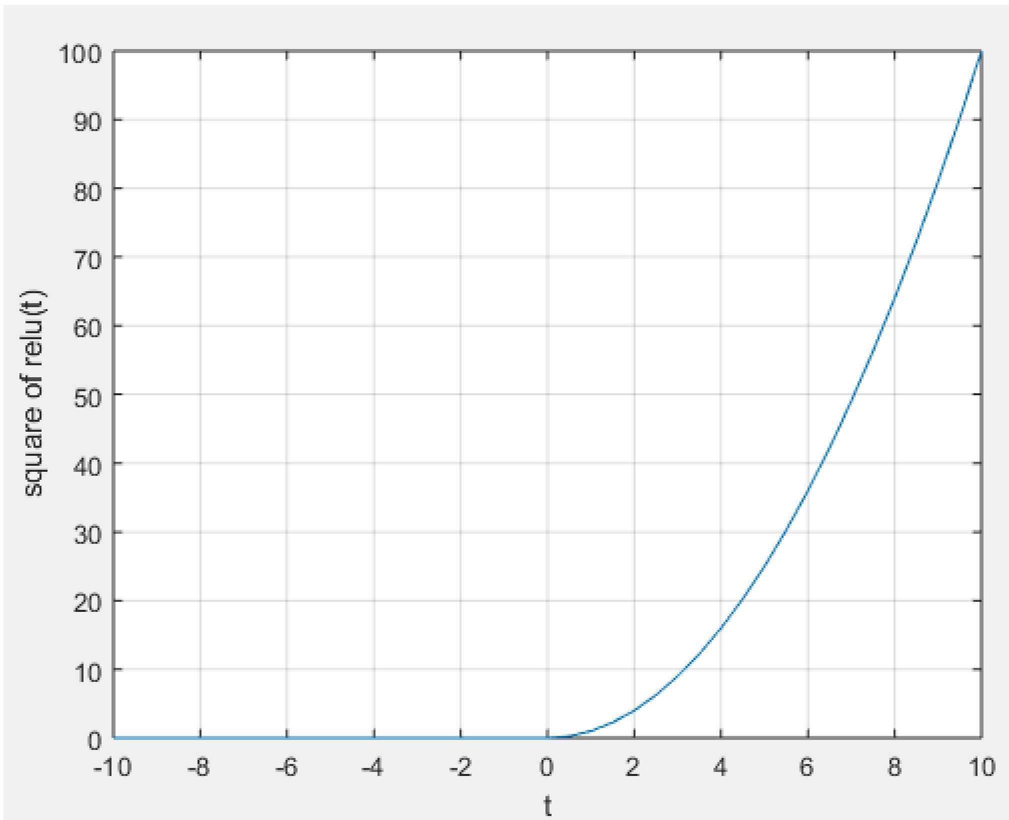


Figure 5: Task 2



square of relu(t)
t
0
10
20
30
40
50
60
70
80
90
100
-10
-8
-6
-4
-2
0
2
4
6
8
10

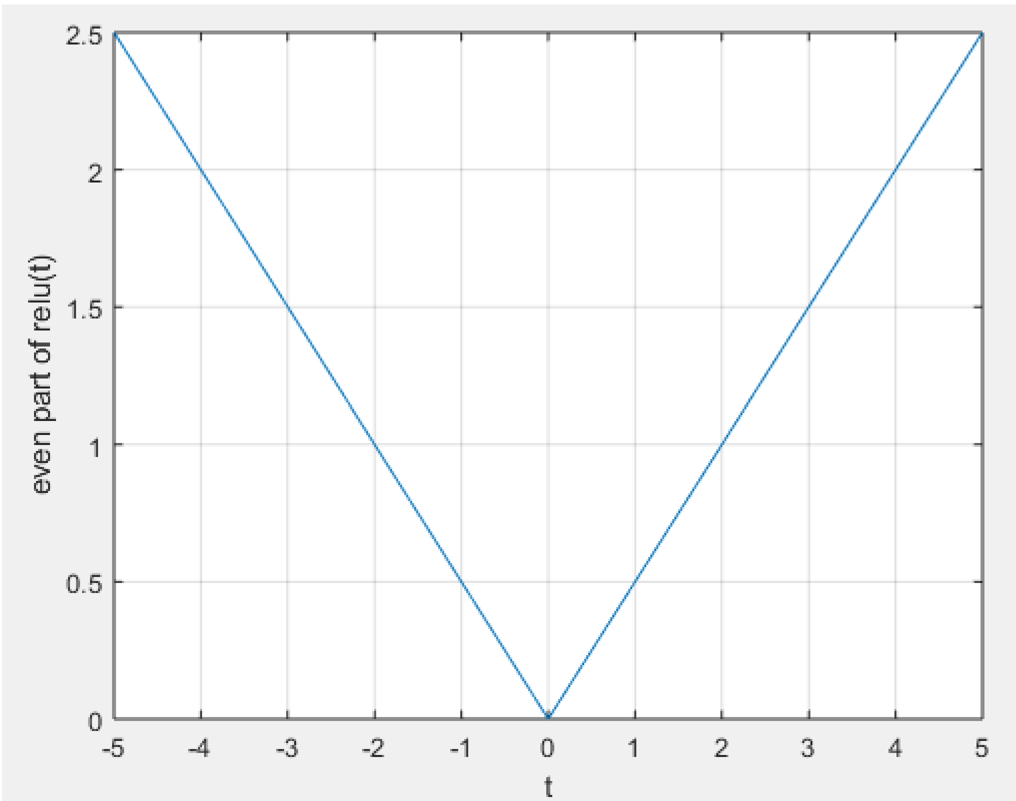


Figure 6: Even component of relu(t)

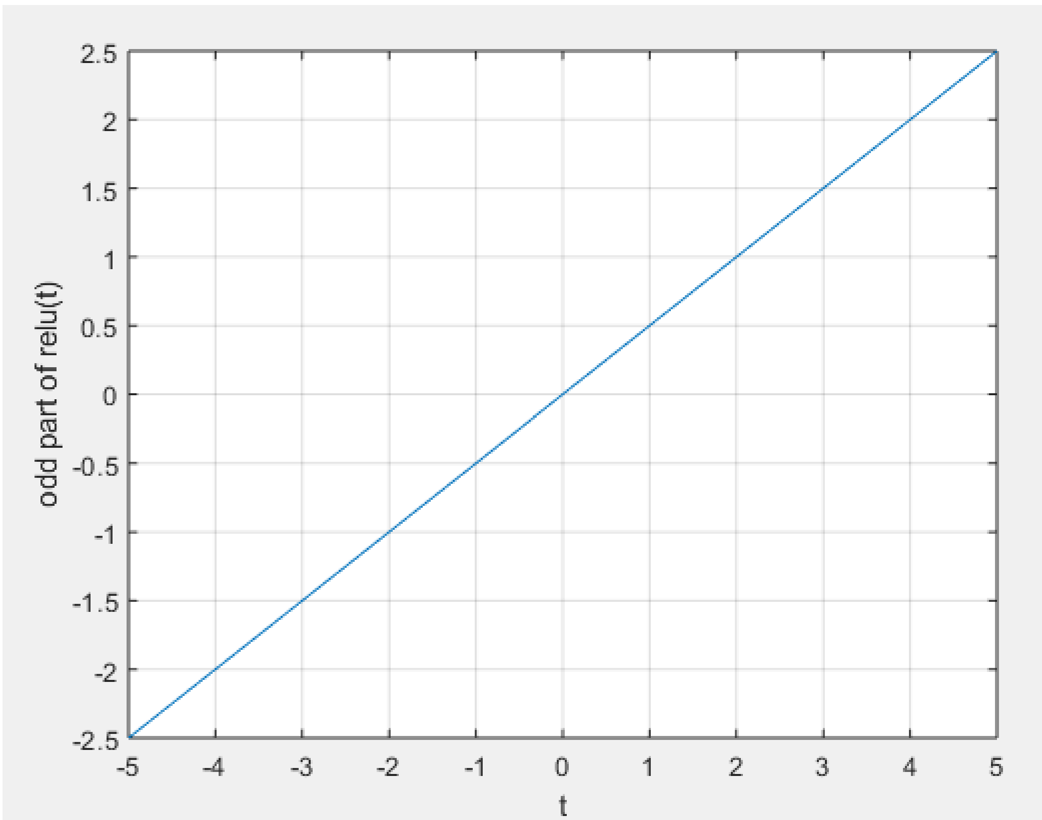


Figure 7: Odd component of relu(t)