ECE102, Fall 2020 **Homework #5**

Signals & Systems Prof. J.C. Kao

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Due Friday, 20 Nov 2020, by 11:59pm to Gradescope.
Covers material up to Lecture 10.
100 points total.
This homework covers questions relate to Fourier series and Fourier transform.

1. (18 points) **Fourier Series**

   (a) (7 points) When the periodic signal $f(t)$ is real, you have seen in class some properties of symmetry for the Fourier series coefficients of $f(t)$ (handout 8, slide 41). How do these properties of symmetry change when $f(t)$ is pure imaginary?

   Solution: Since $f(t)$ is pure imaginary, it can equivalently written as $f(t) = jg(t)$, where $g(t)$ is real. Using the equations in slide 8-40,

$$\begin{aligned}
c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right] dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} jg(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right] dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} jg(t)\cos\left(\frac{2\pi k}{T_0}t\right) + g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt + j\frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\cos\left(\frac{2\pi k}{T_0}t\right) dt
\end{aligned}$$

   Now, because $g(t)$ is real:

$$\begin{aligned}
Re(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt \\
Im(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\cos\left(\frac{2\pi k}{T_0}t\right) dt
\end{aligned}$$

   Therefore,

$$\begin{aligned}
Re(c_k) &= -Re(c_{-k}) \\
Im(c_k) &= Im(c_{-k}) \\
c_k^* &= -c_{-k} \\
|c_k| &= |c_{-k}| \\
\angle c_k &= -\angle c_k^* \\
\angle c_k &= -\angle c_{-k} \pm \pi
\end{aligned}$$

   (b) Suppose we are given the following information about a signal $x(t)$:

- $x(t)$ is real and odd.
- $x(t)$ is periodic with period $T = 2$ and has Fourier coefficients a_k.
- $a_k = 0$ for $|k| > 1$.
- $\frac{1}{2}\int_0^2 |x(t)|^2 dt = 1$

Solution: From the problem description we know that signal $x(t)$ is real and odd. Using the properties of the Fourier series we know that Fourier coefficients are purely imaginary and odd: $a_k = -a_{-k}$ and $a_0 = 0$.
For real signal $x(t)$ it becomes:

$$|a_{-k}| = |a_k|$$

Furthermore, from problem description, we know that $a_k = 0$ for $|k| > 1$, so $a_{-1} \neq 0, a_0 \neq 0, a_1 \neq 0$.
Then we can use Parseval's relation for continuous-time periodic signal.

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = 1$$

Furthermore we can write,

$$|a_{-1}|^2 + |a_0|^2 + |a_1|^2 = 1$$

$$|a_{-1}|^2 + |a_1|^2 = 1$$

Using the fact that the signal is real, we can say that

$$|a_{-1}| = |a_1| \implies |a_1|^2 + |a_1|^2 = 2|a_1|^2 = 1$$

Then there are two sets of possible values for a_1, a_{-1}.

$$-a_{-1} = a_1 = \frac{1}{j\sqrt{2}} \implies a_{-1} = -\frac{1}{j\sqrt{2}}$$

or

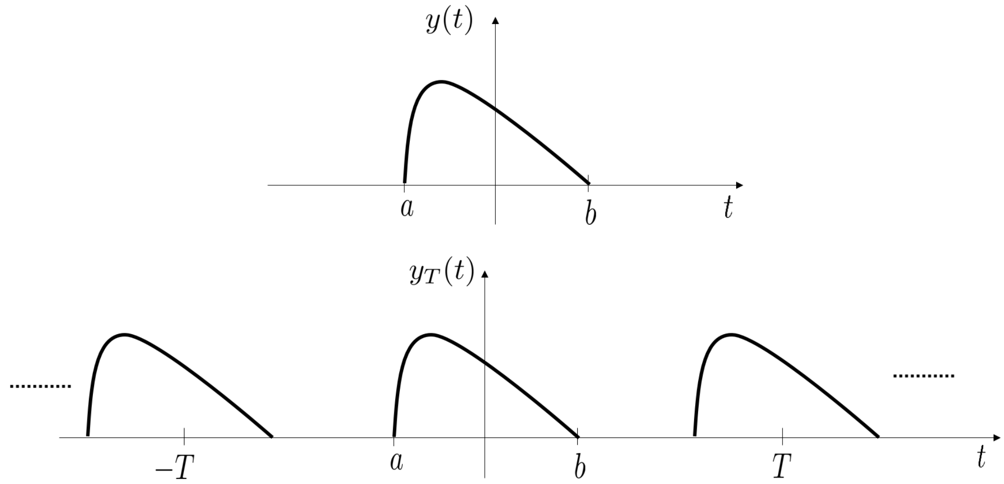
$$-a_{-1} = a_1 = -\frac{1}{j\sqrt{2}} \implies a_{-1} = \frac{1}{j\sqrt{2}}$$

So the first solution is $x(t) = \sqrt{2}sin(\pi t)$ and the second solution is $x(t) = -\sqrt{2}sin(\pi t)$.

(c) (4 points) Consider the signal $y(t)$ shown below and let $Y(j\omega)$ denote its Fourier transform.

Let $Y_T(t)$ denote its periodic extension:

How can the Fourier series coefficients of $y_T(t)$ be obtained from the Fourier transform $Y(j\omega)$ of $y(t)$? (Note that the figures given in this problem are for illustrative purposes,



the question is for any arbitrary $y(t)$).

Solution:
The Fourier transform of $y(t)$ is given by:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{a}^{b} y(t)e^{-j\omega t}dt \tag{1}$$

The coefficients of the Fourier series for $y_T(t)$ are given by:

$$Y_k = \frac{1}{T}\int_{a}^{b} y(t)e^{-j(2\pi k/T)t}\,dt \tag{2}$$

for any integer k. Therefore, by comparing (2) to (1), we conclude:

$$Y_k = \frac{1}{T}Y(j\omega)\big|_{\omega=\frac{2\pi k}{T}}$$

2. (32 points) **Symmetry properties of Fourier transform**

   (a) (16 points) Determine which of the signals, whose Fourier transforms are depicted in Fig. 1, satisfy each of the following:

   i. $x(t)$ is even

   Solution: If $x(t)$ is even, then its Fourier transform should be even. Since $X(j\omega)$ in (a), (d) and (e) are even, signals (a), (d) and (e) are all even in the time domain.

   ii. $x(t)$ is odd

   Solution: If $x(t)$ is odd, then its Fourier transform should be odd. Since $X(j\omega)$ in (f) is odd, signal in (f) is odd in the time domain.

iii. $x(t)$ is real

Solution: If $x(t)$ is real, then $X(j\omega)$ is Hermitian, i.e., $X(-j\omega) = X^*(j\omega)$. This means the real part of $X(j\omega)$ is even and the imaginary part of $X(j\omega)$ is odd. It also means that the magnitude of $X(j\omega)$ is even and the phase of $X(j\omega)$ is odd. Since $X(j\omega)$ in (c) and (e) are both Hermitian, signals (c) and (e) are real in the time domain.

iv. $x(t)$ is complex (neither real, nor pure imaginary)

Solution: For $x(t)$ to be complex (not real neither pure imaginary), $X(j\omega)$ should not be Hermitian or anti-Hermitian. We know from the previous part that $X(j\omega)$ in (c) and (e) are Hermitian. Signals in (d) and (f) are anti-Hermitian. Therefore, signals in (a) and (b) are both complex in the time domain.

v. $x(t)$ is real and even

Solution: If $x(t)$ is real and even, then $X(j\omega)$ is real and even. Therefore, it is (e).

vi. $x(t)$ is imaginary and odd

Solution: If $x(t)$ is imaginary and odd, then $X(j\omega)$ is real and odd. Therefore, it is (f).

vii. $x(t)$ is imaginary and even

Solution: If $x(t)$ is imaginary and even, then $X(j\omega)$ is imaginary and even. Therefore, it is (d).

viii. There exists a non-zero ω_0 such that $e^{j\omega_0 t}x(t)$ is real and even

Solution: If $e^{j\omega_0 t}x(t)$ is real and even, then $X(j(\omega - \omega_0))$ is real and even. None of the signals have this property. However:
Corrections:
In (b), if the signal was starting from 1, then it was symmetric and if we shift $X(j\omega)$ to the left by 2 (i.e., $\omega_0 = -2$), we obtain a real and even Fourier transform. (No points are lost if none or (b) was given as an answer)

(b) (8 points) Using the properties of Fourier transform, determine whether the assertions are true or false.

i. The convolution of a real and even signal and a real and odd signal is odd.

Solution: Let $f(t)$ be a real and even signal, and $g(t)$ be a real and odd signal. Then $F(j\omega)$ is real and even, and $G(j\omega)$ is imaginary and odd. The convolution $h(t) = (f * g)(t)$ has the Fourier transform

$$H(j\omega) = F(j\omega)G(j\omega)$$

If $F(j\omega)$ is real and even, and $G(j\omega)$ is imaginary and odd, then $H(j\omega)$ is imaginary and odd, and $h(t)$ is real and odd. The assertion is true.

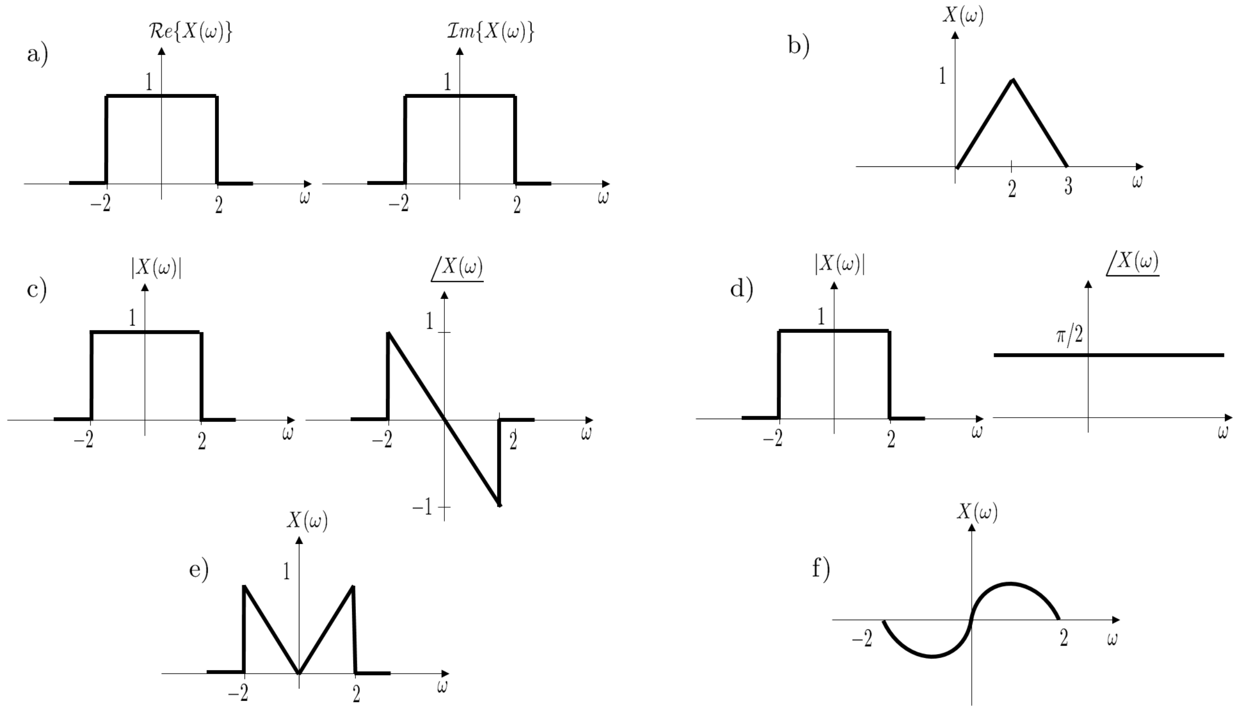


Figure 1: P2.a

ii. The convolution of a signal and the same signal reversed is an even signal.

Solution: Let $f(t)$ be a signal, and $f_R(t) = f(-t)$. Let $h(t) = (f * f_R)(t)$. Then

$$H(j\omega) = F(j\omega)F_R(j\omega) = F(j\omega)F(-j\omega)$$

which is even (replacing ω by $-\omega$ results in the same expression). The assertion is true.

(c) (8 points) Show the following statements:

i. If $x(t) = x^*(-t)$, then $X(j\omega)$ is real.

Solution:

$$\begin{aligned}
X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^* \\
&= \int_{-\infty}^{+\infty} \left[x(t)e^{-j\omega t}\right]^*\,dt \\
&= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt \\
&= \int_{-\infty}^{+\infty} x^*(-\tau)^{-j\omega\tau}\,d\tau, \text{ here we did the variable change } \tau = -t \\
&= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}\,d\tau, \text{ here we used the fact that } x(\tau) = x^*(-\tau) \\
&= X(j\omega),
\end{aligned}$$

Since $X^*(j\omega) = X(j\omega)$, we conclude that $X(j\omega)$ is real.

ii. If $x(t)$ is a real signal with $X(j\omega)$ its Fourier transform, then the Fourier transforms $X_e(j\omega)$ and $X_o(j\omega)$ of the even and odd components of $x(t)$ satisfy the following:

$$X_e(j\omega) = Re\{X(j\omega)\}$$

and

$$X_o(j\omega) = jIm\{X(j\omega)\}$$

Solution:

Since $x(t) = x_e(t) + x_o(t)$, the Fourier transform of $x(t)$ is given by:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \underbrace{\int_{-\infty}^{\infty} x_e(t)e^{-j\omega t}dt}_{X_e(j\omega)} + \underbrace{\int_{-\infty}^{\infty} x_o(t)e^{-j\omega t}dt}_{X_o(j\omega)}$$

Now using Euler,

$$X_e(j\omega) = \int_{-\infty}^{+\infty} x_e(t)(\cos(\omega t) - j\sin(\omega t))dt = \int_{-\infty}^{+\infty} x_e(t)\cos(\omega t)dt$$

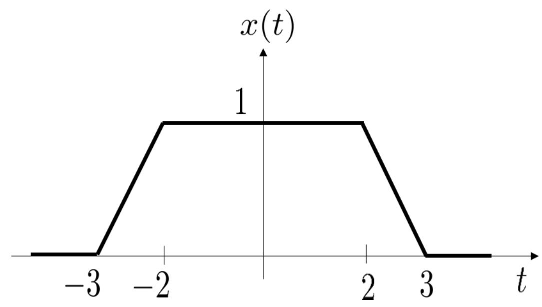
$$X_o(j\omega) = \int_{-\infty}^{+\infty} x_o(t)(\cos(\omega t) - j\sin(\omega t))dt = -j\int_{-\infty}^{+\infty} x_o(t)\sin(\omega t)dt$$

Since $x(t)$ is real, $x_e(t)$ and $x_o(t)$ are both real. Therefore, $X_e(j\omega)$ is real and $X_o(j\omega)$ is pure imaginary. Therefore,

$$\begin{aligned}
\mathcal{R}e\{X(j\omega)\} &= \int_{-\infty}^{+\infty} x_e(t)\cos(\omega t)dt = X_e(j\omega) \\
\mathcal{I}m\{X(j\omega)\} &= -\int_{-\infty}^{+\infty} x_o(t)\sin(\omega t)dt = -jX_o(j\omega)
\end{aligned}$$

3. (15 points) **Fourier transform properties**

(a) (10 points) Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ sketched below:



Evaluate the following quantities without explicitly finding $X(j\omega)$:

i. $\int_0^\infty X(j\omega)d\omega$

Solution:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\, d\omega$$
$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, d\omega = \frac{1}{\pi}\int_{0}^{+\infty} X(j\omega)\, d\omega$$

This is because $x(t)$ is real and even, then $X(j\omega)$ is real and even. This implies: $\int_{-\infty}^{+\infty} X(j\omega)\, d\omega = 2\int_0^{+\infty} X(j\omega)\, d\omega$. Therefore,

$$\int_0^{+\infty} X(j\omega)\, d\omega = \pi x(0) = \pi$$

ii. $X(j\omega)|_{\omega=0}$

Solution:
Since

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\, dt$$

we then have:

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\, dt|_{\omega=0} = \int_{-\infty}^{+\infty} x(t)\, dt = \frac{1}{2} + 4 + \frac{1}{2} = 5$$

iii. $\underline{/X(j\omega)}$

Solution: The Fourier transform of a real and even function is real and even. Therefore the phase of $X(j\omega)$ is either 0 or π. It is zero when $X(j\omega) \geq 0$ and it is π when $X(j\omega) < 0$.

iv. $\int_{-\infty}^{\infty} e^{-j\omega} X(j\omega) d\omega$

Solution:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\, d\omega \\ x(t)|_{t=-1} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\, d\omega|_{t=-1} \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{-j\omega}\, d\omega \end{aligned}$$

Therefore,

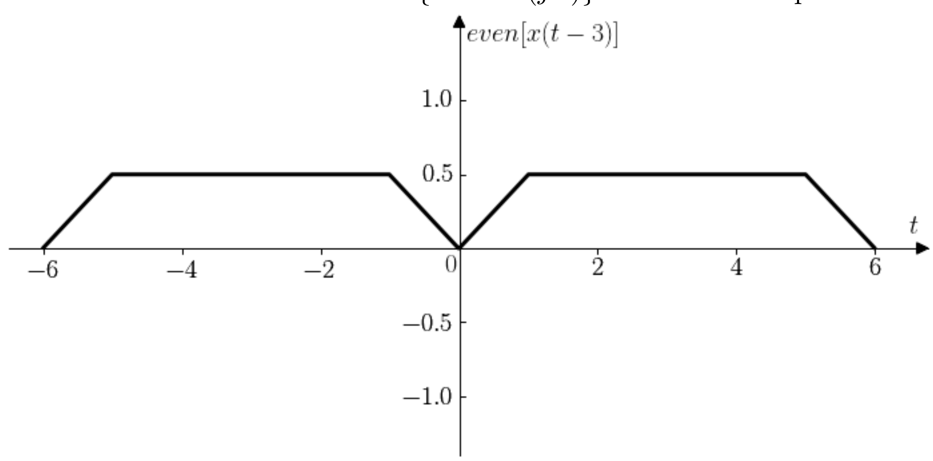
$$\int_{-\infty}^{+\infty} X(j\omega)e^{-j\omega}\, d\omega = 2\pi x(-1) = 2\pi$$

v. Plot the inverse Fourier transform of $\mathcal{R}e\{e^{-3j\omega}X(j\omega)\}$

Solution: Let $Y(j\omega) = e^{-3j\omega}X(j\omega)$, then $y(t) = x(t-3)$. Since $y(t)$ is real,

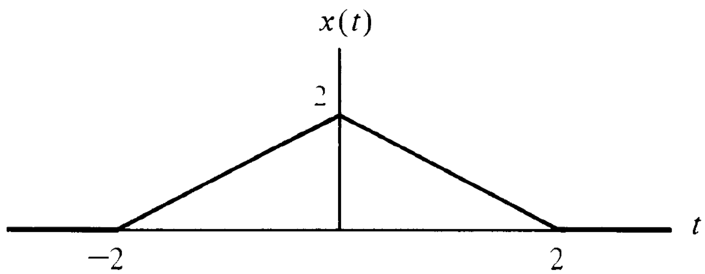
$$\mathcal{R}e\{e^{-3j\omega}X(j\omega)\} = \mathcal{R}e\{Y(j\omega)\} = Y_e(j\omega)$$

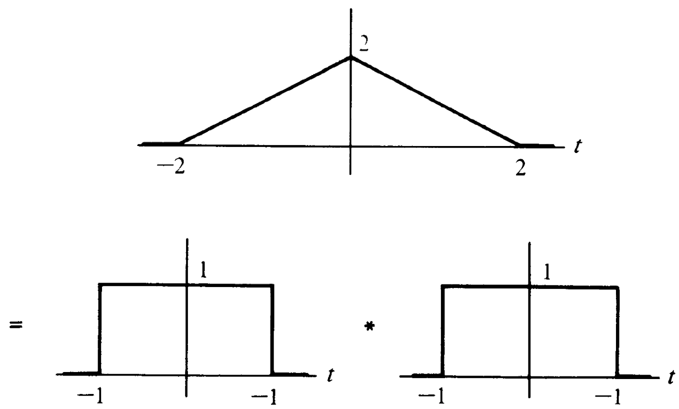
where $Y_e(j\omega)$ is the Fourier transform of the even component of $y(t)$. Therefore, the inverse Fourier transform of $\mathcal{R}e\{e^{-3j\omega}X(j\omega)\}$ is the even component of $x(t-3)$.



(b) (5 points) By first expressing the triangular signal $x(t)$ shown below as the convolution of a rectangular pulse with itself, determine the Fourier transform of $x(t)$.

Solution: A triangular signal can be represented as the convolution of two rectangular pulses, as indicated in figure below. Since each of the rectangular pulses has a Fourier transform given by $\frac{(2\sin\omega)}{\omega}$, the convolution property tells us that the triangular function will have a Fourier transform given by the product of the transforms of the rectangular





pulses:

$$X(jw) = \frac{(2\sin\omega)}{\omega} \times \frac{(2\sin\omega)}{\omega}$$
$$X(jw) = \frac{(4\sin^2\omega)}{\omega^2}$$

4. (35 points) **Fourier transform and its inverse**

   (a) (21 points) Find the Fourier transform of each of the signals given below:
   Hint: you may use Fourier Transforms derived in class.

      i. (**optional**) $x_1(t) = 2\text{rect}\left(\frac{-t-3}{2}\right)\cos(10\pi t)$

      Solution:
      We know that:

$$\text{rect}(t) \longleftrightarrow \text{sinc}\left(\frac{\omega}{2\pi}\right)$$

      Therefore,

$$\text{rect}\left(\frac{-t}{2}\right) \longleftrightarrow 2\text{sinc}\left(-2\frac{\omega}{2\pi}\right) = 2\text{sinc}\left(\frac{\omega}{\pi}\right), \text{sinc is an even function}$$

$$2\text{rect}\left(\frac{-1}{2}(t+3)\right) \longleftrightarrow 4\text{sinc}\left(\frac{\omega}{\pi}\right)e^{j3\omega}$$

$$\text{rect}\left(\frac{-1}{2}(t+3)\right)\cos(10\pi t) \longleftrightarrow 2\text{sinc}\left(\frac{\omega}{\pi}-10\right)e^{j3(\omega-10\pi)}+2\text{sinc}\left(\frac{\omega}{\pi}+10\right)e^{j3(\omega+10\pi)}$$

ii. $x_2(t) = e^{(2+3j)t}u(-t+1)$

Solution:
We can write $x_2(t)$ as follows:

$$x_2(t) = e^{j3t}e^{2t}u(-t+1) = e^{j3t}e^{2}e^{2(t-1)}u(-(t-1))$$

We know:

$$\begin{aligned}
e^{-2t}u(t) &\longleftrightarrow \frac{1}{2+j\omega} \\
e^{2t}u(-t) &\longleftrightarrow \frac{1}{2-j\omega} \\
e^{2(t-1)}u(-(t-1)) &\longleftrightarrow \frac{e^{-j\omega}}{2-j\omega} \\
e^{j3t}e^{2(t-1)}u(-(t-1)) &\longleftrightarrow \frac{e^{-j(\omega-3)}}{2-j(\omega-3)}
\end{aligned}$$

Therefore,

$$X_2(j\omega) = e^{2}\frac{e^{-j(\omega-3)}}{2-j(\omega-3)}$$

iii. $x_3(t) = \begin{cases} 1+\cos(\pi t), & |t|<1 \\ 0, & \text{otherwise} \end{cases}$

Solution:
We can compute $X_3(j\omega)$ by applying the definition of Fourier transform:

$$\begin{aligned}
X_3(j\omega) &= \int_{-1}^{1}[1+\cos(\pi t)]e^{-j\omega t}dt \\
&= -\frac{1}{j\omega}[e^{-j\omega}-e^{j\omega}] + \frac{1}{j2(\pi-\omega)}[e^{j(\pi-\omega)}-e^{-j(\pi-\omega)}] - \frac{1}{j2(\pi+\omega)}[e^{-j(\pi+\omega)}-e^{j(\pi+\omega)}] \\
&= \frac{2\sin(\omega)}{\omega} + \frac{\sin(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega} = 2\text{sinc}\left(\frac{\omega}{\pi}\right) + \text{sinc}\left(\frac{\omega-\pi}{\pi}\right) + \text{sinc}\left(\frac{\omega+\pi}{\pi}\right)
\end{aligned}$$

Or we can see that:

$$x_3(t) = \text{rect}\left(\frac{t}{2}\right) + \cos(\pi t)\text{rect}\left(\frac{t}{2}\right)$$

so that,

$$X_3(j\omega) = 2\text{sinc}\left(\frac{\omega}{\pi}\right) + \text{sinc}\left(\frac{\omega-\pi}{\pi}\right) + \text{sinc}\left(\frac{\omega+\pi}{\pi}\right)$$

iv. $x_4(t) = te^{-2t}u(t)$

Solution:

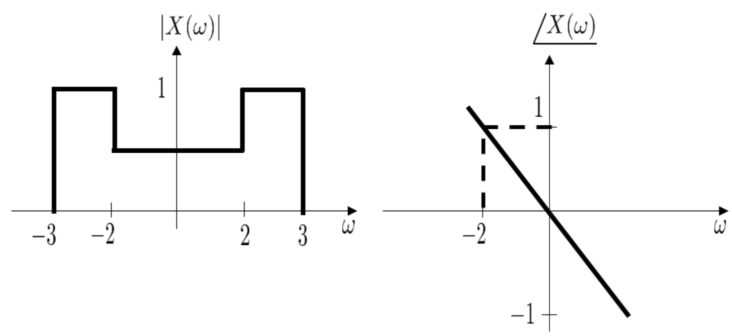
We know that:

$$-jtf(t) \longleftrightarrow F'(j\omega)$$
$$e^{-2t}u(t) \longleftrightarrow \frac{1}{2+j\omega}$$

Therefore,

$$X_4(j\omega) = -\frac{1}{j}\left(\frac{d}{d\omega}\frac{1}{2+j\omega}\right) = \frac{1}{(2+j\omega)^2}$$

(b) (6 points) Find the inverse Fourier transform of the signal shown below:



Solution: We have:

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|e^{j\angle X(\omega)}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-3}^{-2} e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega + \int_{-2}^{2}\frac{1}{2}e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega + \int_{2}^{3} e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega\right) \\
&= \frac{1}{2\pi}\left(\int_{-3}^{-2} e^{j(t-\frac{1}{2})\omega}d\omega + \int_{-2}^{2}\frac{1}{2}e^{j(t-\frac{1}{2})\omega}d\omega + \int_{2}^{3} e^{j(t-\frac{1}{2})\omega}d\omega\right) \\
&= \frac{1}{2\pi}\left(\frac{e^{-j2(t-\frac{1}{2})} - e^{-j3(t-\frac{1}{2})}}{j(t-\frac{1}{2})} + \frac{e^{j2(t-\frac{1}{2})} - e^{-j2(t-\frac{1}{2})}}{j2(t-\frac{1}{2})} + \frac{e^{j3(t-\frac{1}{2})} - e^{j2(t-\frac{1}{2})}}{j(t-\frac{1}{2})}\right) \\
&= \frac{1}{2\pi}\left(\frac{e^{-2j(t-\frac{1}{2})} - e^{2j(t-\frac{1}{2})}}{j2(t-\frac{1}{2})} + \frac{e^{j3(t-\frac{1}{2})} - e^{-j3(t-\frac{1}{2})}}{j(t-\frac{1}{2})}\right) \\
&= \frac{1}{2\pi}\left(-\frac{\sin\left(2(t-\frac{1}{2})\right)}{(t-\frac{1}{2})} + \frac{2\sin\left(3(t-\frac{1}{2})\right)}{(t-\frac{1}{2})}\right)
\end{aligned}$$

(c) (8 points) Two signals $f_1(t)$ and $f_2(t)$ are defined as

$$f_1(t) = \mathrm{sinc}(2t)$$
$$f_2(t) = \mathrm{sinc}(t)\cos(3\pi t)$$

Let the convolution of the two signals be

$$f(t) = (f_1 * f_2)(t)$$

i. Find $F(j\omega)$, the Fourier transform of $f(t)$.

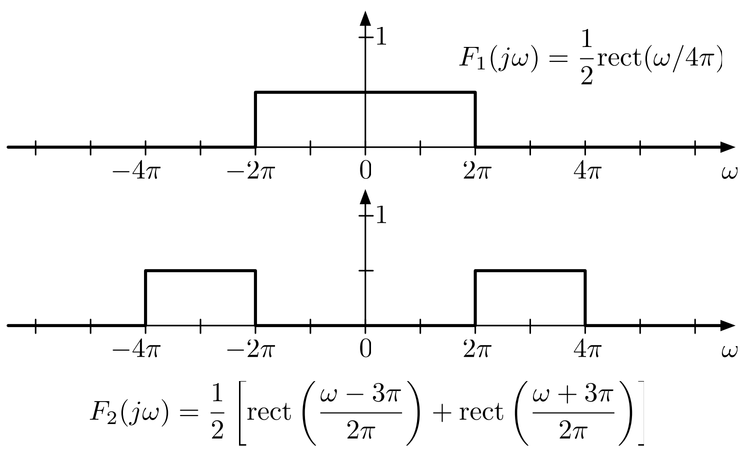
Solution: We know that:

$$f_1(t) = \text{sinc}(2t) \longleftrightarrow F_1(j\omega) = \frac{1}{2}\text{rect}\left(\frac{\omega}{4\pi}\right)$$
$$f_2(t) = \text{sinc}(t)\cos(3\pi t) \longleftrightarrow F_2(j\omega) = \frac{1}{2}\left(\text{rect}\left(\frac{\omega - 3\pi}{2\pi}\right) + \text{rect}\left(\frac{\omega + 3\pi}{2\pi}\right)\right)$$

We then have:

$$f(t) = (f_1 * f_2)(t) \longleftrightarrow F(j\omega) = F_1(j\omega)F_2(j\omega)$$

To see what the multiplication of $F_1(j\omega)$ and $F_2(j\omega)$ gives us, let us first plot them. We clear see that $F_1(j\omega)$ and $F_2(j\omega)$ do not overlap, therefore $F(j\omega) = 0$.



ii. Find $f(t)$.

Solution:
Since $F(j\omega) = 0$, $f(t)$ is then 0