**ECE 102, Fall 2018** **Final Exam**

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UCLA True Bruin academic integrity principles apply.
Open: Four pages of cheat sheet allowed.
Closed: Book, computer, internet.
11:30am-2:30pm, Haines Room 118
Tuesday, 11 Dec 2018.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

Signature: ____________________

ID#: _________________

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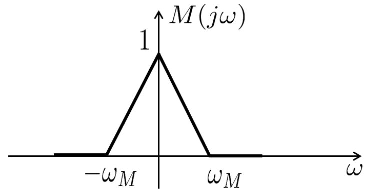
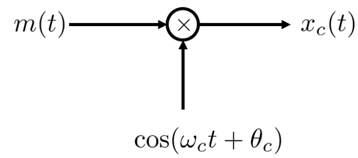
| BONUS | _____ | / | 10 bonus points |
| Total | _____ | / | 200 points + 10 bonus points |

**Problem 1** (25 points)

Consider a bandlimited signal $m(t)$, its frequency spectrum $M(j\omega)$ is shown below. We modulate $m(t)$ with $\cos(\omega_c t + \theta_c)$, where $\theta_c$ is a constant phase but unknown:




(a) (8 points) Express $X_c(j\omega)$, the Fourier transform of $x_c(t)$, in terms of $M(j\omega)$. *Hint: use the fact that* $\cos(u) = \dfrac{e^{ju} + e^{-ju}}{2}$.

**Solution:**

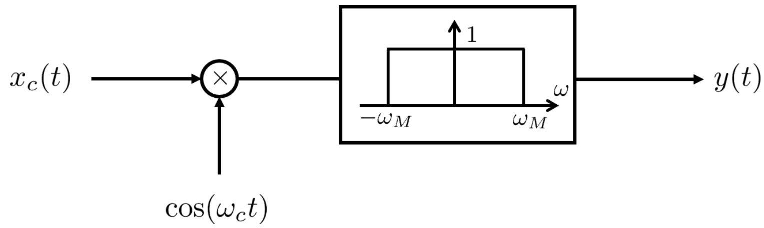
We have:

$$\begin{aligned} x_c(t) &= m(t)\cos(\omega_c t + \theta_c) = m(t)\frac{1}{2}\left(e^{j(\omega_c t + \theta_c)} + e^{-j(\omega_c t + \theta_c)}\right) \\ &= \frac{1}{2}e^{j\theta_c} m(t) e^{j\omega_c t} + \frac{1}{2}e^{-j\theta_c} m(t) e^{-j\omega_c t} \end{aligned}$$

Therefore,

$$X_c(j\omega) = \frac{1}{2}e^{j\theta_c} M(j(\omega - \omega_c)) + \frac{1}{2}e^{-j\theta_c} M(j(\omega + \omega_c))$$

(b) (10 points) We demodulate $x_c(t)$ as follows:




Show that $y(t) = \frac{1}{2}\cos(\theta_c)m(t)$. Assume $\omega_c \gg \omega_M$.

**Solution:** The input of the low pass filter:

$$x_c(t)\cos(\omega_c t)$$

Taking its Fourier transform, we obtain:

$$\begin{aligned}
&\frac{1}{2}X_c(j(\omega-\omega_c)) + \frac{1}{2}X_c(j(\omega+\omega_c)) = \\
&\qquad \frac{1}{4}e^{j\theta_c}M(j(\omega-2\omega_c)) + \frac{1}{4}e^{-j\theta_c}M(j\omega) + \frac{1}{4}e^{j\theta_c}M(j\omega) + \frac{1}{4}e^{-j\theta_c}M(j(\omega+2\omega_c)) = \\
&\qquad \frac{1}{4}e^{j\theta_c}M(j(\omega-2\omega_c)) + \frac{1}{4}\left(e^{-j\theta_c}+e^{j\theta_c}\right)M(j\omega) + \frac{1}{4}e^{-j\theta_c}M(j(\omega+2\omega_c)) = \\
&\qquad \frac{1}{4}e^{j\theta_c}M(j(\omega-2\omega_c)) + \frac{1}{2}\cos(\theta_c)M(j\omega) + \frac{1}{4}e^{-j\theta_c}M(j(\omega+2\omega_c))
\end{aligned}$$

After the low pass-filter, the term that only remains is

$$\frac{1}{2}\cos(\theta_c)M(j\omega)$$

Therefore,

$$y(t) = \frac{1}{2}\cos(\theta_c)m(t)$$

(c) (7 points) Assume that you also know $z(t) = \frac{1}{2}\sin(\theta_c)m(t)$. How can you recover $m(t)$ from $y(t)$ and $z(t)$?
*Hint:* $cos^2(u) + \sin^2(u) = 1$.

**Solution:**
To recover $m(t)$ from $z(t)$ and $y(t)$, we can compute the following:

$$2\sqrt{y^2(t) + z^2(t)}$$

This is because:

$$2\sqrt{y^2(t) + z^2(t)} = 2\sqrt{\left(\frac{1}{4}\cos^2(\theta_c)m^2(t) + \frac{1}{4}\sin^2(\theta_c)m^2(t)\right)} = 2\sqrt{\frac{1}{4}m^2(t)} = m(t)$$

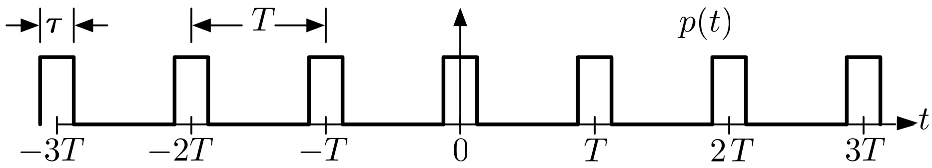
*Note: We noticed that we forgot to mention in the question that $m(t) > 0$. So we assumed all of the following answers as correct: $m(t)$, $\pm m(t)$, or $|m(t)|$. We also accepted answers where the above operation was proposed to be done in the frequency domain, i.e., $\mathcal{F}^{-1}\{2\sqrt{Y(j\omega)^2 + Z(j\omega)^2}\}$. Some of you proposed the following:*

$$2\left(z(t)\sin(\theta_c) + y(t)\cos(\theta_c)\right)$$

*This method is mathematically valid, but it cannot be implemented to recover $m(t)$ because $\theta_c$ is unknown for us, i.e., we cannot multiply $z(t)$ or $y(t)$ by a factor that we do not know. However, we gave full credit for it.*

**Problem 2** (41 points)

Consider the following sequence of short rect$(\cdot)$ pulses, denoted by $p(t)$:




Each rect$(\cdot)$ pulse has width $\tau$, and the pulses are spaced by $T$ as diagrammed above.

(a) (14 points) Find $P(j\omega)$, the Fourier transform of $p(t)$. Express $P(j\omega)$ as a sum, and simplify where possible. *Hint: One approach is to write $p(t)$ as convolution of a rect function with an impulse train.*

**Solution:**

We can write $p(t)$ as follows:

$$p(t) = \text{rect}\left(\frac{t}{\tau}\right) * \delta_T(t)$$

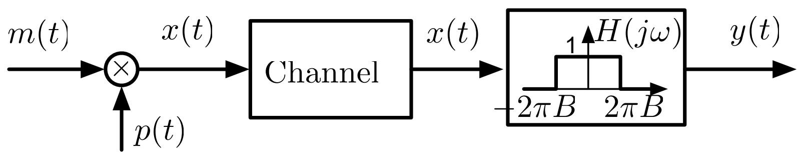
Therefore,

$$P(j\omega) = \tau \text{sinc}\left(\frac{\omega\tau}{2\pi}\right) \cdot \omega_0 \delta_{\omega_0}(\omega)$$

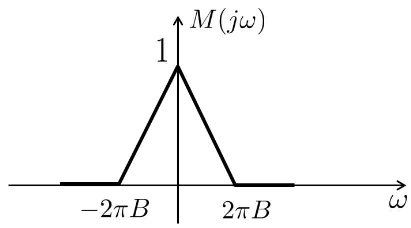
where $\omega_0 = \dfrac{2\pi}{T}$. Therefore,

$$\begin{aligned}
P(j\omega) &= \tau \text{sinc}\left(\frac{\omega\tau}{2\pi}\right) \cdot \omega_0 \delta_{\omega_0}(\omega) \\
&= \tau\omega_0 \text{sinc}\left(\frac{\omega\tau}{2\pi}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \\
&= \tau\omega_0 \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{\omega\tau}{2\pi}\right) \delta(\omega - k\omega_0) \\
&= \tau\omega_0 \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k\omega_0\tau}{2\pi}\right) \delta(\omega - k\omega_0)
\end{aligned}$$

(b) (10 points) Consider the following system:




where the input $m(t)$ is multiplied with the rect pulse train, $p(t)$. The signal $m(t)$ is bandlimited and it has the following frequency spectrum:



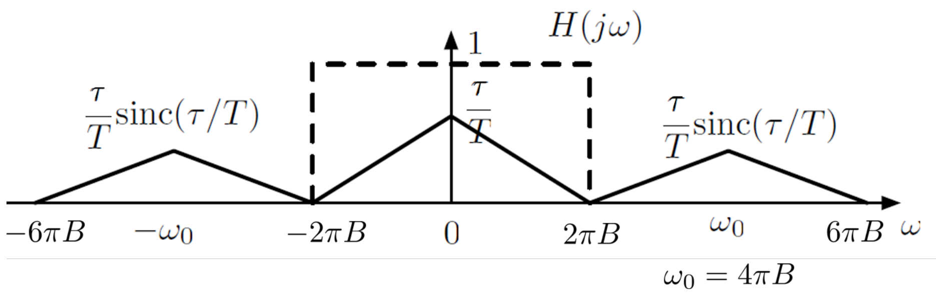

Assume that the rect$(\cdot)$ pulses are spaced by $T = \dfrac{1}{2B}$. Express the spectrum $X(j\omega)$ of $x(t)$ in terms of $M(j\omega)$.

**Solution:** Since $x(t) = p(t)m(t)$, we have:

$$\begin{aligned}
X(j\omega) &= \frac{1}{2\pi} M(j\omega) * P(j\omega) \\
&= \frac{1}{2\pi} M(j\omega) * \tau\omega_0 \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k\omega_0\tau}{2\pi}\right) \delta(\omega - k\omega_0) \\
&= \frac{\tau}{T} M(j\omega) * \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k\omega_0\tau}{2\pi}\right) \delta(\omega - k\omega_0) \\
&= \frac{\tau}{T} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k\tau}{T}\right) M(j(\omega - k\omega_0)) \\
&= \frac{\tau}{T} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k\tau}{T}\right) M(j(\omega - k\omega_0))
\end{aligned}$$

where $\omega_0 = \frac{2\pi}{T} = 4\pi B$ rad/s.

(c) (10 points) Sketch $X(j\omega)$ for $-6\pi B \leq \omega \leq 6\pi B$.




Note that $\mathrm{sinc}(0) = 1$.

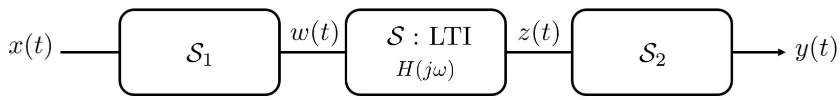
(d) (7 points) Find the spectrum of the signal at the output of the lowpass filter $Y(j\omega)$, i.e., find an expression of $Y(j\omega)$ in terms of $M(j\omega)$.

**Solution:** After the low pass filter, we have:

$$Y(j\omega) = \frac{\tau}{T} M(j\omega)$$

**Problem 3** (30 points)

An LTI system $\mathcal{S}$ is cascaded in series with two other non-LTI systems as follows:




The system $\mathcal{S}_1$ is given by:

$$w(t) = x\left(\frac{t}{2}\right)$$

And the system $\mathcal{S}_2$ is:

$$y(t) = z(2t)$$

The system $\mathcal{S}$ has $H(j\omega)$ as its frequency response.

*(This question continues on the next page.)*

(a) (15 points) Find how $Y(j\omega)$ is related to $X(j\omega)$, in terms of $H(j\omega)$. Deduce the overall frequency response $H_{eq}(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$.

**Solution:**
We have:

$$w(t) = x\left(\frac{t}{2}\right) \rightarrow W(j\omega) = 2X(j2\omega)$$
$$z(t) = h(t) * w(t) \rightarrow Z(j\omega) = H(j\omega)W(j\omega)$$
$$y(t) = z(2t) \rightarrow Y(j\omega) = \frac{1}{2}Z\left(j\frac{\omega}{2}\right)$$

Therefore,

$$Y(j\omega) = \frac{1}{2}Z\left(j\frac{\omega}{2}\right) = \frac{1}{2}H\left(j\frac{\omega}{2}\right)W\left(j\frac{\omega}{2}\right) = H\left(j\frac{\omega}{2}\right)X\left(j\omega\right)$$

Therefore,

$$H_{eq}(j\omega) = H\left(j\frac{\omega}{2}\right)$$

(b) (15 points) If $H(j\omega)$ is given by:

$$H(j\omega) = \frac{2a - j\omega}{2a + j\omega}$$

where $a > 0$, find the impulse response $h(t)$ of the system $\mathcal{S}$. Deduce the overall impulse response $h_{eq}(t)$.

**Solution:** We have:

$$H(j\omega) = \frac{2a}{2a + j\omega} - \frac{j\omega}{2a + j\omega}$$

Therefore,

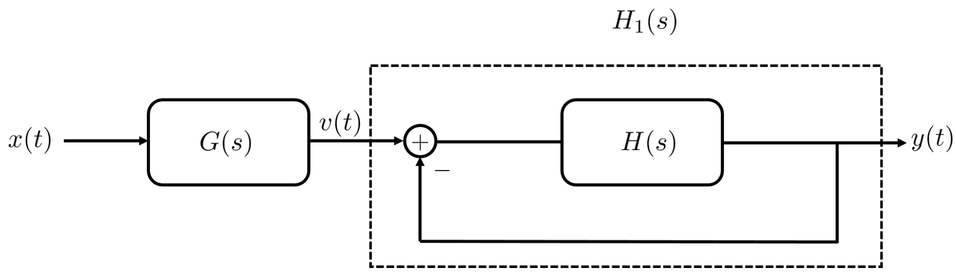
$$h(t) = 2ae^{-2at}u(t) - \frac{d}{dt}\left(e^{-2at}u(t)\right) = 2ae^{-2at}u(t) - \left(-2ae^{-2at}u(t) + \delta(t)\right) = 4ae^{-2at}u(t) - \delta(t)$$

Since $H_{eq}(j\omega) = H\left(j\frac{\omega}{2}\right)$, we have:

$$h_{eq}(t) = 2h(2t) = 8ae^{-4at}u(2t) - 2\delta(2t) = 8ae^{-4at}u(t) - \delta(t)$$

**Problem 4** (40 points)

Consider the following system:




(a) (10 points) Find the transfer function $H_1(s)$ of the system that maps $v(t)$ to $y(t)$.

**Solution:**

$$Y(s) = H(s)(V(s) - Y(s)) \implies \frac{Y(s)}{V(s)} = \frac{H(s)}{1 + H(s)}$$

Therefore,

$$H_1(s) = \frac{H(s)}{1 + H(s)}$$

(b) (5 points) Find the overall transfer function $H_{eq}(s)$.

**Solution:**

$$H_{eq}(s) = G(s)H_1(s) = G(s)\frac{H(s)}{1+H(s)}$$

(c) (10 points) How can we choose $H(s)$ in terms of $G(s)$ so that the overall system has the following impulse response $h_{eq}(t) = \delta(t)$?

**Solution:**

$$h_{eq}(t) = \delta(t) \to H_{eq}(s) = 1$$

Thus, we need to have:

$$G(s)\frac{H(s)}{1+H(s)} = 1 \implies G(s)H(s) = 1 + H(s) \implies H(s) = \frac{1}{G(s)-1}$$

(d) (15 points) Using the relation you found in part (c), find $h(t)$ if $g(t) = e^{-2t}u(t)$.

**Solution:**

$$G(s) = \frac{1}{s+2}$$

Therefore,

$$H(s) = \frac{1}{\frac{1}{s+2} - 1} = -\frac{s+2}{s+1} = -\left(1 + \frac{1}{s+1}\right)$$

Thus,

$$h(t) = -\delta(t) - e^{-t}u(t)$$

**Problem 5** (20 points)

A system is described by the following differential equation:

$$y''(t) + 5y'(t) + 6y(t) = x'(t) + 5x(t)$$

If the input is

$$x(t) = e^{-4t}u(t-2)$$

find the output $y(t)$. Assume all initial conditions are zero.

*There is additional space on the next page if needed.*

**Solution:** Applying the Laplace trasnform to the differential equation:

$$s^2Y(s) + 5sY(s) + 6Y(s) = sX(s) + 5X(s)$$

Therefore,

$$Y(s) = \frac{s+5}{s^2+5s+6}X(s)$$

Now,

$$x(t) = e^{-8}e^{-4(t-2)}u(t-2) \implies X(s) = e^{-8}e^{-2s}\frac{1}{s+4}$$

Therefore,

$$Y(s) = e^{-8}e^{-2s}\frac{s+5}{(s+3)(s+2)(s+4)} = e^{-8}e^{-2s}\left(\frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{s+4}\right)$$

where,

$$\begin{aligned}
A &= \left.\frac{s+5}{(s+2)(s+4)}\right|_{s=-3} = -2 \\
B &= \left.\frac{s+5}{(s+3)(s+4)}\right|_{s=-2} = 3/2 \\
C &= \left.\frac{s+5}{(s+3)(s+2)}\right|_{s=-4} = 1/2
\end{aligned}$$

Therefore,

$$y(t) = e^{-8}\left(-2e^{-3(t-2)} + \frac{3}{2}e^{-2(t-2)} + \frac{1}{2}e^{-4(t-2)}\right)u(t-2)$$

**Problem 6** (44 points)

(a) (24 points) Determine if each of the following four statements is true or false. When the statement is false, a counter example is sufficient. If the statement is true, you must justify your answer to receive full credit.

i. If $x(t) * y(t) = 0$, then $x(t) = 0$ or $y(t) = 0$.

**Solution:**
**False**: We know that:

$$x(t) * y(t) \to X(j\omega)Y(j\omega)$$

Let

$$X(j\omega) = \text{rect}(\omega) \text{ and } Y(j\omega) = \text{rect}(\omega - 2)$$

We then have

$$X(j\omega)Y(j\omega) = 0$$

while $X(j\omega) \neq 0$ and $Y(j\omega) \neq 0$.

ii. If $x(t) * h(t) = x(t)$, then $h(t)$ must be an impulse, i.e., $h(t) = \delta(t)$.

**Solution:**
**False**: We have:

$$x(t) * h(t) \to X(j\omega)H(j\omega)$$

If $x(t)$ is bandlimited to $\pm\frac{1}{2}\omega_c$ and $H(j\omega) = \text{rect}\left(\frac{\omega}{2\omega_c}\right)$, then

$$X(j\omega)H(j\omega) = X(j\omega)$$

However, $h(t)$ is not an impulse.

iii. A signal $x(t)$ is bandlimited where its Fourier transform $X(j\omega) = 0$ for $|\omega| > 2\pi B$ rad/s. The Nyquist rate of $\cos(4\pi Bt)x(t-2) + x(2t)$ is $6B$ Hz.

**Solution:**
**True** The fourier trasnform of the given signal:

$$\frac{1}{2}\left(e^{-j2(\omega-4\pi B)}X(j(\omega - 4\pi B)) + e^{-j2(\omega+4\pi B)}X(j(\omega + 4\pi B))\right) + \frac{1}{2}X(j\omega/2)$$

The highest frequency component is: $6\pi B$ rad/s or $3B$ Hz. Therefore, the Nyquist rate: $2(3B) = 6B$ Hz

iv. If $x(t) = \text{sinc}(t)$, then the energy of $x(3t+2)$ is $\frac{1}{3}$.

**Solution:**
**True:** Let $y(t) = x(3t+2)$, then the energy of $y(t)$ is given by:

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega)|^2 d\omega$$

Now,

$$Y(j\omega) = \frac{1}{3} e^{j2\omega/3} X(j\omega/3) = \frac{1}{3} e^{j2\omega/3} \text{rect}(\omega/6\pi)$$

Therefore,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega)|^2 d\omega = \frac{1}{2\pi} \frac{1}{9} \int_{-3\pi}^{3\pi} 1 d\omega = \frac{6\pi}{2\pi \cdot 9} = \frac{1}{3}$$

(b) (10 points) If $y(t) = x(t) * h(t)$, then show that the following identity holds:

$$\int_{-\infty}^{\infty} y(t)dt = \left(\int_{-\infty}^{\infty} h(t)dt\right) \cdot \left(\int_{-\infty}^{\infty} x(t)dt\right)$$

Hint: One approach is to look at the integral expression for the Fourier transform when $\omega = 0$.

**Solution:**

We have

$$Y(j\omega) = H(j\omega)X(j\omega)$$

Therefore, if we evaluate the above equality at $\omega = 0$, we have:

$$Y(0) = H(0)X(0)$$

Now since

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt \implies Y(0) = \int_{-\infty}^{\infty} y(t)dt$$

we conclude:

$$\int_{-\infty}^{\infty} y(t)dt = \left(\int_{-\infty}^{\infty} h(t)dt\right) \cdot \left(\int_{-\infty}^{\infty} x(t)dt\right)$$

(c) (10 points) An LTI system has the following impulse response: $h(t) = e^t u(-1-t)$. Is the system stable? Is it causal?

**Solution:**

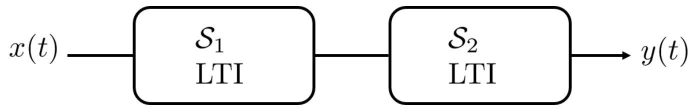
Since $u(-t-1) = 1$ for $t \leq -1$, we have $h(t) \neq 0$ for $t < 0$, therefore the system is not causal.

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{t=-\infty}^{-1} e^t dt = e^{-1} < \infty$$

The system is then stable.

**BONUS** (10 points)

(a) (5 points) Two LTI systems are linearly cascaded as follows:




The impulse response of the first system is $h_1(t) = e^t u(t)$ and the impulse response of the second system is $h_2(t) = e^{2t}\cos(t)$. What is the impulse response of the equivalent system $h_{eq}(t)$?

**Solution:**

Since the two systems are LTI, we can switch the order so that system $\mathcal{S}_2$ comes first and $\mathcal{S}_1$ is the second system. Then in this case, computing $h_{eq}(t)$ is equivalent to compute the output of system $\mathcal{S}_1$ to input $e^{2t}\cos(t)$. To compute the output we are going to use the eigenfunction property. We have the following:

$$H_1(s) = \frac{1}{s-1}, \quad \mathrm{Re}\{s\} > 1$$

and

$$e^{2t}\cos(t) = \frac{1}{2}e^{(2+j)t} + \frac{1}{2}e^{(2-j)t}$$

Therefore,

$$\begin{aligned}
h_{eq}(t) &= \frac{1}{2}H_1(2+j)e^{(2+j)t} + \frac{1}{2}H_1(2-j)e^{(2-j)t} \\
&= \frac{1}{2}\frac{1}{1+j}e^{(2+j)t} + \frac{1}{2}\frac{1}{1-j}e^{(2-j)t} \\
&= \frac{1}{4}(1-j)e^{(2+j)t} + \frac{1}{4}(1+j)e^{(2-j)t} \\
&= \frac{1}{2}e^{2t}\left(\cos(t) + \sin(t)\right)
\end{aligned}$$

(b) (5 points) If $F_s$ is the Nyquist rate of $x(t)$, determine in terms of $F_s$, the Nyquist rate of $x^3(t) * x^2(t)$.

**Solution:**

If $F_s$ is the Nyquist rate of $x(t)$, then the highest frequency component of $x(t)$ is: $F_s/2$ and $x(t)$ is bandlimited to $\pm Fs/2$.

Now,

$$y(t) = x^3(t) \implies Y(j\omega) = \frac{1}{2\pi} X(j\omega) * \left( \frac{1}{2\pi} X(j\omega) * X(j\omega) \right)$$

Therefore, if $x(t)$ is bandlimited to $\pm F_s/2$, $y(t)$ is then bandlimited to $\pm 3F_s/2$.

$$z(t) = x^2(t) \implies Y(j\omega) = \left( \frac{1}{2\pi} X(j\omega) * X(j\omega) \right)$$

Therefore, if $x(t)$ is bandlimited to $\pm F_s/2$, $z(t)$ is then bandlimited to $\pm 2F_s/2$ or $\pm F_s$.

Now

$$y(t) * z(t) \rightarrow Y(j\omega) Z(j\omega)$$

This means that $y(t) * z(t)$ is bandlimited to $\pm Fs$. Therefore the Nyquist rate is $2F_s$.

# Fourier Transform Tables

| Property | Signal | Transform |
|---|---|---|
| Linearity | $\alpha\, x_1(t) + \beta\, x_2(t)$ | $\alpha\, X_1(j\omega) + \beta\ \ X_2(j\omega)$ |
| Duality | $X(t)$ | $2\pi\, x(-\omega)$ |
| Conjugate symmetry | $x(t)$ real | $X^*(j\omega) = \ X(-j\omega)$<br>Magnitude: $\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$<br>Phase: $\Theta(-\omega) = -\Theta(\omega)$<br>Real part: $X_r(-j\omega) = X_r(j\omega)$<br>Imaginary part: $X_i(-j\omega) = -X_i(j\omega)$ |
| Conjugate antisymmetry | $x(t)$ imaginary | $X^*(j\omega) = -X(-j\omega)$<br>Magnitude: $\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$<br>Phase: $\Theta(-\omega) = -\Theta(\omega) \mp \pi$<br>Real part: $X_r(-j\omega) = -X_r(j\omega)$<br>Imaginary part: $X_i(-j\omega) = X_i(j\omega)$ |
| Even signal | $x(-t) = x(t)$ | $X(j\omega)$: even |
| Odd signal | $x(-t) = -x(t)$ | $X(j\omega)$: odd |
| Time shifting | $x(t-\tau)$ | $X(j\omega)\, e^{-j\omega\tau}$ |
| Frequency shifting | $x(t)\ e^{j\omega_0 t}$ | $X(j(\omega-\omega_0))$ |
| Modulation property | $x(t)\ \cos(\omega_0 t)$ | $\frac{1}{2}\left[X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right]$ |
| Time and frequency scaling | $x(at)$ | $\dfrac{1}{\lvert a\rvert} X\left(\dfrac{j\omega}{a}\right)$ |
| Differentiation in time | $\dfrac{d^n}{dt^n}[x(t)]$ | $(j\omega)^n X(j\omega)$ |
| Differentiation in frequency | $(-jt)^n\ x(t)$ | $\dfrac{d^n}{d\omega^n}[X(j\omega)]$ |
| Convolution | $x_1(t) * x_2(t)$ | $X_1(j\omega)\, X_2(j\omega)$ |
| Multiplication | $x_1(t)\ x_2(t)$ | $\dfrac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$ |
| Integration | $\displaystyle\int_{-\infty}^{t} x(\lambda)\, d\lambda$ | $\dfrac{X(j\omega)}{j\omega} + \pi\, X(0)\, \delta(\omega)$ |
| Parseval's theorem | $\displaystyle\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\ dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\, d\omega$ | |

**Table 4.4** – Fourier transform properties.

| Additional properties: | | |
|---|---|---|
| | $x$(t): even and real | $X(j\omega)$: even and real |
| | $x$(t): odd and real | $X(j\omega)$: odd and imaginary |
| | $x$(t): even and imaginary | $X(j\omega)$: even and imaginary |
| | $x$(t): odd and imaginary | $X(j\omega)$: odd and real |

| Name | Signal | Transform |
|---|---|---|
| Rectangular pulse | $x(t) = A\,rect(t/\tau)$ | $X(j\omega) = A\tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$ |
| Triangular pulse | $x(t) = A\,\Lambda(t/\tau)$ | $X(j\omega) = A\tau\,\text{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$ |
| Right-sided exponential | $x(t) = e^{-at}\,u(t)$ | $X(j\omega) = \frac{1}{a+j\omega}$ |
| Two-sided exponential | $x(t) = e^{-a\|t\|}$ | $X(j\omega) = \frac{2a}{a^2+\omega^2}$ |
| Signum function | $x(t) = \text{sgn}(t)$ | $X(j\omega) = \frac{2}{j\omega}$ |
| Unit impulse | $x(t) = \delta(t)$ | $X(j\omega) = 1$ |
| Sinc function | $x(t) = \text{sinc}(t)$ | $X(j\omega) = rect\left(\frac{\omega}{2\pi}\right)$ |
| Constant-amplitude signal | $x(t) = 1,\ \text{all } t$ | $X(j\omega) = 2\pi\,\delta(\omega)$ |
| | $x(t) = \frac{1}{\pi t}$ | $X(j\omega) = -j\,\text{sgn}(\omega)$ |
| Unit-step function | $x(t) = u(t)$ | $X(j\omega) = \pi\,\delta(\omega) + \frac{1}{j\omega}$ |
| Modulated pulse | $x(t) = rect\left(\frac{t}{\tau}\right)\cos(\omega_0 t)$ | $X(j\omega) = \frac{\tau}{2}\,\text{sinc}\left(\frac{(\omega-\omega_0)\tau}{2\pi}\right) + \frac{\tau}{2}\,\text{sinc}\left(\frac{(\omega+\omega_0)\tau}{2\pi}\right)$ |

**Table 4.5** – Some Fourier transform pairs.

Note:

$$\text{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

$$\text{rect}(t/\tau) = u(t+\tau/2) - u(t-\tau/2)$$

## LAPLACE TRANSFORM

### 1. Some Laplace transform pairs

| Signal | Transform | ROC |
|---|---|---|
| $\delta(t)$ | $1$ | All s |
| $u(t)$ | $\frac{1}{s}$ | $\mathcal{R}e\{s\} > 0$ |
| $\cos(\omega_0 t)u(t)$ | $\frac{s}{s^2+\omega_0^2}$ | $\mathcal{R}e\{s\} > 0$ |
| $\sin(\omega_0 t)u(t)$ | $\frac{\omega_0}{s^2+\omega_0^2}$ | $\mathcal{R}e\{s\} > 0$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ | $\mathcal{R}e\{s\} > -a$ |
| $e^{-at}\cos(\omega_0 t)u(t)$ | $\frac{s+a}{(s+a)^2+\omega_0^2}$ | $\mathcal{R}e\{s\} > -a$ |
| $e^{-at}\sin(\omega_0 t)u(t)$ | $\frac{\omega_0}{(s+a)^2+\omega_0^2}$ | $\mathcal{R}e\{s\} > -a$ |
| $te^{-at}u(t)$ | $\frac{1}{(s+a)^2}$ | $\mathcal{R}e\{s\} > -a$ |
| $t^n u(t)$ | $\frac{n!}{s^{n+1}}$ | $\mathcal{R}e\{s\} > 0$ |

## 2. Laplace transform properties

| Signal | Transform | ROC |
|---|---|---|
| $x(t)$ | $X(s)$ | $R_x$ |
| $x_1(t)$ | $X_1(s)$ | $R_1$ |
| $x_2(t)$ | $X_2(s)$ | $R_2$ |
| $ax_1(t) + bx_2(t)$ | $aX_1(s) + bX_2(s)$ | At least $R_1 \cap R_2$ |
| $x(t - t_0)$ | $e^{-st_0} X(s)$ | $R_x$ |
| $e^{s_0 t} x(t)$ | $X(s - s_0)$ | Shifted version of $R_x$<br>($s$ is in the ROC if $s - s_0 \in R_x$) |
| $x(at)$, $a > 0$ | $\frac{1}{a} X\left(\frac{s}{a}\right)$ | Scaled version of $R_x$<br>($s$ is in the ROC if $s/a \in R_x$) |
| $x_1(t) * x_2(t)$ | $X_1(s)X_2(s)$ | At least $R_1 \cap R_2$ |
| $\int_0^t x(\tau) d\tau$ | $\frac{X(s)}{s}$ | At least $R_x \cap \{\mathcal{R}e\{s\} > 0\}$ |
| $\frac{d}{dt} x(t)$ | $sX(s) - x(0)$ | At least $R_x$ |
| $\frac{d^2}{dt^2} x(t)$ | $s^2 X(s) - sx(0) - x'(0)$ | At least $R_x$ |
| $tx(t)$ | $-\frac{d}{ds} X(s)$ | $R_x$ |