Applications of the Laplace transform

This lecture is about applications of the Laplace transform. Topics include:

- Linear constant coefficient ordinary differential equations (LCCODEs).
- Poles and zeros: intuition
- Filters and bode plots
- Feedback

Several examples in these lecture notes, including ODE examples, are thanks to Prof. John Pauly.

LCCODEs

The Laplace transform allows us to straightforwardly solve linear differential equations that have:

- Constant coefficients.
- Initial conditions.
- Input signals

The procedure to solve for such a linear constant coefficient differential equation (LCCODE) is to:

1. Use the Laplace transform to convert the differential equation, initial conditions, and input signals into an algebraic equation.
2. Solve for the Laplace transform of the output.
3. Invert the Laplace transform of the output.

Importantly, the Laplace transform enables us to separate out the contribution of the output from the *initial conditions* of the ODE versus the *inputs* to the system.

LCCODE example

Consider the LCCODE

$$y''(t) + 5y'(t) + 6y(t) = x'(t) + x(t)$$

where $y(t)$ is the output, and the initial conditions are:

- $y(0) = 2$.
- $y'(0) = 1$.

These initial conditions can cause some response in $y(t)$ (imagine a mass-spring ths starts off with some initial displacement, or a resistor-capacitor circuit where the capacitor has some charge stored in it at time $t = 0$).

In addition to this, we supply an input, $x(t) = e^{-4t}u(t)$. The input will also cause some response in $y(t)$.

The Laplace transform will allow us to solve for $y(t)$ and also know which component of $y(t)$ is due to the initial condition, versus which is due to the input.

LCCODE example (cont.)

Contribution from the initial condition. If we want the contribution from the initial condition, then we treat $x'(t) + x(t) = 0$. This is the same as solving for the Laplace transform of the left hand side of the differential equation (which we would have had to do anyways). So, taking the Laplace transform of the left hand side, we have:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 6Y(s) & \\ &= Y(s)(s^2 + 5s + 6) - 2s - 1 - 10 \\ &= Y(s)(s^2 + 5s + 6) - (2s + 11) \end{aligned}$$

Contribution from the input. To find the contribution from the input, we take the Laplace transform of the right hand side, which is:

$$\begin{aligned} sX(s) - x(0) + X(s) &= X(s)(s+1) \\ &= \frac{s+1}{s+4} \end{aligned}$$

since $e^{-4t}u(t) \iff (s+4)^{-1}$.

LCCODE example (cont.)

This gives that the Laplace transform of the LCCODE is:

$$Y(s)(s^2 + 5s + 6) - (2s + 11) = \frac{s+1}{s+4}$$

Hence,

$$Y(s) = \frac{2s + 11}{s^2 + 5s + 6} + \frac{s+1}{(s+4)(s^2 + 5s + 6)}$$

Now, the terms

- $\frac{2s+11}{s^2+5s+6}$ is the contribution to the output due to the initial conditions (since this is the Laplace transform when the inputs are zero).
- $\frac{s+1}{(s+4)(s^2+5s+6)}$ is therefore the contribution to the output due to the input.

We're going to solve this in two ways.

1. First, we'll solve the entire LCCODE giving the total ouptut response due to both initial conditions and inputs.
2. Second, we'll solve for $y(t)$ as a function of the contributions from the initial condition (called **zero input** solution) and from the input (called **zero state** solution).

We'll confirm these give identical solutions.

LCCODE example (cont.)

First, solving for the total output response, ignoring what is from initial condition and what is from input. So, we'll first combine terms.

$$\begin{aligned}
Y(s) &= \frac{2s+11}{s^2+5s+6} + \frac{s+1}{(s+4)(s^2+5s+6)} \\
&= \frac{(2s+11)(s+4)+(s+1)}{(s+4)(s^2+5s+6)} \\
&= \frac{2s^2+20s+45}{(s+4)(s+3)(s+2)}
\end{aligned}$$

We find the inverse Laplace transform through partial fractions.

$$\frac{(2s+11)(s+4)+(s+1)}{(s+4)(s+3)(s+2)} = \frac{r_1}{s+2} + \frac{r_2}{s+3} + \frac{r_3}{s+4}$$

This continues on the next page.

LCCODE example (cont.)

Using Method 2 (cover up),

$$
\begin{aligned}
r_1 &= \frac{2(-2)^2 + 20(-2) + 45}{(-2+4)(-2+3)} = \frac{8 - 40 + 45}{2 \cdot 1} = \frac{13}{2} \\
r_2 &= \frac{2(-3)^2 + 20(-3) + 45}{(-3+4)(-3+2)} = \frac{18 - 60 + 45}{1 \cdot -1} = -3 \\
r_3 &= \frac{2(-4)^2 + 20(-4) + 45}{(-4+3)(-4+2)} = \frac{32 - 80 + 45}{-1 \cdot -2} = -\frac{3}{2}
\end{aligned}
$$

Hence,we then have that

$$Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$

so that

$$y(t) = \left[\frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t}\right] u(t)$$

This is $y(t)$ due to both inputs (zero state) and initial conditions (zero input), but we don't know how each contributes.

LCCODE example (cont.)

We can solve this problem by finding the contribution from the initial condition (zero input solution) and the input (zero state solution). Recall, we derived

$$Y(s) = \frac{2s+11}{s^2+5s+6} + \frac{s+1}{(s+4)(s^2+5s+6)}$$

where the first term is the zero input term (due to no input) and the second term is the zero state term (due to no initial condition).

Zero input: To solve the zero input term, $y_{zi}(t)$, we take the inverse Laplace transform of

$$\frac{2s+11}{(s+2)(s+3)} = \frac{r_1}{s+2} + \frac{r_2}{s+3} = \frac{7}{s+2} - \frac{5}{s+3}$$

giving

$$y_{zi}(t) = [7e^{-2t} - 5e^{-3t}]u(t)$$

This is the zero input response, i.e., if $x(t) = 0$, this is what the output of the ODE would be.

LCCODE example (cont.)

Zero state: To solve for the zero state (i.e., no initial condition) term, $y_{zs}(t)$, we take the inverse Laplace transform of

$$
\begin{aligned}
\frac{s+1}{(s+2)(s+3)(s+4)} &= \frac{r_1}{s+2} + \frac{r_2}{s+3} + \frac{r_3}{s+4} \\
&= -\frac{1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}
\end{aligned}
$$

which gives that

$$
y_{zs}(t) = \left[-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t} \right] u(t)
$$

Hence, the total solution is

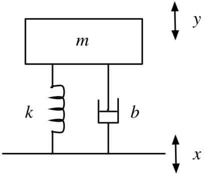
$$
\begin{aligned}
y(t) &= y_{zi}(t) + y_{zs}(t) \\
&= [7e^{-2t} - 5e^{-3t}]u(t) + \left[-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t} \right] u(t)
\end{aligned}
$$

You can confirm that the sum of these terms is equal to the total solution we earlier derived, i.e.,

$$
y(t) = \left[\frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t} \right] u(t)
$$

Example: vehicle suspension

Consider a vehicle suspension system as illustrated below.



- input x is road height (along vehicle path);
- output y is vehicle height

(Fig acknowledgment: Prof. John Pauly)

The vehicle's dynamics, which you don't need to know how to derive, are

$$my'' + by' + ky = bx' + kx$$

where k is the spring stiffness and b is the dampening provided by shock absorbers. Assume zero initial conditions in both the input and output, i.e., $y(0) = y'(0) = x(0) = 0$. Then,

$$(ms^2 + bs + k)Y(s) = (bs + k)X(s)$$

and so

$$H(s) = \frac{bs + k}{ms^2 + bs + k}$$

Example: vehicle suspension (cont.)

Let's choose $m = 1$, $k = 2$ and $b = 3$, and say we want to solve for the step response ($x(t) = u(t)$, i.e., the car is driving onto a curb). Then,

$$\begin{aligned} H(s) &= \frac{bs + k}{ms^2 + bs + k} \\ &= \frac{3s + 2}{s^2 + 3s + 2} \\ &= \frac{3s + 2}{(s + 1)(s + 2)} \end{aligned}$$

and so for $X(s) = 1/s$ and $Y(s) = H(s)X(s)$,

$$\begin{aligned} Y(s) &= \frac{3s + 2}{s(s + 1)(s + 2)} \\ &= \frac{1}{s} + \frac{1}{s + 1} - \frac{2}{s + 2} \end{aligned}$$

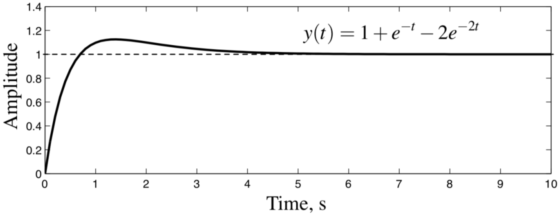
so that

$$y(t) = 1 + e^{-t} - 2e^{-2t}$$

This is illustrated on the next page.

Example: vehicle suspension (cont.)

This is the step response for $m = 1, k = 2, b = 3$.



(Fig acknowledgment: Prof. John Pauly)

We can check some intuitions.

- If $x(t) = u(t)$, i.e., the level of the road changes as a step (like in a curb) then $y(t) = 1$ in steady state. This makes sense; if the road rises, the car should also rise by the same amount (i.e. 1).
- When the car first hits the curb, there is some transient. We first see finite rise time (i.e., it takes some time for the car's level to reach the curb due to the suspension system).
- After that, the car overshoots; this is a result of the spring which allows the car to displace vertically.
- Finally, there is dampening so that the vehicle's level, $y(t)$ does not keep oscillating due to the spring.

Example: vehicle suspension (cont.)

What if we make the spring stiffer ($k = 5$) and reduce the shock absorbance ($b = 2$)? The partial fraction expansion (please work out on your own; we used a quadratic factor) is

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \left(\frac{2s+5}{s^2+2s+5}\right)\left(\frac{1}{s}\right) \\ &= \frac{1}{s} - \frac{s}{(s+1)^2+4} \end{aligned}$$

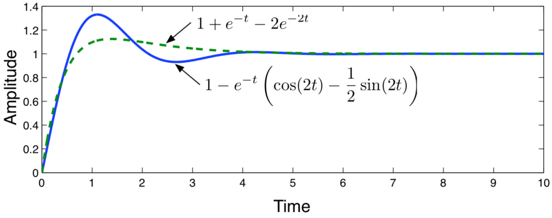
This can be further simplified (to use an inverse Laplace transform table)

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s}{(s+1)^2+4} \\ &= \frac{1}{s} - \frac{(s+1)-1}{(s+1)^2+4} \\ &= \frac{1}{s} - \frac{(s+1)}{(s+1)^2+2^2} + \left(\frac{1}{2}\right)\frac{2}{(s+1)^2+2^2} \end{aligned}$$

Example: vehicle suspension (cont.)

By look up table, the inverse Laplace transform is:

$$
\begin{aligned}
y(t) &= 1 - e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) \\
&= 1 - e^{-t}\left(\cos(2t) - \frac{1}{2}\sin(2t)\right)
\end{aligned}
$$



(Fig acknowledgment: Prof. John Pauly)

Here we see that due to less shock absorbance, the car vibrates more (it has greater over and undershoot) and because of the spring's stiffness increasing, it has a higher frequency of oscillation. This matches intuition.

More on poles and zeros

We talked previously about how all LCCODE's can be written as rational transfer functions. To recap, for zero initial conditions (i.e., zero state), we have that the Laplace transform of the LCCODE (and for $y^{(i)}(t)$ denoting the ith derivative of y w.r.t. t),

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)} + \cdots + a_1 y'(t) + a_0 y(t)$$

$$= b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_1 x(t) + b_0$$

is

$$Y(s)[a^n s^n + a_{n-1} s^{n-1} + \cdots + a_0] = X(s)[b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0]$$

so that

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a^n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

More on poles and zeros (cont.)

LCCODEs can be re-written in terms of their poles and zeros.

$$
\begin{aligned}
H(s) &= \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a^n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \\
&= k \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}
\end{aligned}
$$

where

- $k = b_m / a_n$
- z_i is the ith zero of $H(s)$ (i.e., the roots of $b(s)$).
- p_i is the ith pole of $H(s)$ (i.e., the roots of $a(s)$).

More on poles and zeros (cont.)

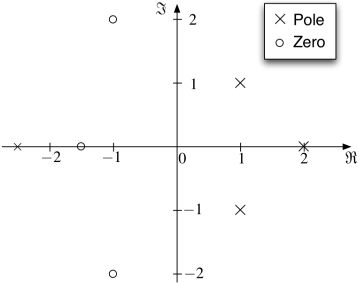
If $a(s)$ and $b(s)$ have real coefficients, then the poles and zeros of $H(s)$, i.e., the roots of $a(s)$ and $b(s)$ must come in complex conjugate pairs, or else be real. To see this, assume λ is a root of $b(s)$. Then,

$$\begin{aligned} 0 &= b_m\lambda^m + b_{m-1}\lambda^{m-1} + \cdots + b_0 \\ &= (b_m\lambda^m + b_{m-1}\lambda^{m-1} + \cdots + b_0)^* \\ &= b_m(\lambda^m)^* + b_{m-1}(\lambda^{m-1})^* + \cdots + b_0 \\ &= b_m(\lambda^*)^m + b_{m-1}(\lambda^*)^{m-1} + \cdots + b_0 \\ &= b(\lambda^*) \end{aligned}$$

and therefore λ^* is also a root of $b(s)$. The same type of approach holds for the roots of $a(s)$.

More on poles and zeros (cont.)

We often illustrate poles ('x') and zeros ('o') through a pole zero plot. We show an example below:



(Fig acknowledgment: Prof. John Pauly)

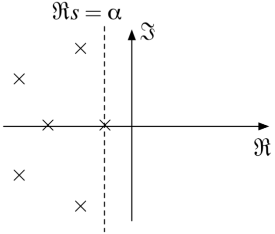
and thus for this transfer function,

$$H(s) = k\frac{(s + 3/2)(s + 1 + 2j)(s + 1 - 2j)}{(s + 5/2)(s - 2)(s - 1 - j)(s - 1 + j)}$$

More on poles and zeros (cont.)

Pole zero plots can provide a lot of intuition, by inspection, as to how the system works. We'll do several examples of this by looking at filters, but first a few notes / terminology.

Dominant poles. If the poles of $H(s)$ are $p_1, \ldots, p_n$, then the asymptotic growth (or decay) rate of $h(t)$ is determined by the maximum real part of all poles, i.e., $\max(\Re(p_1), \ldots, \Re(p_n))$, illustrated below.



(Fig acknowledgment: Prof. John Pauly)

This is because as $t \to \infty$, the pole with the largest real part becomes larger than the other terms no matter how big the residues.

More on poles and zeros (cont.)

Dominant poles (cont). For example, consider

$$H(s) = \frac{100}{s+2} + \frac{1}{s+1}, \qquad h(t) = 100e^{-2t} + e^{-t}$$

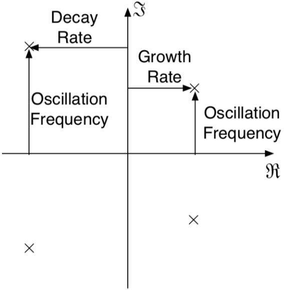
Though the residue of the pole at $s = -2$ is $100\times$ larger than the pole at $s = 1$, after $t > 4.6$, the term associated with the dominant pole, e^{-t}, is larger.

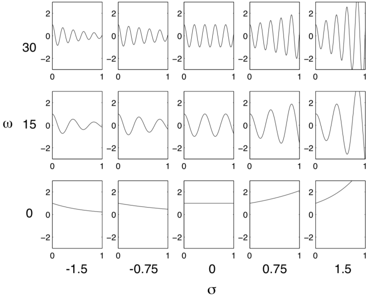
Qualitative characteristics of the output. Poles can be purely real, purely imaginary, or complex (with non-zero real and imaginary parts).

- Purely real poles correspond to exponential decay or growth. This is because the inverse Laplace transform of $1/(s+\sigma)$, with a real, is $\exp(-\sigma t)$.
- Purely imaginary poles correspond to oscillatory behavior. This is because the inverse Laplace transform of $1/(s+j\omega)$, where ω is a real, is $\exp(-j\omega t)$.
- Complex poles have both exponential decay / growth and oscillation. The oscillation is attenuated (or amplified) by an envelope given by the real part of the exponential, $e^{-\sigma t}$.

More on poles and zeros (cont.)

Qualitative characteristics of the output (cont). This is illustrated below.





$h(t) = e^{\sigma t}\cos(\omega t)$

(Fig acknowledgments: Prof. John Pauly)

More on poles and zeros (cont.)

Stability. For a system to be stable, all poles of $H(s)$ have to be on the left hand side of the plane. To see this, consider a zero-input LCCODE.

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)} + \cdots + a_1 y'(t) + a_0 y(t) = 0$$

This system is stable if all solutions converge to zero, regardless of the initial condition. Intuitively, this says all transients die to zero.

Taking the Laplace transform (with arbitrary initial condition), we have:

$$a_n \left[s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - y^{(n-1)}(0) \right]$$

$$+ a_{n-1} \left[s^{n-1} Y(s) - s^{n-2} y(0) - \cdots - y^{n-2}(0) \right] + \cdots + a_0 Y(s) = 0$$

Rearranging, we arrive at

$$Y(s) = \frac{q_{n-1} s^{n-1} + q_{n-2} s^{n-2} + \cdots + q_1 s + q_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

where q_k is a function of the initial conditions.

More on poles and zeros (cont.)

Stability (cont.). This expression for $Y(s)$ is a strictly proper rational fraction and hence can be written in partial fraction form. For $y(t)$ to decay to zero, all poles therefore have to have negative real part, so that the real part of the complex exponentials, $\sigma_i = \Re(p_i) < 0$, causes a decaying envelope to zero, i.e.,

$$e^{\sigma_i t} e^{j\omega t}$$

will always decay to zero as $t \to \infty$ if $\sigma_i < 0$.

Filtering

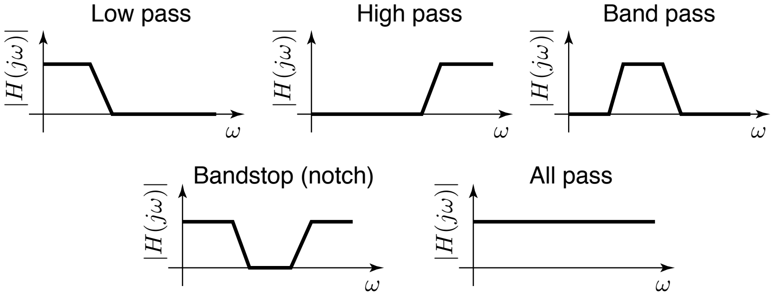
We prior talked about the *frequency response*, $H(j\omega)$. This is a special case of the Laplace transform when $\sigma = 0$ and $s = j\omega$. The frequency response completely describes an LTI system.

However, sometimes, the $j\omega$ axis may not be in the region of convergence, and so the frequency response may not exist. Further, we may be interested in *filter* design: how do we actually physically realize implementable filters?

For this, we turn to the Laplace transform. We will see that we can design filters to achieve certain qualities by choosing its poles and zeros.

Types of filters

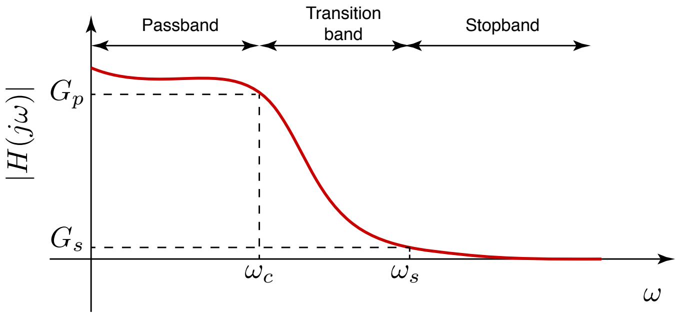
We previously talked about lowpass, highpass and bandpass filters. There are a few more that you may use.



- What might a notch filter be used for?
- What might an all pass filter be used for?

Filter terms

We're familiar with some filter terms; here are a few more as we consider non-ideal filters.



- G_p is the minimum passband gain. The passband is defined where the gain of the filter (i.e., $|H(j\omega)|$) is greater than G_p.
- G_s is the maximum stopband gain. The stopband gain is defined where the gain of the filter is less than G_s.
- ω_c is called the cut-off frequency, where the passband ends.
- ω_c is called the stopband frequency, where the stopband begins.

Filter design

In prior lectures, we talked about filters in terms of their frequency response, and they were e.g., a rect. These were ideal filters that were not implementable.

- Practically, real-time filters need to be implemented as an analog or discrete circuit.
- Circuits implement a rational function.
- Therefore, we need to analyze and design rational functions to arrive at a certain frequency response.

A rational function H can be factorized, as in the partial fraction lecture.

$$H(s) = \frac{b(s)}{a(s)} = k\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

where z_i are the zeros of $H(s)$ and p_i are the poles of $H(s)$.

Filter design (cont.)

With filters, we are concerned about how signals at certain frequencies are attenuated. To do this, a common tool are Bode plots. Bode plots quantify how the magnitude and phase of the transfer function, H change with frequency.

Bode magnitude: The Bode magnitude plots $20\log_{10}|H(j\omega)|$ as a function of ω. This is therefore a log-scale view of $|H(j\omega)|$, where a change in value of 20 corresponds to $H(j\omega)$ changing by one order of magnitude. Note:

$$20\log_{10}|H(j\omega)| \quad = \quad 20\log_{10}|k| + \sum_{i=1}^{m} 20\log_{10}|j\omega - z_i| - \sum_{i=1}^{n} 20\log_{10}|j\omega - p_i|$$

and so the magnitude response decays as we move away from poles, i.e., $|j\omega - p_i|$ large, and increases as we move away from zeros, i.e., $|j\omega - z_i|$ large. Is this what we intuitively expect?

Bode phase: The Bode phase plots $\angle H(j\omega)$ as a function of ω. Note:

$$\angle H(j\omega) \quad = \quad \angle k + \sum_{i=1}^{m} \angle(j\omega - z_i) - \sum_{i=1}^{n} \angle(j\omega - p_i)$$

As $\omega \gg p_i$ and $\omega \gg z_i$, each pole (with negative real part) shifts the signal by -90° while each zero (with negative real part) shifts the signal by $+90^\circ$. These

Filter design (cont.)

Taking the magnitude of $H(s)$, and plugging in $s = j\omega$ (to evaluate the frequency response given poles and zeros at certain locations), we arrive at

$$|H(j\omega)| = |k| \frac{|j\omega - z_1| \cdots |j\omega - z_m|}{|j\omega - p_1| \cdots |j\omega - p_n|}$$

As before, we see that

- $|H(j\omega)|$ gets relatively bigger when $j\omega$ is near a pole.
- $|H(j\omega)|$ gets relatively smaller when $j\omega$ is near a zero.

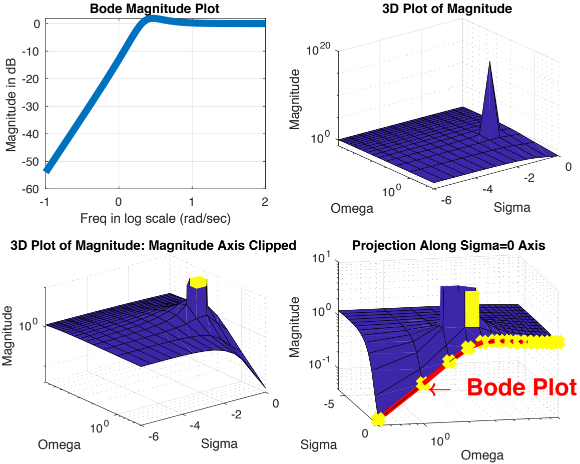
This is illustrated on the next slide.

Filter design (cont.)

(This example is thanks to: `https://tinyurl.com/y75dpem5`) Consider the transfer function

$$H(s) = \frac{s^2}{s^2 + 2s + 5}$$

This has two zeros at $s = 0$ and two poles at $s = 1 \pm 2j$.

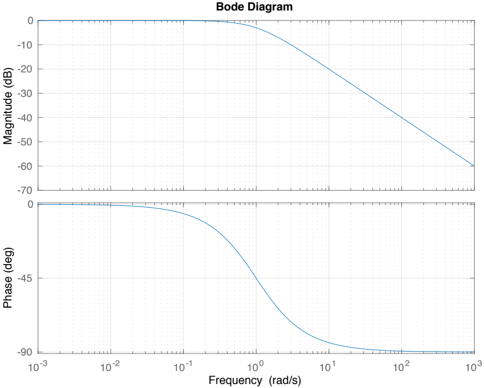


Single pole low pass filter

Consider the transfer function

$$H(s) = \frac{1}{s+a}$$

with $a > 0$. What, intuitively, should its Bode plot look like? Below is a Bode plot for $a = 1$.



Single pole low pass filter (cont.)

What we see is that:

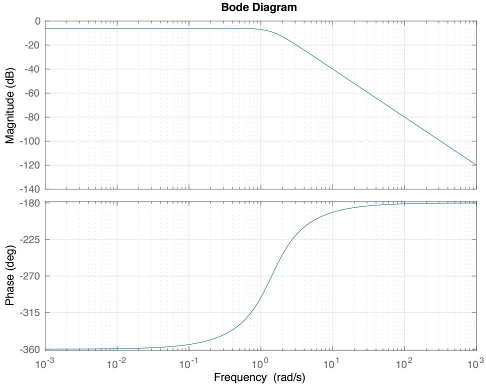
- The single pole low pass filter is far from ideal, but it does attenuate high frequencies.
- At its cut off frequency, which is $\omega_c = a$, it starts to attenuate higher frequencies.
- When $\omega_c > a$, attenuation occurs at 20 dB/decade. dB are the units decibals and decade refers to an order of magnitude of frequency. Recall each pole contributes a 20 dB attenuation every increase in order of magnitude of the frequency. Since there's only one pole, this is why the slope is -20 dB/decade.
- Also centered around $\omega_c = a$ is a frequency-dependent phase shift.

Two pole low pass filter

Consider two complex conjugate poles. Let's say our poles are at $-1 \pm j$. This has transfer function

$$H(s) = \frac{1}{s^2 - 2s + 2}$$

Its Bode plot is:



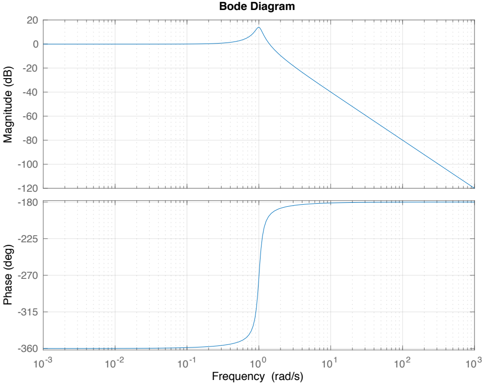
As there are two poles, the attenuation is at -40 dB/decade.

Two pole low pass filter (cont.)

What happens if we move the poles closer to the $j\omega$ axis? Let's say our poles are at $-0.1 \pm j$. This has transfer function

$$H(s) = \frac{1}{s^2 - 0.2s + 1.01}$$

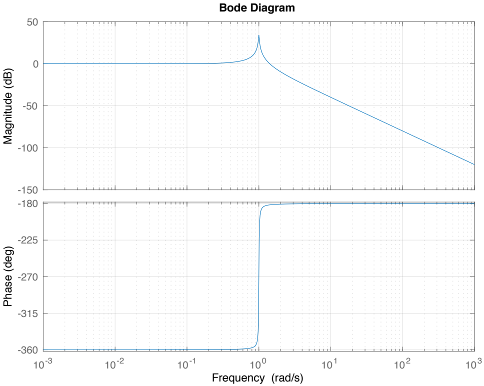
What, intuitively, should happen to the Bode plot?



Two pole low pass filter (cont.)

As the poles are closer to the $j\omega$ axis, they "pull up" the frequency response, providing amplification for a range of frequencies close to the pole. In general, as the poles get closer to the $j\omega$ axis,

- There is more amplification in the frequency response.
- The range of affected frequencies gets narrower – see below for $s = -0.01 \pm j$.



Two zero filter

If we design a filter with only two zeros, what does it qualitatively do?

- Since the zeros add 20 dB per decade, we should see that after the location of the zero in the Bode plot, the magnitude spectrum should increase by 20 dB/decade for each zero.
- Near the zero, the frequency response should dip (as long as the zero is sufficiently close to the $j\omega$ axis.
- An example is shown on the next page for the zeros being at $-0.1 \pm j$, leading to transfer function

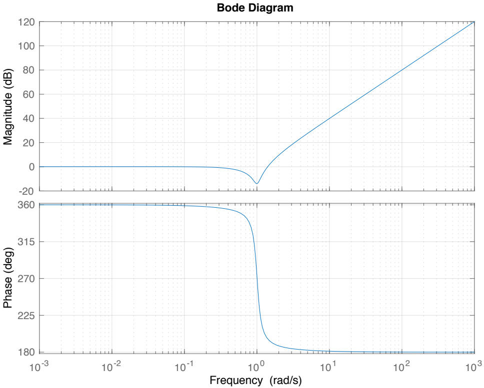
$$H(s) = s^2 - 0.2s + 1.01$$

Two zero filter (cont.)

The bode plot of the two zeros filter:

$$H(s) = s^2 - 0.2s + 1.01$$

is shown below.



Filter design

This gives us ideas for how to implement certain filters.

- How would we place poles and zeros to get a low pass filter?
- How would we place poles and zeros to get a high pass filter?
- How would we place poles and zeros to get a bandpass filter?
- How would we place poles and zeros to get a bandstop filter?

Example high pass filter

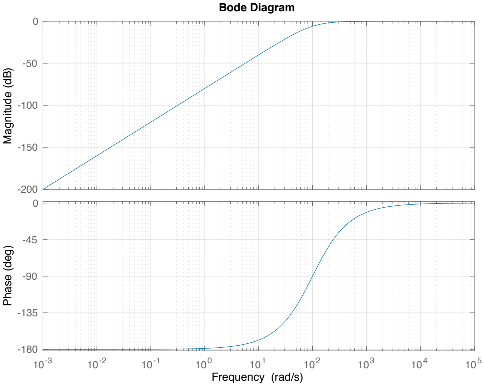
To get a high pass filter, we should have at least one zero at $\omega = 0$ (which starts to give a 20 dB per decade rise for each zero). Then to stop the rise, we should put the same number of poles at the desired passband frequency. For example, let's say we wanted a high pass filter that passed frequencies at $\omega = 100$ radians per second and with a rise of 40 dB per decade. We would put two zeros at the origin, and then place two poles at $s = 100$. This is a transfer function

$$H(s) = \frac{s^2}{(s-100)^2}$$

Note: you could put the poles at different locations in the complex plane (such that their magnitude was 100 which would adjust the phase of the filter).

The Bode plot of this example filter is shown on the next slide.

Example high pass filter (cont.)



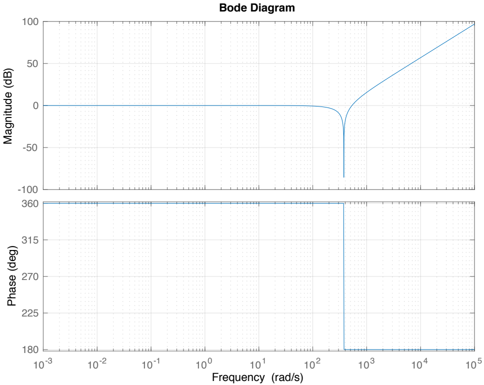
Notch filter

Let's say we were building a system but there was a lot of 60 Hz line noise. We want to pass all frequencies, but zero out the 60 Hz line noise so it doesn't corrupt our signal. This means we need a strong notch filter at 60 Hz.

To implement this, we should use a zero, close to the $j\omega$ axis. Let's say we put it at $-0.01 \pm 2\pi \cdot 60j$. This would cause substantial attenuation around 60 Hz. However, if we only had these zeros, the frequency response would rise at 40 dB per decade around 60 Hz. See below for

$$H(s) = \frac{1}{(2\pi \cdot 60)^2}(s^2 - 0.02s + (2\pi \cdot 60)^2 + 0.01^2)$$

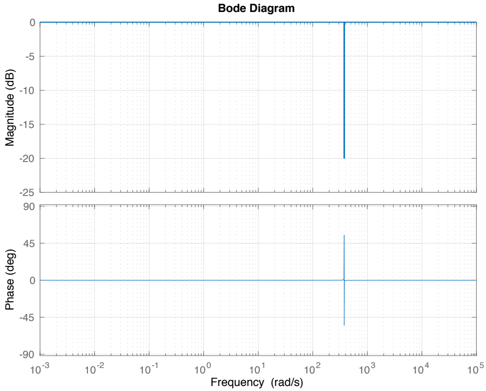
Notch filter (cont.)



Notch filter (cont.)

To ameliorate this, we should stop the rise by placing two poles at a frequency right after 60 Hz to cancel out the 40 dB per decade rise. Let's place two poles at $-0.05 \pm 2\pi \cdot 60j$, so that now:

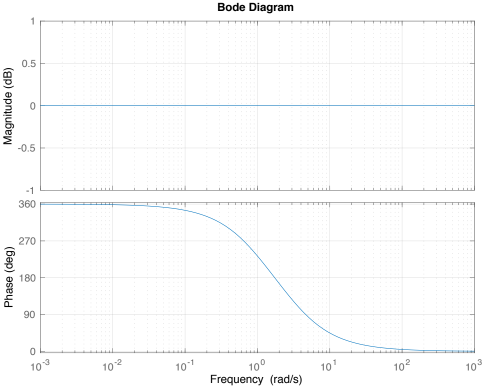
$$H(s) = \frac{s^2 - 0.02s + (2\pi \cdot 60)^2 + 0.01^2}{s^2 - 0.1s + +(2\pi \cdot 60)^2 + 0.05^2}$$



All pass filter

The point of an all pass filter is to let the signal pass, but to change its phase. Therefore, the poles and zeros should mirror each other (i.e., they cancel out) but contribute different amounts of phase. Consider

$$H(s) = \frac{(s-1)(s-3)}{(s+1)(s+3)} = \frac{s^2 - 4s + 3}{s^2 + 4s + 3}$$



Filter design

Filter design encompasses far more, and there are some standard filters that weigh certain properties you could use. Some more common examples are:

- Butterworth filters. These filters provide a maximally flat passband. Butterworth filters only have poles, and so they decrease at 20 dB per decade for each pole.
- Chebyshev Type I and Type II filters. Chebyshev filters allow the transition band to be much steeper, but the cost is that the passband (Type I) or the stopband (Type II) will have some amount of ripple.
- Bessel filters. Bessel filters have a maximally flat group delay (i.e., maximally linear phase response).

Feedback systems

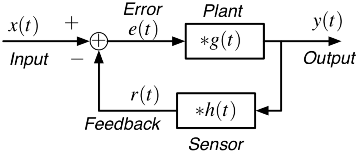
Real systems are sensitive to things like:

- Parameter variation. This means if you slightly change the parameters of the system, it does wildly different things. This is not great, because e.g., when building a circuit, it's very reasonable to have components (e.g., resistors and capacitors) that don't take on exactly the value you want.
- External disturbance. If your system is very sensitive, external disturbances (like noise) can substantially corrupt the output. This is something we want to avoid.
- Nonlinearities. Most real systems have nonlinearity. When we idealize a transistor and do 'small signal analysis,' we're doing local linearization under the assumption that the circuit is linear. But in reality, the circuits are not nonlinear. Not taking this to account can drastically change the performance of your system if it is sensitive.

Feedback helps to reduce these sensitivities, resulting in more robust systems.

Feedback systems (cont.)

The general idea is to feed the output, $y(t)$, back into the system. A block diagram below demonstrates this.



(Fig acknowledgment: Prof. John Pauly)

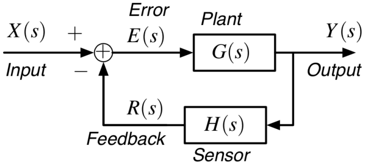
Here,

- The plant, $g(t)$, is the system we want to control.
- The sensor, $h(t)$, is a measurement of the output.
- The system is fedback, and subtracted from the original signal, to produce the error signal, $e(t)$. The plant processes the error, which is given by

$$e(t) = x(t) - y(t) * h(t)$$

Feedback systems (cont.)

We can take the Laplace transform of this system.



(Fig acknowledgment: Prof. John Pauly)

Hence,

$$Y(s) = (X(s) - H(s)Y(s))G(s)$$

and rearranging terms, we have

$$Y(s) + G(s)H(s)Y(s) = G(s)X(s)$$

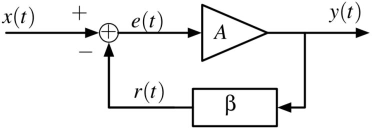
This produces an overall transfer function

$$T(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Feedback amplifier example

$$T(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The term $G(s)H(s)$ is called the *loop gain*. It turns out that this form of the transfer function helps to reduce the sensitivities earlier mentioned. Let's see this with an example.



(Fig acknowledgment: Prof. John Pauly)

Say $A = 1000$ and therefore, without feedback, any sensitivity in A can wildly change the output. The transfer function of this system is

$$T = \frac{A}{1 + \beta A}$$

In this manner, we can choose β to achieve some desired gain, T.

Feedback amplifier example (cont.)

Example: say we wanted a gain of $T = 10$. What should the feedback multiplier, β, be? If we solve for β, we have that

$$\beta = \frac{1}{A}\left(\frac{A}{T} - 1\right)$$

and so for $T = 10$, we have that $\beta = 0.099$.

Let's see how this reduces sensitivity. Say ideally, $A = 1000$. In the open loop system (absent of feedback), say we had variation such that $A = 900$ and we wanted to gain a signal by 10. But

$$T = \frac{900}{1 + (0.099)(900)} = 9.989$$

which is only off by a factor of 0.11%.

This means we traded a factor of 100 in gain (from $A = 1000$ to $T = 10$) in exchange for a factor of 100 reduction in sensitivity to variations in A.

Sensitivity reduction

Let's say

$$T = \frac{G}{1+GH}$$

If G changes by some amount, it changes T. We can calculate the sensitivity, which is

$$S = \frac{\Delta T/T}{\Delta G/G}$$

A change, ΔG, produces a change ΔT, via,

$$\begin{aligned}
\Delta T &= \frac{\partial T}{\partial G}\Delta G \\
&= \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right)\Delta G \\
&= \left(\frac{1}{1+GH} - \frac{GH}{(1+GH)^2}\right)\Delta G \\
&= \frac{1}{(1+GH)^2}\Delta G
\end{aligned}$$

Sensitivity reduction (cont.)

Then,

$$\begin{aligned}\frac{\Delta T}{T} &= \frac{1}{(1+GH)^2}\Delta G \cdot \frac{1+GH}{G} \\ &= \frac{1}{1+GH}\frac{\Delta G}{G}\end{aligned}$$

and hence,

$$S = \frac{\Delta T/T}{\Delta G/G} = \frac{1}{1+GH}$$

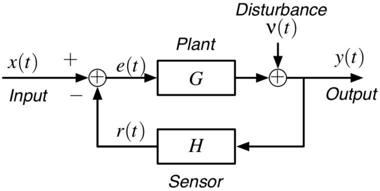
And so, if GH is chosen to be large, this drastically reduces the sensitivity of the system.

Note that if GH is very large, then

$$\begin{aligned}T &\approx \frac{1}{H} \\ S &\approx \frac{1}{GH}\end{aligned}$$

Disturbance reduction

Say the output was corrupted by noise $\nu(t)$.



(Fig acknowledgment: Prof. John Pauly)

The transfer function of this system is:

$$Y = \frac{G}{1+GH}X + \frac{1}{1+GH}V$$

where $\nu(t) \iff V(s)$. If $GH \gg 1$, then

$$Y \approx \frac{1}{H}X + \frac{1}{GH}V$$

hence substantially attenuating V.

Increasing the bandwidth of systems

In systems that vary as a function of time, i.e., convolution with some $g(t) \iff G(s)$, feedback, like before, will decrease the gain of the system, but also increases the bandwidth of the system (making the system lower gain, but faster). For example, consider

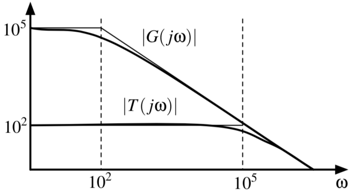
$$G(s) = \frac{10^5}{1 + s/100}$$

This system has a DC gain of 10^5 and a bandwidth of 100 rad/s. When we incorporate feedback, say $H = 0.01$, the system becomes

$$\begin{aligned} T(s) &= \frac{G(s)}{1 + G(s)H(s)} \\ &= \frac{\frac{10^5}{1+s/100}}{1 + \frac{10^5}{1+s/100} \cdot 0.01} \\ &= \frac{99.9}{1 + s/(1.001 \times 10^5)} \end{aligned}$$

This system now has a gain of approximately 100, but it has a bandwidth of 10^5 rad/s. A Bode plot is shown on the next slide.

Increasing the bandwidth of systems (cont.)



(Fig acknowledgment: Prof. John Pauly)

This is commonly used in the design of operational amplifiers. If they must amplify over a large bandwidth, analog designers will often use such feedback compensation through a "compensation capacitor."

Stabilizing an unstable system

Feedback can also be used to stabilize and unstable system. Consider a system with

$$G(s) = \frac{1}{s-a}$$

where $a > 0$. This transfer function $G(s)$ is an increasing exponential, and hence convolution with $g(t)$ will produce an unstable system. If we set $H(s) = K$, we notice that

$$\begin{aligned} T(s) &= \frac{G(s)}{1 + G(s)H(s)} \\ &= \frac{1/(s-a)}{1 + K/(s-a)} \\ &= \frac{1}{s - a + K} \end{aligned}$$

and therefore if $K > a$, this system becomes stable.

Inverting a system

Let's say that we want to recover an input $x(t)$ from its output

$$y(t) = g(t) * x(t) \iff Y(s) = G(s)X(s)$$

To invert this system, we need to implement $1/G(s)$, i.e.,

$$X(s) = \frac{1}{G(s)} Y(s)$$

How do we implement this system? We can use a feedback system with $G = K$ (a constant) and $H(s) = G(s)$. Then,

$$T(s) = \frac{K}{1 + KG(s)}$$

and as long as $KG(s) \gg 1$, this implements the inverse system.