Frequency response

This handout covers the frequency response, which is the Fourier transform of the impulse response. Topics include:

- Transfer function: amplitude and phase
- Low pass filter
- High pass filter
- Band pass filter
- Example: AM radio
- Causal filters
- Amplitude and phase distortion

Motivation for this lecture

We previously discussed the impulse response, $h(t)$, which is the output of a system when the input is an impulse, $\delta(t)$. We saw that $h(t)$ characterized any LTI system, as for any LTI system with input, $x(t)$, we could calculate the output as

$$\begin{aligned} y(t) &= (x * h)(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned}$$

A complication we discussed is that computing the output this way requires evaluating a convolution integral, which can be difficult and time-consuming.

Motivation for this lecture (cont.)

But now, equipped with the convolution theorem, why not just take the Fourier transform of both sides? This turns the convolution into multiplication.

$$Y(j\omega) = H(j\omega)X(j\omega)$$

where $X(j\omega)$ is the Fourier transform of the input, $Y(j\omega)$ is the Fourier transform of the output, and $H(j\omega)$ is the *frequency response*, i.e., the Fourier transform of the impulse response.

- In addition to *frequency response*, $H(j\omega)$ is sometimes called the *transfer function* of the system.
- The reason its called frequency response is that $H(j\omega)$ describes how the input is changed at every single frequency.
- In particular, the frequency response scales the amplitude response by $|H(j\omega)|$, i.e.,
$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$
- The frequency response shifts the phase response by $\angle H(j\omega)$, i.e.,
$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Motivation for this lecture (cont.)

To see this, note that if the input to a system is a complex exponential, $e^{j\omega_0 t}$ (recall, these are the eigenfunctions of an LTI system), then

$$\begin{aligned} X(j\omega) &= \mathcal{F}\left[e^{j\omega_0 t}\right] \\ &= 2\pi\delta(\omega-\omega_0) \end{aligned}$$

Therefore, the output is

$$\begin{aligned} Y(j\omega) &= H(j\omega)(2\pi\delta(\omega-\omega_0)) \\ &= H(j\omega_0)(2\pi\delta(\omega-\omega_0)) \end{aligned}$$

This means that

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}[Y(j\omega)] \\ &= \mathcal{F}^{-1}[H(j\omega_0)(2\pi\delta(\omega-\omega_0))] \\ &= H(j\omega_0)e^{j\omega_0 t} \\ &= \left|H(j\omega_0)\right| e^{j(\omega_0 t+\angle H(j\omega_0))} \end{aligned}$$

Motivation for this lecture (cont.)

To summarize here, we input a complex exponential input, $x(t) = e^{j\omega_0 t}$ to a LTI system, and saw that the output was

$$y(t) = |H(j\omega_0)| \, e^{j(\omega t + \angle H(j\omega_0))}$$

i.e., inputting a complex sinusoid to an LTI system produces an output that:

- is at the same frequency, ω_0.
- is scaled in amplitude by $|H(j\omega_0)|$.
- is phase shifted by $\angle H(j\omega_0)$.

Frequency response example

Consider the input:

$$x(t) = 2\cos(t) + 3\cos(3t/2) + \cos(2t)$$

and system with impulse response

$$h(t) = \frac{2}{\pi}\operatorname{sinc}^2(t/\pi)$$

Find $y(t) = (x * h)(t)$.

Frequency response example (cont.)

First, we find the frequency response (transfer function), which is

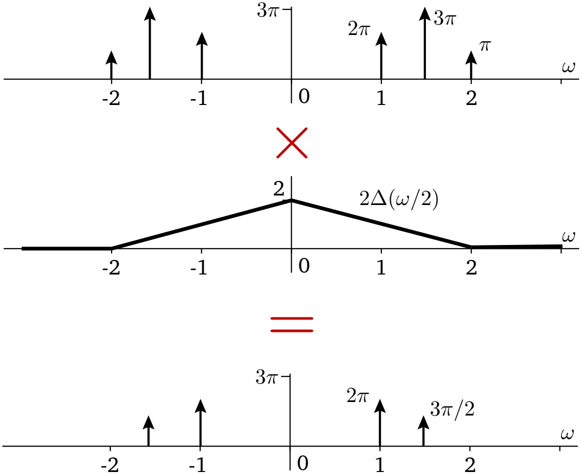
$$\begin{aligned} H(j\omega) &= \mathcal{F}\left[\frac{2}{\pi}\operatorname{sinc}^2(t/\pi)\right] \\ &= \frac{2}{\pi}\pi\Delta(\pi\omega/2\pi) \\ &= 2\Delta(\omega/2) \end{aligned}$$

Second, we recognize that

$$\begin{aligned} X(j\omega) &= 2\pi\left[\delta(\omega-1)+\delta(\omega+1)\right]+3\pi\left[\delta(\omega-3/2)+\delta(\omega+3/2)\right] \\ &\quad +\pi\left[\delta(\omega-2)+\delta(\omega+2)\right] \end{aligned}$$

To get $Y(j\omega)$, all we do is multiply the two together. This is shown below.

Frequency response example (cont.)



Frequency response example (cont.)

This gives that

$$Y(j\omega) = 2\pi\left[\delta(\omega-1)+\delta(\omega+1)\right] + \frac{3\pi}{2}\left[\delta(\omega-3/2)+\delta(\omega+3/2)\right]$$

Taking the inverse Fourier transform, we get that

$$y(t) = 2\cos(t) + \frac{3}{2}\cos(3t/2)$$

Frequency response example 2

Let $x(t) = e^{-t}u(t)$. We input this signal into a system with impulse response:

$$h(t) = 2e^{-2t}u(t)$$

What are $Y(j\omega)$ and $y(t)$?

Like before, we take the spectrum of the input:

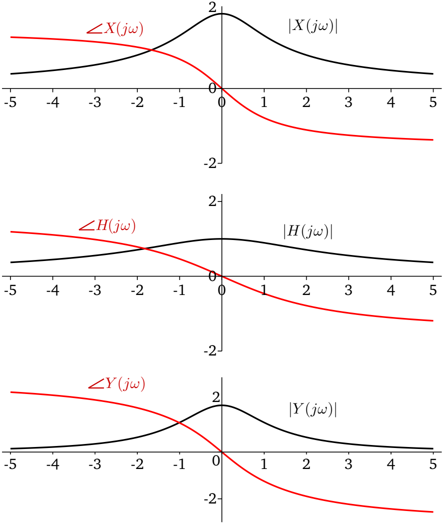
$$\begin{aligned} X(j\omega) &= \mathcal{F}\left[e^{-t}u(t)\right] \\ &= \frac{1}{1+j\omega} \end{aligned}$$

The frequency response is:

$$\begin{aligned} H(j\omega) &= \mathcal{F}[2e^{-2t}u(t)] \\ &= \frac{2}{2+j\omega} \end{aligned}$$

Frequency response example 2 (cont.)

Recall that when we multiply two complex numbers, their magnitudes multiply and their phases add. This is shown below.



Frequency response example 2 (cont.)

We have that

$$Y(j\omega) = \frac{2}{(1+j\omega)(2+j\omega)}$$

To get this into a form that we can take the inverse Fourier transform of, we use the technque of partial fractions, i.e., we set

$$\frac{2}{(1+j\omega)(2+j\omega)} = \frac{A}{1+j\omega} + \frac{B}{2+j\omega}$$

This yields that:

$$2 = A(2+j\omega) + B(1+j\omega)$$

Since the real parts of the l.h.s. and r.h.s., as well as the imaginary parts of the l.h.s. and r.h.s. must be equal, we have that:

$$\begin{aligned} 2A + B &= 2 \\ A + B &= 0 \end{aligned}$$

The solution to this is $A = 2$ and $B = -2$. (We'll get much more practice with this during Laplace transforms.)

Frequency response example 2 (cont.)

Therefore, we have that

$$Y(j\omega) = \frac{2}{1+j\omega} - \frac{2}{2+j\omega}$$

The inverse Fourier transform of this is

$$y(t) = 2\left(e^{-t} - e^{-2t}\right)u(t)$$

This is a good time to check that indeed the following relations hold:

$$\begin{aligned} |Y(j\omega)| &= |H(j\omega)||X(j\omega)| \\ \angle Y(j\omega) &= \angle H(j\omega) + \angle X(j\omega) \end{aligned}$$

as we diagrammed above.

Filters

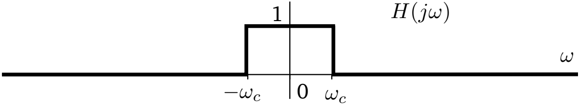
Filters are designed to extract or attenuate certain desired frequencies from a signal. For example, consider a recording of music where the microphones accidentally recorded the sopranos too loudly. It would be possible to rebalance the audio by attenuating higher frequencies in the signal.

We'll first discuss ideal filters, which only pass through certain frequencies. There are three main types of filters we'll discuss:

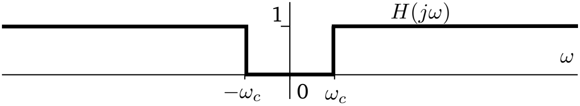
- Low pass filter: suppresses all frequencies that are higher than a specified frequency, ω_c. Its name comes from the fact that it lets frequencies less than ω_c through (i.e., low frequencies).
- High pass filter: supresses all frequencies that are lower than a specified frequency, ω_c.
- Band pass filter: supresses all frequencies outside of a range $\pm\omega_c$ around a chosen frequency ω_0.
- These three filters are illustrated on the next page.

Filter illustration

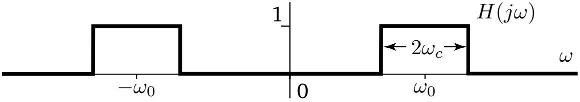
- Low pass filter:



- High pass filter:



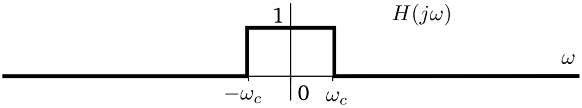
- Band pass filter:



These may alternatively be written as “lowpass” or “low-pass” filter, etc.

Ideal low pass filter

The ideal low pass filter is



We call the region where frequencies are not suppressed (i.e., up to frequency ω_c for this ideal low pass filter) the "passband." This filter can be represented as

$$H(j\omega) = \mathrm{rect}(\omega/(2\omega_c))$$

To find its impulse response, we use the Fourier transform pair we previously derived

$$\mathrm{rect}(t/T) \iff T\,\mathrm{sinc}(\omega T/2\pi)$$

and then apply duality to get that:

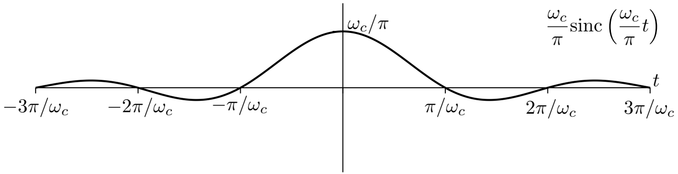
$$T\,\mathrm{sinc}(tT/2\pi) \iff 2\pi\mathrm{rect}(\omega/T)$$

Ideal low pass filter (cont.)

Setting $T = 2\omega_c$, its impulse response is:

$$\begin{aligned}
\mathcal{F}^{-1}[\text{rect}(\omega/(2\omega_c))] &= \mathcal{F}^{-1}\left[\frac{1}{2\pi}2\pi\text{rect}(\omega/(2\omega_c))\right] \\
&= \frac{1}{2\pi}2\omega_c \,\text{sinc}(t2\omega_c/2\pi) \\
&= \frac{\omega_c}{\pi}\,\text{sinc}\left(\frac{\omega_c}{\pi}t\right)
\end{aligned}$$

Thus, the ideal low pass filter's impulse response is:



Note that we've only shown a small interval here, the sinc function is nonzero for t outside of the plotted window.

Ideal low pass filter (cont.)

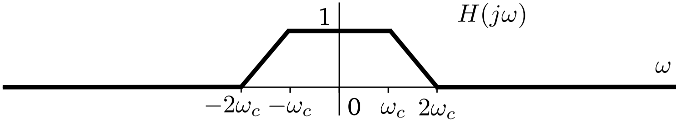
Great, so now we know how to remove all frequencies in a signal above a certain frequency ω_c, right?

What are limitations of actually filtering this signal?

- The impulse response is non-causal, i.e., it is nonzero for $t < 0$. Hence, ideal low pass filtering the signal at time t requires convolution with future signal, which might not be known to us.
- The filter impulse response has an infinite duration, and thus to convolve would take an infinite amount of time. Realistically, $\mathrm{sinc}(\cdot)$ is close to zero as $|t|$ grows, so it could be truncated.
- But even if we truncate it, we see that $\mathrm{sinc}(t)$ decays with an envelope of $1/t$ (since $\mathrm{sinc}(t) = \sin(\pi t)/\pi t$), and so the decay is relatively slow. It would be better if the impulse response decayed faster, like $1/t^2$, etc. This means the impulse response convolution requires less computation time.

A (less ideal) low pass filter

Consider the following low pass filter, which is not ideal because it lets in some frequencies between ω_c and $2\omega_c$, albeit attenuated. This region (from $[\omega_c, 2\omega_c]$) is sometimes called the *transition band*.



How do we write $H(j\omega)$ in a way that is amenable to a simple inverse Fourier transform using things we already know?

- We note this can be written as the convolution of two rects. The total width of the convolution is $4\omega_c$, so the sum of the width of the two rects needs to equal $4\omega_c$. One rect should be of length ω_c to give us the transition band (will discuss this intuition in class). You should check that this filter can indeed be written as

$$H(j\omega) = \mathrm{rect}(\omega/3\omega_c) * \mathrm{rect}(\omega/\omega_c)$$

A (less ideal) low pass filter (cont.)

- We know that

$$\mathrm{rect}(\omega/3\omega_c) \iff \frac{3\omega_c}{2\pi}\,\mathrm{sinc}\left(\frac{3\omega_c}{2\pi}t\right)$$

and

$$\mathrm{rect}(\omega/\omega_c) \iff \frac{\omega_c}{2\pi}\,\mathrm{sinc}\left(\frac{\omega_c}{2\pi}t\right)$$

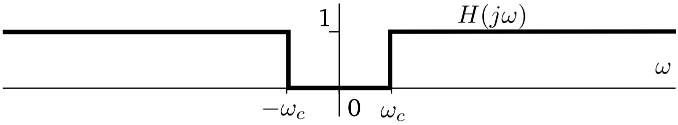
and therefore by the convolution theorem, we have that

$$h(t) = \left(\frac{3\omega_c}{2\pi}\,\mathrm{sinc}\left(\frac{3\omega_c}{2\pi}t\right)\right)\left(\frac{\omega_c}{2\pi}\,\mathrm{sinc}\left(\frac{\omega_c}{2\pi}t\right)\right)$$

- Since this is the product of $\mathrm{sinc}(\cdot)$'s, it decays as $1/t^2$.
- Does this make sense intuitively?
- Hence, the general trend is that a smoother transition band produces a shorter (in time) impulse response.

Ideal high pass filter

The ideal high pass filter is



This can be written as 1 minus the ideal low pass filter, i.e.,

$$H(j\omega) = 1 - \mathrm{rect}(\omega/(2\omega_c))$$

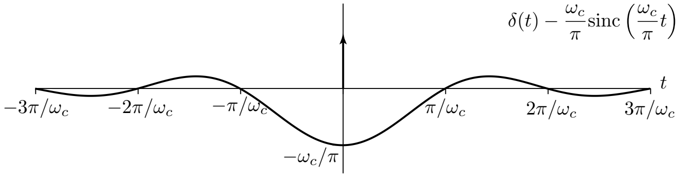
Therefore, its impulse response is

$$h(t) = \delta(t) - \frac{\omega_c}{\pi} \mathrm{sinc}\left(\frac{\omega_c}{\pi} t\right)$$

Its impulse response is plotted below.

Ideal high pass filter (cont.)

The impulse response of the ideal high pass filter is:

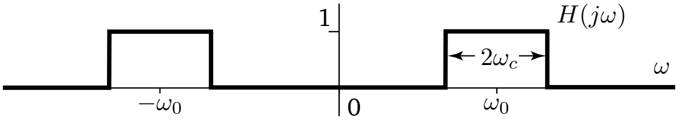


This filter has the same limitations of the ideal low pass filter, with the additional inconvenience that we need to generate an approximate $\delta(t)$.

- Can we get rid of the $\delta(t)$ in the high pass filter?
- How do we get around having a $\delta(t)$ in the impulse response?

Ideal band pass filter

The ideal band pass filter is



The frequency response of this filter is

$$H(j\omega) = \text{rect}\left(\frac{\omega + \omega_0}{2\omega_c}\right) + \text{rect}\left(\frac{\omega - \omega_0}{2\omega_c}\right)$$

We can write this as the convolution of an ideal low pass filter with two delta functions (this should remind us of modulation),

$$H(j\omega) = H_{\mathsf{LPF}}(j\omega) * (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$

Ideal band pass filter (cont.)

$$H(j\omega) = H_{\mathsf{LPF}}(j\omega) * (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$

Since

$$h_{\mathsf{LPF}}(t) = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi} t\right)$$

and

$$\delta(\omega - \omega_0) + \delta(\omega + \omega_0)$$

has inverse Fourier transform

$$\frac{1}{\pi} \cos(\omega_0 t)$$

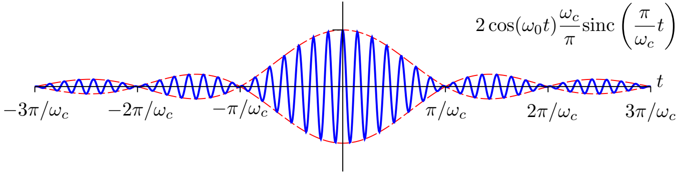
Hence, we have that

$$\begin{aligned} h(t) &= 2\pi \frac{1}{\pi} \cos(\omega_0 t) \left(\frac{\omega_c}{\pi}\right) \operatorname{sinc}\left(\frac{\omega_c}{\pi} t\right) \\ &= 2 \cos(\omega_0 t) \left(\frac{\omega_c}{\pi}\right) \operatorname{sinc}\left(\frac{\omega_c}{\pi} t\right) \end{aligned}$$

This impulse response is drawn on the next slide, but is merely the LPF impulse response multiplied by a cosine and amplified by 2.

Ideal band pass filter (cont.)

The impulse response of the ideal band pass filter is:



Hence, convolution with this impulse response extracts out frequencies in a range $\omega_0 \pm \omega_c$! The same intuitions about using a transition band to make the filter more practical hold.

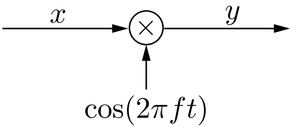
Example: AM radio

We previously talked about amplitude modulation (AM) which is used for e.g. AM radio. We are now equipped to understand how it works. Here is the problem setup:

- Say we want to transmit a message signal, $m(t)$.
- To avoid interference and leverage the electromagnetic spectrum, we are given a frequency range over which we can transmit this signal.
- Anyone with knowledge of what frequency we are transmitting on can tune a receiver to receive the message signal, $m(t)$.

Example: AM radio (cont.)

The fundamental idea of AM radio is to "multiply by a cosine." We know that when we do so, we implement the following system:



so that the transmitted signal is $m(t)\cos(\omega_c t)$ with Fourier transform

$$\frac{1}{2}\left[M(j(\omega+\omega_c)) + M(j(\omega-\omega_c))\right]$$

Hence, multiplying by a cosine creates copies of the signal at $\pm\omega_c$.

Example: AM radio (cont.)

After the message signal is transmitted how should the receiver (e.g., the radio in your car) recover the original signal? i.e., how do we manipulate the spectrum of the transmitted signal, so we can get back $M(j\omega)$?

Intuition: if we want to move either of the modulated copies at $\pm\omega_c$ back to the origin, we need to "de-modulate" the signal. One way to do this is to convolve the modulated signals with delta functions centered at $\pm\omega_c$. This corresponds to multiplying the received signal, $m(t)\cos(\omega_c t)$ by $\cos(\omega_c t)$.

Example: AM radio (cont.)

Let's show this mathematically. If

$$\mathcal{F}[m(t)\cos(\omega_c t)] = \frac{1}{2}M(j(\omega+\omega_c)) + \frac{1}{2}M(j(\omega-\omega_c))$$

Then,

$$\begin{aligned}
\mathcal{F}[m(t)\cos^2(\omega_c t)] &= \frac{1}{2\pi}\left[\frac{1}{2}M(j(\omega+\omega_c)) + \frac{1}{2}M(j(\omega-\omega_c))\right]\cdots \\
&\quad \cdots * \left[\pi\delta(\omega+\omega_c) + \pi\delta(\omega-\omega_c)\right] \\
&= \frac{1}{4}\left[M(j(\omega+\omega_c) + M(j(\omega-\omega_c)\right] * \left[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)\right] \\
&= \frac{1}{4}M(j(\omega+2\omega_c)) + \frac{1}{2}M(j\omega) + \frac{1}{4}M(j(\omega-2\omega_c))
\end{aligned}$$

After this, what kind of filtering do we do to extract $M(j\omega)$?

Example: AM radio (cont.)

What if we demodulated by multiplying with a sine? i.e.,

$$m(t)\cos(\omega_c t)\sin(\omega_c t)$$

(Ans: this would not recover the signal, as two modulated signals would cancel out upon demodulation.)

How about if we demodulated by multiplying by a complex exponential? i.e.,

$$m(t)\cos(\omega_c t)e^{j\omega_c t}$$

Ideal filters are not causal

The impulse responses of the ideal low pass, high pass, and band pass filters are all non-causal. This is obviously not practical in real-time scenarios, where the future is not known.

To this end, it seems we have a problem. If we can't implement non-causal filters, then there has to be some approximations made (beyond introducing transition bands).

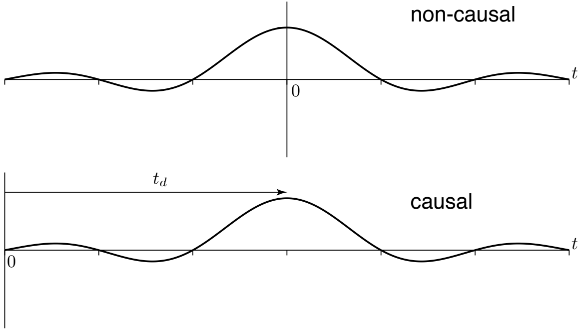
What we ought to recognize is that we will never be able to take a signal at time t, given by $x(t)$, and immediately filter it. However, we could wait a bit of time, and then filter $x(t)$. This enables us to implement a causal filter, with the con being that the signal is delayed.

Causal filter

A practically implementable system that is causal must introduce a delay. For example, we could implement a practical and causal low pass filter by taking the impulse response of the ideal low pass filter, and:

- Truncate it (so that it is zero for some time $|t| > t_d$).
- Delay it by time t_d so that it is causal.

This is illustrated below.



Distortionless LTI systems

When using filters, distortion could be introduced. In the frequency domain, depending on the filter being used, components of the signal at different frequencies may be amplified differently or may be delayed. We know that filtering causes:

- Amplitude scaling by $|H(j\omega)|$
- Phase shifting by $\angle H(j\omega)$

and depending on the frequencies of the signal, this could cause distortion to the system.

A system is without distortion if

$$y(t) = Kx(t - t_d)$$

where K is a scaling factor and t_d is a delay. This tells us that a distortionless signal is one that is identical to the input signal up to amplification and a constant delay.

Distortionless LTI systems (cont.)

What is $H(j\omega)$ for a distortionless LTI system? We can calculate

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Taking the Fourier transform of both sides of the distortionless equation, we have that

$$Y(j\omega) = Ke^{-j\omega t_d}X(j\omega)$$

and therefore

$$H(j\omega) = Ke^{-j\omega t_d}$$

What is its impulse response? Does it make sense? Is it implementable?

Distortionless LTI systems (cont.)

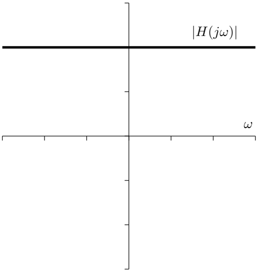
This is a system with

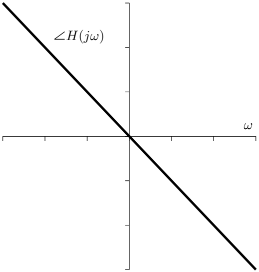
$$|H(j\omega)| = K$$

and

$$\angle H(j\omega) = -\omega t_d$$

It has the following amplitude and phase spectra:



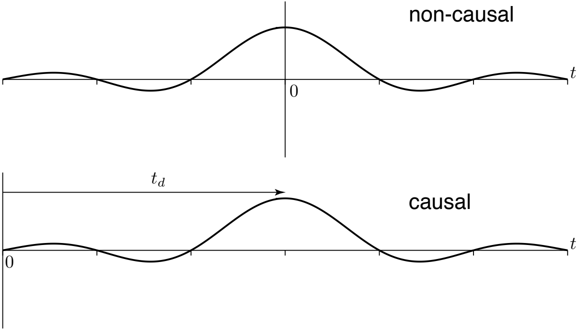


Causal filter

Let's return to our causal, implementable filter:

- Truncate it (so that it is zero for some time $|t| > t_d$).
- Delay it by time t_d so that it is causal.

This is illustrated below.



How distortion relates to implementable systems

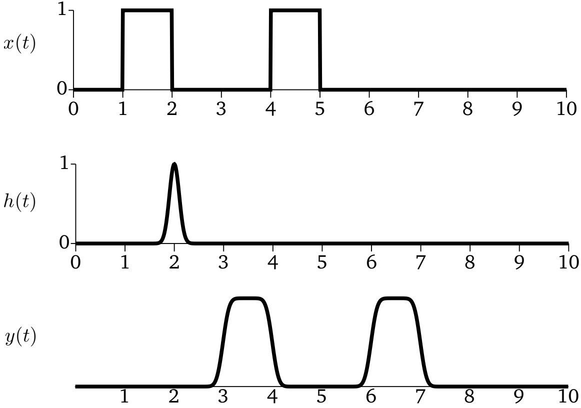
How do these operations distort the system?

- Truncating the impulse response is equivalent to multiplying the signal in time by a rect with width $2t_d$. What does this do to the frequency response?
- Delaying the response by t_d is equivalent to convolution in the frequency domain by $\delta(t - t_d)$. What does this do to the frequency response?

In short, we see that there are two distortions: amplitude and phase distortion.

Amplitude distortion

Amplitude distortion is relatively straightforward. See below.



Phase distortion

We also see that the phase response can have a substantial effect on the output signal. For example, $\delta(t)$ and $\delta(t - t_d)$ have the same amplitude response, but due to different phase response, the latter delays the signal.

There is a more sophisticated example of phase distortion here: `http://eeweb.poly.edu/iselesni/EL6113/matlab_examples/phase_distortion_demo/html/phase_distortion_demo.html`

In general, phase distortion can lead to considerable differences in the output.

Group delay

We know that convolution with $\delta(t - t_d)$ only delays the signal, but introduces no distortion. Further, $\mathcal{F}[\delta(t - t_d)] = e^{-j\omega t_d}$, and so its phase is

$$\angle H(j\omega) = -\omega t_d$$

It turns out that if the phase is linear, the signal is only delayed in time without distortion. Why is this intuitively the case?

Consider a simple example, a signal $x(t) = \cos(\omega t) + \cos(2\omega t)$. Delaying this signal by π/ω, i.e., $y(t) = x(t) * \delta(t - \pi/\omega)$ gives

$$\begin{aligned} y(t) &= \cos(\omega(t - \pi/\omega)) + \cos(2\omega(t - \pi\omega)) \\ &= \cos(\omega t - \pi) + \cos(2\omega - 2\pi) \end{aligned}$$

This makes intuitive sense; if the signal is higher frequency, we need a larger phase shift to gain the same temporal offset.

Group delay (cont.)

When the phase is not linear, different frequencies will experience different temporal delays. Therefore, the amount of delay will be frequency dependent, i.e., t_d will be a function of ω. Since $\angle H(j\omega) = -\omega t_d$, it is straightforward to derive what the frequency-dependent temporal delay is, i.e., it's

$$t_d(\omega) = -\frac{d}{d\omega}\angle H(j\omega)$$

From this, we can see that if $\angle H(j\omega)$ is a line with slope $-k$, then its derivative is simply $-k$, leading to every frequency having the same delay $t_d = k$. This confirms our intuition from the prior slide.

This quantity, $t_d(\omega)$, is called *group delay*.

Amplitude and phase distortion

It turns out that for audio or speech,

- Amplitude distortion is quite noticeable to humans.
- However, phase distortion is not very noticeable.

For images or ivdeo,

- Amplitude distortion tends to be unimportant.
- But phase distortion can change how our eyes perceive the image.