ECE102, Fall 2019 **Final**

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UCLA True Bruin academic integrity principles apply.
Open: Four cheat sheets allowed.
Closed: Book, computer, internet.
8:00-11:00am.
Wednesday, 11 Dec 2019.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

Signature: ____________________

ID#: ____________________

Problem 1	_____	/ 40
Problem 2	_____	/ 45
Problem 3	_____	/ 40
Problem 4	_____	/ 30
Problem 5	_____	/ 45
BONUS	_____	/ 15 bonus points
Total	_____	/ 200 points + 15 bonus points

1. **Signal and System Basics** (40 points)

   (a) (16 points) For each statement below, determine whether it is true or false. You must justify your answer to receive full credit.

      i. (8 points) If $f(t)$ is a real and even signal, and $g(t)$ is a real and odd signal, the convolution of $f(t)$ and $g(t)$ is real and odd.

      ii. (8 points) All LTI systems are stable.

(b) (12 Points) Suppose we have an unknown system (black box). We input

$$x(t) = \mathrm{sinc}(t)$$

into the system, and measure that its output is

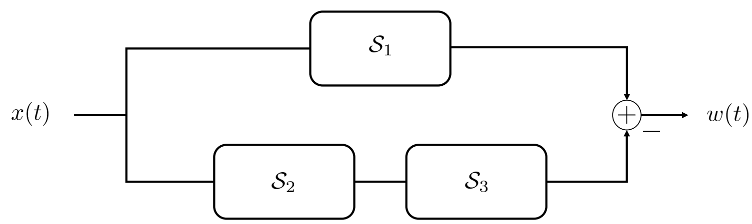
$$y(t) = e^{-t}u(t).$$

Can this system be LTI? You must justify your answer to receive full credit.

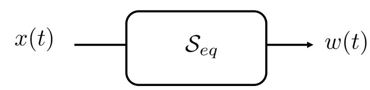
(c) (12 Points) Determine whether the following system is (1) causal, and (2) stable.

$$y(t) = \int_{-\infty}^{t} (x(\tau) + e^{-\tau})u(\tau + 1)d\tau$$

2. **Frequency Response and LTI system** (45 points)
Suppose the three systems are interconnected as shown below.



And we denote the equivalent system as below.



(a) (8 points) Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are all LTI systems. Is the equivalent system $\mathcal{S}_{eq}$ an LTI system? Please justify your answer to receive full credit.

(b) (8 points) Suppose the equivalent system $\mathcal{S}_{eq}$ is an LTI system. Are $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ all necessarily LTI systems? Please justify your answer to receive full credit.

(c) (15 points) Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are each characterized by an LTI system,

- The first system $\mathcal{S}_1$, with frequency response $H_1(j\omega)$, is given by its input-output relationship: $y(t) = x(t-3)$;
- The second system $\mathcal{S}_2$, with frequency response $H_2(j\omega)$, is given by its impulse response: $h_2(t) = u(t-3)$;
- The third system $\mathcal{S}_3$, with frequency response $H_3(j\omega)$, is given by its input-output relationship: $y(t) = \frac{d}{dt}x(t) + \frac{d^2}{dt^2}x(t)$.

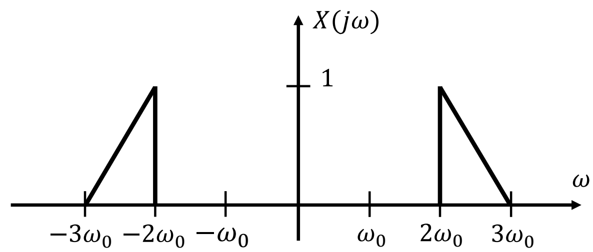
Determine the frequency responses $H_1(j\omega)$, $H_2(j\omega)$ and $H_3(j\omega)$ of each system as well as $H_{eq}(j\omega)$ of the equivalent system.

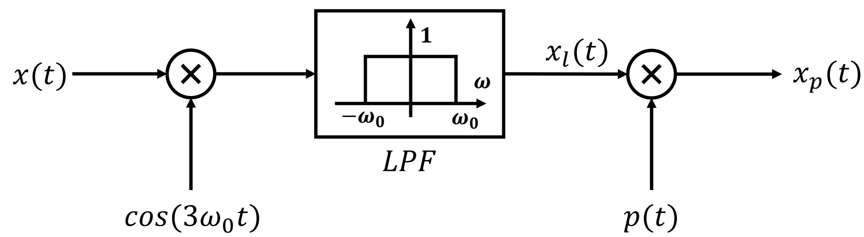
(d) (14 points) For the system in part(c), the output $w(t)$ to an input $x(t) = e^{j\pi t/3}$ can be written as:

$$w(t) = Ae^{j\theta} x(t).$$

Determine A and θ.

3. **Sampling and Modulation** (40 points)
Assume we have a continuous bandpass signal $x(t)$ with frequency spectrum as shown below. We also assume that $x(t)$ is real. The sampling theorem states that, to recover a signal without distortion, a signal must be sampled at a rate greater than twice its bandwidth. However, since $x(t)$ has most of its energy concentrated in a narrow band, it would seem reasonable to expect that a sampling rate lower than the Nyquist rate could be used. Now consider the system shown below where $p(t)$ is the sampling function.





(a) (5 points) What is the Nyquist rate of $x(t)$?

(b) (5 points) What is the Nyquist rate of $x_l(t)$? Sketch the frequency spectrum after the low pass filter, i.e. $X_l(j\omega)$.

(c) (10 points) If the sampling function is an impulse train

$$p(t) = \sum_{k=-\infty}^{k=+\infty} \delta(t - kT)$$

find the maximum sampling period T such that $x(t)$ is recoverable from $x_p(t)$. Sketch the output frequency spectrum $X_p(jw)$.

(d) (20 points) With the $p(t)$ found in part (c), design a system to recover $x(t)$ from $x_p(t)$ without using a bandpass or highpass filter. Note that the recovered signal should have the same amplitude as $x(t)$ in frequency spectrum. Draw a flow diagram of your system and clearly state each component (including cutoff frequencies of any lowpass filter). Write out the explicit mathematical expression of any signal involved.

4. **Laplace Transform** (30 points)

A system can be described by the following differential equation:

$$y''(t) + y'(t) - 2y(t) = 6x'(t) - 3x(t)$$

where the initial conditions are all zero, i.e. $y''(0) = 0$, $y'(0) = 0$ and $y(0) = 0$.

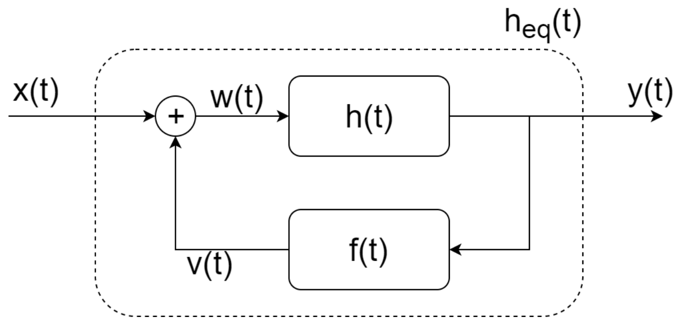
(a) (10 points) Find the transfer function $H(s) = Y(s)/X(s)$. Assume $x(0) = 0$.

(b) (20 points) If the input is

$$x(t) = e^{-t}u(t)$$

then find the output $y(t)$.

5. **Feedback System** (45 points)
Consider the feedback system shown below (all components are LTI):



where $h(t) = e^{-2t}u(t)$ and $y(0) = 0$.

(a) (10 points) Show that

$$H_{eq}(s) = \frac{H(s)}{1 - H(s)F(s)}$$

(b) (10 points) Find the Laplace Transform $H(s)$ of $h(t)$. What is the frequency response $H(j\omega)$? Why is this a low-pass filter?

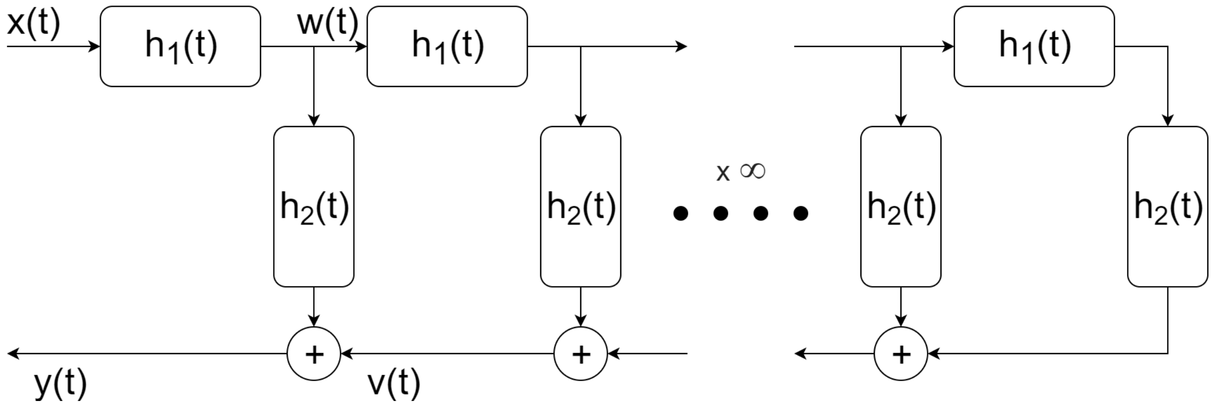
(c) (10 points) $v(t)$ and $y(t)$ satisfy the differential equation

$$v(t) = \frac{d}{dt}y(t) + y(t) - 10\int_0^t y(\tau)d\tau$$

What is $F(s)$?

(d) (15 points) Using $F(s)$ found in part c, what is $h_{eq}(t)$? Is this a low-pass, band-pass, or high-pass filter?

Bonus (15 points) Consider the LTI system S shown below, which is a system ladder with an infinite number of rungs. Let $y(t) = S[x(t)]$.



(a) (8 points) In terms of $H_1(s)$ and $H_2(s)$, what is the equivalent transfer function $H_{eq}(s)$ between $Y(s)$ and $X(s)$? *Hint: how does* $\frac{V(s)}{W(s)}$ *relate to* $\frac{Y(s)}{X(s)}$*?*

(b) (7 points) Suppose $h_1(t) = e^{-a_1 t}u(t)$ and $h_2(t) = e^{-a_2 t}u(t)$, where a_1 and a_2 are real and positive. For what values of a_1 is S BIBO stable?

Property	Signal	Transform
Linearity	$\alpha\, x_1(t) + \beta\, x_2(t)$	$\alpha\, X_1(j\omega) + \beta\; X_2(j\omega)$
Duality	$X(t)$	$2\pi\, x(-\omega)$
Conjugate symmetry	$x(t)$ real	$X^*(j\omega) = X(-j\omega)$ Magnitude: $\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$ Phase: $\Theta(-\omega) = -\Theta(\omega)$ Real part: $X_r(-j\omega) = X_r(j\omega)$ Imaginary part: $X_i(-j\omega) = -X_i(j\omega)$
Conjugate antisymmetry	$x(t)$ imaginary	$X^*(j\omega) = -X(-j\omega)$ Magnitude: $\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$ Phase: $\Theta(-\omega) = -\Theta(\omega) \mp \pi$ Real part: $X_r(-j\omega) = -X_r(j\omega)$ Imaginary part: $X_i(-j\omega) = X_i(j\omega)$
Even signal	$x(-t) = x(t)$	$X(j\omega)$: even
Odd signal	$x(-t) = -x(t)$	$X(j\omega)$: odd
Time shifting	$x(t-\tau)$	$X(j\omega)\, e^{-j\omega\tau}$
Frequency shifting	$x(t)\, e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Modulation property	$x(t)\, \cos(\omega_0 t)$	$\frac{1}{2}\left[X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right]$
Time and frequency scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert} X\left(\dfrac{j\omega}{a}\right)$
Differentiation in time	$\dfrac{d^n}{dt^n}[x(t)]$	$(j\omega)^n X(j\omega)$
Differentiation in frequency	$(-jt)^n\, x(t)$	$\dfrac{d^n}{d\omega^n}[X(j\omega)]$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)\, X_2(j\omega)$
Multiplication	$x_1(t)\, x_2(t)$	$\dfrac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$
Integration	$\displaystyle\int_{-\infty}^{t} x(\lambda)\, d\lambda$	$\dfrac{X(j\omega)}{j\omega} + \pi\, X(0)\, \delta(\omega)$
Parseval's theorem	$\displaystyle\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\, d\omega$	

Table 4.4 – Fourier transform properties.

Additional properties:		
	x(t): even and real	$X(j\omega)$: even and real
	x(t): odd and real	$X(j\omega)$: odd and imaginary
	x(t): even and imaginary	$X(j\omega)$: even and imaginary
	x(t): odd and imaginary	$X(j\omega)$: odd and real

Name	Signal	Transform
Rectangular pulse	$x(t) = A\, rect(t/\tau)$	$X(j\omega) = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$
Triangular pulse	$x(t) = A\,\Lambda(t/\tau)$	$X(j\omega) = A\tau \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$
Right-sided exponential	$x(t) = e^{-at}\,u(t)$	$X(j\omega) = \dfrac{1}{a+j\omega}$
Two-sided exponential	$x(t) = e^{-a\|t\|}$	$X(j\omega) = \dfrac{2a}{a^2+\omega^2}$
Signum function	$x(t) = \operatorname{sgn}(t)$	$X(j\omega) = \dfrac{2}{j\omega}$
Unit impulse	$x(t) = \delta(t)$	$X(j\omega) = 1$
Sinc function	$x(t) = \operatorname{sinc}(t)$	$X(j\omega) = rect\left(\frac{\omega}{2\pi}\right)$
Constant-amplitude signal	$x(t) = 1,\ \text{all } t$	$X(j\omega) = 2\pi\,\delta(\omega)$
	$x(t) = \dfrac{1}{\pi t}$	$X(j\omega) = -j\operatorname{sgn}(\omega)$
Unit-step function	$x(t) = u(t)$	$X(j\omega) = \pi\,\delta(\omega) + \dfrac{1}{j\omega}$
Modulated pulse	$x(t) = rect\left(\frac{t}{\tau}\right)\cos(\omega_0 t)$	$X(j\omega) = \frac{\tau}{2}\operatorname{sinc}\left(\frac{(\omega-\omega_0)\tau}{2\pi}\right) + \frac{\tau}{2}\operatorname{sinc}\left(\frac{(\omega+\omega_0)\tau}{2\pi}\right)$

Table 4.5 – Some Fourier transform pairs.

Note:

$$\operatorname{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

$$\operatorname{rect}(t/\tau) = u(t+\tau/2) - u(t-\tau/2)$$

LAPLACE TRANSFORM

1. Some Laplace transform pairs

Signal	Transform	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathcal{R}e\{s\} > -a$
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -a$
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -a$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\mathcal{R}e\{s\} > -a$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathcal{R}e\{s\} > 0$

2. Laplace transform properties

Signal	Transform	ROC
$x(t)$	$X(s)$	R_x
$x_1(t)$	$X_1(s)$	R_1
$x_2(t)$	$X_2(s)$	R_2
$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
$x(t - t_0)$	$e^{-st_0}X(s)$	R_x
$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R_x (s is in the ROC if $s - s_0 \in R_x$)
$x(at)$, $a > 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$	Scaled version of R_x (s is in the ROC if $s/a \in R_x$)
$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
$\int_0^t x(\tau)d\tau$	$\frac{X(s)}{s}$	At least $R_x \cap \{\mathcal{R}e\{s\} > 0\}$
$\frac{d}{dt}x(t)$	$sX(s) - x(0)$	At least R_x
$\frac{d^2}{dt^2}x(t)$	$s^2X(s) - sx(0) - x'(0)$	At least R_x
$tx(t)$	$-\frac{d}{ds}X(s)$	R_x