Problem 1: Laplace transform

(a) $f(t) = \sum_{k=0}^{\infty} a^k \delta(t-kT), \quad a>0$

$\Rightarrow \mathcal{L}[f(t)] = \mathcal{L}\left[\sum_{k=0}^{\infty} a^k \delta(t-kT)\right]$

Using the linearity property of laplace transform

$$\mathcal{L}[f(t)] = \sum_{k=0}^{\infty} a^k \mathcal{L}[\delta(t-kT)]$$

Now,

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt$$

$$= \int_{0^-}^{\infty} \delta(t)\, dt$$

$$= 1$$

By the t-shift property of laplace transform

$$\mathcal{L}[\delta(t-kT)] = e^{-kTs}$$

$$\mathcal{L}[f(t)] = \sum_{k=0}^{\infty} a^k e^{-kTs} = \sum_{k=0}^{\infty} (ae^{-sT})^k$$

Now the term inside the summation forms a geometric series with common ratio ae^{-sT}. Hence the infinite sum converges to a finite value iff $|ae^{-sT}| < 1$

So,

$$F(s) = \frac{1}{1 - ae^{-sT}}$$

For ROC, we have

$$|ae^{-sT}| < 1$$

$$\Rightarrow a^2 e^{-2sT} < 1$$

$$\Rightarrow 2\log a - 2sT < 0$$

$$\Rightarrow s > \frac{1}{T} \log a$$

(b) $f(t) = \sin(\omega_0 t + b)\, e^{-at} u(t)$

$$\Rightarrow f(t) = \left[\sin(\omega_0 t)\cos(b) + \cos(\omega_0 t)\sin(b)\right] e^{-at} u(t)$$

$$f(t) = \cos(b)\sin(\omega_0 t) e^{-at} u(t) + \sin(b)\cos(\omega_0 t) e^{-at} u(t)$$

Now using the LT table, we have

$$\sin(\omega_0 t)\, e^{-at} u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+a)^2 + \omega_0^2}$$

$$\cos(\omega_0 t)\, e^{-at} u(t) \xleftrightarrow{\mathcal{L}} \frac{s+a}{(s+a)^2 + \omega_0^2}$$

Hence using linearity of LT,

$$F(s) = \frac{\omega_0 \cos(b) + (s+a)\sin(b)}{(s+a)^2 + \omega_0^2}$$

Poles of $F(s)$:

$$(s+a)^2 + \omega^2 = 0$$

$$\Rightarrow (s+a)^2 = -\omega^2$$

$$\Rightarrow s = -a \pm j\omega$$

Since $f(t)$ is a right-sided signal, so ROC is all right-sided and to the right of the rightmost pole.

$\therefore$ ROC: $Re\{s\} > -a$

Problem 2: Inverse Laplace transform

ⓐ $X(s) = \dfrac{s+1}{s^2+5s+6}$, ROC: $\sigma > -2$

$\Rightarrow X(s) = \dfrac{s+1}{(s+3)(s+2)}$,

Now using partial fraction expansion,

$$\frac{s+1}{(s+3)(s+2)} = \frac{r_1}{s+3} + \frac{r_2}{s+2}.$$

using cover up procedure,

$$r_1 = (s+3)X(s)\Big|_{s=-3} = \frac{-2}{-1} = 2$$

$$r_2 = (s+2)X(s)\Big|_{s=-2} = \frac{-1}{1} = -1$$

Hence,

$$X(s) = \frac{2}{s+3} - \frac{1}{s+2}$$

Since the ROC is right-sided, so the time-domain signal is also right-sided.

Using lookup table,

$$x(t) = 2e^{-3t}u(t) - e^{-2t}u(t)$$

(b) $X(s) = \frac{s+1}{(s+1)^2+4}$, ROC: $\sigma > -1$

Let $Y(s) = \frac{s}{s^2+4}$

Then using lookup table,

$$y(t) = \cos(2t)\,u(t)$$

Now,

$$X(s) = Y(s+1)$$

Using the s-shift property of laplace transform,

$$x(t) = e^{-t} y(t)$$

$$x(t) = e^{-t} \cos(2t) u(t)$$

Problem 3: Stability and Causality of LTI systems

$$\ddot{y}(t) - \dot{y}(t) - 2y(t) = x(t)$$

(a) Taking the laplace transforms of both sides

$$s^2 Y(s) - s\,y(0^-) - \dot{y}(0^-) - sY(s) + y(0^-) - 2Y(s) = X(s)$$

$$Y(s)[s^2 - s - 2] - sy(0^-) - \dot{y}(0^-) + y(0^-) = X(s)$$

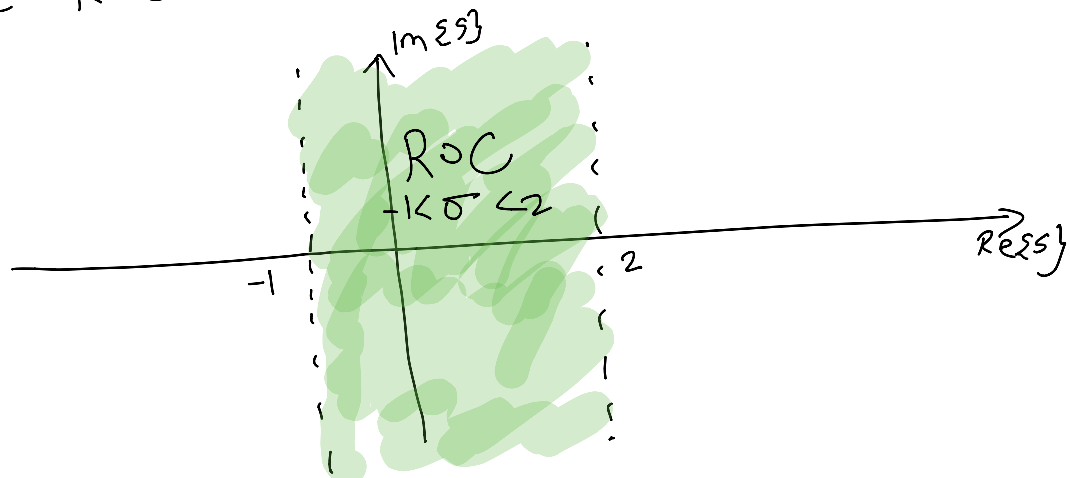
$$\Rightarrow \frac{Y(s)}{X(s)} = H(s) = \frac{1}{s^2 - s - 2}$$

(b) (i) If the system is stable, then the RoC of H(s) should include the imaginary axis. Now,

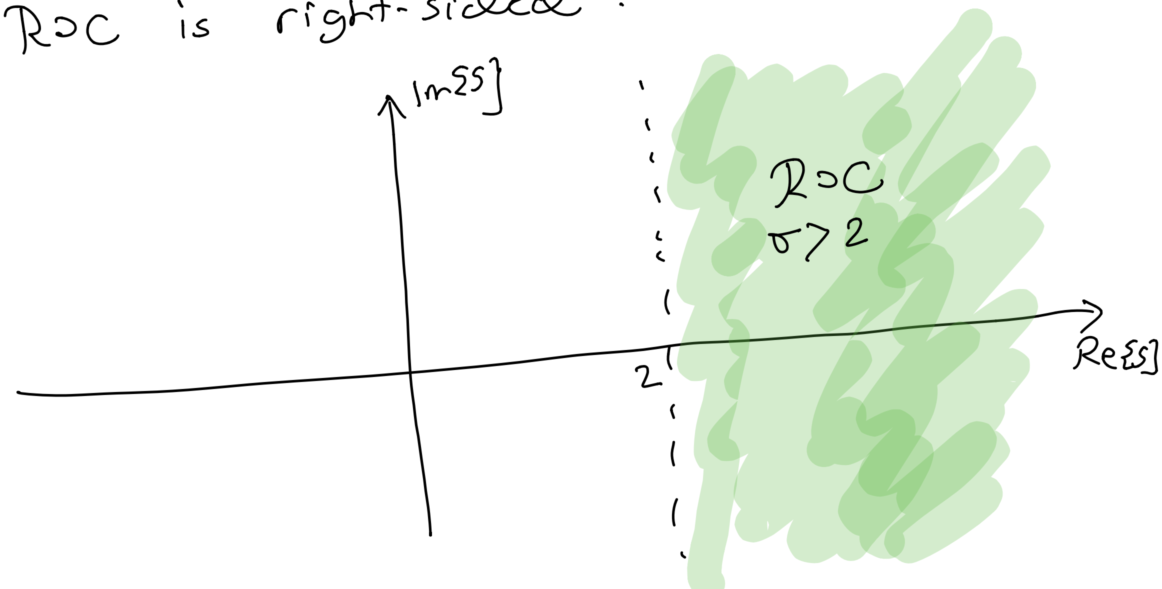
$$H(s) = \frac{1}{(s+1)(s-2)}$$

Hence H(s) has two poles: $s = -1, 2$

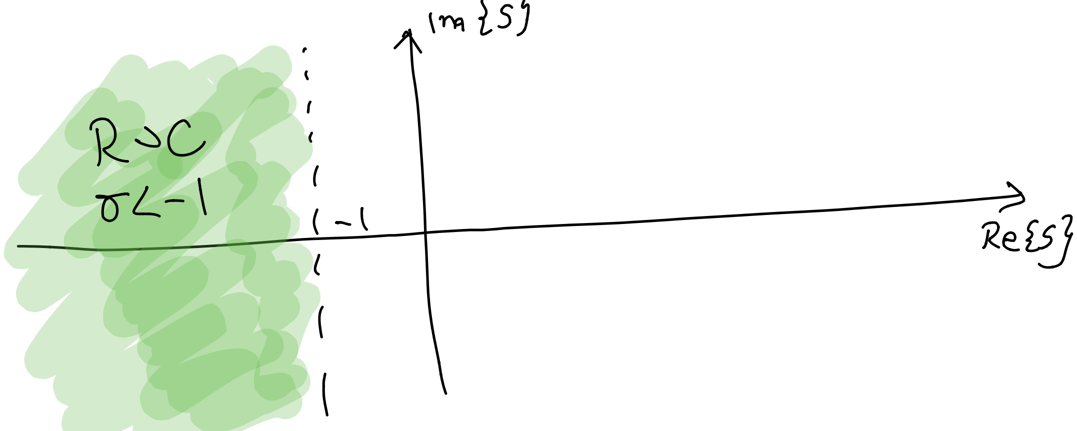
The only scenario in which the RoC of H(s) can contain the imaginary axis is if h(t) is two-sided and the RoC is bounded by the two poles.



ii) The system is causal iff $h(t) = 0$ for $t < 0$. This can only happen if the ROC is right-sided.



iii) If the system is neither stable nor causal, then the only option for the ROC is $\sigma < -1$.



(c) If the system is causal, then the ROC is right-sided and the impulse response is also right-sided. Hence,

$$H(s) = \frac{1}{(s+1)(s-2)}$$

Using partial fraction expansion,

$$\frac{1}{(s+1)(s-2)} = \frac{r_1}{s+1} + \frac{r_2}{s-2}$$

By cover-up procedure,

$$r_1 = (s+1)H(s)\Big|_{s=-1} = -1/3$$

$$r_2 = (s-2)H(s)\Big|_{s=2} = 1/3$$

$$\Rightarrow H(s) = \frac{-1/3}{s+1} + \frac{1/3}{s-2}$$

Using lookup table,

$$h(t) = -\tfrac{1}{3}e^{-t}u(t) + \tfrac{1}{3}e^{2t}u(t)$$