ECE 102, Fall 2018 **Final Exam**

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UCLA True Bruin academic integrity principles apply.
Open: Four pages of cheat sheet allowed.
Closed: Book, computer, internet.
11:30am-2:30pm, Haines Room 118
Tuesday, 11 Dec 2018.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

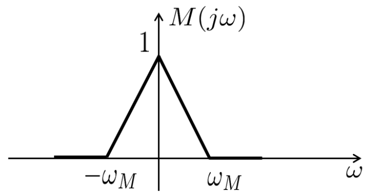
Signature: ____________________

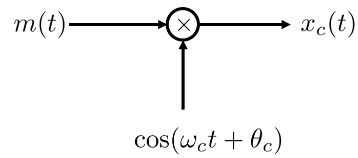
ID#: _________________

Problem 1	_____	/	25
Problem 2	_____	/	41
Problem 3	_____	/	30
Problem 4	_____	/	40
Problem 5	_____	/	20
Problem 6	_____	/	44
BONUS	_____	/	10 bonus points
Total	_____	/	200 points + 10 bonus points

Problem 1 (25 points)

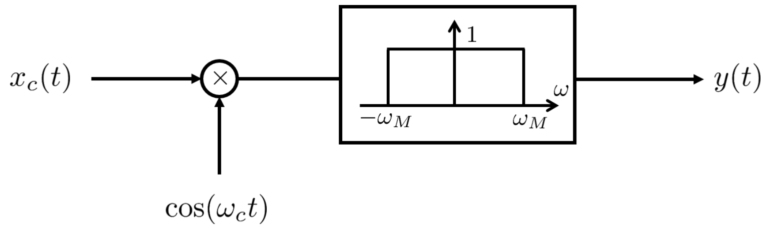
Consider a bandlimited signal $m(t)$, its frequency spectrum $M(j\omega)$ is shown below. We modulate $m(t)$ with $\cos(\omega_c t + \theta_c)$, where θ_c is a constant phase but unknown:





(a) (8 points) Express $X_c(j\omega)$, the Fourier transform of $x_c(t)$, in terms of $M(j\omega)$. *Hint: use the fact that* $\cos(u) = \dfrac{e^{ju} + e^{-ju}}{2}$.

(b) (10 points) We demodulate $x_c(t)$ as follows:



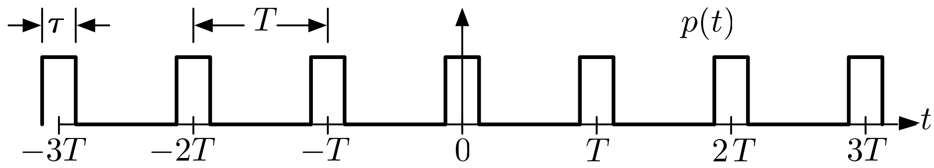
Show that $y(t) = \dfrac{1}{2}\cos(\theta_c)m(t)$. Assume $\omega_c \gg \omega_M$.

(c) (7 points) Assume that you also know $z(t) = \frac{1}{2}\sin(\theta_c)m(t)$. How can you recover $m(t)$ from $y(t)$ and $z(t)$?

Hint: $cos^2(u) + \sin^2(u) = 1$.

Problem 2 (41 points)

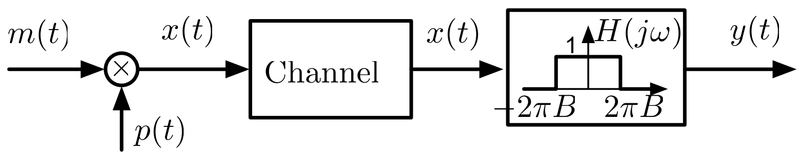
Consider the following sequence of short rect($\cdot$) pulses, denoted by $p(t)$:



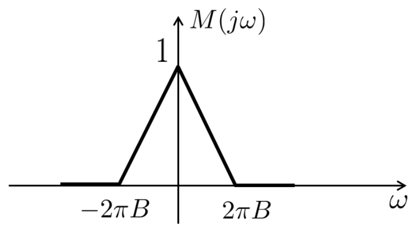
Each rect($\cdot$) pulse has width τ, and the pulses are spaced by T as diagrammed above.

(a) (14 points) Find $P(j\omega)$, the Fourier transform of $p(t)$. Express $P(j\omega)$ as a sum, and simplify where possible. *Hint: One approach is to write $p(t)$ as convolution of a rect function with an impulse train.*

(b) (10 points) Consider the following system:



where the input $m(t)$ is multiplied with the rect pulse train, $p(t)$. The signal $m(t)$ is bandlimited and it has the following frequency spectrum:



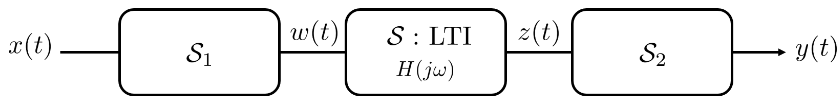
Assume that the rect$(\cdot)$ pulses are spaced by $T = \dfrac{1}{2B}$. Express the spectrum $X(j\omega)$ of $x(t)$ in terms of $M(j\omega)$.

(c) (10 points) Sketch $X(j\omega)$ for $-6\pi B \leq \omega \leq 6\pi B$.

(d) (7 points) Find the spectrum of the signal at the output of the lowpass filter $Y(j\omega)$, i.e., find an expression of $Y(j\omega)$ in terms of $M(j\omega)$.

Problem 3 (30 points)

An LTI system $\mathcal{S}$ is cascaded in series with two other non-LTI systems as follows:



The system $\mathcal{S}_1$ is given by:

$$w(t) = x\left(\frac{t}{2}\right)$$

And the system $\mathcal{S}_2$ is:

$$y(t) = z(2t)$$

The system $\mathcal{S}$ has $H(j\omega)$ as its frequency response.

(This question continues on the next page.)

(a) (15 points) Find how $Y(j\omega)$ is related to $X(j\omega)$, in terms of $H(j\omega)$. Deduce the overall frequency response $H_{eq}(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$.

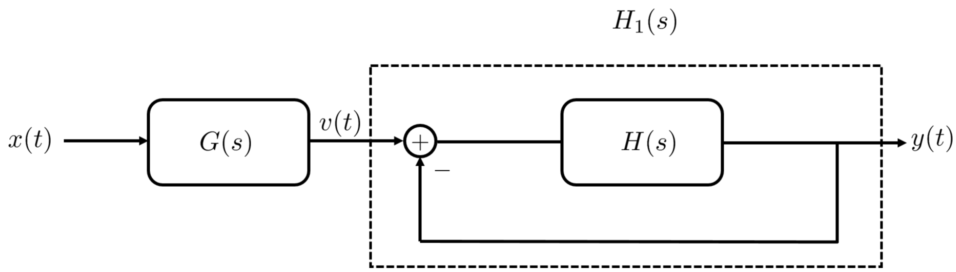
(b) (15 points) If $H(j\omega)$ is given by:

$$H(j\omega) = \frac{2a - j\omega}{2a + j\omega}$$

where $a > 0$, find the impulse response $h(t)$ of the system $\mathcal{S}$. Deduce the overall impulse response $h_{eq}(t)$.

Problem 4 (40 points)

Consider the following system:



(a) (10 points) Find the transfer function $H_1(s)$ of the system that maps $v(t)$ to $y(t)$.

(b) (5 points) Find the overall transfer function $H_{eq}(s)$.

(c) (10 points) How can we choose $H(s)$ in terms of $G(s)$ so that the overall system has the following impulse response $h_{eq}(t) = \delta(t)$?

(d) (15 points) Using the relation you found in part (c), find $h(t)$ if $g(t) = e^{-2t}u(t)$.

Problem 5 (20 points)

A system is described by the following differential equation:

$$y''(t) + 5y'(t) + 6y(t) = x'(t) + 5x(t)$$

If the input is

$$x(t) = e^{-4t}u(t-2)$$

find the output $y(t)$. Assume all initial conditions are zero.

There is additional space on the next page if needed.

(Additional space for problem 5.)

Problem 6 (44 points)

(a) (24 points) Determine if each of the following four statements is true or false. When the statement is false, a counter example is sufficient. If the statement is true, you must justify your answer to receive full credit.

i. If $x(t) * y(t) = 0$, then $x(t) = 0$ or $y(t) = 0$.

ii. If $x(t) * h(t) = x(t)$, then $h(t)$ must be an impulse, i.e., $h(t) = \delta(t)$.

iii. A signal $x(t)$ is bandlimited where its Fourier transform $X(j\omega) = 0$ for $|\omega| > 2\pi B$ rad/s. The Nyquist rate of $\cos(4\pi Bt)x(t-2) + x(2t)$ is $6B$ Hz.

iv. If $x(t) = \text{sinc}(t)$, then the energy of $x(3t+2)$ is $\frac{1}{3}$.

(b) (10 points) If $y(t) = x(t) * h(t)$, then show that the following identity holds:

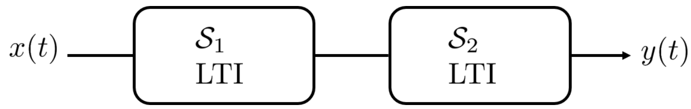
$$\int_{-\infty}^{\infty} y(t)dt = \left(\int_{-\infty}^{\infty} h(t)dt\right) \cdot \left(\int_{-\infty}^{\infty} x(t)dt\right)$$

Hint: One approach is to look at the integral expression for the Fourier transform when $\omega = 0$.

(c) (10 points) An LTI system has the following impulse response: $h(t) = e^t u(-1 - t)$. Is the system stable? Is it causal?

BONUS (10 points)

(a) (5 points) Two LTI systems are linearly cascaded as follows:



The impulse response of the first system is $h_1(t) = e^t u(t)$ and the impulse response of the second system is $h_2(t) = e^{2t}\cos(t)$. What is the impulse response of the equivalent system $h_{eq}(t)$?

(b) (5 points) If F_s is the Nyquist rate of $x(t)$, determine in terms of F_s, the Nyquist rate of $x^3(t) * x^2(t)$.

Fourier Transform Tables

Property	Signal	Transform
Linearity	$\alpha\, x_1(t) + \beta\, x_2(t)$	$\alpha\, X_1(j\omega) + \beta \;\; X_2(j\omega)$
Duality	$X(t)$	$2\pi\, x(-\omega)$
Conjugate symmetry	$x(t)$ real	$X^*(j\omega) = \;\; X(-j\omega)$ Magnitude: $\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$ Phase: $\Theta(-\omega) = \;\; -\Theta\;(\omega)$ Real part: $X_r(-j\omega) = \;\; X_r(j\omega)$ Imaginary part: $X_i(-j\omega) = \;\; -X_i(j\omega)$
Conjugate antisymmetry	$x(t)$ imaginary	$X^*(j\omega) = -X(-j\omega)$ Magnitude: $\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$ Phase: $\Theta(-\omega) = \;\; -\Theta\;(\omega) \mp \pi$ Real part: $X_r(-j\omega) = \;\; -X_r(j\omega)$ Imaginary part: $X_i(-j\omega) = \;\; X_i(j\omega)$
Even signal	$x(-t) = x(t)$	$X(j\omega)$: even
Odd signal	$x(-t) = -x(t)$	$X(j\omega)$: odd
Time shifting	$x(t-\tau)$	$X(j\omega)\, e^{-j\omega\tau}$
Frequency shifting	$x(t)\; e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Modulation property	$x(t)\;\cos(\omega_0 t)$	$\frac{1}{2}\;[X(j(\omega-\omega_0)\,) + \; X(j(\omega+\omega_0))]$
Time and frequency scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert}\, X\left(\dfrac{j\omega}{a}\right)$
Differentiation in time	$\dfrac{d^n}{dt^n}\,[x(t)]$	$(j\omega)^n\, X(j\omega)$
Differentiation in frequency	$(-jt)^n\;x(t)$	$\dfrac{d^n}{d\omega^n}\,[X(j\omega)]$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)\, X_2(j\omega)$
Multiplication	$x_1(t)\;\, x_2(t)$	$\dfrac{1}{2\pi}\, X_1(j\omega) * X_2(j\omega)$
Integration	$\displaystyle\int_{-\infty}^{t} x(\lambda)\, d\lambda$	$\dfrac{X(j\omega)}{j\omega} + \pi\, X(0)\,\delta(\omega)$
Parseval's theorem	$\displaystyle\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\; dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\, d\omega$	

Table 4.4 – Fourier transform properties.

Additional properties:		
	$x(\mathrm{t})$: even and real	$X(j\omega)$: even and real
	$x(\mathrm{t})$: odd and real	$X(j\omega)$: odd and imaginary
	$x(\mathrm{t})$: even and imaginary	$X(j\omega)$: even and imaginary
	$x(\mathrm{t})$: odd and imaginary	$X(j\omega)$: odd and real

Name	Signal	Transform
Rectangular pulse	$x(t) = A\, rect(t/\tau)$	$X(j\omega) = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$
Triangular pulse	$x(t) = A\,\Lambda(t/\tau)$	$X(j\omega) = A\tau \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$
Right-sided exponential	$x(t) = e^{-at}\, u(t)$	$X(j\omega) = \frac{1}{a + j\omega}$
Two-sided exponential	$x(t) = e^{-a\|t\|}$	$X(j\omega) = \frac{2a}{a^2 + \omega^2}$
Signum function	$x(t) = \operatorname{sgn}(t)$	$X(j\omega) = \frac{2}{j\omega}$
Unit impulse	$x(t) = \delta(t)$	$X(j\omega) = 1$
Sinc function	$x(t) = \operatorname{sinc}(t)$	$X(j\omega) = rect\left(\frac{\omega}{2\pi}\right)$
Constant-amplitude signal	$x(t) = 1,\ \text{all } t$	$X(j\omega) = 2\pi\,\delta(\omega)$
	$x(t) = \frac{1}{\pi t}$	$X(j\omega) = -j \operatorname{sgn}(\omega)$
Unit-step function	$x(t) = u(t)$	$X(j\omega) = \pi\,\delta(\omega) + \frac{1}{j\omega}$
Modulated pulse	$x(t) = rect\left(\frac{t}{\tau}\right) \cos(\omega_0 t)$	$X(j\omega) = \frac{\tau}{2} \operatorname{sinc}\left(\frac{(\omega-\omega_0)\,\tau}{2\pi}\right) + \frac{\tau}{2} \operatorname{sinc}\left(\frac{(\omega+\omega_0)\,\tau}{2\pi}\right)$

Table 4.5 – Some Fourier transform pairs.

Note:

$$\operatorname{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

$$\operatorname{rect}(t/\tau) = u(t+\tau/2) - u(t-\tau/2)$$

LAPLACE TRANSFORM

1. Some Laplace transform pairs

Signal	Transform	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathcal{R}e\{s\} > -a$
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -a$
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -a$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\mathcal{R}e\{s\} > -a$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathcal{R}e\{s\} > 0$

2. Laplace transform properties

Signal	Transform	ROC
$x(t)$	$X(s)$	R_x
$x_1(t)$	$X_1(s)$	R_1
$x_2(t)$	$X_2(s)$	R_2
$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	At least $R_1 \cap R_2$
$x(t-t_0)$	$e^{-st_0}X(s)$	R_x
$e^{s_0t}x(t)$	$X(s-s_0)$	Shifted version of R_x (s is in the ROC if $s-s_0 \in R_x$)
$x(at)$, $a>0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$	Scaled version of R_x (s is in the ROC if $s/a \in R_x$)
$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
$\int_0^t x(\tau)d\tau$	$\frac{X(s)}{s}$	At least $R_x \cap \{\mathcal{R}e\{s\} > 0\}$
$\frac{d}{dt}x(t)$	$sX(s)-x(0)$	At least R_x
$\frac{d^2}{dt^2}x(t)$	$s^2X(s)-sx(0)-x'(0)$	At least R_x
$tx(t)$	$-\frac{d}{ds}X(s)$	R_x