Convolution

This lecture discusses convolution, what it is, and how we do it. Topics include:

- Definition of convolution
- Intuition behind what convolution calculates
- Doing convolution graphically: flip and drag
- Properties of convolution
  - Commutativity
  - Associativity
  - Distributivity
  - Identity element
  - Delay
  - LTI
  - Composition
- Properties of convolution extend to LTI systems
- Calculating impulse response from step response

Convolution

The concept of convolution is *critical* to signal processing. It is one of the most basic operations and makes its way to several applications. Even beyond signal processing, an operation close to convolution is used in *convolutional neural networks*, which are a modern machine learning algorithm that has achieved superhuman performance on various tasks including image recognition.

To this end, it's very important to gain a solid understanding of convolution, what it means intuitively, and how to do it without a computer.

A brief review

Recall that at the end of last lecture, we saw that we could calculate a system's output if we knew the impulse response. To do so, we would have to compute this expression:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

where $h(t)$ is the impulse response of the system. This integral is a *convolution integral*.

Convolution notation

The convolution of an input signal, $x(t)$, and an impulse response, $h(t)$ is denoted as:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= (x * h)(t) \end{aligned}$$

We may also omit the t's and write this shorthand as

$$y = x * h$$

Occasionally, you will see this notation,

$$y(t) = x(t) * h(t)$$

We will use both notation, though when possible we will try to use the earlier notation. The reason why is that in the latter notation, the t's on the l.h.s. and r.h.s. of the equation have different meanings.

Convolution block diagram

Convolution is so frequently encountered in signals and systems that we reserve the simple notation of a signal entering a block (denoted by a function, $h(t)$) to mean convolution. You may sometimes see the convolution operator, $*$ used (see below) but this is less common.

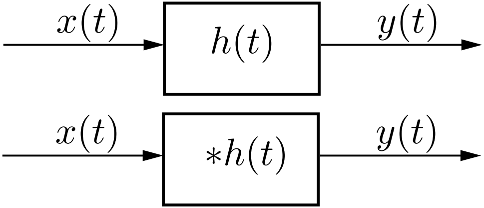
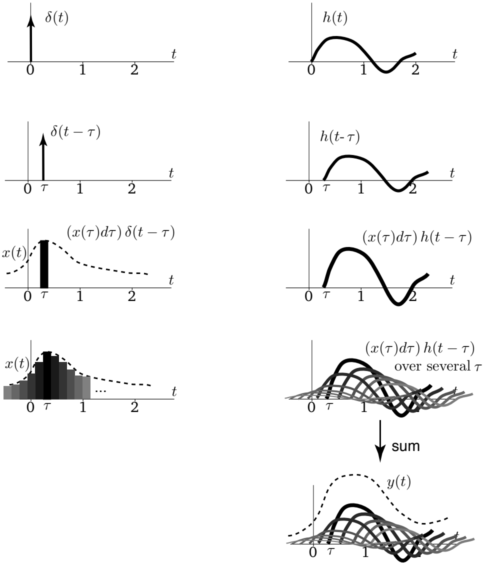


Illustration of convolution

This is a conceptual illustration of what convolution is doing.



Convolution for a causal system

In a causal system, $h(t) = 0$ for $t < 0$. (Why? Hint: what happens if $h(t) \neq 0$ for some $t < 0$?)

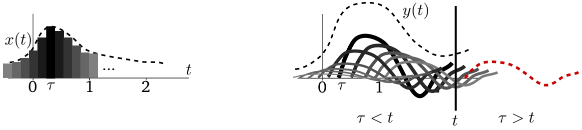
This means that $h(t - \tau) = 0$ if $\tau > t$. Hence, there is no need to integrate if τ exceeds t, since $h(t - \tau) = 0$. We can use this to simplify the convolution integral.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \end{aligned}$$

This equation tells us that only past and present values of $x(\tau)$ contribute to $y(t)$.

Convolution for a causal system (cont.)

The idea of causal convolution is illustrated below. Here, we show a scaled impulse response that occurs at $\tau > t$ in dotted red. This scaled impulse response would not contribute to $y(t)$ since it occurs for $\tau > t$.



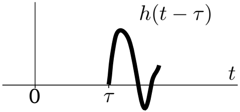
Convolution without a computer: flip and drag

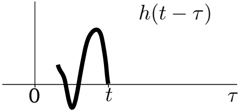
So far, the convolution integral may still seem opaque. Let's approach convolution in another way, which will give us a technique to do convolution without a computer. The convolution integral is:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Let's break this integral down piece by piece.

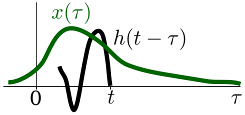
- The term $h(t-\tau)$, w.r.t. t, is the impulse response delayed to time τ.
- However, our integral is over τ, and so we should consider how h varies with τ.
- The term $h(t-\tau)$, w.r.t. τ, tells us that we should first delay the signal to time t and then reverse the signal. This operation, which we colloquially call "flipping," is illustrated below.

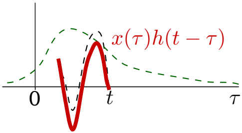




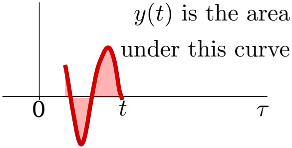
Convolution without a computer: flip and drag (cont.)

- Next, convolution tells us to multiply $h(t-\tau)$, our flipped impulse response, with $x(\tau)$ and do it for all τ. This means we simply multiply $x(\tau)$ and $h(t-\tau)$ together pointwise. This is illustrated below in red.





- Finally, to get $y(t)$ for this particular value of t, we integrate this curve over all τ. This is illustrated below.



- Now, to get $y(t)$ for all values of t, we repeat this process, "dragging" $h(t-\tau)$ across different delays t.

Summary of the flip and drag technique

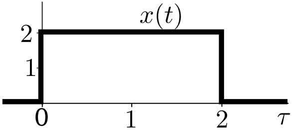
To calculate $y(t) = (x * h)(t)$,

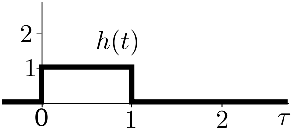
- Flip (i.e., reverse in time) the impulse response. This changes $h(\tau)$ to $h(-\tau)$.
- Begin to drag the reversed time response by some amount, t. This results in $h(t-\tau)$.
- For a given t, multiply $h(t-\tau)$ pointwise by $x(\tau)$. This produces $x(\tau)h(t-\tau)$.
- Integrate this product over τ. This produces $y(t)$ at this particular time t.

This technique is referred to as the "flip-and-drag" technique.

Flip and drag example

Let's do the convolution below using the flip and drag technique.





Other examples, and communication channel example

To be done in class.

Properties of convolution

We'll next derive some properties about convolution. These are:

- Commutativity
- Associativity
- Distributivity
- LTI
- Composition
- And a few others

Commutativity

Convolution is commutative, so that $\boxed{(x * h)(t) = (h * x)(t)}$. Let's go ahead and derive this. Say that,

$$\begin{aligned} y(t) &= (x * h)(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned}$$

We define $\tau' = t - \tau$. Then $d\tau' = -d\tau$ and the integration bounds change to be ∞ to $-\infty$, i.e.,

$$\begin{aligned} y(t) &= \int_{\infty}^{-\infty} x(t-\tau')h(\tau')(-d\tau') \\ &= \int_{-\infty}^{\infty} h(\tau')x(t-\tau')d\tau' \\ &= (h * x)(t) \end{aligned}$$

where the second line uses the result from calculus that

$$\int_a^b f(t)dt = -\int_b^a f(t)dt$$

Commutativity in flip and drag

Because convolution is commutative, note that we can decide which signal (i.e., x or h) to flip-and-drag. Usually, one will be easier than the other.

Commutativity example

Using commutativity, we can show the following stability property of the impulse response: if the impulse response is absolutely integrable (i.e., integrating the absolute value of the impulse response results in a finite value), then the system is BIBO stable.

$$\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right| \\
&= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \\
&\leq \int_{-\infty}^{\infty} \left| h(\tau)x(t-\tau) \right| d\tau \\
&= \int_{-\infty}^{\infty} \left| h(\tau) \right| \left| x(t-\tau) \right| d\tau \\
&\leq M_x \int_{-\infty}^{\infty} |h(\tau)| d\tau
\end{aligned}$$

Hence, if $\int_{-\infty}^{\infty} |h(\tau)|d\tau \leq A < \infty$, then $|y(t)| \leq M_x \cdot A$.

Associativity

If we convolve three functions, $f(t)$, $g(t)$, and $h(t)$, we can do this in any order, i.e.,

$$\boxed{(f * (g * h))(t) = ((f * g) * h)(t)}$$

To show this, we use the definition of integration.

$$\begin{aligned}(f * (g * h))(t) &= \int_{-\infty}^{\infty} f(\tau_1)[(g * h)(t - \tau_1]d\tau_1 \\ &= \int_{-\infty}^{\infty} f(\tau_1)\left[\int_{-\infty}^{\infty} g(\tau_2)h(t - \tau_1 - \tau_2)d\tau_2\right] d\tau_1\end{aligned}$$

To simplify this expression and show associativity, our strategy is to interchange the order of the integration operations. We will try to get out a $(f * g)(t)$ first. To do so, we need to relate τ_2 to τ_1. Hence, a reasonable thing to do is to introduce a new variable, $\tau_3 = \tau_1 + \tau_2$. Note that $d\tau_3 = d\tau_2$. This proof continues on the next page.

Associativity (cont.)

$$\begin{aligned}
(f*(g*h))(t) &= \int_{-\infty}^{\infty} f(\tau_1)\left[\int_{-\infty}^{\infty} g(\tau_2)h(t-\tau_1-\tau_2)d\tau_2\right]d\tau_1 \\
&= \int_{-\infty}^{\infty} f(\tau_1)\left[\int_{-\infty}^{\infty} g(\tau_3-\tau_1)h(t-\tau_3)d\tau_3\right]d\tau_1 \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau_1)g(\tau_3-\tau_1)d\tau_1\right]h(t-\tau_3)d\tau_3 \\
&= \int_{-\infty}^{\infty}(f*g)(\tau_3)h(t-\tau_3)d\tau_3 \\
&= ((f*g)*h)(t)
\end{aligned}$$

- In the third equality, we used the fact that we can interchange the order of integration as long as the integral of the absolute value of the integrand is finite (Fubini's theorem). (Rigorous proof of this is beyond the scope of this class, but you should have had exposure to this in calculus and frequently used this trick before.)

Associativity and commutivity

Combining the commutative and associative properties, we have that:

$$\begin{aligned} f * g * h &= f * h * g \\ &= g * f * h \\ &= \vdots \\ &= h * g * f \end{aligned}$$

Distributivity

Convolution is distributive, meaning that:

$$\boxed{f * (g + h) = f * g + f * h}$$

To prove this, we write out the definition of convolution:

$$\begin{aligned}(f * (g + h))(t) &= \int_{-\infty}^{\infty} f(\tau)\left[g(t-\tau) + h(t-\tau)\right] d\tau \\ &= \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau + \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \\ &= (f * g)(t) + (f * h)(t)\end{aligned}$$

Identity element of convolution

Here, we have something that looks like an "algebra of signals," with addition like in ordinary algebra, and multiplication is replaced by convolution. In standard algebra, the multiplicative identity is 1. In signals, the convolution identity is the Dirac delta function, $\delta(t)$.

In particular, note that:

$$\boxed{x(t) * \delta(t) = x(t)}$$

because

$$\begin{aligned}
x(t) * \delta(t) &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \delta(\tau)x(t)d\tau \\
&= x(t)\int_{-\infty}^{\infty} \delta(\tau)d\tau \\
&= x(t)
\end{aligned}$$

Delaying signals with the impulse

Convolution with the impulse can also be used to delay signals, i.e.,

$$\boxed{x(t) * \delta(t - t_d) = x(t - t_d)}$$

To prove this, note that:

$$x(t) * \delta(t - t_d) = \int_{-\infty}^{\infty} x(\tau)\delta(t - t_d - \tau)d\tau$$

i.e., $x(\tau)$ is being multiplied by an impulse that occurs at $\tau = t - t_d$. From what we know about convolution, this extracts out the value of $x(\tau)$ at $t - t_d$. So,

$$\begin{aligned} x(t) * \delta(t - t_d) &= \int_{-\infty}^{\infty} x(\tau)\delta(t - t_d - \tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t - t_d)\delta(t - t_d - \tau)d\tau \\ &= x(t - t_d)\int_{-\infty}^{\infty} \delta(t - t_d - \tau)d\tau \\ &= x(t - t_d) \end{aligned}$$

Examples

Calculate the following expressions:

- $\delta(t-t_1) * \delta(t-t_2)$
- Let $x(t) = \delta(t-5) + \delta(t+5)$ and $h(t) = \delta(t-0.5) - 3\delta(t)$. Calculate $x * h$.
- Let $x(t) = u(t) - u(t-2)$ and $h(t) = \delta(t - 1 + \delta(t-2)$.

Implementing integration with convolution

Convolution can be used to implement integration. In particular, to integrate a signal x from $-\infty$ to t, we integrate it with a unit step.

$$\begin{aligned} x(t) * u(t) &= \int_{-\infty}^{\infty} x(t)u(t-\tau)d\tau \\ &= \int_{-\infty}^{t} x(\tau)d\tau \end{aligned}$$

where we used the fact that $u(t-\tau)$ is zero for when $\tau > t$.

Properties of convolution systems

Given these properties of convolution, there are now a few properties we can derive regarding convolution.

- **Linearity**: Convolution is **linear**, since for all signals x_1, x_2 and all $\alpha, \beta \in \mathbb{R}$,
$$h * (\alpha x_1 + \beta x_2) = \alpha(h * x_1) + \beta(h * x_2)$$
- **Time-invariance**: if $y(t) = x(t) * h(t)$, then if we delay the input by T, i.e., the new input is $x(t - T)$, then the output is $y(t - T)$. How would you prove this?

Properties of convolution systems (cont.)

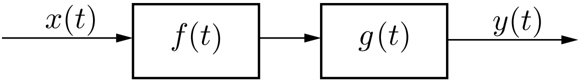
- **Cascade (composition)**: Due to the associativity of convolution, the cascade connection of two convolution systems,

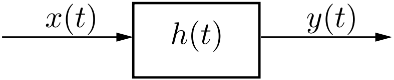
$$y = (x * f) * g$$

is equivalent to a single systsem

$$y = x * h$$

where $h = f * g$. That is, the following two block diagrams are equivalent:





Properties of convolution systems (cont.)

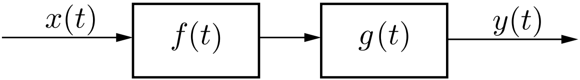
- **Swapping (composition II)**: If

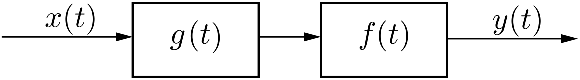
$$y = (x * f) * g$$

then, due to the commutivity of convolution, this is equivalent to

$$y = (x * g) * f$$

This means that you can swap the order of convolutions, as illustrated in the block diagram below:





Many operations can be written as convolutions (integration, delays, differentiation, etc.) and these operations all commute.

Properties of convolution systems (cont.)

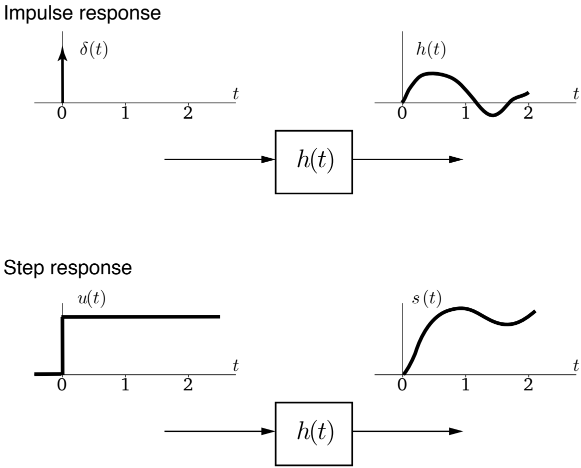
A practical application of the commutativity of these operations is in finding the impulse response. But we previously talked before about how generating an impulse is not straightforward; it is infinitely large around $t = 0$ and zero elsewhere, but has area 1. However, we know that

$$\delta(t) = \frac{du(t)}{dt}$$

i.e., the Dirac delta is the derivative of the unit step. Since convolution (and LTI) operations are interchangeable, we can find the impulse response by using a unit step.

Properties of convolution systems (cont.)

Here, we first find the *step-response* by convolving the system with a step input (instead of a Dirac delta). To arrive at the impulse response, we then take the derivative of the step response. The step response is illustrated below:

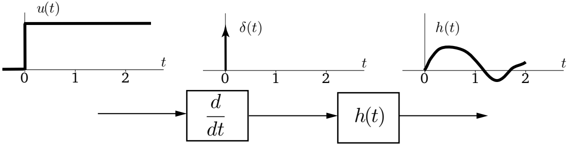


Properties of convolution systems (cont.)

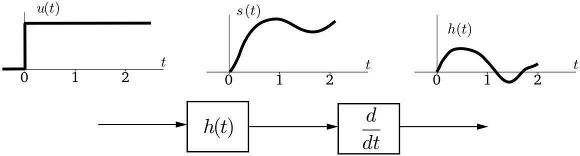
Due to commutivity, we can now find the impulse response by differentiating the step response, i.e.,

$$h(t) = \frac{ds(t)}{dt}$$

This is illustrated below.



is equivalent to



All properties of convolution extend to LTI systems

- Last lecture, we discussed how every LTI system can be written as the convolution of an input, $x(t)$, with an impulse response, $h(t)$.
- Therefore, every LTI system can be written as a convolution integral.
- Since convolution has properties here, including associativity, distributivity, commutativity, composition, etc., *all* of these properties extend to LTI systems.
- e.g., in a block diagram with LTI operations, you can interchange the order of the blocks however you like.

Examples

To be done in class.

Summary

- Every LTI system can be characterized as the convolution of some input, $x(t)$, with an impulse response, $h(t)$.
- Because convolution is associative and commutative, and all LTI systems can be written as convolutions, these properties carry over to LTI systems.
- For example, it is possible to change the order of system blocks or combine system blocks together.
- Great! Have we solved linear systems? Where to now? (Convolution is conceptually straightforward, but to do it can be practically difficult...)