ECE102, Fall 2020 **Homework #4**
Signals & Systems Prof. J.C. Kao
UCLA; Department of ECE TAs: A. Ghosh, T. Monsoor, G. Zhao

Due Friday, 6 Nov 2020, by 11:59pm to Gradescope.
Covers material up to Lecture 8.
100 points total.
This homework covers questions relate to Fourier series and LTI systems.

1. (28 points) **Fourier Series**

   (a) (18 points) Find the Fourier series coefficients for each of the following periodic signals:

   i. (9 points) $f(t) = \cos(5\pi t) + \frac{1}{2}\sin(4\pi t)$

   Solution: We first find the period of $f(t)$. The first term $\cos(5\pi t)$ is periodic with period $T_1 = \frac{2\pi}{5\pi} = \frac{2}{5}$. The second term $\sin(4\pi t)$ is periodic with period $T_2 = \frac{2\pi}{4\pi} = \frac{1}{2}$. Since $\frac{T_1}{T_2} = \frac{4}{5}$, $f(t)$ is then periodic with fundamental period $T_0 = 5T_1 = 4T_2 = 2$ sec, and fundamental frequency $\omega_0 = \frac{2\pi}{\omega_0} = \pi$ rad/s.

   Using Euler's identity, $f(t)$ can be equivalently written as:

$$f(t) = \cos(5\pi t) + \frac{1}{2}\sin(4\pi t) = \frac{1}{2}\left(e^{j5\pi t} + e^{-j5\pi t}\right) + \frac{1}{4j}\left(e^{j4\pi t} - e^{-j4\pi t}\right)$$
$$= \frac{1}{2}e^{j5\pi t} + \frac{1}{2}e^{-j5\pi t} + \frac{-j}{4}e^{j4\pi t} + \frac{j}{4}e^{-j4\pi t}$$

   The fundamental frequency of $f(t)$ is $\omega_0 = \pi$, and since any periodic signal can be written as:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt}$$

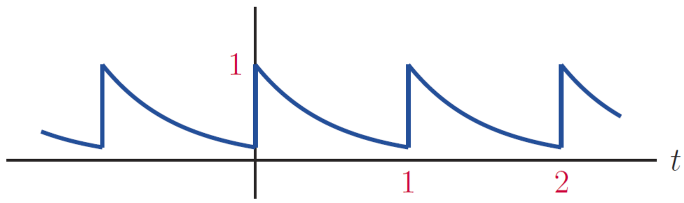
   we deduce for $f(t)$ the following Fourier series coefficients:

$$c_k = \begin{cases} \frac{-j}{4}, & \text{if } k = 4 \\ \frac{j}{4}, & \text{if } k = -4 \\ \frac{1}{2}, & \text{if } k = -5,\ 5 \\ 0, & \text{otherwise} \end{cases}$$

   ii. (9 points) $f(t)$ is a periodic signal with period $T = 1$ s, where one period of the signal is defined as e^{-2t} for $0 < t < 1$ s, as shown below.

   Solution:
   Since $f(t)$ is periodic with period $T_0 = 1$ s, we can rewrite it as:

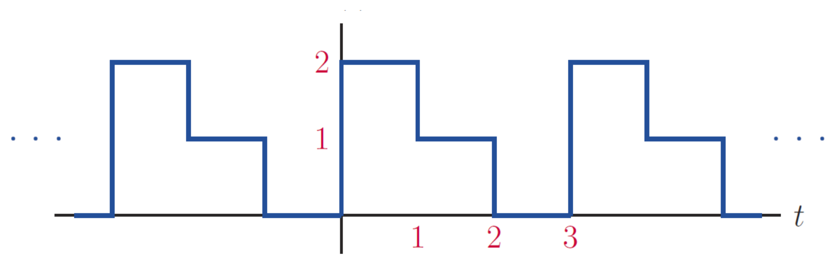
$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$



where $\omega_0 = \frac{2\pi}{T_0} = 2\pi$ rad/s and the coefficients $c_k{}'s$ are as follows:

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_0^T f(t)e^{-jk\omega_0 t}dt = \int_0^1 e^{-2t}e^{-j2k\pi t}dt \\
&= \frac{1-e^{-(2+j2\pi k)}}{2+j2\pi k} = \frac{1-e^{-2}}{2+j2\pi k}
\end{aligned}
$$

iii. (optional) (0 points) $f(t)$ is the periodic signal shown below:



Solution: Since $f(t)$ is periodic with period $T_0 = 3$ s, we can rewrite it as:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{3}$ rad/s and the coefficients $c_k{}'s$ are as follows:

$$c_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{3}\left(\int_0^1 2dt + \int_1^2 1dt\right) = 1$$

and for $k \neq 0$, we have:

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_0^T f(t)e^{-j\omega_0 kt}dt = \frac{1}{3}\left(\int_0^1 2e^{-j(2\pi/3)kt}dt + \int_1^2 e^{-j(2\pi/3)kt}dt\right) \\
&= \frac{1}{3}\left(2\frac{1-e^{-j(2\pi/3)k}}{j(2\pi/3)k} + \frac{e^{-j(2\pi/3)k}-e^{-j(4\pi/3)k}}{j(2\pi/3)k}\right) = \frac{2-e^{-j(2\pi/3)k}-e^{-j(4\pi/3)k}}{j2\pi k} \\
&= \frac{2-e^{-j(2\pi/3)k}-e^{j(2\pi/3)k}}{j2\pi k} = \frac{2-2\cos\left(\frac{2\pi k}{3}\right)}{j2\pi k} = \frac{1-\cos\left(\frac{2\pi k}{3}\right)}{j\pi k}
\end{aligned}
$$

(b) (10 points) Suppose you have two periodic signals $x(t)$ and $y(t)$, of periods T_1 and T_2 respectively. Let x_k and y_k be the Fourier series coefficients of $x(t)$ and $y(t)$.

i. (5 points) If $T_1 = T_2$, express the Fourier series coefficients of $z(t) = x(t) + y(t)$ in terms of x_k and y_k.

Solution:
If $T_1 = T_2$, then $y(t)$ is also periodic with period $T_0 = T_1 = T_2$. If $\omega_0 = \frac{2\pi}{T_0}$, then

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t}$$

and

$$y(t) = \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t}$$

Therefore,

$$z(t) = \sum_{k-\infty}^{\infty} x_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (x_k + y_k) e^{jk\omega_0 t}$$

Therefore, the Fourier series coefficients of $z(t)$ are:

$$z_k = x_k + y_k$$

ii. (5 points) If $T_1 = \frac{1}{2}T_2$, express the Fourier series coefficients of $w(t) = x(t) + y(t)$ in terms of x_k and y_k.

Solution: First of all, $w(t)$ is periodic with period $T_0 = T_1 = \frac{1}{2}T_2$, and frequency $\omega_1 = 2\omega_2 = 2\omega_0$. Let,

$$x(t) = \sum_{m=-\infty}^{\infty} x_m e^{jm\omega_1 t} = \sum_{m=-\infty}^{\infty} x_m e^{2jm\omega_0 t}$$

and

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_2 t} = \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_0 t}$$

Therefore, $w(t)$ can be written as:

$$w(t) = x(t) + y(t) = \sum_{m=-\infty}^{\infty} x_m e^{j2m\omega_0 t} + \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_0 t}$$

Let $m' = 2m$, then

$$\begin{aligned} w(t) &= \sum_{m=-\infty}^{\infty} y_n e^{jn\omega_0 t} + \sum_{\text{even } m'} x_{\frac{m'}{2}} e^{jm'\omega_0 t} \\ &= \sum_{\text{even } n} y_n e^{jn\omega_0 t} + \sum_{\text{odd } n} y_n e^{jn\omega_0 t} + \sum_{\text{even } m'} x_{\frac{m'}{2}} e^{jm'\omega_0 t} \end{aligned}$$

Therefore,

$$w_n = \begin{cases} y_n, & \text{for } n \text{ odd} \\ y_n + x_{\frac{n}{2}}, & \text{for } n \text{ even} \end{cases}$$

2. (20 points) **Fourier series of transformation of signals**

Suppose that $f(t)$ is a periodic signal with period T_0, with the following Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Determine the period of each of the following signals, then express its Fourier series in terms of c_k:

(a) (5 points) $g(t) = 2f(t)$

Solution:
The function $g(t)$ has the same period of $f(t)$. Scaling the signal will affect the Fourier coefficient c_k:

$$g(t) = 2f(t) = 2\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} 2c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega_0 t}$$

Therefore $c'_k = 2c_k$

(b) (5 points) $g(t) = f(-2t)$

Solution:
The period of $g(t)$ is $T'_0 = \frac{T_0}{2}$ and its corresponding frequency is $\omega'_0 = 2\omega_0$

$$g(t) = f(-2t) = \sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 2t} = \sum_{k=-\infty}^{\infty} c_{-k} e^{jk2\omega_0 t} = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega'_0 t}$$

Therefore $c'_k = c_{-k}$

(c) (5 points) $g(t) = f(t - t_0)$

Solution:
$g(t)$ has the same period of $f(t)$.

$$g(t) = f(t - t_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 (t-t_0)} = \sum_{k=-\infty}^{\infty} (c_k e^{-jk\omega_0 t_0}) e^{jk\omega_0 t}$$

Therefore $c'_k = c_k e^{-jk\omega_0 t_0}$

(d) (5 points) $g(t) = f(t/a)$, where a is positive real number

Solution:
The period of $g(t)$ is $T'_0 = aT_0$, and its corresponding frequency is: $\omega'_0 = \frac{\omega_0}{a}$. Therefore,

$$g(t) = f(t/a) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 (t/a)} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega'_0 t}$$

Therefore, the Fourier series coefficients of $f(t)$ and $g(t)$ are the same.

3. (10 points) **Eigenfunctions and LTI systems**

   (a) (5 points) Show that $f(t) = \cos(\omega_0 t)$ is not an eigenfunction of an LTI system.

   Solution:
   Assume that $h(t)$ is the impulse response of the system. Then the output $y(t)$ to input $f(t) = \cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$is as follows:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{j\omega_0(t-\tau)}h(\tau)d\tau + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j\omega_0(t-\tau)}h(\tau)d\tau \\ &= \frac{1}{2}e^{j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{-j\omega_0\tau}h(\tau)d\tau}_{=a_1} + \frac{1}{2}e^{-j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{j\omega_0\tau}h(\tau)d\tau}_{=a_2} \end{aligned}$$

   For $f(t)$ to be an eigenfunction for the system, its corresponding output should be of the form $af(t)$, where a is constant. The output to $\cos(\omega_0 t)$ is:

$$y(t) = \frac{1}{2}a_1 e^{j\omega_0 t} + \frac{1}{2}a_2 e^{-j\omega_0 t}$$

   Since, in general $a_1 \neq a_2$, we cannot construct again $cos(\omega_0 t)$ in $y(t)$. For instance, suppose $f(t) = \delta(t-4)$, then $a_1 = e^{-j4\omega_0}$ and $a_2 = e^{j4\omega_0}$. Therefore,

$$y(t) = \frac{1}{2}e^{j\omega_0(t-4)} + \frac{1}{2}e^{-j\omega_0(t-4)} = \cos\left(\omega_0(t-4)\right)$$

   We then see the output is not of the form $a\cos(\omega_0 t)$, therefore $\cos(\omega_0 t)$ is not an eigenfunction for an LTI system. (We will accept a counterexample as correct, since complex exponentials are eigenfunctions of all LTI systems.)

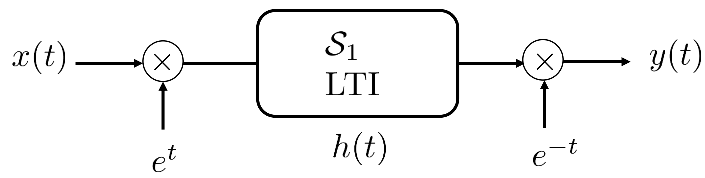
   (b) (5 points) Show that $f(t) = t$ is not an eigenfunction of an LTI system.

   Solution:
   Assume that $h(t)$ is the impulse response of the system. Then the output $y(t)$ to input $f(t) = t$ is as follows:

$$y(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} (t-\tau)h(\tau)d\tau = t\underbrace{\int_{-\infty}^{\infty} h(\tau)d\tau}_{=a_1} - \underbrace{\int_{-\infty}^{\infty} \tau h(\tau)d\tau}_{=a_2}$$

   $y(t)$ is of the form $a_1 t + a_2$, therefore the function $f(t) = t$ is not an eigenfunction of an LTI system.

4. (29 points) **LTI systems**



(a) Consider the following system:
The system takes as input $x(t)$, it first multiplies the input with e^t, then sends it through an LTI system. The output of the LTI system gets multiplied by e^{-t} to form the output $y(t)$.

i. (5 points) Show that we can write $y(t)$ as follows:

$$y(t) = \left[\left(e^t x(t)\right) * h(t)\right] e^{-t} \tag{1}$$

Solution:
The input $x(t)$ gets first multiplied by e^t and forms the intermediate signal:

$$y_1(t) = e^t x(t)$$

Next, $y_1(t)$ is fed to the LTI system, the output $y_2(t)$ is then the convolution of $y_1(t)$ with $h(t)$:

$$y_2(t) = y_1(t) * h(t) = (e^t x(t)) * h(t)$$

Finally, $y_2(t)$ gets finally multiplied by e^{-t}:

$$y(t) = e^{-t} y_2(t) = [(e^t x(t)) * h(t)]e^{-t}$$

ii. (5 points) Use the definition of convolution to show that (1) can be equivalently written as:

$$y(t) = \int_{-\infty}^{\infty} h'(\tau) x(t-\tau) d\tau \tag{2}$$

where $h'(t)$ is a function to define in terms of $h(t)$.

Solution:
By applying the definition of convolution, we obtain:

$$\begin{aligned} y(t) &= [(e^t x(t)) * h(t)]e^{-t} \\ &= e^{-t} \int_{-\infty}^{\infty} h(\tau) e^{t-\tau} x(t-\tau) d\tau \\ &= e^{-t} e^{t} \int_{-\infty}^{\infty} h(\tau) e^{-\tau} x(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) e^{-\tau} x(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} h'(\tau) x(t-\tau) d\tau \end{aligned}$$

where $h'(\tau) = h(\tau)e^{-\tau}$.

iii. (5 points) Equation (2) represents a description of the equivalent system that maps $x(t)$ to $y(t)$. Show using (2) that the equivalent system is LTI and determine its impulse response $h_{eq}(t)$ in terms of $h(t)$.

Solution:
Linearity:
Suppose that for inputs $x_1(t)$ and $x_2(t)$, we have respectively the corresponding outputs $y_1(t)$ and $y_2(t)$ outputs. Now, let $x(t) = ax_1(t) + bx_2(t)$, we then have the following:
Method 1: Using the equation from part b:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h'(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h'(\tau)(ax_1(t-\tau) + bx_2(t-\tau))d\tau \\
&= \int_{-\infty}^{\infty} (ah'(\tau)x_1(t-\tau) + bh'(\tau)x_2(t-\tau))d\tau \\
&= \int_{-\infty}^{\infty} ah'(\tau)x_1(t-\tau)d\tau + bh'(\tau)x_2(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} ah'(\tau)x_1(t-\tau)d\tau + \int_{-\infty}^{\infty} bh'(\tau)x_2(t-\tau)d\tau \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

Method 2:

$$\begin{aligned}
y(t) &= [(e^t x(t)) * h(t)]e^{-t} \\
&= [(e^t(ax_1(t) + bx_2(t))) * h(t)]e^{-t} \\
&= [(ae^t x_1(t) + be^t x_2(t))) * h(t)]e^{-t} \\
&= [(ae^t x_1(t)) * h(t) + (be^t x_2(t)) * h(t)]e^{-t} \\
&= [(ae^t x_1(t)) * h(t)]e^{-t} + [(be^t x_2(t)) * h(t)]e^{-t} \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

Therefore system is linear.
Time invariance:
Using result from part b, if we delay the input for t_0:

$$\begin{aligned}
y_{t_0}(t) &= \int_{-\infty}^{\infty} h'(\tau)x(t-\tau-t_0)d\tau \\
&= \int_{-\infty}^{\infty} h'(\tau)x(t-t_0-\tau)d\tau \\
&= y(t-t_0)
\end{aligned}$$

Therefore system is TI. From part b, we know that $h'(t) = h(t)e^{-t}$. Therefore, the impulse response of the equivalent system is:

$$h_{eq}(t) = h(t)e^{-t}$$

iv. (Optional) (0 points) Suppose that system $\mathcal{S}_1$ is given by its step response $s(t) = r(t-1)$. Find the impulse response $h(t)$ of $\mathcal{S}_1$. What can you say about the causality and stability of system $\mathcal{S}_1$? What can you say about the causality and stability of the overall equivalent system?

Solution:
The impulse response of system $\mathcal{S}_1$ is:

$$h(t) = \frac{d}{dt}s(t) = u(t-1)$$

Since $h(t) = 0$ for $t < 0$, the system $\mathcal{S}_1$ is causal. However, this same system is not stable because

$$\int_{-\infty}^{\infty} |h(t)|dt \to \infty$$

The equivalent system has the following equivalent impulse response:

$$h_{eq}(t) = e^{-t}u(t-1)$$

Since $h_{eq}(t) = 0$ for $t < 0$, the system is causal. It is also stable, because:

$$\int_{-\infty}^{\infty} |h_{eq}(t)|dt = \int_{t=1}^{\infty} e^{-t}dt = e^{-1} < \infty$$

(b) Suppose $x(t)$ is periodic with period T and is specified in the interval $0 < t < T/4$ as shown in figure 1.

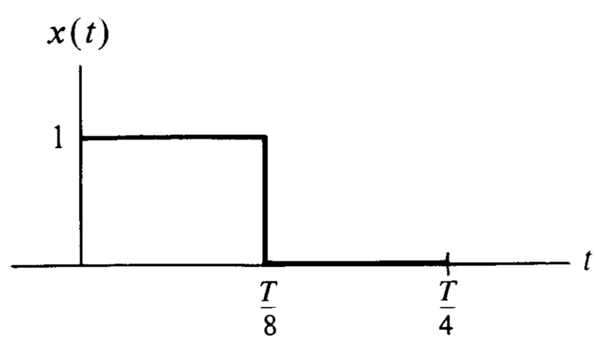


Figure 1: $x(t)$ in the interval $0 < t < T/4$

Sketch $x(t)$ in the interval $0 < t < T$ if

i. (7 points) the Fourier series has only odd harmonics and $x(t)$ is an even function

Solution: Since $x(t)$ is even, we can extend figure 1 as indicated in figure 2. Since $x(t)$ has only odd harmonics, it must have the property $x(t-T/2) = -x(t)$. So we have $x(t)$ as in figure 3.

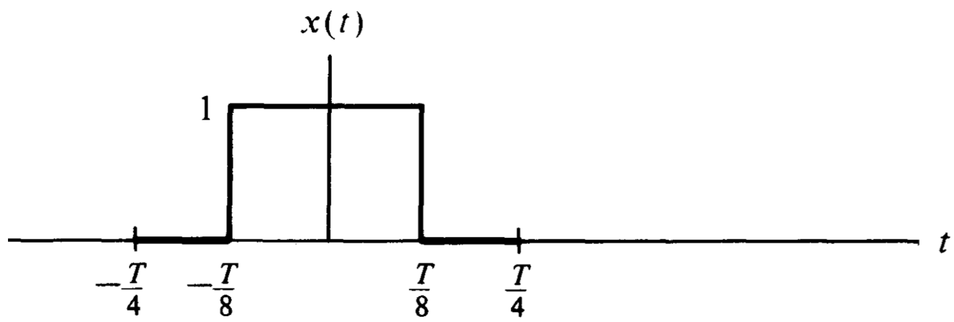


Figure 2: $x(t)$ (even) in the interval $-T/4 < t < T/4$

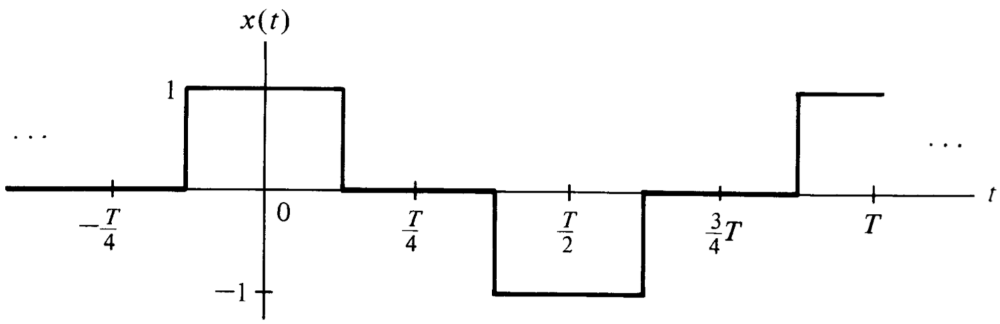


Figure 3: $x(t)$ (even) in the interval $-T < t < T$

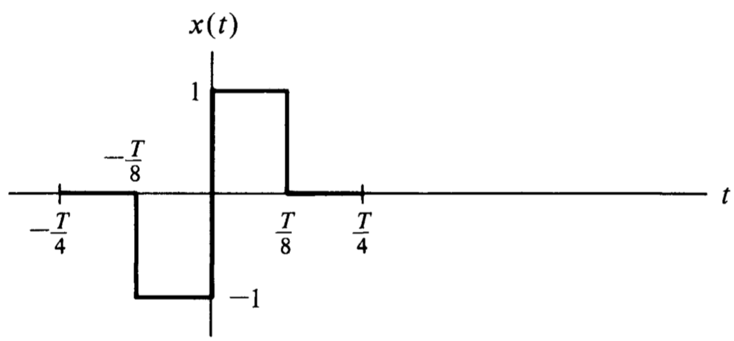


Figure 4: $x(t)$ (odd) in the interval $-T/4 < t < T/4$

ii. (7 points) the Fourier series has only odd harmonics and $x(t)$ is an odd function

Solution: Since $x(t)$ is odd, for $-T/4 < t < T/4$ it must be as indicated in figure 4. Since $x(t)$ has odd harmonics, so $x(t - T/2) = -x(t)$. Consequently $x(t)$ is as shown in figure 5.

5. (13 points) **MATLAB**

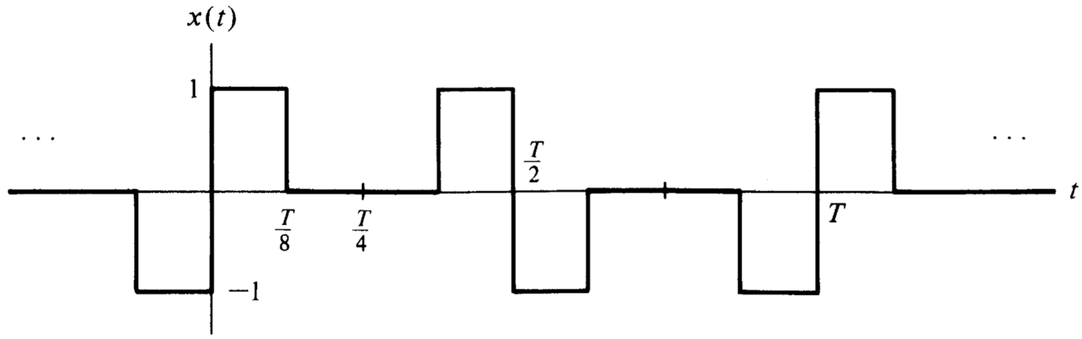


Figure 5: $x(t)$ (odd) in the interval $-T < t < T$

(a) (6 points) **Task 1**
Write an m-file that takes a set of Fourier series coefficients, a fundamental frequency, and a vector of output times, and computes the truncated Fourier series evaluated at these times. The declaration and help for the m-file might be:

```
 function fn = myfs(Dn,omega0,t)
%
% fn = myfs(Dn,omega0,t)
% % Evaluates the truncated Fourier Series at times t
%
% Dn -- vector of Fourier series coefficients
%
% omega0 -- fundamental frequency
% t -- vector of times for evaluation
%
% fn -- truncated Fourier series evaluated at t
```

The output of the m-file should be

$$f_N(t) = \Sigma_{n=-N}^{N} D_n e^{j\omega_0 nt}$$

The length of the vector Dn should be $2N + 1$. You will need to calculate N from the length of Dn.

Solution:

```
function fn = myfs(Dn,omega0,t)
% fn = myfs(Dn,omega0,t)
% Evaluates the truncated Fourier Series at times t
% Dn -- vector of Fourier series coefficients
% assumed to run from -N:N, where length(Dn) is 2N+1
% omega0 -- fundamental frequency
% t -- vector of times for evaluation
```

```
% fn -- truncated Fourier series evaluated at t
N = (length(Dn)-1)/2;
fn = zeros(size(t));
for n = -N:N
D_n = Dn(n+N+1);
fn = fn + D_n*exp(j*omega0*n*t);
end
```

(b) (7 points) **Task 2**

Verify the output of your routine by checking the Fourier series coefficients for the sawtooth waveform. The sawtooth signal is given by f(t) = t mod 1 described in the class notes. Try for $N = 10$, $N = 50$. Use the MATLAB subplot command to put multiple plots on a page.

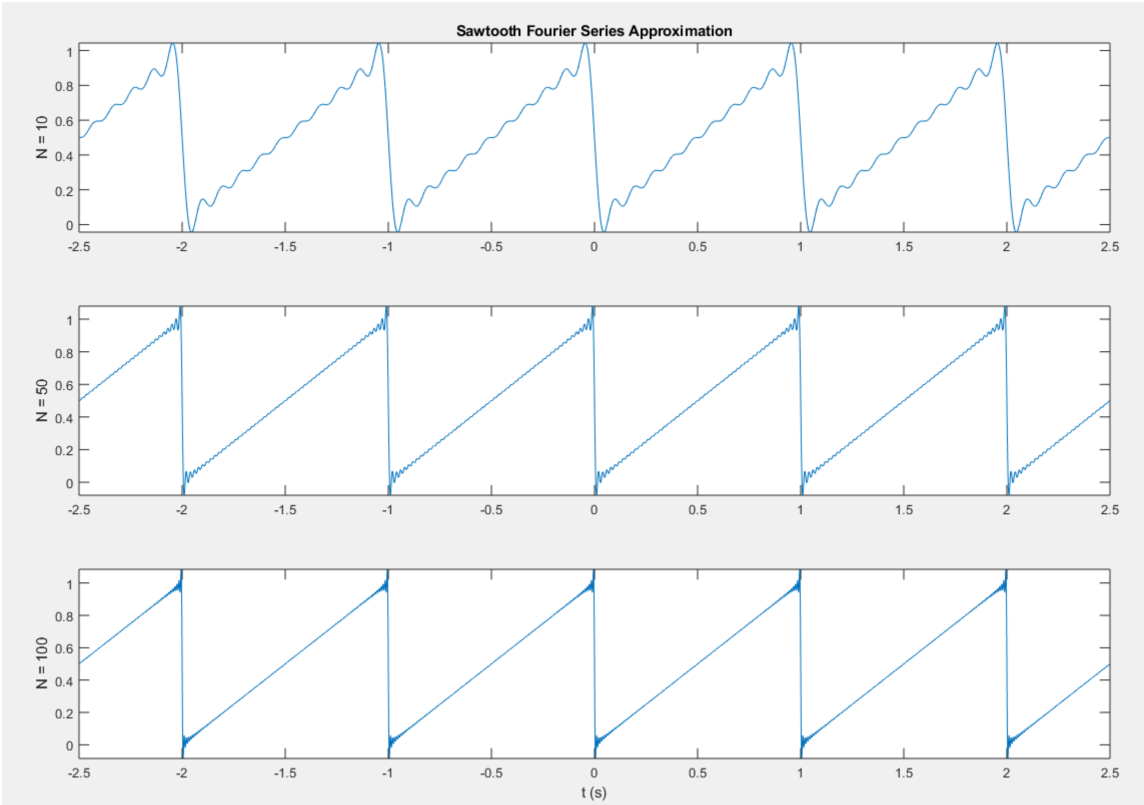
Solution:

```
N=10;n1=1:1:N;n2=-N:1:-1;
D1=j./(2*pi*n1);D2=j./(2*pi*n2);D=[D2 0.5 D1];
omega0=2*pi;
t=-2.5:0.001:2.5;
fn = myfs(D,omega0,t);
subplot(3,1,1);
plot(t,fn);
xlabel('t (s)'); ylabel('N=10');
title('Sawtooth Fourier Series Approximation');

N=50;n1=1:1:N;n2=-N:1:-1;
D1=j./(2*pi*n1);D2=j./(2*pi*n2);D=[D2 0.5 D1];
fn = myfs(D,omega0,t);
subplot(3,1,2);
plot(t,fn);
xlabel('t (s)'); ylabel('N=50');

N=100;n1=1:1:N;n2=-N:1:-1;
D1=j./(2*pi*n1);D2=j./(2*pi*n2);D=[D2 0.5 D1];
omega0=2*pi;
t=-2.5:0.001:2.5;
fn = myfs(D,omega0,t);
subplot(3,1,3);
plot(t,fn);
xlabel('t (s)'); ylabel('N=100');
```

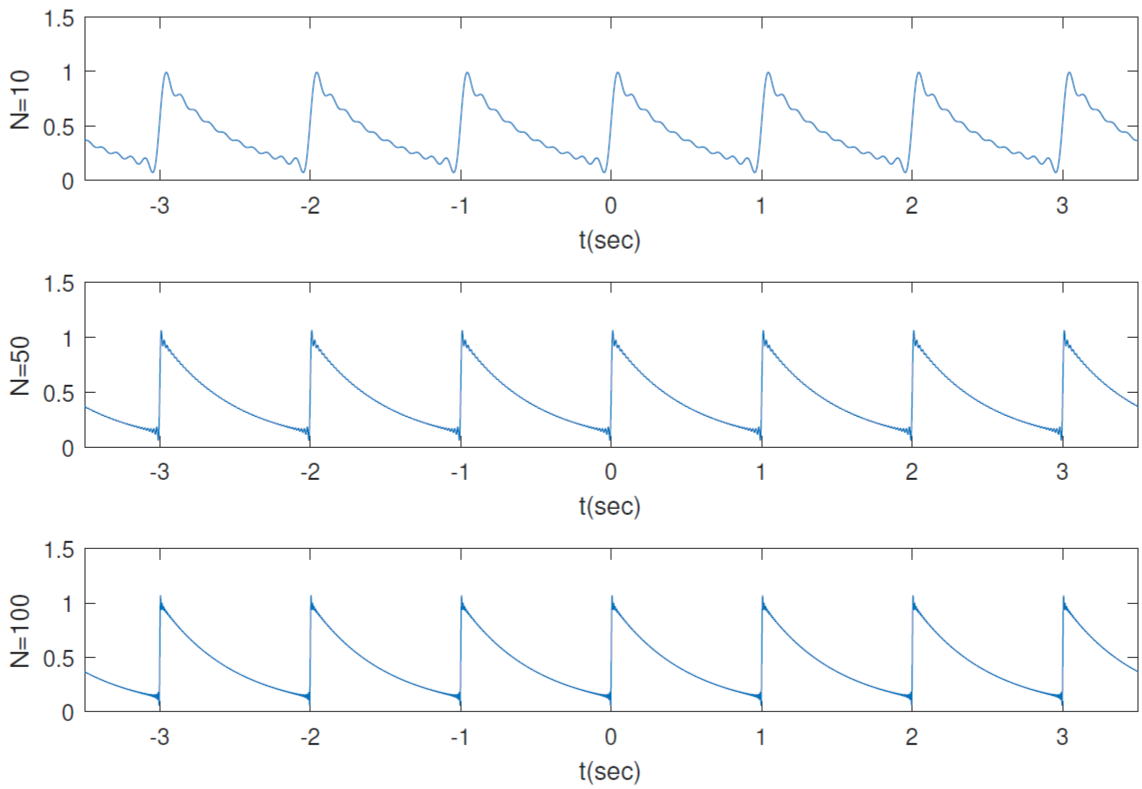
(c) (Optional) (0 points) **Task 3**

Repeat the steps of Task 2 for the case of the signal from Problem 1-a-ii.

Solution:




```
 N=10;n=-N:1:N;D=(1-exp(-2))./(2+2*j*pi*n);
omega0=2*pi;t=-3.5:0.001:3.5;
fn = myfsHs(D,omega0,t);
subplot(3,1,1);
plot(t,fn);
xlabel('t(sec)'); ylabel('N=10');

N=50;n=-N:1:N;D=(1-exp(-2))./(2+2*j*pi*n);
omega0=2*pi;t=-3.5:0.001:3.5;
fn = myfsHs(D,omega0,t);
subplot(3,1,2);
plot(t,fn);
xlabel('t(sec)'); ylabel('N=50');

N=100;n=-N:1:N; D=(1-exp(-2))./(2+2*j*pi*n);
omega0=2*pi;t=-3.5:0.001:3.5;fn = myfsHs(D,omega0,t);
subplot(3,1,3);
plot(t,fn);
xlabel('t(sec)'); ylabel('N=100');
```


N=10
N=50
N=100
1.5
1
0.5
0
-3
-2
-1
0
1
2
3
t(sec)