Inversion of the Laplace transform

This lecture is about inverting the Laplace transform. Topics include:

- Poles and zeros
- Partial fraction expression
- Methods to find residues of partial fractions
- Partial fractions with repeated poles
- Quadratic factors
- Partial fractions for nonproper rational functions

Motivation

The inverse of the Laplace transform is given by

$$f(t) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} F(s) e^{st} ds$$

where σ is large enough that $F(s)$ is defined for $\Re(s) \geq c$.

Calculating this integral is not straightforward (topic for a complex analysis class). It is a contour integral in the complex plane. Instead, a standard way to invert the Laplace transform is to rewrite the Laplace transform into simpler terms that can be inverted by inspection (i.e., a look up table). This is achieved through *partial fraction expansion*.

Partial fraction expansion

Let

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

- The zeros of $b(s)$, i.e., the s such that $b(s) = 0$ are called the *zeros* of $F(s)$ because at these s, $F(s) = 0$.
- The zeros of $a(s)$, i.e., the s such that $a(s) = 0$ are called the *poles* of $F(s)$ because at these s, $F(s)$ tends to infinity.

Let's first assume that no poles are repeated and that $m < n$ (i.e., more poles than zeros).

Then, $F(s)$ can be written in its *partial fraction expansion*:

$$F(s) = \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}$$

where

- $\lambda_1, \ldots, \lambda_n$ are the poles of F.
- The numbers $r_1, \ldots, r_n$ are called residues.
- It turns out when $\lambda_k = \lambda_l^*$, then $r_k = r_l^*$.

Inversion of a partial fraction

In partial fraction form, inverting the Laplace transform is easy because

$$\begin{aligned}\mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left[\frac{r_1}{s-\lambda_1}+\cdots+\frac{r_n}{s-\lambda_n}\right] \\ &= r_1\mathcal{L}^{-1}\left[\frac{1}{s-\lambda_1}\right]+\cdots+r_n\mathcal{L}^{-1}\left[\frac{1}{s-\lambda_n}\right] \\ &= r_1 e^{\lambda_1 t}+\cdots+r_n e^{\lambda_n t}\end{aligned}$$

The inverse Laplace transform of a partial fraction is always real, since whenever poles are conjugate, so are the corresponding residues.

How to find the partial fraction expansion

To find the partial fraction expansion, we

- Find the poles $\lambda_1, \ldots, \lambda_n$, which means we find the zeros of $a(s)$.
- Find the residues of $r_1, \ldots, r_n$.

There are several methods to calculate partial fraction expansions. We'll go over examples for all of these.

Method 1: partial fractions via solving linear equations

In this method, we factor $a(s)$ to find the poles and then solve linear equations to find the residues. Say $m = 2$ and $n = 3$. Then,

$$\frac{b_0 + b_1 s + b_2 s^2}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \frac{r_3}{s-\lambda_3}$$

First, we clear the denominators by multiplying both sides by

$$(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)$$

This gives:

$$b_0 + b_1 s + b_2 s^2 = r_1(s-\lambda_2)(s-\lambda_3) + r_2(s-\lambda_1)(s-\lambda_3) + r_3(s-\lambda_1)(s-\lambda_2)$$

At this point, we equate the coefficients of each power of s. (This continues on the next slide.)

Method 1: partial fractions via solving linear equations (cont.)

Equate coefficients:

- Coefficients of s^0.

$$b_0 = (\lambda_2\lambda_3)r_1 + (\lambda_1\lambda_3)r_2 + (\lambda_1\lambda_2)r_3$$

- Coefficients of s^1.

$$b_1 = (-\lambda_2 - \lambda_3)r_1 + (-\lambda_1 - \lambda_3)r_2 + (-\lambda_1 - \lambda_2)r_3$$

- Coefficients of s^2.

$$b_2 = r_1 + r_2 + r_3$$

After this, we solve for r_1, r_2, r_3, which is possible because we have three equations and three unknowns.

Method 2: partial fractions via the "cover-up" procedure

Here, we solve for each residual individually in the following way. E.g., to get r_1, we first multiply both sides by $(s - \lambda_1)$.

$$\frac{(s-\lambda_1)(b_0 + b_1 s + b_2 s^2)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = r_1 + \frac{r_2(s-\lambda_1)}{s-\lambda_2} + \frac{r_3(s-\lambda_1)}{s-\lambda_3}$$

On the left hand side, we can cancel $(s - \lambda_1)$ terms. This equation must hold for all s, in particular $s = \lambda_1$. So we set $s = \lambda_1$ to eliminate the r_2 and r_3 terms, i.e.,

$$r_1 = \frac{b_0 + b_1\lambda_1 + b_2\lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}$$

This is an explicit formula for r_1.

It is straightforward to solve for r_2 and r_3 in the same way. Generally,

$$r_k = (s - \lambda_k)\ F(s)\big|_{s=\lambda_k}$$

Method 2 example

Let's find the following partial fraction expansion:

$$\frac{s^2-2}{s(s+1)(s+2)} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

The poles are $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -2$.

- r_1:

$$\begin{aligned} r_1 &= \left. \left(r_1 + \frac{r_2 s}{s+1} + \frac{r_3 s}{s+2} \right) \right|_{s=0} \\ &= \left. \frac{s^2-2}{(s+1)(s+2)} \right|_{s=0} \\ &= -1 \end{aligned}$$

- r_2:

$$\begin{aligned} r_2 &= \left. \left(\frac{r_1(s+1)}{s} + r_2 + \frac{r_3 s}{s+2} \right) \right|_{s=-1} \\ &= \left. \frac{s^2-2}{s(s+2)} \right|_{s=-1} \\ &= 1 \end{aligned}$$

Method 2 example (cont.)

- r_3:

$$\begin{aligned} r_3 &= \left(\frac{r_1(s+1)}{s} + \frac{r_2(s+2)}{s+1} + r_3 \right)\Bigg|_{s=-2} \\ &= \frac{s^2-2}{s(s+1)}\Bigg|_{s=-2} \\ &= 1 \end{aligned}$$

Hence,

$$\frac{s^2-2}{s(s+1)(s+2)} = \frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2}$$

Method 3: partial fractions via l'Hopital's rule

Another way to find the kth residual is to calculate:

$$r_k = \frac{b(\lambda_k)}{a'(\lambda_k)}$$

The idea behind this approach is to still use the cover-up method (i.e., multiply the partial fraction expansion by $(s - \lambda_k)$) and set s to λ_k. This technique finds another formula for the residual.

$$r_k \quad = \quad \lim_{s \to \lambda_k} \frac{(s - \lambda_k) b(s)}{a(s)}$$

To simplify this expression, we use l'Hopital's rule to differentiate both the numerator and denominator w.r.t. s.

$$\begin{aligned} r_k \quad &= \quad \lim_{s \to \lambda_k} \frac{b(s) + b'(s)(s - \lambda_k)}{a'(s)} \\ &= \quad \frac{b(\lambda_k)}{a'(\lambda_k)} \end{aligned}$$

Method 3 example

Let's do the same example as in Method 2,

$$\frac{s^2 - 2}{s(s+1)(s+2)} = \frac{s^2 - 2}{s^3 + 3s^2 + 2s}$$

with poles $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -2$. We have that $a'(s) = 3s^2 + 6s + 2$. Therefore,

$$r_1 = \left. \frac{s^2 - 2}{3s^2 + 6s + 2} \right|_{s=0} = -1$$

Similarly,

$$\begin{aligned} r_2 &= \left. \frac{s^2 - 2}{3s^2 + 6s + 2} \right|_{s=-1} = 1 \\ r_3 &= \left. \frac{s^2 - 2}{3s^2 + 6s + 2} \right|_{s=-2} = 1 \end{aligned}$$

Ordinary differential equation example

Let's solve the following ODE:

$$v'''(t) - v(t) = 0$$

with initial conditions $v(0) = 1$, $v'(0) = v''(0) = 0$. We'll do this in steps:

1. Take the Laplace transform. Since

$$\begin{aligned}\mathcal{L}[v'''(t)] &= s^3V(s) - s^2v(0) - sv'(0) - v''(0) \\ &= s^3V(s) - s^2\end{aligned}$$

   we have that

$$s^3V(s) - s^2 - V(s) = 0$$

2. Solve for $V(s)$, i.e.,

$$V(s) = \frac{s^2}{s^3 - 1}$$

3. Obtain the poles of $V(s)$ by taking the cuberoot of 1. The cuberoots of 1 are $e^{j2\pi k/3}$ for $k = 0, 1, 2$. Therefore,

$$s^3 - 1 = (s-1)\left(s + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(s + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

Ordinary differential equation example (cont.)

4. Obtain the partial fraction expansion of $V(s)$. We'll use method 3.

$$V(s) = \frac{r_1}{s-1} + \frac{r_2}{\left(s + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)} + \frac{r_3}{\left(s + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)}$$

The residues are

$$\begin{aligned} \lambda_k &= \left.\frac{b(s)}{a'(s)}\right|_{s=\lambda_k} \\ &= \left.\frac{s^2}{3s^2}\right|_{s=\lambda_k} \\ &= \frac{1}{3} \end{aligned}$$

for any λ_k, since the s^2 terms cancel. Hence,

$$V(s) = \frac{1/3}{s-1} + \frac{1/3}{\left(s + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)} + \frac{1/3}{\left(s + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)}$$

You can always double check your work via algebra.

Ordinary differential equation example (cont.)

6. Take the inverse Laplace transform.

$$\begin{aligned} v(t) &= \frac{1}{3}e^t + \frac{1}{3}e^{\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)t} + \frac{1}{3}e^{\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)t} \\ &= \frac{1}{3}e^t + \frac{2}{3}e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) \end{aligned}$$

And that's it. With ODE's, you can check your work by evaluating if $v'''^{(t)} - v(t) = 0$ and if the initial conditions, i.e., $v(0) = 1$, $v'(0) = v''(0) = 0$.

Repeated poles

Our prior results were for non-repeated poles. Sometimes, poles will be repeated. Here,

$$F(s) = \frac{b(s)}{(s-\lambda_1)^k (s-\lambda_2)\cdots(s-\lambda_\ell)}$$

The poles λ_i are distinct, but λ_1 has multiplicity k.

The partial fraction expansion for $F(s)$ will still have n residues (where n is the order of the polynomial $a(s)$ as before) but will involve higher powers of $(s-\lambda_1)$. In particular, the expansion is:

$$\begin{aligned} F(s) \quad &= \quad \frac{r_{1,k}}{(s-\lambda_1)^k} + \frac{r_{1,k-1}}{(s-\lambda_1)^{k-1}} + \cdots + \frac{r_{1,1}}{(s-\lambda_1)} \\ &\qquad + \frac{r_2}{s-\lambda_2} + \cdots + \frac{r_\ell}{s-\lambda_\ell} \end{aligned}$$

After achieving this expansion, we use the following inverse Laplace transform:

$$\mathcal{L}^{-1}\left[\frac{r}{(s-\lambda)^k}\right] = \frac{r}{(k-1)!} t^{k-1} e^{\lambda t}$$

Calculating the residuals for repeated poles

To calculate the residuals, we can use Method 1 as before.

We can also extend Method 2. To get $r_{i,k}$, where k is the multipilcity of the pole, we multiply both sides by $(s - \lambda_i)^k$ and evaluate at $s = \lambda_i$. Hence,

$$r_{i,k} = F(s)(s - \lambda_i)^k \Big|_{s=\lambda_i}$$

To get the other residues, $r_{i,k-j}$, we differentiate with respect to s, j times, leading to the formula:

$$\frac{1}{j!}\frac{d^j}{ds^j}\left(F(s)(s - \lambda_i)^k\right)\Big|_{s=\lambda_i} = r_{i,k-j}$$

To get an intuition for this, let's do an example.

Repeated poles example

Let's find the partial fraction expansion of

$$\frac{1}{s^2(s+1)} = \frac{r_{1,2}}{s^2} + \frac{r_{1,1}}{s} + \frac{r_2}{s+1}$$

- We calculate r_2 in the standard Method 2 way:

$$r_2 = F(s)(s+1)\big|_{s=-1} = 1$$

- We use the extended Method 2 to first calculate $r_{1,2}$.

$$r_{1,2} = F(s)s^2\big|_{s=0} = 1$$

- Next, we multiply through by s^2, yielding:

$$\frac{1}{s+1} = r_{1,2} + sr_{1,1} + \frac{r_2 s^2}{s+1}$$

To get rid of the $r_{1,2}$ term, we differentiate w.r.t. s to get an equation in terms of $r_{1,1}$ and r_2, then evaluate at $s=0$, the pole of $r_{1,1}$. This continues on the next page.

Repeated poles example (cont.)

- Finding $r_{1,1}$, continued...

$$\frac{-1}{(s+1)^2} = r_{1,1} + \frac{d}{ds}\frac{r_2 s^2}{s+1}$$

Evaluating at $s = 0$, this yields that $r_{1,1} = -1$.

- As an aside, you can combine Methods / other algebraic techniques at your disposal, as long as you follow rules of algebra. For example, after knowing $r_{1,2}$ and r_3, we could have found $r_{1,1}$ through solving an algebraic equation:

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} + \frac{r_{1,1}}{s} + \frac{1}{s+1}$$

This must hold for all s, so let's choose an easy one that leads to a well-defined equation. Set $s = 1$, so that:

$$\frac{1}{2} = 1 + r_2 + \frac{1}{2}$$

Then $r_2 = -1$.

Quadratic factors (optional, potentially easier)

Again, we can use algebraic tricks to find the partial fractions. We're going to demonstrate another trick on how to do partial fractions with quadratic factors. First, if we define a partial fraction with a quadratic denominator, we will use the following Laplace transforms.

$$
\begin{aligned}
\mathcal{L}[\cos(\omega t)] &= \frac{s}{s^2+\omega^2} \\
\mathcal{L}[\sin(\omega t)] &= \frac{\omega}{s^2+\omega^2} \\
\mathcal{L}[e^{-at}\cos(\omega t)] &= \frac{(s+a)}{(s+a)^2+\omega^2} \\
\mathcal{L}[e^{-at}\sin(\omega t)] &= \frac{\omega}{(s+a)^2+\omega^2}
\end{aligned}
$$

A quadratic factor in the partial fraction expansion will have the form:

$$
\frac{r_1 s + r_2}{a_2 s^2 + a_1 s + a_0}
$$

Quadratic factors example

Let's consider the example we did earlier:

$$V(s) = \frac{s^2}{s^3 - 1} = \frac{s^2}{(s-1)(s^2+s+1)}$$

The partial fraction expansion is:

$$\frac{s^2}{(s-1)(s^2+s+1)} = \frac{r_1}{s-1} + \frac{r_2 s + r_3}{s^2+s+1}$$

- To find r_1, we use Method 2.

$$r_1 = \left.\frac{s^2}{s^2+s+1}\right|_{s=1} = \frac{1}{3}$$

Hence, we now have that

$$\frac{s^2}{(s-1)(s^2+s+1)} = \frac{1/3}{s-1} + \frac{r_2 s + r_3}{s^2+s+1}$$

Quadratic factors example (cont.)

- For r_2 and r_3, we could use Method 1. However, let's go about this an easier way.
- Solve for r_3 by setting $s = 0$ to eliminate r_2 yields:

$$0 = \frac{1/3}{-1} + r_3$$

so that $r_3 = 1/3$.

- Finally, we can solve for r_2:

$$\frac{s^2}{(s-1)(s^2+s+1)} = \frac{1/3}{s-1} + \frac{r_2 s + 1/3}{s^2+s+1}$$

Let's set $s = -1$ (since setting $s = 1$ would make one term undefined).

$$\frac{1}{-2 \cdot 1} = -\frac{1}{6} + \frac{-r_2 + 1/3}{1}$$

Solving for r_2 gives $r_2 = 2/3$.

Quadratic factors example (cont.)

Though it was more straightforward to calculate r_1, r_2 and r_3, we need to now factor the term

$$\frac{\frac{2}{3}s + \frac{1}{3}}{s^2 + s + 1}$$

into an expression where we can take the inverse Laplace transform. We do this by completing the square.

First, we recognize that the numerator can be re-written as

$$\frac{2}{3}s + \frac{1}{3} = \left(\frac{2}{3}\right)\left(s + \frac{1}{2}\right)$$

Getting this into the form of an inverse Laplace transform we know, i.e.,

$$\mathcal{L}[e^{-at}\cos(\omega t)] = \frac{(s+a)}{(s+a)^2 + \omega^2}$$

means that we need to complete the square by getting a $(s + 1/2)^2$ in the denominator.

Quadratic factors example (cont.)

To complete the square, we recognize the quadratic denominator can be re-written as:

$$\begin{aligned} s^2 + s + 1 &= s^2 + 2\left(\frac{1}{2}\right)s + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2 \\ &= \left(s + \frac{1}{2}\right)^2 + \frac{3}{4} \\ &= \left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \end{aligned}$$

and therefore,

$$\frac{\frac{2}{3}s + \frac{1}{3}}{s^2 + s + 1} = \frac{\frac{1}{3}}{s - 1} + \left(\frac{2}{3}\right)\frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

By look up table, the inverse Laplace transform is:

$$v(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

What to use? (Informal)

We've given you a bunch of techniques you could use to do partial fractions. Here are some guidelines that I personally use; you should adopt what is most comfortable for you. These are completely informal, and you don't have to follow these.

- The summary is that I use Method 2 almost all the time.
- I will (at times) first see if Method 3 can apply to minimize any algebra. If Method 3 doesn't obviously give a simple calculation, I will avoid it.
- If not, I will almost always use Method 2, the cover up approach.
- I almost never use Method 1 as the algebra can get involved.
- In the case of complex roots, I feel our first approach of finding the complex roots is more straight-forward. However, complex arithmetic is more prone to errors.
- Quadratic factors may be more straightforward for you; while the residues are easy to calculate, however, it requires the extra step of completing the square to get the final answer.

Nonproper rational functions

The partial fraction expansion that we've talked about earlier is for a *strictly proper* $F(s)$, with $m < n$.

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

There may be $F(s)$ for which this is not true and we want to find the Laplace transform of. To do so, we perform:

$$\begin{aligned} F(s) &= \frac{b(s)}{a(s)} \\ &= c(s) + \frac{d(s)}{a(s)} \end{aligned}$$

where

$$c(s) = c_0 + c_1 s + \cdots + c_{m-n} s^{m-n}$$

and

$$d(s) = d_0 + \cdots + d_k s^k$$

where $k < n$.

Nonproper rational functions (cont.)

In this manner, we have split $F(s)$ into a component $c(s)$ and then a strictly proper component $d(s)/a(s)$. For notation, we denote $\delta^{(k)}$ to be the kth derivative of the δ function. The inverse Laplace transform of $F(s)$ is therefore:

$$\mathcal{L}^{-1}[F(s)] = c_0\delta(t) + c_1\delta^{(1)}(t) + \cdots + c_{m-n}\delta^{(m-n)}(t) + \mathcal{L}^{-1}\left[\frac{d(s)}{a(s)}\right]$$

where we use partial fractions to represent $d(s)/a(s)$.

In this class, we haven't talked about what derivatives of delta functions mean. You will not be responsible for them, and if we ever give a nonproper rational function, we will have $m = n$ so at most there is a δ function in the inverse Laplace transform. We have presented this for the sake of "completeness."

Nonproper rational function example

Let

$$F(s) = \frac{5s+3}{s+1}$$

This is not strictly proper. Therefore, we factor it into $c(s) + d(s)/a(s)$, via

$$\begin{aligned} F(s) &= \frac{5(s+1) - 5 + 3}{s+1} \\ &= 5 - \frac{2}{s+1} \end{aligned}$$

Therefore,

$$\mathcal{L}^{-1}[F(s)] = 5\delta(t) - 2e^{-t}$$

We'll likely do one more involved nonproper rational function example in class.