ECE 113 HW #7

1. $h_1(n) = \left(\frac{1}{2}\right)^n u(n)$ $\quad h_2(n) = \left(\frac{1}{3}\right)^n u(n-1) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1} u(n-1)$

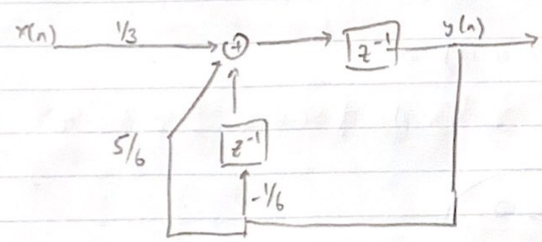
a. $H_1(z) = \frac{z}{z-1/2}$ ROC $|z| > 1/2$

$H_2(z) = \frac{1}{3} z^{-1} \cdot \frac{z}{z-1/3} = \frac{1}{3z-1}$ $\quad |z| > 1/3$ $\quad$ cascade $\rightarrow$ $H_T(z) = H_1(z) H_2(z)$

$$= \frac{z}{z-1/2} \cdot \frac{1}{3z-1} = \frac{z}{(z-1/2)(z-1/3)} \quad |z| > 1/2$$

$$\frac{z}{3z^2 - \frac{5}{2}z + 1/2} \rightarrow \frac{z^{-1}}{3 - \frac{5}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{Y(z)}{X(z)} \rightarrow Y(z)\left(3 - \frac{5}{2}z^{-1} + \frac{1}{2}z^{-2}\right) = X(z)\, z^{-1}$$

$$\boxed{3y(n) - \frac{5}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n-1)}$$

b. 

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + \frac{1}{3}x(n-1)$$

c. $\frac{z}{(z-1/2)(3z-1)} = \frac{A}{z-1/2} + \frac{B}{3z-1}$ $\quad |z| > 1/2$ $\quad$ $\frac{1}{2} \rightarrow A\,\frac{1}{2} \rightarrow A = 1$

$\frac{1}{3} = B(-1/6)$ $\quad B = -2$

$$\frac{1}{z-1/2} - \frac{2}{3z-1} \quad |z| > 1/2$$

$$h(n) = \left(\frac{1}{2}\right)^{n-1} u(n-1) - \frac{2}{3}\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

$$= \left[\left(\frac{1}{2}\right)^{n-1} - \frac{2}{3}\left(\frac{1}{3}\right)^{n-1}\right] u(n-1)$$

d. $|z| > 1/2$ includes $z = 1$ $\rightarrow$ stable system $\quad$ modes @ $z = 1/3, 1/2$.

e. Mode cancellation @ $X(z) = 3z - 1$ causing $Y(z) = \frac{z}{z-1/2}$

$$\boxed{x(n) = 3\delta(n+1) - \delta(n)}$$

c. $3y(n) - \frac{5}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n-1)$

unilateral z-transform $\rightarrow 3Y^+(z) - \frac{5}{2}\left[z^{-1}Y^+(z) + y(-1)\right] + \frac{1}{2}\left[z^{-2}Y^+(z) + y(-1)z^{-1} + y(-2)\right]$

$= z^{-1}X^+(z) + x(-1)$

$x(n) = \delta(n)$ $x(-1) = 0$ $X^+(z) = 1$

$Y^+(z)\left[1 - \frac{5}{2}z^{-1} + \frac{1}{2}z^{-2}\right] - \frac{5}{2}y(-1) + \frac{1}{2}z^{-1}y(-1) + \frac{1}{2}y(-2) = z^{-1}$

$$Y^+(z) = \frac{z\left(1 - \frac{1}{2}y(-1)\right) + \frac{1}{2}z^2\left(5y(-1) - y(-2)\right)}{(3z-1)(z-\frac{1}{2})}$$

$\frac{1}{2}\left(1 - \frac{1}{2}y(-1)\right) + \frac{1}{8}\left(5y(-1) - y(-2)\right) = 0$

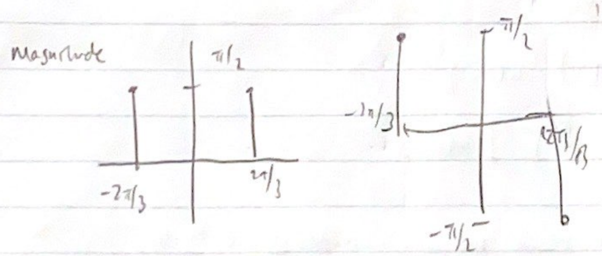
$y(-2) = 0 \rightarrow \frac{1}{2} - \frac{1}{4}y(-1) + \frac{5}{8}y(-1) = 0$ $\quad \frac{3}{8}y(-1) = -\frac{1}{2}$

$y(-1) = -\frac{4}{3}$

IC: $y(-1) = -\frac{4}{3}$ $\quad y(-2) = 0$

2 a) $x(n) = \cos(\frac{\pi}{3}n)\sin(\frac{\pi}{3}n) = \frac{1}{2}\sin(\frac{2\pi}{3}n)$

$X(e^{j\omega}) = \frac{1}{2}\left[-j\pi\left(\delta(\omega - \frac{2\pi}{3}) - \delta(\omega - \frac{2\pi}{3}\right)\right] = \frac{-j\pi}{2}\left(\delta(\omega - \frac{2\pi}{3}) - \delta(\omega + \frac{2\pi}{3})\right)$



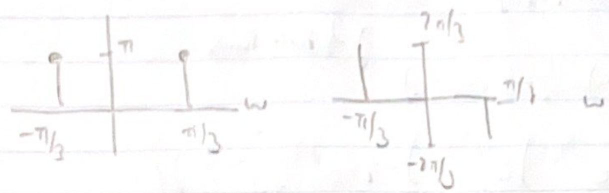
b. $x(n) = \cos(\frac{\pi}{3}n - \frac{2\pi}{3})$ $\quad X(e^{j\omega}) = e^{-2j\omega}\pi\left[\delta(\frac{\pi}{3} + \omega) + \delta(\omega - \frac{\pi}{3})\right]$

DTFT $\cos(\frac{\pi}{3}(n-2))$

$-\pi/3$ phase $= 2\pi/3$

$\pi/3$ phase $= -2\pi/3$

mag



3. $x(n) = \dfrac{\sin\left(\frac{\pi}{6}n - \frac{\pi}{3}\right)}{2n-4} = \dfrac{\sin\left(\frac{\pi}{6}n - \frac{\pi}{3}\right)}{\frac{12}{\pi}\left[\frac{\pi}{6}n - \pi/3\right]} = \dfrac{\pi}{12}\,\mathrm{sinc}\left(\frac{\pi}{6}(n-2)\right)$

$$\frac{\pi}{12}\,\mathrm{sinc}(\pi/6) \longleftrightarrow \begin{cases} \pi/2 & |\omega| < \pi/6 \\ 0 & \frac{\pi}{6} \le |\omega| < \pi \end{cases}$$

time shift (−2) $\rightarrow$

$$\boxed{\begin{cases} \frac{\pi}{2} e^{-j2\omega} & |\omega| < \pi/6 \\ 0 & \frac{\pi}{6} \le |\omega| < \pi \end{cases}}$$

4. a) $X(e^{j\omega}) = \cos^2\left(\frac{\pi}{2}\omega\right) = \dfrac{1 + \cos(\pi\omega)}{2} = \frac{1}{2} + \frac{1}{2}\cos(\pi\omega) = \frac{1}{2} + \frac{1}{4}\left[e^{j\pi\omega} + e^{-j\pi\omega}\right]$

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} \frac{1}{2} + \frac{1}{4}\left[e^{j\pi\omega} + e^{-j\pi\omega}\right] e^{j\omega n}\, d\omega$$

$$= \frac{1}{8\pi}\left[\frac{2}{jn}e^{j\omega n} + \frac{1}{j(n+\pi)}e^{j\omega(n+\pi)} + \frac{1}{j(n-\pi)}e^{j\omega(n-\pi)}\right]_0^{2\pi}$$

$$= \frac{1}{8\pi}\left[\frac{2}{jn} + \frac{1}{j(n+\pi)}e^{j2\pi^2} + \frac{1}{j(n-\pi)}e^{-j2\pi^2} - \frac{2}{jn} - \frac{1}{j(n+\pi)} - \frac{1}{j(n-\pi)}\right]$$

$$= \boxed{\frac{1}{j8\pi(n+\pi)}\left(e^{j2\pi^2} - 1\right) + \frac{1}{j(8\pi(n-\pi))}\left(e^{-j2\pi^2} - 1\right)}$$

d. $X(e^{j\omega}) = \cos^2(\frac{\pi}{6}\omega) + \sin^2(\frac{\pi}{4}\omega)$

$$= \frac{1}{2} + \frac{1}{2}\cos(\frac{\pi}{3}\omega) + \frac{1}{2} - \frac{1}{2}\cos(\frac{\pi}{2}\omega) \quad \Big\} \ \tfrac{1}{2} \text{ angle formula}$$

$$= 1 + \frac{1}{2}\left[\frac{1}{2}\left(e^{j\pi/3\,\omega} + e^{-j\pi/3\,\omega}\right) - \frac{1}{2}\left(e^{j\frac{\pi}{2}\omega} + e^{-j\frac{\pi}{2}\omega}\right)\right]$$

$$x(n) = \frac{1}{2\pi}\int_{2\pi}\left(1 + \frac{1}{4}\left(e^{j\pi/3\,\omega} + e^{-j\pi/3\,\omega} - e^{j\pi/2\,\omega} - e^{-j\pi/2\,\omega}\right)\right)e^{j\omega n}\,d\omega$$

Some terms $\int$ to 0

$$= \frac{1}{8\pi}\int_{2\pi} e^{j\omega(n+\pi/3)} + e^{j\omega(n-\pi/3)} - e^{j\omega(n+\pi/2)} - e^{j\omega(n-\pi/2)}\,d\omega$$

$$= \frac{1}{8\pi}\left[\frac{1}{j(n+\frac{\pi}{3})}e^{j\frac{2\pi^2}{3}} + \frac{1}{j(n-\pi/3)}e^{j\frac{-2\pi^2}{3}} - \frac{1}{j(n+\pi/2)}e^{j\pi^2} - \frac{1}{j(n+\pi/2)}e^{-j\pi^2}\right.$$

$$\left. - \frac{1}{j(n+\frac{\pi}{3})} - \frac{1}{j(n-\pi/3)} + \frac{1}{j(n+\pi/2)} + \frac{1}{j(n-\pi/2)}\right]$$

$$= \frac{1}{j8\pi(n+\pi/3)}\left(e^{j\frac{2\pi^2}{3}} - 1\right] + \frac{1}{j8\pi(n-\pi/3)}\left(e^{-j2\pi^2/3} - 1\right]$$

$$- \frac{1}{j8\pi(n+\pi/2)}\left(e^{j\pi^2} - 1\right) - \frac{1}{j8\pi(n-\pi/2)}\left(e^{-j\pi^2} - 1\right)$$