Discussion 6

Lecture 9 Review

Transform Methods

– Characteristic Function ($\approx$ Fourier Transform)

$$\Phi_X(w) = E[e^{jwX}]$$

$$= \begin{cases} \sum_{x_i} e^{jwx_i} P(X=x_i) & \text{Discrete} \\ \int e^{jwx} f(x)\,dx & \text{Continuous} \end{cases}$$

– Moment Theorem

$$\boxed{E[X^n] = \frac{1}{j^n} \cdot \frac{d^n \Phi_X(w)}{dw^n}\Big|_{z=0}}$$

– Sum of Independent R.V.

$Y = \sum_{i=1}^{n} X_i$ $\quad X_i$ are independent

$$\Phi_Y(w) = \prod_{i=1}^{n} \Phi_{X_i}(w)$$

$$= (\Phi_{X_1}(w))^n \qquad X_i \text{ are identical}$$

$$E[Y] = \frac{1}{j}\frac{d}{dw}(\Phi_X(w))^n\Big|_{w=0} = \frac{n}{j}(\Phi_X(w))^{n-1}\frac{d\Phi_X(w)}{dw}\Big|_{w=0}$$

$X_i \sim \text{Bern}(p)$

$$\Phi_X(w) = E[e^{jwX}] = e^{jw}p + (1-p)e^{jw0}$$

$$= e^{jw}p + (1-p)$$

– Moment-Generating Function (Laplace Transform) $a, b \in R$

$$\boxed{M_X(s) = E[e^{sX}]} \leftarrow s = a + bi$$

$$\Phi_X(w) = E[e^{jwX}] \leftarrow$$

– Probability Generating Function (Z-transform)

$$G_X(z) = E[z^X] = \sum_{n=0}^{\infty} z^n P(X=n)$$

X is non-negative and integer-valued

$$\frac{d}{dz}G_X(z)\Big|_{z=1} = E[X] \qquad Var(X) = E[X^2] - (E[X])^2$$

$$\frac{d^2}{dz^2}G_X(z)\Big|_{z=1} = E[X^2] - E[X]$$

$$\frac{d^n G_X(z)}{dz^n}\Big|_{z=0} = P(X=n)\cdot n!$$

Lecture 10

Inequalities

– Markov Inequality

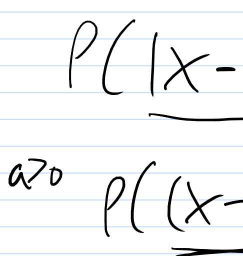
X is a non-negative RV $\Rightarrow P(X \geq a) \leq \boxed{\frac{E[X]}{a}}$

– Chebyshev Inequality

$$P(|X - E[X]| > a) \leq \frac{Var(X)}{a^2}$$

$a > 0$

$$P((X - E(X))^2 \geq a^2) \leq \frac{E(X - E(X))^2}{a^2}$$



$a \geq b$
$\Updownarrow$
$a^2 \geq b^2$

– Chernoff Bound

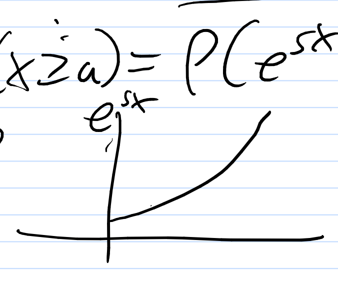
$$P(X \geq a) \leq e^{-sa} \cdot E[e^{sX}] \quad \forall s > 0, \ s \in R$$

$$\boxed{P(X \geq a) \leq \min_{s>0} e^{-sa} \cdot E[e^{sX}]}$$

HW: $Q(x) \leq e^{-\frac{x^2}{2}}$

$$P(X \geq a) = P(e^{sX} \geq e^{sa}) \leq \frac{E[e^{sX}]}{e^{sa}}$$

$s > 0$



$$P(X \geq a) = P(g(X) > g(a)) \leq \frac{E[g(X)]}{g(a)}$$

$g(x)$ is non-negative and monotonically increasing

Problem 1

a) $X \sim \text{Uniform}\{1, 2, 3 \cdots L\}$ $\quad c \in \{1 \cdots L\}$

$$P(X > c) = 1 - P(X \leq c)$$

$$= 1 - \frac{c}{L}$$

$$P(X > c) = P(X \geq c) - P(X = c)$$

$$\leq \frac{E[X]}{c} - P(X = c)$$

$$= \frac{L+1}{2} \cdot \frac{1}{c} - \frac{1}{L}$$

$$\boxed{1 - \frac{c}{L} \leq \frac{L+1}{2c} - \frac{1}{L}}$$

b) $X \sim \text{Geometric}(p)$ $\quad c \in \{1, 2, 3, \cdots \infty)$

$$P(X > c) = (1-p)^c$$

$$P(X > c) = P(X \geq c) - P(X = c)$$

$$\leq \frac{E[X]}{c} - P(X = c)$$

$$= \frac{1}{pc} - p(1-p)^{c-1}$$

$$\boxed{(1-p)^c \leq \frac{1}{pc} - p(1-p)^{c-1}}$$

Problem 2 $\qquad E[X] = 1(1-p) + 3 \cdot p = 2p + 1$

$$P(X = k) = \begin{cases} p & k = 3 \\ 1-p & k = 1 \end{cases}$$

a) $G_X(z) = E[z^X]$

$$= \sum_{x_i} z^{x_i} P(X = x_i)$$

$$= z^1(1-p) + z^3 p$$

$$\frac{d}{dz}G_X(z) = 3z^2 p + (1-p) \qquad z = 1$$

$$= 3p + 1 - p$$

$$= 2p + 1$$

$$\frac{d^2}{dz^2}G_X(z) = 6zp$$

$$= 6p \qquad z = 1$$

$$= E[X^2] - E[X]$$

$$Var(X) = G''(1) + G'(1) - (G'(1))^2$$

$$= E[X^2] - E[X] + E[X] - (E[X])^2$$

$$= 1 - (2p-1)^2$$

b) $f(z) = (G_X(z))^2 = G_X(z) \cdot G_X(z)$

Is $f(z)$ a pgf?

$f(z)$ is pgf of $X_1 + X_2$ $\quad X_1, X_2$ iid

$$E[z^{X_1+X_2}] = E[z^{X_1} \cdot z^{X_2}]$$

$$= E[z^{X_1}] \cdot E[z^{X_2}] = G_{X_1}(z) \cdot G_{X_2}(z)$$

c) $g(z) = \dfrac{G_{X_1}(z) + G_{X_2}(z)}{2}$

$g(z)$ is the pgf of Y r.v.

$$P(Y = k) = \frac{P(X_1 = k) + P(X_2 = k)}{2}$$

Problem 3

$X \sim N(0,1)$ $\quad M_X(s) = e^{\frac{s^2}{2}}$

$X \sim N(m, \sigma^2)$ $\quad M_X(s) = e^{ms + \frac{s^2\sigma^2}{2}}$

$$E[e^{sX}] = \int_{-\infty}^{\infty} f_X(x) e^{sx}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \underbrace{e^{-\frac{(x-m)^2}{2\sigma^2}} e^{sx}}_{e^{sx - \frac{(x-m)^2}{2\sigma^2}}} dx$$

$k = m + s\sigma^2$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-k)^2 + 2ms\sigma^2 + s^2\sigma^4}{2\sigma^2}} dx$$

$$= e^{\frac{2ms\sigma^2 + s^2\sigma^4}{2\sigma^2}} \underbrace{\cancel{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-k)^2}{2\sigma^2}} dx}}_{N(k, \sigma^2)} = 1$$

$\int f_Y(y)\,dx$

$$= \boxed{e^{ms + \frac{s^2\sigma^2}{2}}}$$

$$\frac{d}{ds}M_X(s)\Big|_{s=0} = E[X]$$