EE 131A
Probability
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Discussion Set 4 solutions
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Chapter 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *PMF of 2 RVs.* The input X to a communication channel is "-1" or "1", with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$.The output of the channel Y is given by

$$Y = \begin{cases} X & \text{wp } 1-p-p_e \\ -X & \text{wp } p \\ 0 & \text{wp } p_e \end{cases}$$

   (a) Find S_{XY}, the range of the pair (X, Y).
   Solution:
   Input $X \in \{-1, 1\}$ and output $Y \in \{-1, 0, 1\}$, then S_{XY} is all possible combinations of (X, Y) pairs:

$$S_{XY} = \{(-1,-1), (-1,0), (-1,1), (1,-1), (1,0), (1,1)\}$$

   (b) Find the probabilities for all values of (X, Y).
   Solution:
   The probabilities for all values of (X, Y) are given by:
   $P[X = -1, Y = -1] = (1 - p - p_e)/4$
   $P[X = -1, Y = 0] = p_e/4$
   $P[X = -1, Y = 1] = p/4$
   $P[X = 1, Y = -1] = 3p/4$
   $P[X = 1, Y = 0] = 3p_e/4$
   $P[X = 1, Y = 1] = 3(1 - p - p_e)/4$

   (c) Find $P[X \neq Y]$, $P[Y = 0]$.
   Solution:
   $P[X \neq Y] = p_e/4 + p/4 + 3p/4 + 3p_e/4 = p + p_e$
   $P[Y = 0] = p_e/4 + 3p_e/4 = p_e$

2. Let X be a discrete random variable with the following PMF

$$P_X(x) = \begin{cases} 0.3 & \text{for } x = 3 \\ 0.2 & \text{for } x = 5 \\ 0.3 & \text{for } x = 8 \\ 0.2 & \text{for } x = 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the CDF of X, $F_X(x)$.
Solution:
The CDF is defined as $F_X(x) = P(X \leq x)$. We have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 3 \\ P_X(3) = 0.3 & \text{for } 3 \leq x < 5 \\ P_X(3) + P_X(5) = 0.5 & \text{for } 5 \leq x < 8 \\ P_X(3) + P_X(5) + P_X(8) = 0.8 & \text{for } 8 \leq x < 10 \\ 1 & \text{for } x \geq 10 \end{cases}$$

(b) Find $P(2 < X \leq 5)$ and $P(X > 4)$ using the CDF.
Solution:

$$P(2 < X \leq 5) = F_X(5) - F_X(2) = 0.5$$
$$P(X > 4) = 1 - F_X(4) = 1 - 0.3 = 0.7$$

3. *Exponential RV.* Problem 4.61, page 221 of ALG
Let X be an exponential random variable with parameter λ. The cdf and pdf of X are given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x \geq 0 \end{cases}$$

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \lambda e^{-\lambda x} & \text{for } x \geq 0. \end{cases}$$

(a) Show that $F_X(x)$ is a valid cdf and $f_X(x)$ is a valid pdf.
Solution:
$F_X(x)$ is a valid cdf as it satisfies the following properties of cdfs

i. $0 \leq F_X(x) \leq 1$
ii. $\lim_{x\to\infty} F_X(x) = 1$
iii. $\lim_{x\to-\infty} F_X(x) = 0$
iv. $F_X(x)$ is non decreasing
v. $F_X(x)$ is continuous from right (since it is continuous everywhere)

$f_X(x)$ is a valid pdf as it satisfies the following properties

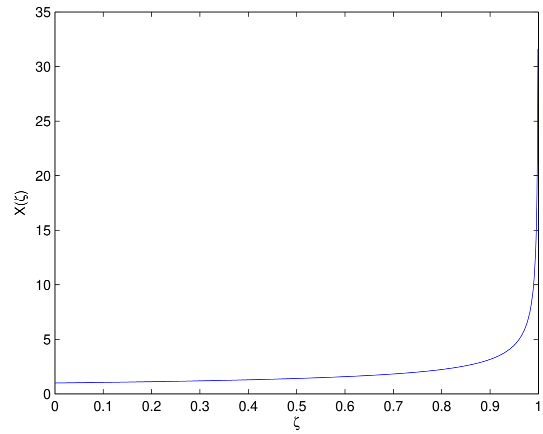
i. $f_X(x) \geq 0$
ii. $\int_{-\infty}^{\infty} f_X(x)dx = 1$

(b) For $d > 0$ and k a nonnegative integer, find $P[kd < X < (k+1)d]$.
Solution:
We have:

$$P[kd < X < (k+1)d] = F_X((k+1)d) - F_X(kd) = e^{-\lambda kd}(1 - e^{-\lambda d}).$$

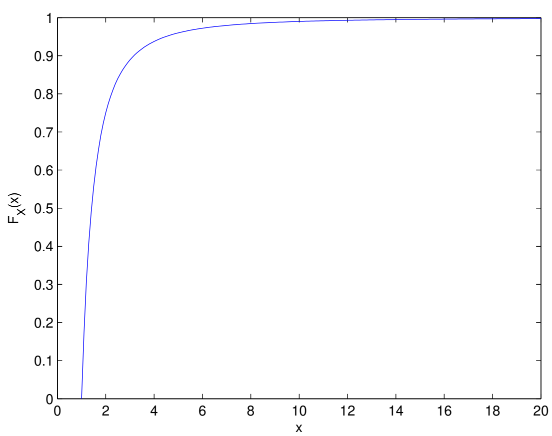
4. *Cdf and pdf calculations.* Let ζ be a point selected at random from the unit interval. Consider the random variable $X = (1-\zeta)^{-\frac{1}{2}}$.

(a) Sketch X as a function of ζ.
Solution:
Sketch X as a function of ζ as following.



(b) Find and plot the cdf of X.
Solution:
Note $S_x = \{x : 1 \le x < \infty\}$. Then the cdf of X is obtained and plotted as follows:

$$\begin{aligned} P[X(\zeta) \le x] &= P[\frac{1}{\sqrt{1-\zeta}} \le x] = P[\frac{1}{1-\zeta} \le x^2] \\ &= P[\zeta \le 1 - \frac{1}{x^2}] = 1 - \frac{1}{x^2} \end{aligned}$$

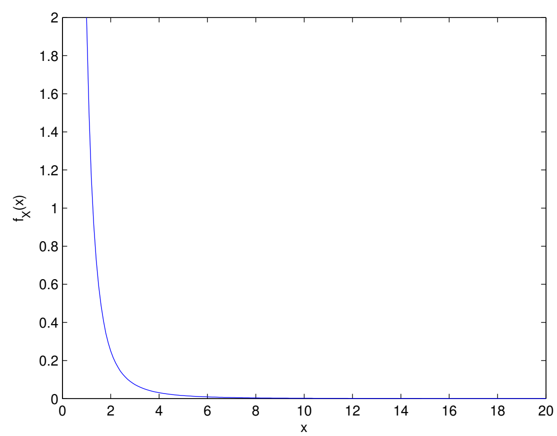


(c) Find the probability of the events $\{X > 1\}, \{5 < X < 7\}, \{X \le 20\}$.
Solution:

$P\{X > 1\} = 1 - F_X(1) = 1 - (1 - 1/1^2) = 1$
$P\{5 < X < 7\} = F_X(7) - F_X(5) = (1 - 1/7^2) - (1 - 1/5^2) = 1/25 - 1/49 = 0.01959$
$P\{X \leq 20\} = 1 - 1/20^2 = 0.9975$

(d) Find and plot the pdf of X.
Solution:
We know $F_X(x) = 1 - \frac{1}{x^2}$ for $x \geq 1$, then the pdf is obtained and plotted as follows:

$$f_x(x) = 2/x^3 \text{ for } \; x \geq 1$$



(e) Use the pdf to find the probabilities of the events: $\{X > a\}$ and $\{X > 2a\}$.
Solution:
For $a \geq 1$, we can find the probability of the event:

$$P\{X > a\} = \int_a^\infty \frac{2}{x^3} \mathrm{d}x = \frac{1}{a^2}$$

For $2a \geq 1$, we can find the probability of the event:

$$P\{X > 2a\} = \int_{2a}^\infty \frac{2}{x^3} \mathrm{d}x = \frac{1}{4a^2}$$

5. *Bonus:* Alice and Bob play a series of games with Alice winning each game with probability p. The overall winner is the first player to have won two or more games than the other. Find the probability that Alice is the overall winner.
Solution: Let A be the event that Alice is the overall winner. Also let W_r and L_r denote the events that Alice wins and loses the r^{th} match respectively. Conditioning on the result of the first two games, we get

$$\begin{aligned} P(A) = &P(A|W_1W_2)P(W_1W_2) + P(A|W_1L_2)P(W_1L_2) \\ &+ P(A|L_1W_2)P(L_1W_2) + P(A|L_1L_2)P(L_1L_2) \end{aligned}$$

Now $P(A|W_1W_2) = 1$, $P(A|L_1L_2) = 0$, $P(A|W_1L_2) = P(A|L_1W_2) = P(A)$. Thus,

$$P(A) = p^2 + 2p(1-p)P(A) + 0$$
$$\implies P(A) = \frac{p^2}{1 - 2p(1-p)}$$