EE 131A
Probability
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Discussion Set 1 Solution
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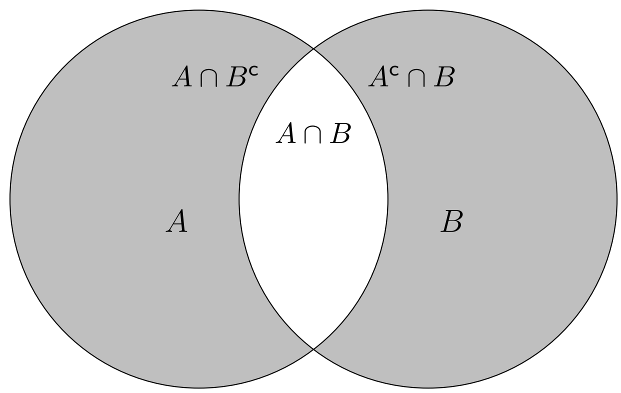
Chapter 2.1-2.3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *Venn diagram.* Let A and B be events. Find an expression for the event "exactly one of the events A and B occurs." Draw a Venn diagram for this event.
   Solution: $(A \cap B^c) \cup (A^c \cap B)$



2. Find $P(A \cup (B^c \cup C^c)^c)$ in each of the following cases:

   (a) A, B, C are mutually exclusive events and $P(A) = 4/7$.
   Solution: according to DeMorgan's rules: $(B^c \cup C^c)^c = B \cap C$, then $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$. Since B, C are mutually exclusive events, $B \cap C = \emptyset$. Therefore, $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C)) = P(A \cup \emptyset) = 4/7$.

   (b) $P(A) = 1/2, P(B \cap C) = 1/3, P(A \cap C) = 0$.
   Solution: From previous part: $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap B \cap C) = 1/2 + 1/3 - 0 = 5/6$

   (c) $P(A^c \cap (B^c \cup C^c)) = 0.65$.
   Solution: according to DeMorgan's rules: $(A \cup (B^c \cup C^c)^c)^c = A^c \cap (B^c \cup C^c)$, then $P(A \cup (B^c \cup C^c)^c) = 1 - P(A^c \cap (B^c \cup C^c)) = 1 - 0.65 = 0.35$

3. Suppose A and B are two events. Prove the following:

   (a) $P(A)P(B) = P(A \cap B) \Longleftrightarrow P(A^c)P(B) = P(A^c \cap B)$
   Solution:
   left to right:

$$\begin{aligned} P(A^c)P(B) &= (1 - P(A))P(B) = P(B) - P(A)P(B) \\ &= P(B) - P(A \cap B) = P(A^c \cap B). \end{aligned}$$

right to left:

$$\begin{aligned}P(A)P(B) &= (1 - P(A^c))P(B) = P(B) - P(A^c)P(B)\\ &= P(B) - P(B \cap A^c) = P(B \cap (A^c)^c) = P(A \cap B).\end{aligned}$$

(b) $P(A \cap B) = 0 \Longrightarrow P(A) \leq P(B^c)$
Solution:

$$\begin{aligned}P(A) &= P(A \cup B) - P(B) + P(A \cap B) = P(A \cup B) - P(B)\\ &\leq 1 - P(B)\\ &\leq P(B^c).\end{aligned}$$

(c) $P(A) = P(B) = P(A \cap B) \Longrightarrow P((A \cap B^c) \cup (B \cap A^c)) = 0$
Solution:
$(A \cap B^c)$ and $(B \cap A^c)$ are two disjoint event since their intersection is empty set. As a result:

$$\begin{aligned}P((A \cap B^c) \cup (B \cap A^c)) &= P(A \cap B^c) + P(B \cap A^c)\\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) = 0.\end{aligned}$$

4. An urn contains 40 red balls and 60 green balls. What is the probability of getting exactly k red balls in a sample of size 30 if the sampling is done without replacement? Assume $0 \leq k \leq 30$. How does the answer change if we have a sample size of 50?
Solution: Note that the ordering of the balls in not important here. The total number of ways of selecting a sample size of 30 when sampling is done without replacement is $\binom{100}{30}$. Now, the total number of ways of getting exactly k red balls in a sample size of 30 is the number of ways of selecting k red balls and $30 - k$ green balls from 40 red and 60 green balls respectively. The number of ways is $\binom{40}{k}\binom{60}{30-k}$. Thus the probability of getting exactly k red balls in a sample of size 30 is

$$\text{P(exactly } k \text{ red balls)} = \frac{\binom{40}{k}\binom{60}{30-k}}{\binom{100}{30}}.$$

The probability of getting exactly k red balls when we have a sample size of 50 can be calculated using the same procedure as above:

$$\text{P(exactly } k \text{ red balls)} = \frac{\binom{40}{k}\binom{60}{50-k}}{\binom{100}{50}}.$$

5. *Inclusion-exclusion principle.* Let $A_1, A_2, \ldots, A_n$ be a set of n events. Then prove that

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2}) + \sum_{1 \le i_1 < i_2 < i_3 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) \\ - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

Proof: We prove the above using induction. For $n = 2$, We have $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$ which we know is true from the lecture. Suppose that the formula is true for n, we show it for $n + 1$.

$$\begin{aligned} &P(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) \\ &\quad = P((A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}) \\ &\quad = P((A_1 \cup A_2 \cup \cdots \cup A_n)) + P(A_{n+1}) - P((A_1 \cup A_2 \cup \cdots \cup A_n) \cap A_{n+1}) \\ &\quad = P((A_1 \cup A_2 \cup \cdots \cup A_n)) + P(A_{n+1}) - P((A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \cdots \cup (A_n \cap A_{n+1})) \end{aligned}$$

The first and the last terms are union of n events for which the formula is true using induction. Thus we get,

$$\begin{aligned} &P(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) \\ &\quad = \Big[\sum_{i=1}^{n} P(A_i) - \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2}) + \sum_{1 \le i_1 < i_2 < i_3 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) \\ &\quad - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n) \Big] + P(A_{n+1}) \\ &\quad - \Big[\sum_{i=1}^{n} P(A_i \cap A_{n+1}) - \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{n+1}) \\ &\quad + \cdots - (-1)^{n} \sum_{1 \le i_1 < i_2 < \cdots < i_{n-1} \le n} P(A_{i_1} \cap A_{i_2} \cap \cdots A_{i_{n-1}} \cap A_{n+1}) \\ &\quad + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) \Big]. \end{aligned}$$

We can view the above summation as follows:

$$\begin{aligned} &P(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) \\ &\quad = \Big[{\color{red}\sum_{i=1}^{n} P(A_i)} {\color{blue}- \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2})} {\color{green}+ \sum_{1 \le i_1 < i_2 < i_3 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3})} \\ &\quad {\color{purple}- \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)} \Big] + {\color{red}P(A_{n+1})} \\ &\quad + \Big[{\color{blue}- \sum_{i=1}^{n} P(A_i \cap A_{n+1})} {\color{green}+ \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{n+1})} \\ &\quad {\color{purple}- \cdots + (-1)^{n+1} \sum_{1 \le i_1 < i_2 < \cdots < i_{n-1} \le n} P(A_{i_1} \cap A_{i_2} \cap \cdots A_{i_{n-1}} \cap A_{n+1})} \Big] \\ &\quad {\color{orange}(-1)^{n+2} P(A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1})} \Big]. \end{aligned}$$

The terms in red account for all the probabilities of single events from 1 to $n+1$. The first summation in blue includes all the two intersection probabilities from 1 to n. The second summation in blue includes all the two-intersection probabilities where the higher index equals $n+1$. These two sums in blue together account for all possible two-intersection probabilities from 1 to $n+1$. Similarly the terms in green account for all possible three-intersection probabilities from 1 to $n+1$. This process continues till all the terms in violet. The final term shown in orange is the last term of intersection of all the $n+1$ events. Clubbing the terms with the same color gives us the following:

$$
\begin{aligned}
&P(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) \\
&\qquad = \sum_{i=1}^{n+1} P(A_i) - \sum_{1 \leq i_1 < i_2 \leq n+1} P(A_{i_1} \cap A_{i_2}) + \sum_{1 \leq i_1 < i_2 < i_3 \leq n+1} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) \\
&\qquad - \cdots + (-1)^{n+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n+1} P(A_{i_1} \cap A_{i_2} \cap \cdots A_{i_n}) \\
&\qquad + (-1)^{n+2} P(A_1 \cap A_2 \cap \cdots \cap A_{n+1}).
\end{aligned}
$$

which is the inclusion-exclusion principle formula for $n+1$ events. Thus we have proved the inclusion-exclusion principle using induction.