EE 131A
Probability and Statistics
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Homework 1 Solutions
Monday, January 4, 2021
Due: Monday, January 11, 2021
before class begins
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Please upload your homework to Gradescope by January 11, 3:59 pm.
Please submit a single PDF directly on Gradescope
You may type your homework or scan your handwritten version. Make sure all the work is discernible.

Reading: Chapter 2.1-2.3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

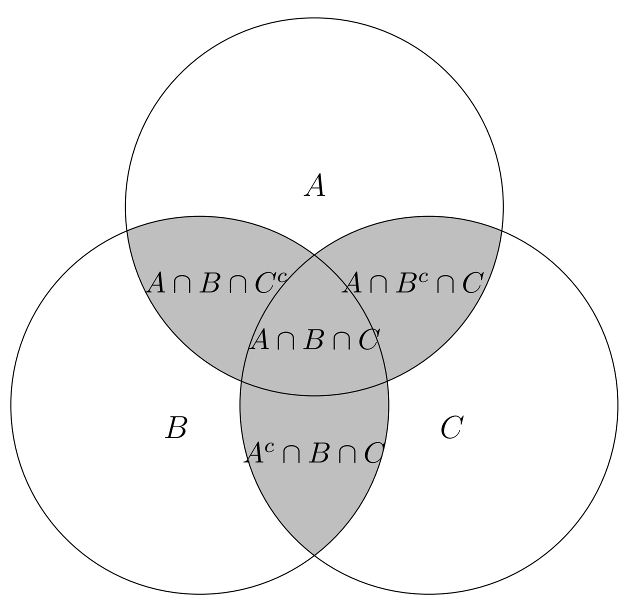
1. Two dice are tossed and the magnitude of the difference in the number of dots facing up in the two dice is noted.

   (a) What is the sample space of the event of tossing of the two dice?
   Solution: $\mathcal{S} = \{(a_1, a_2) : a_1, a_2 \in \{1, 2, \ldots, 6\}\}$

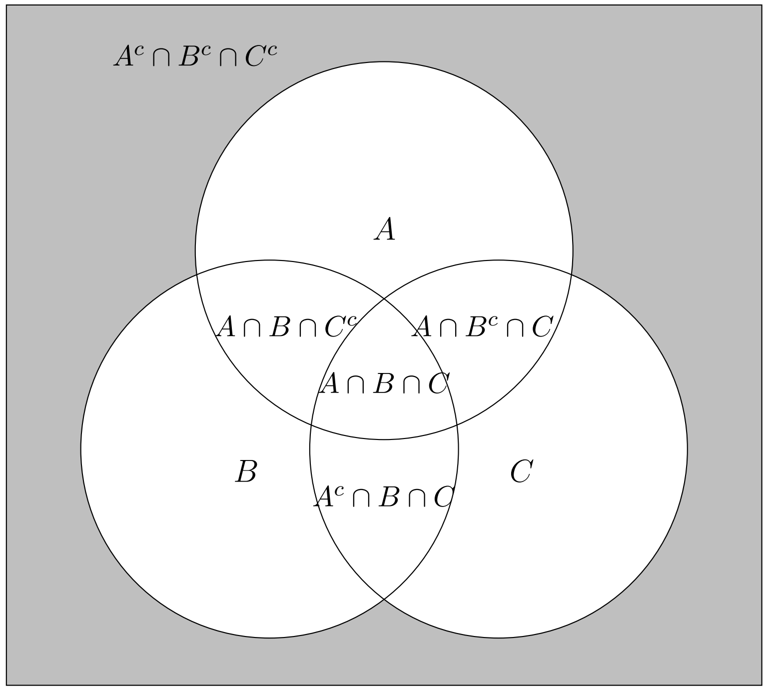
   (b) Find the subset A of the sample space corresponding to the event "magnitude of difference is 3." Also, find the probability of the event corresponding to set A.
   Solution: $A = \{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\}$. $P(A) = \frac{6}{36} = \frac{1}{6}$.

2. Let A, B, and C be three events. Find an expression and draw a Venn diagram for the following events:

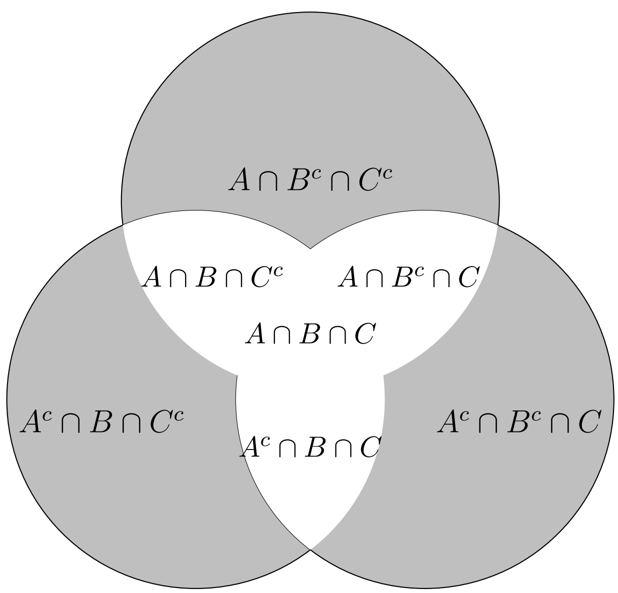
   (a) Two or more of the events occur.
   Solution: The event we want is $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C)$. The Venn diagram is shown below:



(b) None of the events occur.
Solution: We want the event $A^c \cap B^c \cap C^c$. The Venn diagram is:



(c) Exactly one of the three events occurs.
Solution: We want the event$(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$. The Venn diagram is:



3. Suppose A and B are two events. Use the axioms of probability to prove the following:

(a) $P(A \cap B) \geq P(A) + P(B) - 1$.
Proof:We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

We also know that $P(A \cup B) \leq 1$
Therefore, $P(A) + P(B) - P(A \cap B) \leq 1 \implies P(A \cap B) \geq P(A) + P(B) - 1$.

(b) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.
Proof: There are multiple ways to prove the above inequality. Here are two possible methods.
We know that $P(A) \geq 0$. Also using axioms of probability, we know that $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$ for any events X and Y. Thus using $X = A$ and $Y = B \cup C$,

$$\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\
&\leq P(A) + P(B \cup C) \\
&= P(A) + P(B) + P(C) - P(B \cap C) \\
&\leq P(A) + P(B) + P(C)
\end{aligned}$$

Another method is using the following: when $A_1, A_2, \ldots, A_n$ are disjoint sets then $P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$. To use this axiom, let us create some auxiliary sets D, E, and F which are defined as $D = A$, $E = B \setminus A$, and $F = C \setminus (A \cup B)$. Note that D, E, and F are disjoint and that $P(A \cup B \cup C) = P(D \cup E \cup F)$. Additionally, we use corollary 7 from ALG to get $P(D) \leq P(A), P(E) \leq P(B)$, and $P(F) \leq P(C)$. Putting them all together, we get

$$\begin{aligned}
P(A \cup B \cup C) &= P(D \cup E \cup F) \\
&= P(D) + P(E) + P(F) \\
&\leq P(A) + P(B) + P(C)
\end{aligned}$$

4. Five balls are placed at random in five buckets. What is the probability that each bucket has a ball?
Solution: Let N_1 be the number of ways to place the 5 balls into 5 buckets and N_2 be the total number of ways to place the balls such that each bucket has a ball. Clearly, $N_1 = 5^5$ as each ball has 5 choices. The total number of choices when each bucket has a ball is the number of ways of permuting the 5 balls among the buckets. Thus, $N_2 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$. Thus $P(\{\text{each bucket has a ball}\}) = \frac{5!}{5^5} \approx 0.0384$.

5. There is a deck of 4 white cards, 3 blue cards, and 5 red cards.

(a) If we randomly take 5 cards from this deck, what is the probability that exactly 2 of them are white?
Solution:

$$P(\text{Exactly 2 white cards}) = \frac{\binom{4}{2}\binom{8}{3}}{\binom{12}{5}}$$

(b) If we randomly take 5 cards from this deck, what is the probability that at least 2 of them are white?
Solution:

$$P(\text{At least 2 white cards}) = \frac{\binom{4}{2}\binom{8}{3} + \binom{4}{3}\binom{8}{2} + \binom{4}{4}\binom{8}{1}}{\binom{12}{5}}$$

6. A fair coin is tossed four times. Find the probabilities of the following events:

   (a) seeing at least two successive tails.
   Solution: There are totally 16 possible outcomes for this experiment. Desired outcomes are $A = \{HHTT, HTTH, HTTT, THTT, TTHH, TTHT, TTTH, TTTT\}$. Thus, $P(A) = \frac{8}{16} = \frac{1}{2}$.

   (b) seeing exactly one head in the first three tosses.
   Solution: Set of desired outcomes is $B = \{HTTH, HTTT, THTH, THTT, TTHH, TTHT\}$. So, $P(B) = \frac{6}{16} = \frac{3}{8}$.

   Do the answers to (a) and (b) change if we toss the coin more than four times?
   Solution: The answer to (a) depend on the exact number of coin tosses and hence changes if we toss the coin more than 4 times. However answer to (b) depends only on the outcome of the first three coin tosses and is independent of the rest of the tosses. Hence the answer to (b) remains the same.