EE 131A
Probability and Statistics
Instructor: Lara Dolecek

Winter 2021 Assessment 1 Solutions
January 25, 2021
Maximum score: 100 points

You have 2 hours to submit your work **directly on Gradescope under the Exam_1 submission link**.
Please read and carefully follow all the instructions.

Instructions

- The exam is accessible from 10 am PST on January 25th to 10 am PST on January 26th. Once you open the exam, you will have 2 hours to upload your work (therefore open the exam at least 2 hours before the closing time).
- This exam is open book, open notes. You are allowed to consult your own class notes (homework, discussion, lecture notes, textbook). You are not allowed to consult with each other or solicit external sources for help (e.g., an online forum).
- For each question, start a new sheet of paper. Therefore, the number of pages of your scan should be at least the number of questions. It is ok to write multiple parts of a question on one sheet. Properly erase or cross out any scratch work that is not part of the answer.
- Please submit your exam through the corresponding submission link on Gradescope.
- Make sure to include your **full name** and **UID** in your submitted file.
- Make sure to **show all your work**. Unjustified answers will be at a risk of losing points.
- **Policy on the Academic Integrity**
"During this exam, you are **disallowed** to contact with a fellow student or with anyone outside the class who can offer a solution e.g., web forum."
Please write the following statement on the first page of your answer sheet. You will **lose 10 points** if we can not find this statement.

I $\underline{\quad YourName \quad}$ with UID ____ have read and understood the policy on academic integrity.

1. (15 pts) Consider three events A, B, and C defined on the same sample space S. Draw a Venn diagram and write the expression for the following events. Use only complements, intersections, and unions operations.

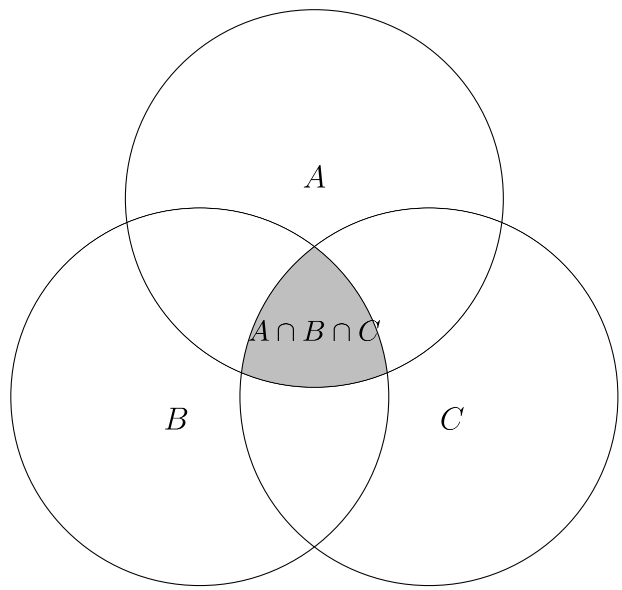
   (a) All three events occur.

   (b) None of the events occur.

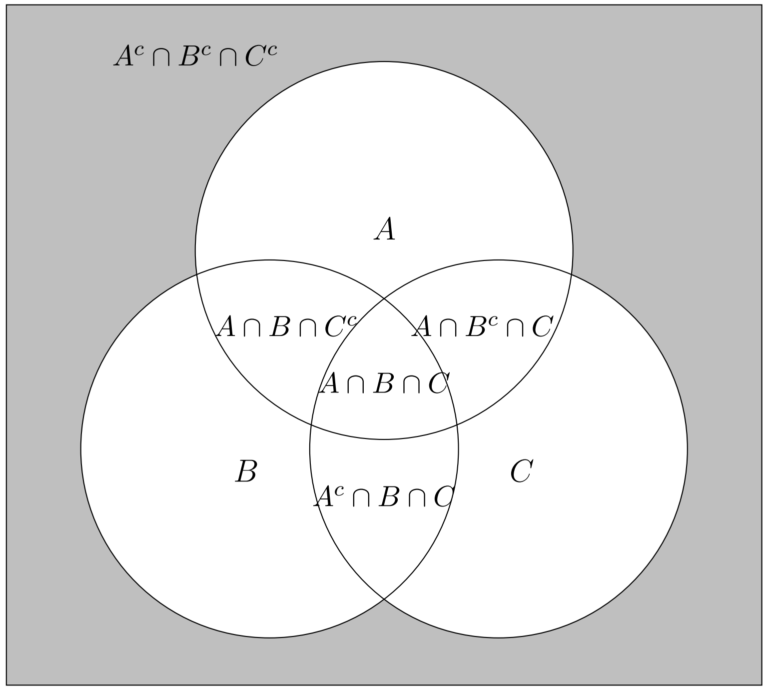
   (c) Events A and B occur, but C does not.

Solution:

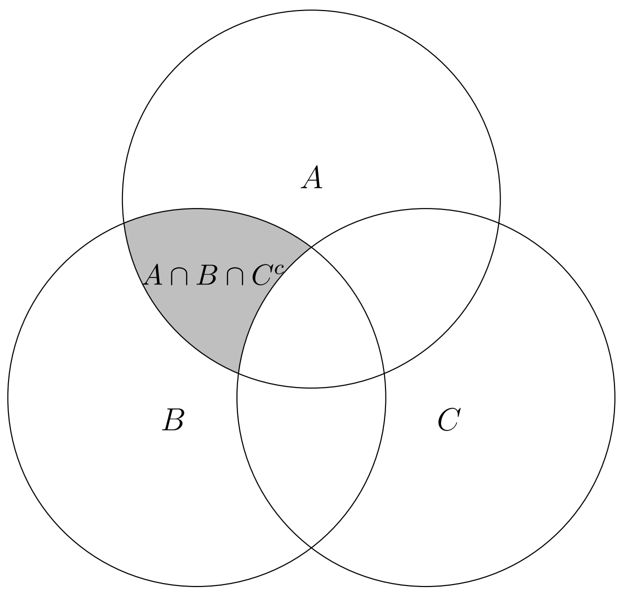
(a) The event we want is $(A \cap B \cap C)$. The Venn diagram is shown below:



(b) The event we want is $A^c \cap B^c \cap C^c$. The Venn diagram is shown below:



(c) The event we want is $A \cap B \cap C^c$. The Venn diagram is shown below:



2. (15 pts) Suppose $P(A)$ is 2/10, $P(B)$ is 3/10, and $P(A \cup B) = 4/10$. **Show all your work**. Compute:

   (a) $P(A \cap B)$.

   (b) $P(A|B)$.

   (c) $P(B|A)$.

Solution:

(a)

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{2}{10} + \frac{3}{10} - \frac{4}{10} = \frac{1}{10}.$$

(b)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{10}}{\frac{3}{10}} = \frac{1}{3}.$$

(c)

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{10}}{\frac{2}{10}} = \frac{1}{2}.$$

3. (40 pts) Suppose X has the following PMF. **Show all your work**.

$$X = \begin{cases} -2, & \text{with probability } 1/4, \\ 0, & \text{with probability } 1/2, \\ 2, & \text{with probability } 1/4. \end{cases} \quad (1)$$

(a) Compute the mean and variance of X.

(b) Compute the mean and variance of $Y = X^2$.

(c) Compute $\mathbf{E}[XY]$.

(d) Are X and Y uncorrelated ? Justify your answer.

Solution:

(a)

$$\begin{aligned} \mathbf{E}[X] &= (-2)\frac{1}{4} + (0)\frac{1}{2} + (2)\frac{1}{4} = 0 \\ Var(X) &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 \\ &= \mathbf{E}[X^2] \\ &= (-2)^2\frac{1}{4} + (0)^2\frac{1}{2} + (2)^2\frac{1}{4} = 2 \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{E}[Y] &= \mathbf{E}[X^2] = 2 \\ Var(Y) &= \mathbf{E}[(X^2)^2] - \mathbf{E}[X^2]^2 \\ &= \mathbf{E}[X^4] - 4 \\ &= (-2)^4\frac{1}{4} + (0)^4\frac{1}{2} + (2)^4\frac{1}{4} - 4 = 8 - 4 = 4 \end{aligned}$$

(c)

$$\begin{aligned} \mathbf{E}[XY] &= \mathbf{E}[X^3] \\ &= (-2)^3\frac{1}{4} + (0)^4\frac{1}{2} + (2)^3\frac{1}{4} = 0 \end{aligned}$$

(d) X and Y are uncorrelated because $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$.

$$0 = \mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y] = 0 \times 2 = 0$$

4. (10 pts) Suppose a box contains 10 balls, 3 of which are red and 7 of which are blue. We draw 3 balls one at the time without replacement. Compute the probability that all three drawn balls are blue. **Show all your work**.

Solution:

One way to solve this is by treating the ball draws in a sequential manner. First, let us get the total number of possible draws. There are 10 ways to select the first ball and remove it, 9 ways to select the second ball and remove it, and 8 ways to select the third ball and remove it. Now, we get the total number of ways to get 3 blue balls. There are 7 ways to select a blue ball as the first ball and remove it, 6 ways to select a blue ball as the second ball and remove it, and 5 ways to select a blue ball as the third ball and remove it. As such,

$$P(\text{ Drawing 3 blue balls }) = \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \approx 0.29167$$

5. (10 pts) Suppose we have a biased coin with probability of heads being 1/4. We throw this biased coin 3 times. **Show all your work**.

   (a) Specify a sample space for this random experiment.

   (b) What is the probability of seeing exactly one head in 3 coin tosses ?

Solution:

(a)

$$S = \{(a_1, a_2, a_3) : a_i \in \{H, T\}, i \in \{1, 2, 3\}\}$$

(b) The event that exactly one head appears in 3 coin tosses is equivalent to the probability that a Binomial random variable with parameters $n = 3$ and $p = \frac{1}{4}$ equals 1.

Thus, the answer is $\binom{3}{1}p(1-p)^2 = 3(1/4)(3/4)^2 = \frac{27}{64} = 0.421875$.

6. (10 pts) Suppose you are trying to get a toy from a box of cereal. The probability that a cereal box has a toy is $1/4$. **Show all your work**.

   (a) What is the expected number of cereal boxes you have to open until you get your first toy (including the last box) ?

   (b) What is the expected number of cereal boxes you have to open until you get your second toy (including the last box) ?

Solution:

(a) Let X be the random variable of the number of boxes needed to open in order to get the first toy. We can treat every opening of a box as a Bernoulli random variable where it is 1 if the toy is inside. As such, X is counting the number of Bernoulli trials until a 1 occurs which is the definition of a geometric random variable. Hence, X is a Geometric random variable with parameter $\frac{1}{4}$.

To get the expectation of X, we use the formula for the expectation of a geometric random variable which is $\frac{1}{p}$. Thus, the answer is $\frac{1}{\frac{1}{4}} = 4$.

(b) As shown in part b, we can treat every box opening as a Bernoulli trial. This scenario is exactly the same as for problem 6 in homework 2. Thus, we can re-use the result that the mean is $\frac{2}{p}$. As such, the answer is $\frac{2}{\frac{1}{4}} = 8$.