EE 131A
Probability
Instructor: Lara Dolecek

Discussion Set 2 Solution
January 15, 2021
TAs: Lev Tauz, Debarnab Mitra

Chapters 2.4-2.5 & 3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Let G be a geometric random variable with parameter p. Recall that the pmf for G is $P(G=k)=(1-p)^{k-1}p$ for $k \geq 1$. Find the variance of G, i.e. $Var(G)$.

   Solution:

   Recall that $Var(G)=\mathbb{E}[G^2]-\mathbb{E}[G]^2$. We already know that $\mathbb{E}[G]=\frac{1}{p}$. Now, we find the second moment $\mathbb{E}[G^2]$.

$$
\begin{aligned}
\mathbb{E}[G^2] &= \mathbb{E}[G(G-1)+G] \\
&= \mathbb{E}[G]+\mathbb{E}[G(G-1)] \\
&= \frac{1}{p}+\sum_{n=1}^{\infty} n(n-1)\cdot P(G=n) \\
&= \frac{1}{p}+\sum_{n=1}^{\infty} n(n-1)(1-p)^{n-1}p \\
&= \frac{1}{p}+p(1-p)\sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2} \\
&= \frac{1}{p}+p(1-p)\frac{d^2}{dp^2}(\sum_{n=1}^{\infty}(1-p)^n) && \text{Differentiation Trick} \\
&= \frac{1}{p}+p(1-p)\frac{d^2}{dp^2}(\frac{(1-p)}{1-(1-p)}) && \text{Geometric Sum Formula} \\
&= \frac{1}{p}+p(1-p)\frac{d^2}{dp^2}(\frac{1}{p}-1) \\
&= \frac{1}{p}+p(1-p)\frac{2}{p^3} \\
&= \frac{1}{p}+\frac{2-2p}{p^2}=\frac{2-p}{p^2}
\end{aligned}
$$

   Thus, the variance is $Var(G)=\mathbb{E}[G^2]-\mathbb{E}[G]^2=\frac{2-p}{p^2}-\frac{1}{p^2}=\frac{1-p}{p^2}$.

2. Ninety students, including Joe and Jane, are to be split into three classes of equal size, and this is to be done at random. What is the probability that Joe and Jane end up in the same class?

   Solution:

   Suppose we label the classes A, B, and C. The probability that Joe and Jane will both be in class A is the number of possible combinations for class A that involve both Joe and Jane, divided by the total number of combinations for class A. Therefore, this probability is

$$\frac{\binom{88}{28}}{\binom{90}{30}}$$

Since there are three classes, the probability that Joe and Jane end up in the same class is

$$3\frac{\binom{88}{28}}{\binom{90}{30}} = \frac{29}{89}$$

Another way to get the solution is as follows. We place Joe in one class. Regarding Jane,there are 89 possible slots, and only 29 of them place her in the same class as Joe.Thus, the answer is $\frac{29}{89}$, which turns out to agree with the answer obtained earlier.

3. *Pairwise independence and overall independence.* Alice, Bob and Claire each throw a fair die once. Show that the events A, B and C where A: "Alice and Bob roll the same face", B: "Alice and Claire roll the same face" and C: "Bob and Claire roll the same face" are pairwise independent but not independent.

   Solution: The events of any two people having the same face is $6 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{6}$. So we have:

$$P(A) = P(B) = P(C) = \frac{1}{6} \tag{1}$$

   On the other hand, the event of any two events of A, B, and C jointly occuring is $6 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. Therefore, we have:

$$P(A \cap B) = P(B \cap C) = P(C \cap A) = \frac{1}{36} \tag{2}$$

   By using (1) and (2), we can prove that A and B are independent by the following:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{6} = P(A)$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{6} = P(B)$$

   Similarly, we can show that all the events A, B, and C are pairwise independent.

   Now we calculate the event of A, B and C jointly occuring. The event $A \cap B$ shows that Alice and Bob roll the same faces, and also Bob and Claire roll the same faces, so we know Alice and Claire roll the same faces. This implies that:

$$P(A \cap B \cap C) = P(A \cap B) = \frac{1}{36} \tag{3}$$

Using (2) and (3), we get:

$$P(A|B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = 1 \neq \frac{1}{6} = P(A)$$

Therefore, we conclude that events A, B and C are pairwise independent, but not independent in general.

4. Each of k jars contains m white and n black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $\frac{m}{m+n}$.

   Solution: We derive a recursion for the probability p_i that a white ball is chosen from the i^{th} jar. We have, using the total probability theorem,

$$\begin{aligned} p_i &= P(\text{White Ball chosen from } i^{\text{th}} \text{ jar}) \\ &= P(\text{White Ball chosen from } i^{\text{th}} \text{ jar}|\text{White Ball put into } i^{\text{th}} \text{ jar})P(\text{White Ball put into } i^{\text{th}} \text{ jar}) \\ &+ P(\text{White Ball chosen from } i^{\text{th}} \text{ jar}|\text{Black Ball put into } i^{\text{th}} \text{ jar})P(\text{Black Ball put into } i^{\text{th}} \text{ jar}) \\ &= \frac{m+1}{m+n+1}p_{i-1} + \frac{m}{m+n+1}(1-p_{i-1}) \end{aligned}$$

   Clearly, the initial condition is $p_1 = \frac{m}{m+n}$. Now that we have the recursion step and the initial condition, we can calculate the values for p_i. Starting with p_2, we get

$$\begin{aligned} p_2 &= \frac{m+1}{m+n+1}p_1 + \frac{m}{m+n+1}(1-p_1) \\ &= \frac{m+1}{m+n+1}\frac{m}{m+n} + \frac{m}{m+n+1}\frac{n}{n+m} \\ &= \frac{m(m+n+1)}{(m+n+1)(m+n)} = \frac{m}{m+n} \end{aligned}$$

   Thus, $p_2 = p_1$ and we can continue this process to show that $p_i = p_1$ for all i.

5. *Bonus:Monty Hall Problem.* You are a contestant on a game show, and you are given a choice of three doors. Behind one door is a car; behind the other two doors are goats. You pick a door, for example, No. 1, and the game show host, who knows what is behind the doors, opens another door, say No. 3, which has a goat. The host asks you,"Do you want to switch to door No. 2?" Should you switch your choice? Why?

   Solution: Initially, it may seem that it does not matter whether you switch, since there are two doors (the door you selected and the remaining door), and it may seem like these two have an equal probability of containing the car. However, this logic is **not correct**.

   The probability that the door you selected has the car is just 1/3. This remains true throughout the problem. The door that the host picks and opens has a probability of

1 of containing a goat (and thus 0 of containing a car). The remaining door, therefore, must have probability 2/3 of containing the car, so you should **switch**.

To see this more clearly, let us enumerate the sample space S. We have the following equiprobable events for what is behind the 3 doors.

$$S = \{(C, G, G), (G, C, G), (G, G, C)\}.$$

In this scenario, you select door 1 (the other cases where you start with another door are identical.) Then, for each event, the following happens:

(a) (C,G,G): the host opens either door 2 or 3, and if you switch, you lose, getting the car with probability 0.

(b) (G,C,G): the host **must** open door 3, and if you switch to door 2, you win, with probability 1.

(c) (G,G,C): the host **must** open door 2, and if you switch to door 3, you win, with probability 1.

Each of these events are equiprobable, so they have probability 1/3. The total probability of winning the car if you switch is $1/3 \times 0 + 1/3 \times 1 + 1/3 \times 1 = 2/3$.

The key idea here is the fact that the host **always reveals a door that does not have a car**. Therefore, you are computing the probability of winning when switching to another door, given that the remaining door did not have a car. This conditional probability is precisely why the probability of winning when switching is 2/3.