EE 131A
Probability and Statistics
Instructor: Lara Dolecek

Winter 2021 Final Exam
March 19, 2021
Maximum score: 100 points

You have 3 hours to submit your work **directly on Gradescope under the Final_Exam submission link**.
**Please read and carefully follow all the instructions.**

## Instructions

- The exam is accessible from 10 am PST on March 19th to 10 am PST on March 20th. Once you open the exam, you will have 3 hours to upload your work (therefore open the exam at least 3 hours before the closing time).
- This exam is open book, open notes. You are allowed to consult your own class notes (homework, discussion, lecture notes, textbook). You are not allowed to consult with each other or solicit external sources for help (e.g., an online forum).
- For each question, start a new sheet of paper. Therefore, the number of pages of your scan should be at least the number of questions. It is ok to write multiple parts of a question on one sheet. Properly erase or cross out any scratch work that is not part of the answer.
- Please submit your exam through the corresponding submission link on Gradescope.
- Make sure to include your **full name** and **UID** in your submitted file.
- Make sure to **show all your work**. Unjustified answers will be at a risk of losing points.
- **Policy on the Academic Integrity**
  During this exam, you are **disallowed** to contact with a fellow student or with anyone outside the class who can offer a solution e.g., web forum.
  **Please write the following statement on the first page of your answer sheet.** You will **lose 10 points** if we can not find this statement.

I $\underline{\quad YourName \quad}$ with UID ____ have read and understood the policy on academic integrity.

1. You have two coins. One of them is a fair coin and the other has heads on both sides. You select a coin at random and toss it.

   (a) What is the probability that the toss results in a heads?

   (b) What is the probability that you had selected the fair coin given that the toss resulted in a heads?

   (c) Suppose you toss the selected coin $n$ times. What is the probability that you had selected the fair coin given that each of the $n$ tosses result in a heads? Write your answer in terms of $n$.

   **Solution:**
   Let $F$ be the event that you select a fair coin and let $H$ be the event that the coin results in a heads.

   (a) Clearly, $P(F) = P(F^c) = 0.5$. By total law of probability we get:

   $$P(H) = P(H|F)P(F) + P(H|F^c)P(F^c) = (0.5)(0.5) + 1(0.5) = 0.75$$

   (b) We use the Bayes' Theorem to find

   $$P(F|H) = \frac{P(H|F)P(F)}{P(H)} = \frac{(0.5)(0.5)}{0.75} = \frac{1}{3}$$

   (c) Let $H_n$ be the event that all the $n$ tosses result in heads. Then,

   $$P(F|H_n) = \frac{P(H_n|F)P(F)}{P(H_n|F)P(F) + P(H_n|F^c)P(F^c)} = \frac{(0.5)^n(0.5)}{(0.5)^n(0.5) + (1)(0.5)} = \frac{1}{2^n + 1}$$

2. Let $X_1$ and $X_2$ be two independent Bernoulli random variables with parameters $p_1$ and $p_2$ respectively.

   (a) Find the characteristic function of $Y = 3X_1 + 5X_2$.

   (b) Using the characteristic function from part (a), find $E[Y]$.

   **Solution:**

   (a)

   $$\begin{aligned}\Phi_Y(\omega) &= \mathrm{E}[e^{j\omega Y}] \\ &= \mathrm{E}[e^{j\omega(3X_1+5X_2)}] \\ &= \mathrm{E}[e^{3j\omega X_1}]\mathrm{E}[e^{5j\omega X_2}] \\ &= \left((1-p_1) + p_1 e^{3j\omega}\right)\left((1-p_2) + p_2 e^{5j\omega}\right)\end{aligned}$$

(b)

$$\frac{d}{d\omega}\Phi_Y(\omega) = \left(3jp_1e^{3j\omega}\right)\left((1-p_2)+p_2e^{5j\omega}\right) + \left((1-p_1)+p_1e^{3j\omega}\right)\left(5jp_2e^{5j\omega}\right)$$

$$\begin{aligned}\mathrm{E}[Y] &= \frac{1}{j}\frac{d}{d\omega}\Phi_Y(\omega)|_{\omega=0} \\ &= \frac{1}{j}(3jp_1+5jp_2) \\ &= 3p_1+5p_2\end{aligned}$$

3. State the weak law of large numbers.
**Solution:**
The weak law of large numbers states the following:
Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d random variables with finite mean $E[X_i] = \mu$.
Let $M_n = \frac{1}{n}\sum_{i=1}^n X_i$. Then for any $\epsilon > 0$,

$$\lim_{n\to\infty} P[|M_n - \mu| < \epsilon] = 1.$$

4. Consider the jointly Gaussian random variables $X$ and $Y$ that have the following joint PDF:

$$f_{X,Y}(x,y) = \frac{5}{2\pi}\exp\left\{-\left[\frac{25}{32}x^2 - \frac{15}{4}xy + \frac{25}{2}y^2\right]\right\}$$

   (a) Find $E[X], E[Y], VAR[X], VAR[Y]$, and $COV[X,Y]$
   (b) Find the marginal PDF of $Y$, $f_Y(y)$
   (c) Find $f_{X|Y}(x|y)$
   (d) Find $E[X|Y=y]$ and $E[X|Y]$

**Solution:**
First, we note this is a joint Gaussian distribution. We just have to unravel the terms to get the parameters. Recall that the pdf of a pair of jointly Gaussian RVs is

$$f_{X,Y}(x,y) = \frac{\exp[\frac{-1}{2(1-\rho_{X,Y}^2)}((\frac{x-m_x}{\sigma_X})^2 - 2\rho_{X,Y}(\frac{x-m_x}{\sigma_X})(\frac{y-m_y}{\sigma_Y}) + (\frac{y-m_y}{\sigma_Y})^2)]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

Comparing the coefficients of the exponent, we get $m_x = 0, m_y = 0$

$$\frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2} = \frac{25}{32}$$

$$\frac{1}{2(1-\rho_{X,Y}^2)\sigma_Y^2} = \frac{25}{2}$$

$$\frac{\rho_{X,Y}}{(1-\rho_{X,Y}^2)\sigma_X\sigma_Y} = \frac{15}{4}$$

Solving for $\rho_{X,Y}, \sigma_X$ and $\sigma_Y$ we get $\rho_{X,Y} = \frac{3}{5}$, $\sigma_X^2 = 1$, $\sigma_Y^2 = \frac{1}{16}$.

(a) Thus $E[X] = 0, VAR(X) = 1, E[Y] = 0, VAR(Y) = \frac{1}{16}$, $COV(X,Y) = \frac{3}{20}$

(b) We know that marginally $Y$ is Gaussian with mean $m_y$ and variance $\sigma_Y^2$. Thus

$$f_Y(y) = \frac{4}{\sqrt{2\pi}} e^{-8y^2}$$

(c) We know that conditioned on $Y = y$, $X$ is Gaussian with mean $m_x + \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(y - m_y)$ and variance $\sigma_X^2(1-\rho_{X,Y}^2)$. Thus in this problem, given $Y = y$, $X$ is Gaussian with mean $\frac{12y}{5}$ and variance $\frac{16}{25}$. Thus,

$$f_{X|Y}(x|y) = \frac{5}{4\sqrt{2\pi}}\exp\left[-\frac{25}{32}\left(x - \frac{12y}{5}\right)^2\right]$$

(d) $E[X|Y = y] = m_x + \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(y - m_y) = \frac{12y}{5}$. $E[X|Y] = \frac{12Y}{5}$.

5. Suppose $X$ is a Gaussian RV with mean 0 and variance 16. Let RV $Y$ be defined as follows:

$$Y = \begin{cases} X, & \text{if } X \geq 1, \\ 2 - X, & \text{if } X < 1. \end{cases}$$

Compute the pdf of $Y$.

**Solution:**

First, note that $Y$ can only take values $y \geq 1$. Thus, for $y \geq 1$,

$$\begin{aligned}
P(Y \leq y) &= P(Y \leq y, X \geq 1) + P(Y \leq y, X < 1) \\
&= P(X \leq y, X \geq 1) + P(2 - X \leq y, X < 1) \\
&= P(1 \leq X \leq y) + P(2 - y \leq X < 1) \\
&= P(2 - y \leq X \leq y) = P(\frac{2-y}{4} \leq \frac{X}{4} \leq \frac{y}{4}) \\
&= \Phi(\frac{y}{4}) - \Phi(\frac{2-y}{4})
\end{aligned}$$

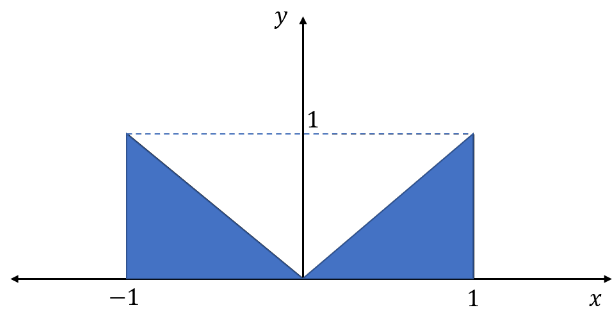
where $\Phi(\cdot)$ is the cdf for the standard normal distribution.

Thus, for $y \geq 1$,

$$\begin{aligned}
f_Y(y) &= \frac{1}{4}\Phi'(\frac{y}{4}) + \frac{1}{4}\Phi'(\frac{2-y}{4}) \\
&= \frac{1}{4\sqrt{2\pi}}(e^{-\frac{y^2}{32}} + e^{-\frac{(2-y)^2}{32}})
\end{aligned}$$

and $f_Y(y) = 0$ for $y < 1$.

6. Let $X$ and $Y$ be continuous random variables that are uniformly distributed on the shaded area in the following figure:




(a) Find the joint PDF of $X$ and $Y$.
**Solution:**
Since $(X, Y)$ are uniform on the region, $f(x, y) = \frac{1}{\text{Area of region}}$. Thus,

$$f(x,y) = \begin{cases} 1 & 0 \leq y \leq |x| \leq 1 \\ 0 & else \end{cases}$$

(b) Compute the marginal PDFs of $X$ and $Y$.
**Solution:**
First, we shall calculate $f_X(x)$. Note that $X$ can only take values between $-1$ and 1. Thus,

$$f_X(x) = \begin{cases} |x| & -1 \leq x \leq 1 \\ 0 & else \end{cases}$$

Now, note that

$$f_Y(y) = \begin{cases} 2 - 2y & 0 \leq y \leq 1 \\ 0 & else \end{cases}$$

(c) Find the conditional pdf of $X$ given $Y$.
**Solution:**

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f(y)} \\ &= \begin{cases} \frac{1}{2-2y} & 0 \leq y \leq 1, y \leq |x| \leq 1 \\ 0 & else \end{cases} \end{aligned}$$

(d) Find $E[X|Y]$
**Solution:**

Visually, we can see that $X$ is always centered on zero regardless of the value of $Y$. Thus, $E[X|Y] = 0$.

We can also confirm this by performing the calculation.

$$
\begin{aligned}
E[X|Y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \\
&= \int_{y}^{1} x \frac{1}{2-2y} dx + \int_{-1}^{-y} x \frac{1}{2-2y} dx \\
&= \frac{1}{2-2y} \left( \frac{1-y^2}{2} + \frac{y^2-1}{2} \right) = 0
\end{aligned}
$$

7. Let $X$ and $Y$ be continuous random variables with the following joint PDF:

$$
f_{X,Y}(x,y) = \begin{cases} k(x+y) & 0 \le y \le x < 1, \\ 0 & \text{otherwise,} \end{cases}
$$

Let $Z = X + Y$.

(a) Find the constant $k$.

**Solution:**

$$
\begin{aligned}
1 &= \int \int f(x,y) dy dx \\
&= \int_0^1 \int_0^x k(x+y) dy dx \\
&= k \int_0^1 xy + \frac{y^2}{2} |_{y=0}^{x} dx \\
&= k \int_0^1 \frac{3}{2} x^2 dx \\
&= k \frac{x^3}{2} |_{x=0}^{1} \\
&= k \frac{1}{2} \implies k = 2
\end{aligned}
$$

(b) Are $X$ and $Y$ independent? Explain your answer.

**Solution:**

No. Note that in the sample space that $Y$ can take values between zero and 1. Conditioned on $X$, $Y$ can only take values between 0 and $X$. Since $X$ restricts the values $Y$ can take, $X$ and $Y$ cannot be independent.

(c) Find the PDF of $Z$.

**Solution:**
Recall from Homework 6, that the pdf of $Z$ can be written in terms of a general $f(x, y)$ as

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)dx$$

Now, note that for this joint pdf that $f(x, z-x) = 2(x+z-x) = 2z$ for $0 \leq z-x \leq x \leq 1$.
Thus, when $0 \leq z \leq 1$, $f(x, z-x)$ is non-zero when $\frac{z}{2} \leq x \leq z$. Hence,

$$f_Z(z) = \int_{\frac{z}{2}}^{z} 2zdx = z^2.$$

Similarly, when $1 \leq z \leq 2$, $f(x, z-x)$ is non-zero when $\frac{z}{2} \leq x \leq 1$. Hence,

$$f_Z(z) = \int_{\frac{z}{2}}^{1} 2zdx = 2z - z^2$$

Thus, the final pdf is

$$f_Z(z) = \begin{cases} z^2 & \text{if } 0 \leq z \leq 1 \\ 2z - z^2 & \text{if } 1 \leq z \leq 2 \\ 0 & \text{else} \end{cases}$$

8. A sequence of 1000 bits are transmitted over a channel. Each bit gets flipped independently of others with a probability 0.2. Let $X$ be the total number of bit flips that occur.

   (a) What type of random variable is $X$?
   **Solution:**
   $X$ is a Binomial Random Variable with $n = 1000$ and $p = 0.2$.

   (b) Find $P(230 < X < 370)$. You can leave the final answer as a summation or integral.
   **Solution:**
   Note that $X$ is discrete. As such,

$$P(230 < X < 370) = \sum_{k=231}^{369} P(X = k) = \sum_{k=231}^{369} \binom{1000}{k} 0.2^k 0.8^{1000-k}$$

   Note that because the bounds are not inclusive, we do note include $P(X = 230)$ and $P(X = 370)$.

   (c) Estimate (approximate) the probability in part (a) using the central limit theorem. You can leave your answer in terms of the Q function.

**Solution:**
Note that $\mathbf{E}[X] = np = 1000*0.2 = 200$ and $Var(x) = np(1-p) = 1000*0.2*0.8 = 160$.

Thus,

$$\begin{aligned}
P(230 < X < 370) &= P(\frac{230-200}{\sqrt{160}} < \frac{X-200}{\sqrt{160}} < \frac{370-200}{\sqrt{160}}) \\
&= P(\frac{30}{\sqrt{160}} < \frac{X-200}{\sqrt{160}} < \frac{170}{\sqrt{160}}) \\
&\approx Q(\frac{30}{\sqrt{160}}) - Q(\frac{170}{\sqrt{160}})
\end{aligned}$$