I, NEVIN LIANG WITH UID 705575353 HAVE READ AND UNDERSTOOD THE POLICY OF ACADEMIC INTEGRITY

NEVIN LIANG
705515353

1\. a) $P(H) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1$

$= \frac{1}{4} + \frac{1}{2} = \boxed{3/4}$

b) $P(\text{fair coin} \mid H) = \dfrac{\frac{1}{2} \cdot \frac{1}{2}}{3/4} = \boxed{1/3}$

c) $P(\text{fair coin} \mid n \text{ Heads}) = \dfrac{P(\text{fair coin} \cap n\ H\text{'s})}{P(n\ H\text{'s})}$

$\xrightarrow{1/2}$ fair coin $\longrightarrow$ n H's probability $= \left(\frac{1}{2}\right)^n$

$\xrightarrow{1/2}$ 2H coin $\longrightarrow P(n\ H\text{'s}) = 1$

$$\frac{\left(\frac{1}{2}\right)^n \cdot \cancel{\frac{1}{2}}}{\left(\frac{1}{2}\right)^n \cdot \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}}$$

$$= \frac{\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^n + 1} = \boxed{\frac{1}{1+2^n}}$$

2.

(a) $Y = 3X_1 + 5X_2$ where $X_1 = \begin{cases} 1 & p_1 \\ 0 & 1-p_1 \end{cases}$ $X_2 = \begin{cases} 1 & p_2 \\ 0 & 1-p_2 \end{cases}$

$$\Phi_y[Y] = \mathbb{E}[e^{j\omega Y}] = \mathbb{E}[e^{j\omega(3X_1+5X_2)}] = \mathbb{E}[e^{3j\omega X_1} + e^{5j\omega X_2}]$$

$$= \mathbb{E}[e^{3j\omega X_1}] + \mathbb{E}[e^{5j\omega X_2}]$$

$$= (1-p_1) + p_1 \cdot e^{3j\omega} + (1-p_2) + p_2 \cdot e^{5j\omega}$$

(b)
$$E[Y] = \frac{1}{j} \cdot \frac{d^{n'} \Phi_Y(\omega)}{d\omega^{n'}}\Bigg|_{\omega=0} = \frac{1}{j}\left(3j \cdot p_1 \cdot e^{3j\omega} + 5j\, p_2 \cdot e^{5j\omega}\right)\Bigg|_{\omega=0}$$

$$= \frac{1}{j}(3j\, p_1 + 5j \cdot p_2) = \boxed{3p_1 + 5p_2}$$

3: Given a sequence of i.i.d RV with mean $\mathbb{E}[X] = \mu$

X_1 X_2 X_3 ...

For a small # $\varepsilon > 0$,

$$\lim_{n \to \infty} P\left(|M_n - \mu| < \varepsilon \right) = 1.$$

btw mean must be finite.

SUMMARY: when n is a big enough #, the sample mean of n ~~trials~~ samples ≈ the true mean.

8.4.

$$f_{XY}(x,y) = \frac{5}{2\pi} \exp\left(-\left(\frac{25}{32}x^2 - \frac{15}{4}xy + \frac{25}{2}y^2\right)\right)$$

no value of x or y so μ_x and $\mu_y = 0$.

$$= \frac{\exp\left\{\frac{-1}{2}\left(\frac{25}{16}x^2 - \frac{15}{2}xy + 25y^2\right)\right\}}{2\pi \cdot \frac{1}{5}}$$

$(1-\rho_{XY}^2)\cdot\sigma_1^2 = \frac{16}{25}$; $(1-\rho_{XY}^2)\cdot\sigma_2^2 = \frac{1}{25}$

$\frac{\sigma_1 \sigma_2 (1-\rho_{XY}^2)}{2\rho_{XY}} = \frac{2}{15}$; $\sigma_1\sigma_2\sqrt{1-\rho_{XY}^2} = \frac{1}{5}$

$\sigma_1^2\sigma_2^2(1-\rho_{XY}^2) = \frac{1}{25}$

$\sigma_1^2 = \boxed{1}$ $\sigma_2^2 = \cancel{1/4}\ \boxed{\frac{1}{16}}$

$1-\rho_{XY}^2 = \frac{1}{25}\cdot 1\cdot 16$

$\rho_{XY}^2 = \frac{9}{25}$

$\rho_{XY} = 3/5$

$\text{Cov}(X,Y) = \sigma_X \cdot \sigma_Y \cdot \rho_{XY}$

$= 1\cdot\frac{1}{4}\cdot\frac{3}{5} = \boxed{3/20}$

a) $E[X] = 0$

$E[Y] = 0$

$VAR[X] = \sigma_1^2 = 1$

$VAR[Y] = \sigma_2^2 = \cancel{1/4}\ \frac{1}{16}$

$\text{cov}[X,Y] = 3/20$

4 b) $f_Y(y) = \dfrac{e^{-(y-m_2)^2/2\sigma_2^2}}{\sqrt{2\pi}\,\sigma_2}$

$$= \frac{e^{-(y-0)^2/2\cdot(1/16)}}{\sqrt{2\pi}\;1/4} = \frac{4\cdot e^{-y^2\cdot 8}}{\sqrt{2\pi}}$$

c) $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{\frac{5}{2\pi}}{\frac{4}{\sqrt{2\pi}}}\, e^{-\frac{25}{32}x^2+\frac{15}{4}xy-\frac{25}{2}y^2+8y^2}$

$$= \frac{5}{4\sqrt{2\pi}}\cdot e^{-\frac{25}{32}x^2+\frac{15}{4}xy-\frac{9}{2}y^2}$$

d). $E[X|Y=y] = \int_{-\infty}^{\infty} x\cdot f_{X|Y}(x|y)\,dx = \cancel{\int_{-\infty}^{\infty} x\cdot\frac{5}{4\sqrt{2\pi}}\, e^{-\left(\frac{25}{32}x^2-\frac{15}{4}xy+\frac{9}{2}y^2\right)}}\,dx$

$$= \cancel{\frac{5}{4\sqrt{2\pi}}\int_{-\infty}^{\infty} x\cdot e^{-\frac{1}{32}(5x-12y)^2}}\,dx$$

textbook equation:

mean of joint gaussian $X|Y$ $= m_1 + \rho_{XY}\,(\sigma_1/\sigma_2)\,(y-m_2)$

$$= 0 + \overset{3/5}{\cancel{\tfrac{3}{20}}}\cdot\frac{1}{1/4}\cdot(y-0)$$

$cov = 3/20$

$\rho = 3/5$

$$= \frac{3}{5}\,\cancel{4}\;4\cdot y = \frac{12y}{5}$$

$\mathbb{E}[X|Y=y] = \dfrac{12y}{5}$ ← scaler

$\mathbb{E}[X|Y] = \dfrac{12Y}{5}$ ← RV

5. Note that Y must be >1. We will use CDF to find Φ then PDF.

$F_Y(y) = P(Y \le y)$

$P(X \le y \text{ if } X \ge 1)$ $\quad$ $P(2-X \le y \text{ if } X < 1)$

$P(1 \le X \le y)$ $\quad$ $P(2-y \le X < 1)$

we use CDF of $\mathcal{N}(0,1)$ Φ

$P(2-y \le X \le y)$

$\mathrm{CDF}(y) - \mathrm{CDF}(2-y)$

S~~TD DEV~~: VAR=16 $\sigma = 4$

$\Phi\left(\frac{y}{4}\right) - \Phi\left(\frac{2-y}{4}\right)$

$$F_X(a) = \int_{-\infty}^{a} \frac{1}{2\pi\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \, da$$

$\Phi(x)$: CDF of $\mathcal{N}(0,1)$

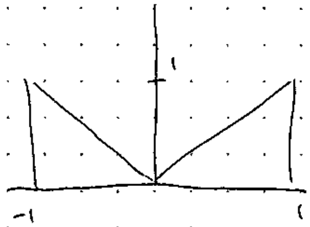
$F_X(a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$

$$\frac{d}{dy}\left(\frac{2-y}{4}\right) = -1$$

$$f_Y(y) = \frac{1}{4}\Phi'\left(\frac{y}{4}\right) + \frac{1}{4}\Phi'\left(\frac{2-y}{4}\right)$$

$$= \boxed{\frac{1}{4\sqrt{2\pi}} e^{-y^2/32} + \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{32}(2-y)^2}}$$

6. (a)



since uniform: $f_{XY}(x,y) = \begin{cases} k & \text{in shaded region} \\ 0 & \text{else.} \end{cases}$

$k \cdot [\text{shaded region}] = 1$

$[\text{shaded region}] = \frac{1 \cdot 1}{2} \cdot 2 = 1. \quad k = 1.$

$$f_{XY}(x,y) = \begin{cases} 1 & 0 \le y \le |x| \text{ AND } |x| \le 1 \\ 0 & \text{else.} \end{cases}$$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

$= \int_0^{|x|} 1\, dy = |x| \qquad f_X(x) = \begin{cases} |x| & |x| \le 1 \\ 0 & \text{else} \end{cases}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \int_{y}^{1} 1 \cdot dx + \int_{-1}^{-y} 1 \cdot dx = (1-y) + (-y+1)$

$= 2 - 2y$

$$= \begin{cases} 2-2y & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

(c) $f_{X|Y}(x|Y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \boxed{\dfrac{1}{2-2y}}$ for $0 \le y \le |x|$

(d) $\mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx = \int_{-1}^{1} \frac{x}{2-2y}\, dx = \frac{1}{2-2y}\left(\frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 1\right) = \boxed{0}$

#7)

a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$\int_0^1\int_0^x k(x+y)\,dy\,dx = \int_0^1 kx^2 + \tfrac{1}{2}kx^2\,dx = \frac{kx^3}{2}\Big|_0^1 = \frac{k}{2} = 1$$

$$\boxed{k=2}$$

b) $$f_X(x) = \int_x^1 2(x+y)\,dy = 2x(1-x) + (1^2-x^2) = -3x^2+2x+1 \quad \text{for } 0\le x\le 1$$

$$f_Y(y) = \int_0^y 2x+2y\,d\cancel{y}x = 3y^2 \quad \text{for } 0\le y\le 1$$

$$2(x+y) \neq (-3x^2+2x+1)\,(3y^2)$$

$$\boxed{\text{NOT INDEP}}$$

c) $$\cancel{f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{XY}(x,z-x)\,dx} = \int_0^1 f_{XY}(x,z-x)\,dx$$

$$F(z) = P[Z\le z]$$

$$= P[X+Y\le Z]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx$$

$$\cancel{2\int_0^1 x+y\,dx}$$

$$\cancel{= \tfrac{1}{2}x^2 + yx\Big|_0^1 =} \ \tfrac{1}{2}+y$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty}\left[\frac{d}{dz}\int_{-\infty}^{z-x} f_{\cancel{XY}}(x,y)\,dy\right]dx = \int_{-\infty}^{\infty} f_{XY}(x,z-x)\,dx$$

$$f_{XY}(x,z-x) = 2z \quad \text{for } 0\le z-x\le x\le 1$$

when $0\le z\le 1$: $f(x,z-x)\ne 0$, $\frac{z}{2}\le x\le z$, $$\int_{z/2}^{z} 2z\,dx = z/2\cdot 2 = \boxed{z^2}$$

$1\le z\le 2$: $f(x,z-x)\ne 0$, $\frac{z}{2}\le x\le 1$, $$\int_{z/2}^{1} \cancel{2x(1-x)}\,z\,dx = 2z(1-z/2) = \boxed{2z-z^2}$$

8. (a) technically X is binomial with 1000 trials and $P(\text{success}) = 0.2$.

all transmissions are $\perp\!\!\!\perp$.

(b) for any given # of successes K.

$$\binom{n}{K} \cdot p^K \cdot (1-p)^{n-k} \quad \cancel{\text{ways}} \ \text{probability}$$

$$P(230 < X < 370) = \sum_{K=231}^{369} \binom{n}{K} \cdot p^K \cdot (1-p)^{n-K}$$

normalize 230, 370

$$E[X] = 0.2 \qquad VAR[X] = \frac{1}{5} \cdot \frac{4}{5} = 4/25$$
$$= 0.16$$

$0.2 \times 1000 = 200$ avg

$$Z_n = \frac{\text{Sum} - 200}{2/5 \cdot \sqrt{1000}} \qquad Z_n(230) = \frac{230-200}{2/5\sqrt{1000}} = 2.372$$

$$Z_n(370) = \frac{370-200}{2/5 \cdot \sqrt{1000}} = 13.440$$

$$Q(13.44) - Q(2.372) = \boxed{0.885\%}$$