ECE 131A | Homework 5
Probability and Statistics | Monday, February 8, 2021
Instructor: Lara Dolecek | Due: Wednesday, February 17, 2021 before class begins

TAs: Lev Tauz levtauz@ucla.edu
Debarnab Mitra debarnabucla@g.ucla.edu

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You may type your homework or scan your handwritten version. Make sure all the work is discernible.

Chapters 4.6,4.7, and 4.9 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Let X be the number of successes in n Bernoulli trials where the probability of success is p. Let $Y = \frac{X}{n}$ be the average number of successes per trial. Apply the Chebyshev inequality to the event $\{|Y - p| \geq a\}$. What happens as $n \to \infty$? What does this result imply about the distribution of Y as n goes to ∞?

2. Compare the Chebyshev inequality and the exact probability for the event $\{|X| > c\}$ as a function of c when X is a continuous uniform random variable in the interval $[-b, b]$ with $b > 0$. For $b = 1$, plot both the Chebyshev inequality and the exact probability for values of c that satisfy $0.2 \leq c \leq 1$.

3. For each of the following random variables, find the characteristic function.

   (a) X is a discrete random variable that counts the number of failures when flipping a coin with success probability p before a success comes up. We can relate X to a geometric random variable A with parameter p by writing $X = A - 1$.

   (b) Y is a continuous uniform random variable and takes values in $[a, b]$ for real values a, b.

   (c) Z is a discrete uniform random variable where Z can only take values $\{c, c+1, c+2, \cdots, d\}$ for integers c, d.

4. Let X be a continuous random variable. We define the random variable Y as $Y = aX+b$ for a, b such that $a > 0$. Let $\Phi_X(w)$ be the characteristic function of X.

   (a) Let $\Phi_Y(w)$ be the characteristic function of Y. Determine $\Phi_Y(w)$ in terms of $\Phi_X(w), a$, and b. Show the process for determining this relationship.

   (b) Assume that $X \sim Exp(\lambda)$ and that $b = 0$. Using the characteristic function of Y, determine what kind of random variable Y is.

   (c) Again, assume that $X \sim Exp(\lambda)$. First, use the properties of expectation to find $\mathbb{E}[Y]$. Now, use the characteristic function of Y to find $\mathbb{E}[Y]$. Answers are in terms of a, b, and λ.

5. In this problem, you will be using the Chernoff bound to determine some useful upper bounds.

   (a) Use the Chernoff bound to prove that $Q(x) \leq e^{\frac{-x^2}{2}} \quad \forall x \geq 0$. Recall that $Q(x) = P(X > x)$ where X is a normal distribution.

   (b) Let $Y = \sum_{i=1}^{n} X_i$ where the X_i are i.i.d. Bernoulli random variables with parameter p. Using the Chernoff bound, prove that

   $$P(Y \geq (1+\delta)np) \leq (\frac{e^{\delta}}{(1+\delta)^{1+\delta}})^{np} \quad \forall \delta > 0.$$

   Hint: The following inequality may prove useful: $1 + y \leq e^y$ for all y.

   What does this bound imply about the distribution of Y?

6. (Generating Random Variables)

   In this question, you will use MATLAB to generate random variables using the transform method. Let U be a uniform random variable from 0 to 1. For $n > 0$, let $\{X_i \mid i \in [1, \cdots, n]\}$ be exponential random variables with parameter λ. Let $F(x)$ be the cdf of a single exponential random variable. You may use the MATLAB function *rand* to generate U.

   (a) Determine the inverse of $F(x)$, i.e. $F^{-1}(u)$.

   (b) Now, you will generate X_i by applying $F^{-1}(u)$ on U. For $\lambda = 0.5$, generate $n = 5000$ exponential random variables and plot a histogram of the points. Compare the histogram to a plot of the pdf of X.

   (c) For $\lambda = 0.5$, generate $n = 5000$ exponential random variables and take the average of the random variables. How does it compare to the expecation of X?