EE 131A
Probability
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Discussion Set 6 Solutions
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Chapters 4.6,4.7, and 4.9 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *Markov inequality.* Problem 4.98, parts (a), (b) and (d), page 224 of ALG
   Compare the Markov inequality and the exact probability for the event $\{X > c\}$ as a function of c for:

   (a) X is a uniform random variable in $\{1, 2, \ldots, L\}$.
   Solution:
   For the discrete uniform random variable, the Markov inequality states :

$$P[X > c] = P[X \geq c] - P[X = c] \leq \frac{E[X]}{c} - \frac{1}{L} = \frac{L+1}{2c} - \frac{1}{L}$$

   The exact probability for event $\{X > c\}$ is given by:

$$P[X > c] = 1 - P[X \leq c] = 1 - \frac{c}{L}$$

   For $c \in \{1, 2, ..., L\}$, we always have:

$$1 - \frac{c}{L} \leq \frac{L+1}{2c} - \frac{1}{L}$$

   (b) X is a geometric random variable with parameter p.
   Solution:
   For the geometric random variable, the Markov inequality states :

$$P[X > c] = P[X \geq c] - P[X = c] \leq \frac{E[X]}{c} - p(1-p)^{c-1} = \frac{1}{pc} - p(1-p)^{c-1}$$

   The exact probability for event $\{X > c\}$ is given by:

$$P[X > c] = (1-p)^c$$

   For $c \in \{0, 1, 2, ...\}$, we always have:

$$(1-p)^c \leq \frac{1}{pc} - p(1-p)^{c-1}$$

   (c) X is a binomial random variable with parameters n and $p = 0.5$.
   Solution:
   For the binomial random variable with $p = 0.5$, the Markov inequality states :

$$P[X > c] = P[X \geq c] - P[X = c] \leq \frac{E[X]}{c} - \binom{n}{c}(0.5)^n = \frac{n}{2c} - \binom{n}{c}(0.5)^n$$

The exact probability for event $\{X > c\}$ is given by:

$$P[X > c] = 1 - P[X \leq c] = 1 - \sum_{j=0}^{c} \binom{n}{j} (0.5)^n$$

For $c \in \{0, 1, 2, ...n\}$, we always have:

$$1 - \sum_{j=0}^{c-1} \binom{n}{j} (0.5)^n \leq \frac{n}{2c}$$

2. *Probability Generating Function*

   (a) Let X be a discrete random variable defined by the following pmf:

$$P(X = k) = \begin{cases} p & k = 3 \\ 1 - p & k = 1. \end{cases}$$

   For simplicity, we define the notation $R(p)$ to refer to this distribution. Find the probability generating function $G_X(z)$. Using the probability generating function, find the mean and variance of X. **Solution:**

$$G_X(z) = \mathbb{E}[z^X] = z^3 p + z(1 - p)$$

$$\begin{aligned} E[X] = G_X^{'}[z]\Big|_{z=1} &= 3z^2 p + (1 - p)\Big|_{z=1} \\ &= 2p + 1 \end{aligned}$$

$$G_X^{''}[z]\Big|_{z=1} = 6zp\Big|_{z=1} = 6p = \mathbb{E}[X^2] - \mathbb{E}[X]$$

$$\begin{aligned} Var(X) &= G_X^{''}[1] + G_X^{'}[1] - (G_X^{'}[1])^2 \\ &= 6p + 2p + 1 - (2p + 1)^2 \\ &= 8p + 1 - (2p + 1)^2 = 1 - (2p - 1)^2 \end{aligned}$$

   (b) Consider the function $f(z) = (G_X(z))^2$. Is $f(z)$ a probability generating function? If so, describe the random variable it generates and find the mean and variance of this new random variable.

   Solution:
   Yes it is. This is the generating function for a sum of two independent and identically distributed random variables. As such, $G_Y(z) = (G_X(z))^2 \implies Y = X_1 + X_2$ where $X_1 \sim R(p)$ and $X_2 \sim R(p)$.
   To calculate the mean and variance, we can re-use the derivatives of the previous part. Let Y be the random variable of this generating function.

$$\begin{aligned} f^{'}(z) &= 2G_X(z)G^{'}_X(z) \\ \implies f^{'}(1) &= 2G_X(1)G^{'}_X(1) \\ &= 2(1)(2p+1) = 4p+2 = \mathbb{E}[Y] \end{aligned}$$

$$\begin{aligned} f^{''}(z) &= 2G^{'}_X(z)G^{'}_X(z) + 2G_X(z)G^{''}_X(z) \\ \implies f^{''}(1) &= 2G^{'}_X(1)G^{'}_X(1) + 2G_X(1)G^{''}_X(1) \\ &= 2(2p+1)(2p+1) + 2(1)(6p) = 2(2p+1)^2 + 12p. \end{aligned}$$

$$\begin{aligned} Var(Y) &= f^{''}(1) + f^{'}(1) - (f^{'}(1))^2 \\ &= 2(2p+1)^2 + 12p + 2(2p+1) - (4p+2)^2 \\ &= 2 - 2(2p-1)^2 \end{aligned}$$

(c) Let X_1 and X_2 be 2 random variables such that $X_1 \sim R(p)$ and $X_2 \sim R(1-p)$. Consider the function $g(z) = \frac{G_{X_1}(z)+G_{X_2}(z)}{2}$. Is $g(z)$ a probability generating function? If so, describe the random variable it generates and find the mean and variance of this new random variable.

Solution:

Yes it is. This is the generating function of a distribution where its pmf is the average of the pmfs of X_1 and X_2. As such, let Y be the random variable associated with this distribution. Hence,

$$P(Y=m) = \frac{P(X_1=m)+P(X_2=m)}{2} = \begin{cases} \frac{1}{2} & k=3 \\ \frac{1}{2} & k=1 \end{cases}$$

You can see this by the fact that the probability generating function is just the Z-Transform on the pmfs and that the Z-transform is linear.

Again, we can re-use previous parts to get the mean and variance.

$$\begin{aligned} g^{'}(z) &= \frac{G^{'}_{X_1}(z)+G^{'}_{X_2}(z)}{2} \\ \implies g^{'}(1) &= \frac{G^{'}_{X_1}(1)+G^{'}_{X_2}(1)}{2} \\ &= \frac{2p+1+2(1-p)+1}{2} = 2 = \mathbb{E}[Y] \end{aligned}$$

$$\begin{aligned} g^{''}(z) &= \frac{G^{''}_{X_1}(z)+G^{''}_{X_2}(z)}{2} \\ \implies g^{''}(1) &= \frac{G^{''}_{X_1}(1)+G^{''}_{X_2}(1)}{2} \\ &= \frac{6p+6(1-p)}{2} = 3. \end{aligned}$$

$$
\begin{aligned}
Var(Y) &= g^{''}(1) + g^{'}(1) - (g^{'}(1))^2 \\
&= 3 + 2 - (2)^2 \\
&= 1
\end{aligned}
$$

3. *Moment-Generating Function of Normal R.V.*

   (a) Let X be a Gaussian distribution with mean m and variance σ^2. Find $M_x(s) = \mathbb{E}[e^{sX}]$. Hint: Let $k = m + s\sigma^2$, then $sx - \frac{(x-m)^2}{2\sigma^2} = \frac{-(x-k)^2+2ms\sigma^2+s^2\sigma^4}{2\sigma^2}$.

   Solution:

   The characteristic function of a random variable X is defined as following:

$$
M_X(s) = \mathbb{E}[e^{sX}] = \int_{-\infty}^{\infty} f_X(x)e^{sx}dx
$$

   Thus, the characteristic function of a normal distribution with mean m and variance σ^2 is:

$$
\begin{aligned}
M_x(s) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{sx-\frac{(x-m)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-(x-k)^2+2ms\sigma^2+s^2\sigma^4}{2\sigma^2}} dx \quad \text{(from hint)} \\
&= e^{\frac{2ms\sigma^2+s^2\sigma^4}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-k)^2}{2\sigma^2}} dx \\
&= e^{ms+\frac{1}{2}s^2\sigma^2} \quad \text{(Integrating Gaussian pdf with mean } k \text{ and variance } \sigma^2\text{)}
\end{aligned}
$$

   (b) Confirm the mean and variance by applying the moment theorem to $M_x(s)$.

   Solution:

$$
\begin{aligned}
E[X] &= \frac{d}{ds}(M_X(s))\Big|_{s=0} = \frac{d}{ds}\left(e^{ms+\frac{1}{2}s^2\sigma^2}\right)\Big|_{s=0} \\
&= e^{ms+\sigma^2\frac{s^2}{2}} \times \left(m + \frac{2\sigma^2 s}{2}\right)\Big|_{s=0} \\
&= \left[e^{ms+\sigma^2\frac{s^2}{2}}\left(m + \sigma^2 s\right)\right]\Big|_{s=0} \\
&= (1)(m - 0) \\
&= m
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \frac{d^2}{ds^2}(M_X(s))\Big|_{s=0} \\
&= \left[\frac{d}{ds}\left[e^{ms+\sigma^2\frac{s^2}{2}} \times (m+\sigma^2 s)\right]\right]\Bigg|_{s=0} \\
&= \left[e^{ms+\sigma^2\frac{s^2}{2}}\left[(m+\sigma^2 s)^2+\sigma^2\right]\right]\Big|_{s=0} \\
&= m^2+\sigma^2 \\
Var[X] &= E[X^2]-E[X]^2 = \sigma^2
\end{aligned}
$$

4. *Chernoff Bound*

In class you learned that the Chernoff bound guarantees

$$P(X \geq a) \leq e^{-sa}\mathbb{E}[e^{sX}]$$

for all a and every $s \geq 0$.

Now, show that

$$P(X \leq a) \leq e^{-sa}\mathbb{E}[e^{sX}]$$

for all a and every $s \leq 0$.

Solution:

First, we note that the following equality holds for all $s \leq 0$:

$$P(X \leq a) = P(e^{sX} \geq e^{sa})$$

By applying the Markov Inequality, we get

$$
\begin{aligned}
P(X \leq a) &= P(e^{sX} \geq e^{sa}) \\
&\leq \frac{\mathbb{E}[e^{sX}]}{e^{sa}}
\end{aligned}
$$