EE 131A
Probability and Statistics
Instructor: Lara Dolecek

Winter 2021 Assessment 2 solutions
February 22, 2021
Maximum score: 100 points

You have 2 hours to submit your work **directly on Gradescope under the Exam_2 submission link**.

Please read and carefully follow all the instructions.

Instructions

- The exam is accessible from 10 am PST on February 22nd to 10 am PST on February 23rd. Once you open the exam, you will have 2 hours to upload your work (therefore open the exam at least 2 hours before the closing time).
- This exam is open book, open notes. You are allowed to consult your own class notes (homework, discussion, lecture notes, textbook). You are not allowed to consult with each other or solicit external sources for help (e.g., an online forum).
- For each question, start a new sheet of paper. Therefore, the number of pages of your scan should be at least the number of questions. It is ok to write multiple parts of a question on one sheet. Properly erase or cross out any scratch work that is not part of the answer.
- Please submit your exam through the corresponding submission link on Gradescope.
- Make sure to include your **full name** and **UID** in your submitted file.
- Make sure to **show all your work**. Unjustified answers will be at a risk of losing points.
- **Policy on the Academic Integrity**
  During this exam, you are **disallowed** to contact with a fellow student or with anyone outside the class who can offer a solution e.g., web forum.
  Please write the following statement on the first page of your answer sheet.
  You will **lose 10 points** if we can not find this statement.

I $\underline{\quad YourName \quad}$ with UID ____ have read and understood the policy on academic integrity.

1. (15 pts) Suppose X is a standard Gaussian RV. Express $P(-1 < X < 3)$ in terms of the Q-functions.
   Solution:

$$\begin{aligned} P(-1 < X < 3) &= P(X > -1) - P(X \geq 3) \\ &= Q(-1) - Q(3) \\ &= 1 - Q(1) - Q(3) \end{aligned}$$

2. (15 pts) Suppose an RV X has the following pdf:

$$f(x) = \begin{cases} Ce^{-x}(1+x) & x \geq 0 \\ 0 & else \end{cases}$$

   (a) Find the constant C.
   Solution:

   Note that e^{-x} is the pdf for an exponential RV with parameter 1 resulting in mean 1 and variance 1.
   Using $\int_{-\infty}^{\infty} f(x)dx = 1$ and the fact that $\int_{-\infty}^{\infty} xf(x)dx = \mathbb{E}[X]$, we get

$$\begin{aligned} 1 &= \int_0^\infty Ce^{-x}(1+x)dx \\ &= \int_0^\infty Ce^{-x}dx + \int_0^\infty Cxe^{-x}dx \\ &= C + C \\ \implies C &= \frac{1}{2} \end{aligned}$$

   (b) Compute the expectation of X.
   Solution:
   Again, we note that $\int_{-\infty}^{\infty} x^2 f(x)dx = \mathbb{E}[X^2]$. Thus,

$$\begin{aligned} \mathbf{E}[X] &= \frac{1}{2}\int_0^\infty xe^{-x}(1+x)dx \\ &= \frac{1}{2}\left(\int_0^\infty xe^{-x}dx + \int_0^\infty x^2e^{-x}dx\right) \\ &= \frac{1}{2}(1+2) \\ &= \frac{3}{2} \end{aligned}$$

3. (15 pts) Consider a Geometric RV X with parameter p that counts the number of trials till the first success in a series of Bernoulli(p) trials including the final trial. Compute

the characteristic function of X and find its second moment (i.e. $\mathbf{E}[X^2]$) using the characteristic function.

Solution:
Note that this is the Geometric RV with sample space $\{1, 2, 3, 4, \dots\}$. Thus,

$$\begin{aligned}
\Phi_X(\omega) &= \mathbf{E}(e^{j\omega X}) \\
&= \sum_{k=1}^{\infty} e^{j\omega k}(1-p)^{k-1}p \\
&= pe^{j\omega} \sum_{k=1}^{\infty} \left(e^{j\omega}(1-p)\right)^{k-1} \\
&= \frac{pe^{j\omega}}{1-(e^{j\omega}(1-p))} \\
&= \frac{p}{e^{-j\omega}-(1-p)}
\end{aligned}$$

$$\frac{d}{d\omega}\Phi_X(\omega) = \frac{jpe^{-j\omega}}{\left(e^{-j\omega}-(1-p)\right)^2}$$

$$\frac{d^2}{d\omega^2}\Phi_X(\omega) = p\left(\frac{2j^2e^{-2j\omega}}{\left(e^{-j\omega}-(1-p)\right)^3} - \frac{j^2e^{-j\omega}}{\left(e^{-j\omega}-(1-p)\right)^2}\right)$$

$$\begin{aligned}
\mathbf{E}[X^2] &= \frac{1}{j^2}\frac{d^2}{d\omega^2}\Phi_X(\omega)|_{\omega=0} \\
&= \frac{1}{j^2}\left(\frac{2j^2}{p^2} - \frac{j^2}{p}\right) \\
&= \frac{2}{p^2} - \frac{1}{p} = \frac{2-p}{p^2}
\end{aligned}$$

This is the result that we expect since $\mathbb{E}[X^2] = Var(X) + \mathbb{E}[X]^2 = \frac{1-p}{p^2} + \frac{1}{p^2} = \frac{2-p}{p^2}$.

4. (20 pts) Suppose X is a Gaussian RV with mean 0 and variance 4. Let RV Y be defined as follows:

$$Y = \begin{cases} X, & \text{if } X \geq 0, \\ X^2, & \text{if } X < 0. \end{cases}$$

Compute the pdf of Y.
Solution:

For $y \geq 0$,

$$\begin{aligned} F_y(y) &= P(Y \leq y) \\ &= P(Y \leq y, X \geq 0) + P(Y \leq y, X < 0) \\ &= P(X \leq y, X \geq 0) + P(X^2 \leq y, X < 0) \\ &= P(0 \leq X \leq y) + P(-\sqrt{y} \leq X < 0) \\ &= P(-\sqrt{y} \leq X \leq y) \\ &= \Phi(\frac{y}{2}) - \Phi(\frac{-\sqrt{y}}{2}) \end{aligned}$$

Thus for $y \geq 0$,

$$\begin{aligned} f_Y(y) &= \frac{1}{2}\Phi'(\frac{y}{2}) + \frac{1}{4\sqrt{y}}\Phi'(\frac{-\sqrt{y}}{2}) \\ &= \frac{1}{2\sqrt{2\pi}}e^{\frac{-y^2}{8}} + \frac{1}{8\pi\sqrt{y}}e^{\frac{-y}{8}} \end{aligned}$$

Where $\Phi(.)$ is the Standard normal CDF function. $f_Y(y) = 0$ for $y < 0$.

5. (15 pts) Suppose X is a uniform RV distributed on the interval $[-a, a]$ for some positive a.

   (a) Compute $P(X > c)$ exactly for all values of $c \in \mathbb{R}$.
   Solution:

$$P(X > c) = \begin{cases} 1 - \frac{a+c}{2a} = \frac{a-c}{2a} & -a \leq c \leq a \\ 0 & c \geq a \\ 1 & c \leq -a \end{cases}$$

   (b) Assume c is a constant such that $0 \leq c \leq a$. Bound the probability $P(X > c)$ using the Chebyshev inequality.
   Solution:
   Now, we bound it using the Chebyshev inequality. First, note that $Var(X) = \frac{a^2}{3}$. Thus,

$$\begin{aligned} P(X > c) &\leq P(|X| > c) \\ &= P(|X - \mathbf{E}[X]| > c) \\ &\leq \frac{\mathbf{VAR}(X)}{c^2} \\ &= \frac{a^2}{3c^2} \end{aligned}$$

   where the second line comes from the fact that the mean of X is zero.

Note there there is a better bound since $P(X > c) = P(X < -c)$ due to the distribution being symmetric about zero. As such,

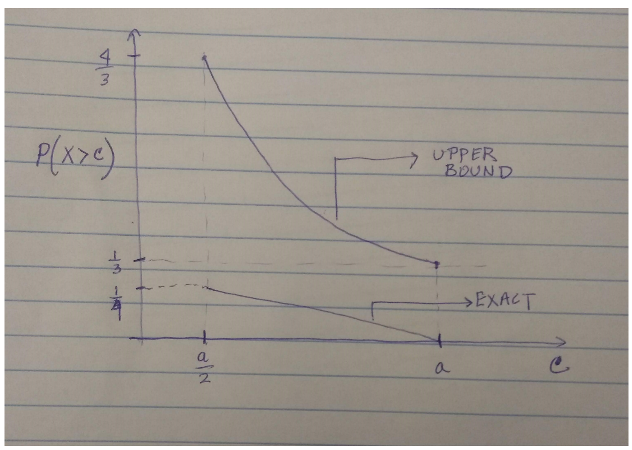
$$P(|X| > c) = P(X > c) + P(X < -c) = 2P(X > c)$$
$$\implies P(X > c) \leq \frac{a^2}{6c^2}$$

You will get full points even if you got the weaker bound. The following sketch will use the weaker bound.

(c) Hand sketch the plots of both the exact probability and the Chebyshev upper bound as a function of c for the range $\frac{a}{2} \leq c \leq a$. Thoroughly label your graphs.

Solution:

Hand sketched plot is shown below



6. (20 pts) Let X be an exponential RV with parameter λ. Let $Y = X + 1$. RV Y is known as a Shifted Exponential.

   (a) Compute the mean and variance of Y.
   Solution:

$$\begin{aligned}\mathbf{E}[Y] &= \mathbf{E}[X+1] \\ &= \mathbf{E}[X] + 1 \\ &= \frac{1}{\lambda} + 1 \\ \mathbf{VAR}(Y) &= \mathbf{VAR}(X+1) \\ &= \mathbf{VAR}(X) = \frac{1}{\lambda^2}\end{aligned}$$

   (b) Find the CDF of Y, i.e. $F_Y(y)$.
   Solution:

$$F_Y(y) = P(Y \leq y) = \begin{cases} 0 & y < 1 \\ 1 - e^{-\lambda(y-1)} & y \geq 1 \end{cases} \tag{1}$$

   (c) Find the inverse CDF of Y, i.e $F_Y^{-1}(z)$ for $0 \leq z \leq 1$.
   Solution:

$$F_Y^{-1}(z) = -\frac{\ln(1-z)}{\lambda} + 1 \tag{2}$$

   valid for only $0 \leq z \leq 1$.

   (d) Suppose we wish to generate samples from a Shifted Exponential using the method for generating random variables covered in the class. Using the result from part c, specify the steps of the procedure.
   Solution:
   Steps:

   i. Generate a uniform random variable U with the range $[0, 1]$.
   ii. Apply the function $F_Y^{-1}(a)$ to U to get $Y = F_Y^{-1}(U)$.

   From these steps, Y has the distribution of a shifted exponential.