I, Nevin Liang, with UID 705575353 have read and understood the policy on Academic Integrity

1.

Q function: $1 - \Phi(a) = Q(a) = P(X > a)$

$$P(-1 < X < 3) = P(X > -1) - P(X > 3)$$

$$= \boxed{Q(-1) - Q(3)}$$

Note: technically $P(X \geq 3)$ but continuous so $P(X=3) = 0$.

2.

a) $$f(x) = \begin{cases} Ce^{-x}(1+x) & x \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^{\infty} C\cdot e^{-x}(1+x)\,dx = 1$$

$$C\left[\int_0^{\infty} e^{-x}(1+x)\,dx\right] = 1 \Rightarrow \lim_{t\to\infty} \int_0^t e^{-x}(1+x)\,dx = \frac{1}{C}$$

$$\lim_{t\to\infty}\left[-e^{-x}(1+x) - \int -e^{-x}\,dx\right]\Bigg|_0^t = \lim_{t\to\infty}\left(-e^{-x}(1+x) - e^{-x}\Big|_0^t\right)$$

Integration by parts

$$= \lim_{t\to\infty}\left(-e^{-x}(x+2)\Big|_0^t\right) = \lim_{t\to\infty}\left(-\frac{t+2}{e^t} + \frac{2}{1}\right)$$

$$= 2 - \lim_{t\to\infty}\frac{t+2}{e^t} = 2 - \lim_{t\to\infty}\frac{1}{e^t} = \boxed{2}$$

l'Hopital's.

$$2 = \frac{1}{C} \Rightarrow \boxed{C = \frac{1}{2}}$$

b) $$E[X] = \int_0^{\infty} x\cdot f(x)\,dx = \int_0^{\infty} x\cdot Ce^{-x}(1+x)\,dx = \frac{1}{2}\int_0^{\infty} x\cdot e^{-x}(1+x)\,dx$$

$$= \frac{1}{2}\cdot\lim_{t\to\infty}\int_0^t e^{-x}(x+x^2)\,dx \longrightarrow$$

Integrate by parts again

$\frac{1}{2}$ a. $\lim_{t \to \infty} \int_0^t e^{-x}(x+x^2)\,dx$ .... let's ignore limit and bounds for now...

$$\int e^{-x}(x+x^2)\,dx = -(x+x^2)e^{-x} - \int -e^{-x}(2x+1)\,dx$$

$$= -(x+x^2)e^{-x} + \int e^{-x}(2x+1)\,dx$$

integrate by parts again

$$= -(x+x^2)e^{-x} - (2x+1)e^{-x} - \int -e^{-x}(2)\,dx$$

$$= -(x+x^2+2x+1)e^{-x} + \int e^{-x} \cdot 2\,dx$$

$$= -(x^2+3x+1)e^{-x} + 2 \cdot (-e^{-x})$$

$$= -(x^2+3x+3)e^{-x}$$

$\frac{1}{2}$ a. $\lim_{t \to \infty} \left(-(t^2+3t+3) \cdot e^{-t} + (3) \right) = \left[3 - \lim_{t \to \infty} \left(\frac{t^2+3t+3}{e^t} \right) \right] \cdot \frac{1}{2}$

double l'Hopital's

$$= [3-0] \cdot \frac{1}{2} = \cancel{\boxed{6/1}} = \boxed{3/2}$$

3.

Discrete: $\Phi_x(w) = \sum_{x_i} e^{jwx_i} \cdot P(X=x_i)$

$$= e^{jw\cdot 1} \cdot P(X=1) + e^{jw\cdot 2} \cdot P(X=2) \dots$$

$$= e^{jw} \cdot p + e^{jw\cdot 2} \cdot (1-p)\cdot p \dots$$

$$= \boxed{\frac{e^{jw}\cdot p}{1-e^{jw}\cdot(1-p)}}$$ char func.

2nd deriv wrt w:

$$\frac{\partial}{\partial w}\Bigg(\underbrace{\frac{\partial}{\partial w}\left(\frac{p}{e^{-jw}-(1-p)}\right)}\Bigg)$$

$$\downarrow$$

$$\frac{d}{dw}\left[p\left(e^{-jw}-(1-p)\right)^{-1}\right]$$

$$= -p\left(e^{-jw}-(1-p)\right)^{-2}\cdot\left(-j\cdot e^{-jw}\right)$$

$$= \frac{pj\left(e^{-jw}\right)}{\left(e^{-jw}-(1-p)\right)^2} = \frac{pj\cdot e^{jw}}{\left(1-e^{jw}(1-p)\right)^2}$$

$$\frac{\partial}{\partial w}\left(\frac{pj\cdot e^{jw}}{\left(1-e^{jw}(1-p)\right)^2}\right) = pj\cdot \cancel{je^{jw}\left(1-e^{jw}(1-p)\right)^2 - e^{jw}}$$

$$= pj\cdot\frac{je^{jw}\left(1-e^{jw}(1-p)\right)^2 - e^{jw}\left(2\left(1-e^{jw}(1-p)\right)\right)\cdot\left(-(1-p)\cdot je^{jw}\right)}{\left(1-e^{jw}(1-p)\right)^4}$$

when $w=0$, $pj\cdot\dfrac{j\left(1-(1-p)\right)^2-\left(2\cdot(1-1+p)\right)\cdot\left(-(1-p)\cdot j\right)}{\left(1-(1-p)\right)^4}$

$$= \frac{pj\cdot\left(jp^2 + 2p\cdot(1-p)j\right)}{p^4} = \frac{-p^3 - 2p^2(1-p)}{p^4}$$

$$= \frac{-2p^2+p^3}{p^4} = \cancel{\boxed{\frac{p-2}{p^2}}} \quad \frac{p-2}{p^2}.$$

HOWEVER

$$E[X^2] = \frac{1}{j^2}\cdot\frac{\partial^2 \Phi_x(w)}{dw^2} = -1\cdot\frac{p-2}{p^2}$$

$$= \boxed{\frac{2-p}{p^2}}.$$

4.

$$Y = \begin{cases} X & X \geq 0 \\ X^2 & X < 0 \end{cases}$$

$$f_{Y}(y) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

b/c $\sigma^2 = 4$

if $X \geq 0$: $Y = X \Rightarrow f_Y(y) = \frac{1}{4\sqrt{2\pi}} \cdot e^{-y^2/\cancel{2\sigma^2}\, 8}$

$$\frac{1}{4\sqrt{2\pi}} \cdot e^{-y^2/8}$$

if $X < 0$: $Y = X^2 \Rightarrow$ chi-squared with $k = 1$:

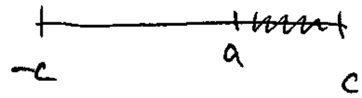
$$f_Y(y) = \frac{y^{-1/2} \cdot e^{-y/2}}{2^{1/2} \cdot \Gamma(1/2)}$$ chi-square

$$= \frac{e^{-y/2}}{\sqrt{y} \cdot \sqrt{2} \cdot \sqrt{\pi}} = \boxed{\frac{e^{-y/2}}{\sqrt{2\pi y}}}$$

$$\text{PDF} = \begin{cases} \frac{1}{4\sqrt{2\pi}} \cdot e^{-y^2/8} + \frac{e^{-y/2}}{\sqrt{2\pi y}} & Y \geq 0 \\ 0 & Y < 0 \end{cases}$$

5.

(a) $P(X>c) = \boxed{\dfrac{a-c}{2a}} \longrightarrow \boxed{\begin{cases} 1 & c<a \\ \dfrac{a-c}{2a} & -a \le c \le a \\ 0 & c>a \end{cases}}$



(b) $P\left(|X-m| \geq a\right) \leq \dfrac{\sigma^2}{a^2}$ ... cheby

uniform RV $[-a,a]$ so $m=0$

$a=c$

$$P(|X| \geq c) \leq \frac{\sigma^2}{c^2}$$

$$VAR = \frac{(a-(-a))^2}{12} = \frac{a^2}{3}$$

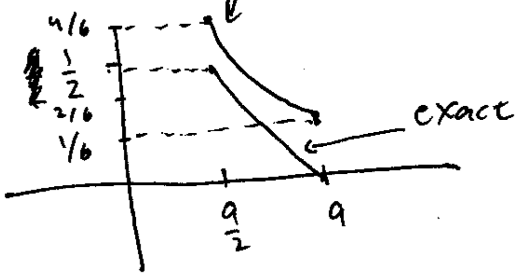
$$P(|X| \geq c) \leq \frac{a^2}{3c^2}$$

BUT we want $P(X \geq c) = P(X>c) = \frac{1}{2}P(|X|>c)$

continuous

$= \boxed{\dfrac{a^2}{6c^2}}$

c)



$$\frac{a^2}{6 \cdot \frac{a^2}{4}} = \frac{2}{3}$$

$$\frac{a^2}{6 \cdot a^2} = \frac{1}{6}$$

6.

(a) $\mathbb{E}[Y] = \mathbb{E}[X+1] = \mathbb{E}[X]+1 = \boxed{\frac{1}{\lambda}+1}$

$\mathrm{VAR}[Y] = \mathrm{VAR}[X+1] = \mathbb{E}[(X+1)^2] - \mathbb{E}[X+1]^2$

$= \mathbb{E}[X^2] + 2\cdot\mathbb{E}[X] + 1 - \mathbb{E}[X+1]^2$

$= \cancel{\frac{1}{\lambda}} + 2\cdot\frac{1}{\lambda} + 1 - \left(\frac{1}{\lambda}+1\right)^2$

$E[X^2]$

makes sense. when shifting, σ does not change

$= E[X^2] + \cancel{\frac{2}{\lambda}} + \cancel{1} - \frac{1}{\lambda^2} - \cancel{\frac{2}{\lambda}} - \cancel{1} =$

$E[X^2] - \frac{1}{\lambda^2}$

$= \frac{1}{\lambda^2} + E[X]^2 - \frac{1}{\lambda^2} = E[X]^2 = \boxed{\frac{1}{\lambda^2}}$

(b) $P(Y>y) = P(X+1>y)$

$= P(X>y-1) = e^{-\lambda(y-1)}$

$\mathrm{CDF} = 1 - P(Y>y) = 1 - e^{-\lambda(y-1)}$

$$\mathrm{CDF} = \begin{cases} 1-e^{-\lambda(y-1)} & y \geq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(c)

$$z = 1 - e^{-\lambda(y-1)}$$

$$e^{-\lambda(y-1)} = 1 - z$$

$$-\lambda(y-1) = \ln(1-z)$$

$$y - 1 = \frac{-1}{\lambda} \cdot \ln(1-z)$$

$$y = 1 - \frac{1}{\lambda} \cdot \ln(1-z)$$

we only want

$0 \le z \le 1$

$$\begin{cases} \cancel{1 \qquad z<0} \\ 1 - \frac{1}{\lambda} \cdot \ln(1-z) & 0 \le z < 1 \\ \infty & z \ge 1 \quad (z=1) \end{cases}$$

(d) Step 1: generate a pseudorandom # in range [0, 1].

2: Use the answer from (c) to get $F_Y^{-1}(z)$.

$0 \le z < 1$ case

3: the output of F_Y^{-1} will have distribution of

the shifted exponential F_Y