ECE 131A Homework 7
Probability and Statistics Wednesday, March 3, 2021
Instructor: Lara Dolecek Due: Wednesday, March 10, 2021 before class begins

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Please upload your homework to Gradescope by March 10, 3:59 pm.
Please submit a single PDF directly on Gradescope
You may type your homework or scan your handwritten version. Make sure all the work is discernible.

Chapters 6.4 and 7.1-7.3 *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Consider two random variables X and Y. Prove that the correlation coefficient $\rho_{X,Y}$ satisfies $-1 \leq \rho_{X,Y} \leq 1$. **Hint:** Consider the function $\mathbb{E}[(\frac{X-\mathbb{E}[X]}{\sigma_X} \pm \frac{Y-\mathbb{E}[Y]}{\sigma_Y})^2]$.

2. Two points are picked uniformly at random in the interval $[0, L]$. What is the expected distance between these points?

3. Consider the jointly Gaussian random variables X and Y that have the following joint PDF:
$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right].$$
   (a) Prove that Y is a Gaussian random variable by deriving its marginal PDF, $f_Y(y)$. Find the mean and variance of Y.
   (b) Prove that $f_{X|Y}(x|y)$ corresponds to another Gaussian random variable, then find its mean and variance.

4. Let X and Y be jointly Gaussian random variables with $\mathbb{E}[Y] = 0$, $\sigma_X = 1$, $\sigma_Y = 2$ and $\mathbb{E}[X|Y] = \frac{Y}{4} + 1$.
   (a) Find the joint pdf of X and Y.
   (b) Find the conditional pdf of $f_{X|Y}(x|y)$.

5. Assume that $X_1, X_2, ..., X_n$ are independent random variables with possibly different distributions and let S_n be their sum. Let $m_k = E(X_k)$, $\sigma_k^2 = VAR(X_k)$, and $M_n = m_1 + m_2 + \cdots + m_n$. Assume that $\sigma_k^2 < R$ and $m_k < T$ for all k. Prove that, for any $\epsilon > 0$,
$$P(|\frac{S_n}{n} - \frac{M_n}{n}| < \epsilon) \to 1$$
as $n \to \infty$.

6. *Application of CLT.*

(a) A fair coin is tossed 100 times. Estimate the probability that the number of heads is between 40 and 60. Estimate the probability that the number is between 50 and 55.

(b) Repeat part (a) for if we toss the coin 1000 times and for the intervals [400,600] and [500,550].

(c) Suppose that 20% of voters are in favor of certain legislation. A large number n of voters are polled and a relative frequency estimate $f_A(n)$ for the above proportion is obtained. Use central limit theorem to estimate how many voters should be polled in order that the probability is at least .95 that $f_A(n)$ differs from 0.20 by less than 0.02.