EE 131A
Probability
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Discussion Set 8 solutions
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Chapter 5.6-5.10 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *The Joint pdf of Two Continuous Random Variables.* Problem 5.26, page 291 of ALG. Let X and Y have joint pdf:

$$f_{X,Y}(x,y) = k(x+y) \quad \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1.$$

   (a) Find k.
   Solution:
   The constant k is found from the normalization condition specified by the rule of integration must equal to 1.

$$\int_{y=0}^{1}\int_{x=0}^{1} f_{X,Y}(x,y)dxdy = 1$$

$$\int_{y=0}^{1}\int_{x=0}^{1} k(x+y)dxdy = 1$$

$$\int_{y=0}^{1} k\left[\frac{x^2}{2}+xy\right]_{x=0}^{x=1} dy = 1$$

$$\int_{y=0}^{1} k\left[\frac{1}{2}+y\right] dy = 1$$

$$k = 1$$

   (b) Find the joint cdf of (X,Y).
   Solution:
   The joint cdf of (X,Y) is,

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_{y=0}^{y}\int_{x=0}^{x} (x+y)dxdy \\
&= \int_{y=0}^{y}\left(\left[\frac{x^2}{2}+xy\right]_{x=0}^{x=x}\right) dy \\
&= \int_{y=0}^{y}\left(\frac{x^2}{2}+xy\right) dy \\
&= \left[\frac{x^2y}{2}+\frac{xy^2}{2}\right]_{y=0}^{y=y} \\
&= \frac{xy(x+y)}{2} \quad \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1.
\end{aligned}$$

(c) Find the marginal pdf of X and of Y.
Solution:
The marginal density of X is,

$$\begin{aligned} f_X(x) &= \int_{y=0}^{1} f_{X,Y}(x,y)dy \\ &= \int_{y=0}^{1} (x+y)dy \\ &= \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=1} \\ &= x + \frac{1}{2} \end{aligned}$$

Therefore, the marginal density of X is $f_X(x) = x + \frac{1}{2}$ for $0 \le x \le 1$.
Similarly, the marginal density of Y is $f_Y(y) = y + \frac{1}{2}$ for $0 \le y \le 1$.

2. *Conditional Probability and Conditional Expectation.* Problem 5.80, page 296 of ALG.

(a) Find $f_Y(y|x)$ in Problem 1.
Solution:

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(x+y)}{x+\frac{1}{2}} = \frac{2(x+y)}{2x+1}$$

(b) Find $P[Y > X|x]$.
Solution:

$$\begin{aligned} P[Y > X|x] &= \frac{P[Y > X, X = x]}{f_X(x)} \\ &= \frac{P[Y > x]}{f_X(x)} \\ &= \frac{\int_x^1 f_{X,Y}(x,y)dy}{x+\frac{1}{2}} \\ &= \frac{\int_x^1 (x+y)dy}{x+\frac{1}{2}} \\ &= \frac{\left(xy + \frac{y^2}{2}\right)_x^1}{x+\frac{1}{2}} \\ &= \frac{x + \frac{1}{2} - \frac{3}{2}x^2}{x+\frac{1}{2}} \\ &= \frac{2x+1-3x^2}{2x+1} \end{aligned}$$

(c) Find $P[Y > X]$ using part b.
Solution:

$$\begin{aligned}
P[Y > X] &= \int_0^1 P[Y > X|x] f_X(x) dx \\
&= \int_0^1 \frac{2x+1-3x^2}{2x+1}\left(x+\frac{1}{2}\right) dx \\
&= \frac{1}{2}\int_0^1 (2x+1--3x^2) dx \\
&= \frac{1}{2}(x^2+x-x^3)_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

(d) Find $E[Y|X = x]$.
Solution:

$$\begin{aligned}
E[Y|X = x] &= \int_0^1 y f_Y(y|x) dy \\
&= \int_0^1 y\frac{2(x+y)}{2x+1} dy \\
&= \frac{2}{2x+1}\int_0^1 (xy+y^2) dy \\
&= \frac{2}{2x+1}\left(x\frac{y^2}{2}+\frac{y^3}{3}\right)_0^1 \\
&= \frac{2}{2x+1}\left(\frac{x}{2}+\frac{1}{3}\right) \\
&= \frac{3x+2}{3(2x+1)}
\end{aligned}$$

3. *(Problem 5.58 of ALG)*
Find $E[X^2e^Y]$ where X and Y are independent random variables, X is a zero-mean, unit-variance Gaussian random variable, and Y is a uniform random variable in the interval $[0, 3]$.
Solution:
Since X and Y are independent random variables, we have:

$$\begin{aligned}
E[X^2e^Y] &= E[X^2]E[e^Y] \\
&= [VAR[X] + E[X]^2)]E[e^Y] \\
&= [1 + 0]E[e^Y] \\
&= \int_0^3 \frac{1}{3}e^y \mathrm{d}y \\
&= \frac{e^3-1}{3}
\end{aligned}$$

4. For two random variables X and Y,

   (a) Express $E[(X+Y)^2]$ in terms of means, variances, and covariances for X,Y.
   Solution:
   $E[(X+Y)^2] = E[X^2] + E[Y^2] + 2E[XY]$. Besides, $E[X^2] = VAR[X] + E[X]^2$, and $E[XY] = COV[X,Y] + E[X]E[Y]$. So, the result is

   $$E[(X+Y)^2] = E[X]^2 + VAR[X] + E[Y]^2 + VAR[Y] + 2COV[X,Y] + 2E[X]E[Y].$$

   (b) Find the variance of $X+Y$ in terms of means, variances, and covariances of X,Y.
   Solution:
   $VAR[X+Y] = E[((X+Y) - E[X+Y])^2] = E[(X+Y)^2 - 2(X+Y)E[X+Y] + E[X+Y]^2) = E[(X+Y)^2] - E[X+Y]^2$. Now, $E[X+Y]^2 = (E[X]+E[Y])^2$, so we can write this as

   $$\begin{aligned}
   &E[X^2] + E[Y^2] + 2E[XY] - (E[X]+E[Y])^2 \\
   &= E[X^2] + E[Y^2] + 2E[XY] - E[X]^2 - E[Y]^2 - 2E[X]E[Y] \\
   &= VAR[X] + VAR[Y] - 2E[XY] - 2E[X]E[Y] \\
   &= VAR[X] + VAR[Y] + 2COV[X,Y]
   \end{aligned}$$

   (c) When is the variance of the sum of RVs the same as the sum of the individual variances of each RV?
   Solution:
   For the variance of the sum of RVs to be the same as the sum of the individual variances of each RV, you need $COV[X,Y] = 0$. This is true when X and Y are uncorrelated.

5. *Sum of uniform RVs.* Suppose X and Y are independent uniform random variables in the interval $[0,1]$, and $Z = X+Y$. Find $f_Z(z)$.
   Solution:
   One solution to this problem is to try to reach the CDF then get the PDF.

   $$F_Z(z) = P[Z \le z] = P[X+Y \le z] = P[Y \le z - X]$$

   It is clear that $F_Z(z) = 0$ if $z < 0$ or if $z > 2$. The region where $0 \le z \le 2$ can be divided into two regions as follows:

   Region 1: $0 \le z \le 1$. Figure 1 shows the area we need to integrate upon (The red colored lines indicate the required area that we need to integrate. The blue lines indicate vertical lines with constant x coordinates: these lines go from $x = 0$ to $x = z$ and for each x, the value of y ranges from 0 to $z - x$): Thus, we can get the CDF as

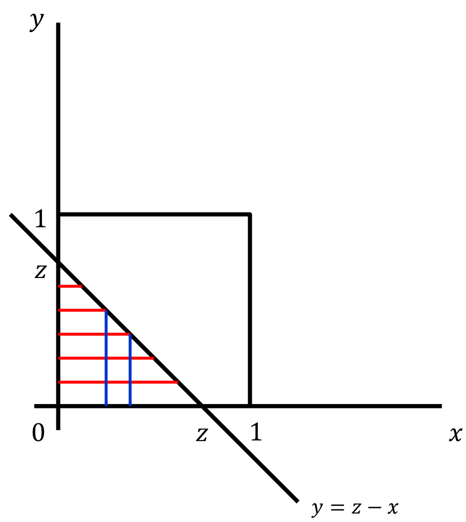

Figure 1: Area of integration for $0 \leq z \leq 1$.

follows:

$$\begin{aligned} F_Z(z) &= \int_{x=0}^{z} \int_{y=0}^{z-x} (1)(1) dy dx \\ &= \int_{x=0}^{z} [y]_0^{z-x} \, dx = \int_{x=0}^{z} (z-x) dx \\ &= \left[zx - \frac{x^2}{2} \right]_{x=0}^{z} = z^2 - \frac{z^2}{2} = \frac{z^2}{2} \end{aligned}$$

Therefore,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = z, \; 0 \leq z \leq 1$$

Region 2: $1 < z \leq 2$. Figure 2 shows the area we need to integrate upon (The red colored lines indicate the required area. Here, instead of integrating the red region, we integrate the remaining portion of the square and subtract the probability from 1. The blue lines indicate vertical lines with constant x coordinates: these lines go from $x = z - 1$ to $x = 1$ and for each x, the value of y ranges from $z - x$ to 1):

Thus, we can get the CDF as follows:

$$\begin{aligned} F_Z(z) &= 1 - \int_{x=z-1}^{1} \int_{y=z-x}^{1} (1)(1) dy dx \\ &= 1 - \int_{x=z-1}^{1} [y]_{z-x}^{1} \, dx = \int_{x=z-1}^{1} (1 - z + x) dx \\ &= 1 - \left[(1-z)x - \frac{x^2}{2} \right]_{x=z-1}^{1} = -1 + 2z - \frac{z^2}{2} \end{aligned}$$

Therefore,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = 2 - z, \; 1 < z \leq 2$$

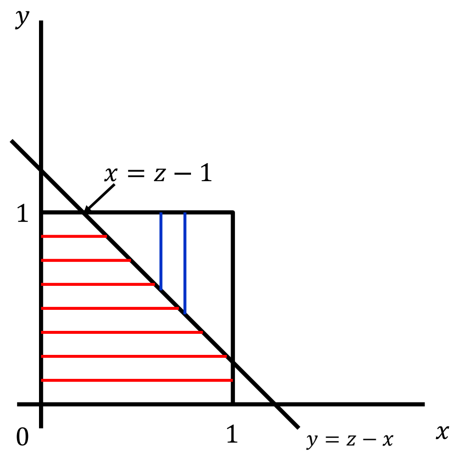

Figure 2: Area of integration for $1 < z \leq 2$.

Overall, the PDF of Z is:

$$f_Z(z) = \begin{cases} z, \ 0 \leq z \leq 1 \\ 2 - z, \ 1 < z \leq 2 \\ 0, \ otherwise \end{cases}$$

The PDF $f_Z(z)$ is also plotted in Figure 3.

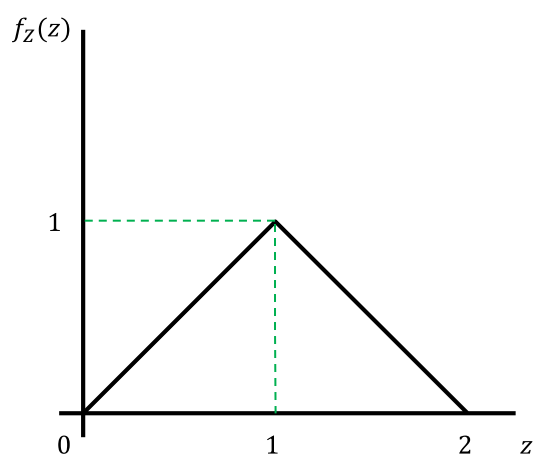

Figure 3: The PDF in Problem 5.

An easier solution is to do direct convolution between the two PDFs $f_X(x)$ and $f_Y(y)$ because they are independent. The result of convoluting two squares from 0 to 1 is the same triangle in Figure 3.

6. Suppose X and Y are independent exponential random variables with common parameter λ, and $Z = \frac{X}{X+Y}$. Find the PDF $f_Z(z)$.
Solution:
We have $Z = \frac{X}{X+Y}$, $x \geq 0$, and $y \geq 0$. Thus, the range of z (the values Z can take) is

between 0 and 1.
We start by getting the CDF $F_Z(z)$.

$$F_Z(z) = P[Z \leq z] = P\left[\frac{X}{X+Y} \leq z\right] = P\left[Y \geq \frac{1-z}{z}X\right].$$

The area we need to integrate upon is shown in Figure 4. The red region is the required area. The blue lines represent vertical lines with constant x coordinates: these lines go from $x = 0$ to $x = \infty$, and for each x, the value of y ranges from $\frac{1-z}{z}x$ to ∞.

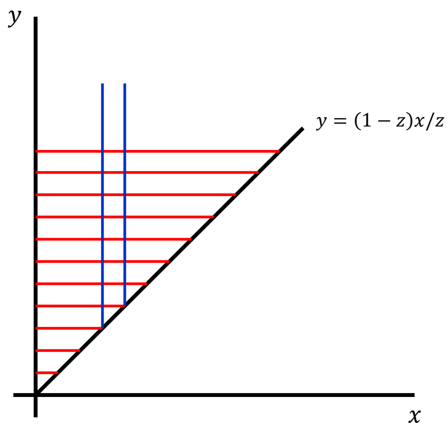

Figure 4: Area of integration for the CDF of $X/(X+Y)$.

Thus, the CDF can be derived as follows:

$$\begin{aligned} F_Z(z) &= \int_{x=0}^{\infty}\int_{y=\frac{1-z}{z}x}^{\infty} f_{X,Y}(x,y)dydx = \int_{x=0}^{\infty}\int_{y=\frac{1-z}{z}x}^{\infty} f_X(x)f_Y(y)dydx \\ &= \int_{x=0}^{\infty}\int_{y=\frac{1-z}{z}x}^{\infty} \lambda^2 e^{-\lambda x}e^{-\lambda y}dydx \\ &= \int_{x=0}^{\infty} \lambda^2 e^{-\lambda x}\int_{y=\frac{1-z}{z}x}^{\infty} e^{-\lambda y}dydx \\ &= \int_{x=0}^{\infty} \lambda^2 e^{-\lambda x}\left[\frac{e^{-\lambda y}}{\lambda}\right]_{y=\frac{1-z}{z}x}^{\infty}. \end{aligned}$$

Thus, we can see that:

$$\begin{aligned} F_Z(z) &= \lambda\int_{x=0}^{\infty} e^{-\lambda x}e^{-\lambda(\frac{1}{z}-1)x}dx \\ &= \lambda\int_{x=0}^{\infty} e^{-\frac{\lambda}{z}x}dx = \lambda\left[\frac{z}{\lambda}e^{-\frac{\lambda}{z}x}\right]_{x=0}^{\infty}. \end{aligned}$$

This means that $F_Z(z) = z$, where $0 \leq z \leq 1$.
Now, we are ready to get the PDF:

$$f_Z(z) = \frac{d}{dz}F_Z(z) = 1,\ 0 \leq z \leq 1.$$

Therefore, the random variable $Z = \frac{X}{X+Y}$ is uniform in the interval $[0, 1]$.