Discussion 9

Conditional Probability

① X is discrete Y is discrete.

Conditional PMF of Y given $X = x_k$

$$P(Y = y_j \mid X = x_k) = \frac{P(X = x_j, Y = y_j)}{P(X = x_k)}$$

$$E[Y \mid X = x_k] = \sum_j y_j \, P(Y = y_j \mid X = x_k)$$

$$\boxed{E[Y] = \sum_j y_j \, P(Y = y_j)}$$

② Y continuous X discrete.

conditional CDF

conditional PDF of Y given $X = x_k$

$$F_{Y|X}(y \mid x_k) = P(Y \leq y \mid X = x_k)$$

$$= \frac{P(Y \leq y, X = x_k)}{P(X = x_k)}$$

$$\underline{f_{Y|X}(y|x_K)} = \frac{d}{dy} F_{Y|X}(y|x_K).$$

$$E[Y|X=x_K] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x_K)\, dy.$$

(3) $\underline{\begin{matrix} Y & \text{cont.} \\ X & \text{cont.} \end{matrix}}$ $P(X=x) = 0.$

$$F_{Y|X}(y|x) = \lim_{h \to 0} P(Y \le y \mid x \le X \le x+h)$$

$$\downarrow \frac{d}{dy}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}.$$

<u>Conditional Expectation</u> $E[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy$

$$E[Y|X=x] = g(x) \checkmark$$

$$E[Y|X] = \underline{g(X)}$$

$$\boxed{E[Y|X] \text{ is a RV in } \underline{X}}$$

1 0

$E[Y|X=1]$ $E[Y|X=0]$

$$E\left[E[Y|X] \right] = E\left[g(X) \right]$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$E[Y|X=x]$

$$= \int_{-\infty}^{\infty} E[Y|X=x] f_X(x)\, dx = E[Y]$$

$$\boxed{E[Y] = \int_{-\infty}^{\infty} E[Y|X=x]\, f_X(x)\, dx}$$

$$\boxed{E\left[E[Y|X] \right] = E[Y]}$$

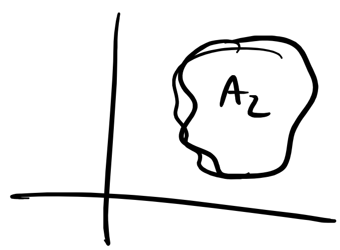
Functions of 2 RVs

(X, Y cont.)

$$Z = g(X, Y)$$

PDF of Z

Start with CDF $F_Z(z)$



$$= P(Z \leq z)$$

$$= P(\underbrace{g(X,Y) \leq z})$$

$$= P((X,Y) \in A_z)$$

$$A_z = \{ (x,y) \text{ s.t } g(x,y) \leq z \}$$

$$= \iint_{(x,y) \in A_z} f_{XY}(x,y)\, dx\, dy$$

$\frac{d}{dz}$

$f_Z(z)$.

$Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx$$

$X \perp\!\!\!\perp Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$f_Z(z) = f_X(x) * f_Y(y)$$

Jointly Gaussians

(X, Y) : if joint PDF is as follows

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-m_1)^2}{\sigma_1^2} - 2\rho\frac{(x-m_1)(y-m_2)}{\sigma_1\sigma_2} + \frac{(y-m_2)^2}{\sigma_2^2}\right\}}$$

$E[X] = m_1$ $\quad VAR(X) = \sigma_1^2$

$E[Y] = m_2$ $\quad VAR(Y) = \sigma_2^2$

$$\underbrace{\frac{Cov(X,Y)}{\sqrt{VAR(X)}\sqrt{VAR(Y)}}}_{\text{Correlation coef. bet}^n\ X \text{ and } Y} = \rho$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\, e^{\frac{-(x-m_1)^2}{2\sigma_1^2}}$$

$X \sim N(m_1, \sigma_1^2)$

$Y \sim N(m_2, \sigma_2^2)$

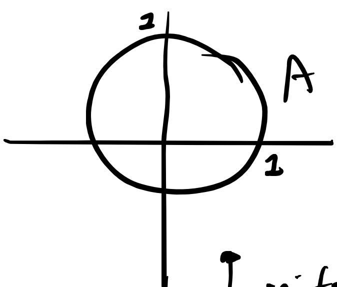
⊛ $\rho = 0$

$$\underline{f_{XY}(x,y)} = \left(\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-m_1)^2}{2\sigma_1^2}}\right) \cdot \left(\frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-m_2)^2}{2\sigma_2^2}}\right)$$

$$= \underline{f_X(x)\, f_Y(y)} \Rightarrow X \perp\!\!\!\perp Y$$

$\rho = 0$ (X uncorrelated to Y) $\Rightarrow$ $X \perp\!\!\!\perp Y$] For jointly Gaussians.

Q1.



Area $= \pi$

(X, Y) uniform in A.

$$f_{XY}(x,y) = \begin{cases} K & (x,y) \in A \\ 0 & \text{otherwise.} \end{cases}$$

a) Find K

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

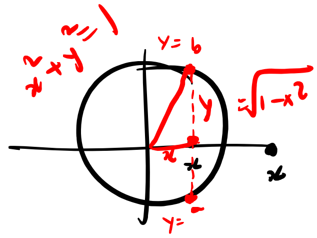
$$K \underbrace{\boxed{\iint_{(x,y)\in A} dx\,dy}}_{\text{area.}} = 1$$

$$k = \frac{1}{\text{Area of A.}} = \frac{1}{\pi}.$$

Marginal of X

b) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$\frac{1}{\pi}$



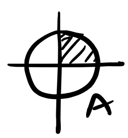
$$= \begin{cases} \int_a^b \frac{1}{\pi}\,dy & -1 \le x \le 1 \\ 0 & x > 1 \\ & x < -1 \end{cases}$$

$a = -\sqrt{1-x^2}$

$b = \sqrt{1-x^2}$

$$= \begin{cases} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy & -1 \le x \le 1 \\ 0 & x > 1 \\ & x < -1 \end{cases}$$

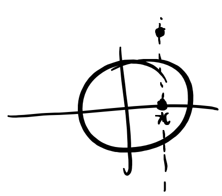
$$f_X(x). = \begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

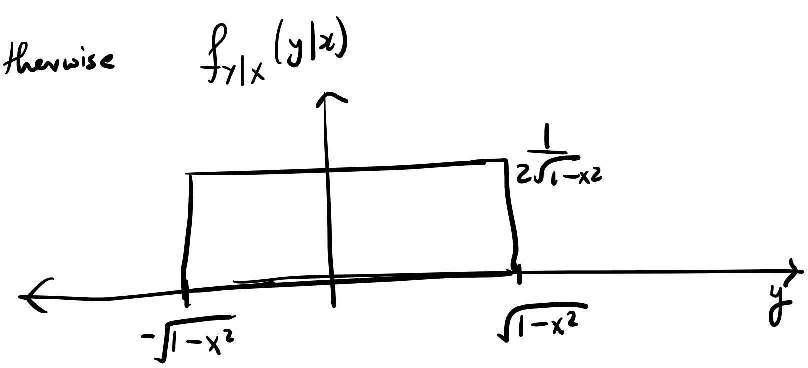


c) $P(X>0, Y>0) = \frac{1}{4}.$

d) $$f_{Y|X}(y|x). = \begin{cases} \frac{1}{2\sqrt{1-x^2}} & \left.\begin{matrix} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{matrix}\right. \\ 0 & \left.\begin{matrix} x > 1 \\ x < -1. \end{matrix}\right. \text{ or } \underbrace{\begin{matrix} -1 \le x \le 1 \\ y > \sqrt{1-x^2} \\ y < -\sqrt{1-x^2} \end{matrix}}\end{cases}$$

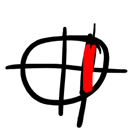
$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$





Given $X = x$

$Y \sim \text{Uniform}\left(-\sqrt{1-x^2}, \sqrt{1-x^2}\right)$ $\quad -1 < x < 1$



Uniform(a,b)

$E[X] = \frac{a+b}{2}$

$E[Y|X=x] = 0$

$E[Y] = 0$

$$\boxed{E[Y] = \int_{-\infty}^{\infty} E[Y|X=x] f_X(x)\, dx}$$

e.g $\boxed{E[Y|X=x] = \frac{1-x}{2}}$ $\longrightarrow$ $f_X(x) = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$E[Y]\,? = \int_0^1 \left(\frac{1-x}{2}\right) \cdot 2(1-x)\, dx$$

Q4. $$f_{XY}(x,y) = \frac{e^{-\frac{1}{2}\left[x^2 + 4y^2 - 3xy + 3y - 2x + 1\right]}}{2\pi c}$$

jointly gaussian.

a) $E[X]$, $E[Y]$, $VAR[X]$, $VAR[Y]$, $COV(X,Y)$

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\left(\frac{(x-m_1)^2}{\sigma_1^2} - \frac{2\rho(x-m_1)(y-m_2)}{\sigma_1\sigma_2} + \frac{(y-m_2)^2}{\sigma_2^2}\right)}$$

$c = \sigma_1\sigma_2\sqrt{1-\rho^2} = \frac{2}{\sqrt{7}}$

$$-\frac{1}{2}\left(x^2 + 4y^2 - 3xy + 3y - 2x + 1\right)$$

$$(x-\ )^2 - -(x-)(y-) + (y-\ )^2$$

$$\left(-\frac{1}{2}\right)\left((x-1)^2 - 3(x-1)y + 4y^2\right)$$

$E[X] = 1$

$E[Y] = 0$

σ_1

$$\begin{cases} \sigma_1^2 = 16/7 \\ \sigma_2^2 = 4/7 \\ \rho = 3/4. \end{cases} \qquad \begin{cases} \dfrac{1}{(1-\rho^2)\sigma_1^2} = 1 \\ \dfrac{1}{(1-\rho^2)\sigma_2^2} = 1 \\ \dfrac{\cancel{2}\rho}{(1-\rho^2)\sigma_1\sigma_2} = 3 \end{cases}$$

$N(m_2, \sigma_2^2)$

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{2\pi c} e^{-\frac{1}{2}\left((x-1)^2 - 3(x-1)y + 4y^2\right)} dx$$

$$\cancel{\theta} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(x-m)^2}{\cancel{\sigma}^2}} dx}_{=\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}f(y)}$$

$$-\frac{1}{2}\left[(x-1)^2 - 3(x-1)y + 4y^2\right]$$

$$-\frac{1}{2}\left[(x-1)^2 - 2\cdot\frac{3}{2}(x-1)\cdot y + \frac{9y^2}{4} - \frac{9y^2}{4} + 4y^2\right]$$

$$-\frac{1}{2}\left[\left(x-1-\frac{3y}{2}\right)^2 + \frac{7y^2}{4}\right]$$

$$f_Y(y) = \frac{1}{2\pi c} \cdot \underbrace{\left(\theta\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x-1-\frac{3y}{2}\right)^2} dx\right)}_{\sqrt{2\pi}}_{\sqrt{2\pi}\, N\left(1+\frac{3y}{2}, 1\right)} \cdot e^{-\frac{1}{2}\frac{7y^2}{4}}$$

$$f_Y(y) = \frac{1}{2\pi c} \cdot \sqrt{2\pi}\, e^{-\frac{1}{2}\frac{7y^2}{4}}.$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\, c} e^{-\frac{y^2}{2\left(\left(\frac{2}{\sqrt{7}}\right)\right)^2}}$$

$$= \frac{1}{\sqrt{2\pi}\, \frac{2}{\sqrt{7}}} e^{-\frac{y^2}{2\left(\left(\frac{2}{\sqrt{7}}\right)^2\right)}} \quad e^{-\frac{y^2}{2\sigma^2}}$$

$$= N\left(0, \overset{\sigma^2}{\frac{4}{7}}\right)$$

$$Y \sim N(m_2, \sigma_2^2)$$