ECE 131A | Discussion Set 6
Probability | February 12, 2021
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Chapters 4.6,4.7, and 4.9 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *Markov inequality.* Problem 4.98, parts (a), (b) and (d), page 224 of ALG
   Compare the Markov inequality and the exact probability for the event $\{X > c\}$ as a function of c for:
   (a) X is a uniform random variable in $\{1, 2, \ldots, L\}$.
   (b) X is a geometric random variable with parameter p.
   (c) X is a binomial random variable with parameters n and $p = 0.5$.

2. *Probability Generating Function*
   (a) Let X be a discrete random variable defined by the following pmf:
   $$P(X = k) = \begin{cases} p & k = 3 \\ 1 - p & k = 1. \end{cases}$$
   For simplicity, we define the notation $R(p)$ to refer to this distribution. Find the probability generating function $G_X(z)$. Using the probability generating function, find the mean and variance of X.
   (b) Consider the function $f(z) = (G_X(z))^2$. Is $f(z)$ a probability generating function? If so, describe the random variable it generates and find the mean and variance of this new random variable.
   (c) Let X_1 and X_2 be 2 random variables such that $X_1 \sim R(p)$ and $X_2 \sim R(1 - p)$. Consider the function $g(z) = \frac{G_{X_1}(z)+G_{X_2}(z)}{2}$. Is $g(z)$ a probability generating function? If so, describe the random variable it generates and find the mean and variance of this new random variable.

3. *Moment-Generating Function of Normal R.V.*
   (a) Let X be a Gaussian distribution with mean m and variance σ^2. Find $M_x(s) = \mathbb{E}[e^{sX}]$. Hint: Let $k = m + s\sigma^2$, then $sx - \frac{(x-m)^2}{2\sigma^2} = \frac{-(x-k)^2+2ms\sigma^2+s^2\sigma^4}{2\sigma^2}$.
   (b) Confirm the mean and variance by applying the moment theorem to $M_x(s)$.

4. *Chernoff Bound*
   In class you learned that the Chernoff bound guarantees
   $$P(X \geq a) \leq e^{-sa}\mathbb{E}[e^{sX}]$$

for all a and every $s \geq 0$.

Now, show that

$$P(X \leq a) \leq e^{-sa}\mathbb{E}[e^{sX}]$$

for all a and every $s \leq 0$.