Discussion 10

Lectures 12-14 Review

- Jointly defined RVs
  - Joint cdf
  - Joint pmf/pdf
  - Marginal pmf/pdf
  - conditional pmf/pdf
  - Independence of RVs

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad \forall (x,y) \in S$$

- Expectations of Multiple RVs
  - Joint Moment: $E[X^a Y^b]$
    - Covariance: $E[(X-E[X])(Y-E[Y])] = Cov(X,Y)$
    - Pearson-correlation coefficient $\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$
    - Independence $\Rightarrow$ Uncorrelated
  - conditional Expectation $E[X|Y]$
    - $E_X[X] = E_Y[E_{X|Y}[X|Y]]$
    - $E[X|Y=y] = \int x \cdot f(x|y)\, dx$ for Continuous X.

- Jointly Gaussian
  - Bivariate
  - Multivariate
  - Jointly Gaussian are defined by means covariance (i.e. first and second joint moments)
  - Marginal pdfs from the joint pdf
  - Conditional pdfs from the joint pdf

- Functions of 2 RVs

$$Z = g(X,Y)$$

$$F_Z(z) = P(g(X,Y) \le z) = \iint_{g(x,y)\le z} f(x,y)\,dx\,dy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

- Limit Theorems
  - Weak Law of Large Numbers

$$S_n = \frac{\sum_{i=1}^{n} X_i}{n} \qquad X_i \text{ are all i.i.d.} \qquad E[X_i] = \mu$$

$\forall \varepsilon > 0$

$$\lim_{n\to\infty} P(|S_n - \mu| < \varepsilon) = 1$$

equivalently,

$$\lim_{n\to\infty} P(|S_n - \mu| > \varepsilon) = 0$$

  - Central Limit Theorem

$$A_n = \sum_{i=1}^{n} X_i \qquad \sigma = \sqrt{Var(X_i)} \qquad \mu = E[X_i] \qquad X_i \text{ are iid}$$

$$Z_n = \frac{A_n - n\mu}{\sqrt{n}\,\sigma}$$

$$\lim_{n\to\infty} P(Z_n > a) = Q(a)$$

$$Q(a) = P(X > a) \text{ when } X \sim N(0,1)$$

Lectures 15 and 16 Recap

- Proof of the CLT
- Applications of CLT
- Multivariate Jointly Gaussian
  - Covariance Matrix: K

$$K_{ij} = Cov(X_i, X_j)$$

$$M = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix}$$

  - K is positive definite. $\Rightarrow$ K is invertible

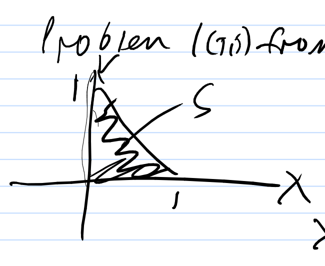
M is a matrix, x is a vector

Positive definite $x^T M x > 0 \quad \forall x$

Positive Semi-definite $x^T M x \ge 0$

- Statistics
  - Confidence intervals
  - Hypothesis Testing
  - Chi-square test

Problem 1(a,b) from Discussion 9



$f(x,y) = K \quad \forall (x,y) \in S$

$x + y = 1 \qquad y \le 1 - x$

a) $K = 2 = \frac{1}{Area}$

b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$0 \le x \le 1$

$= \int_0^{1-x} 2\, dy$

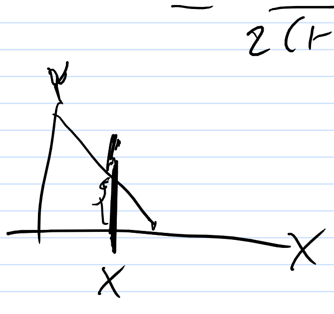
$= \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

c) $P(X>0, Y>0) = P(X \ge 0, Y \ge 0) - \cancel{P(X=0 \text{ or } Y=0)}$ (0)

$= 1$

d) $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$

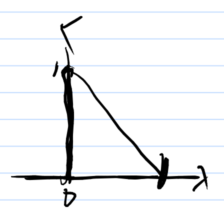
$= \frac{2}{2(1-x)} = \begin{cases} \frac{1}{1-x} & 0 \le x \le 1, \quad 0 \le y \le 1-x \\ 0 & \text{else} \end{cases}$



e) $E[Y|X=x]$

$= \int y \cdot f(y|x)\, dy$

$= \int_0^{1-x} \frac{y}{1-x}\, dy = \frac{1-x}{2}$

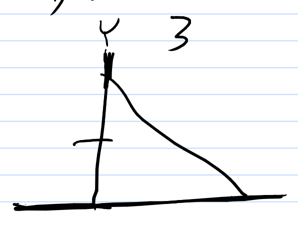


$E_Y[Y] = E_X[E_{Y|X}[Y|X]]$

$= E_X\left[\frac{1-X}{2}\right]$

$= \int_0^1 \frac{1-x}{2} f_X(x)\, dx$

$= \int_0^1 \frac{(1-x)}{2} \cdot 2(1-x) = \frac{1}{3}$



Problem 1 from Discussion 10

$\boxed{\mu = 24 \\ \sigma = 20}$ $= E[X]$, $= \sqrt{Var(X)}$

X_i = # of tickets student i bought

$T = \sum_{i=1}^{100} X_i$

100 students, 250 tickets

P(All 100 students can buy tickets)

$= P(T \le 250) \qquad P\left(\frac{T - E[T]}{\sqrt{Var(T)}}\right)$

$= P\left(\frac{T - (100)(24)}{\sqrt{100} \cdot 2} \le \frac{250 - 100 \cdot 24}{\sqrt{100} \cdot 2}\right)$

$= P(Z \le 0.5)$

$= 1 - Q(0.5)$

$\sqrt{Var(T)} = \sqrt{n\, Var(X_i)}$

$Var\left(\frac{T}{\sqrt{n}}\right) = \frac{Var(T)}{n} = \frac{\cancel{n}\, Var(X_i)}{\cancel{n}} = Var(X_i)$