Discussion 3

→ Sample Space ; Event

↳ Outcomes related to the event.

↳ $P(\text{Event})$

→ Set theory / Properties.

↳ Prove identities based on the properties.

→ Axioms of Probability

1) $P(A) \geq 0$

2) $P(S) = 1$

3) $A \cap B = \phi$

$P(A \cup B) = P(A) + P(B)$

Corollaries

e.g $P(A^c) = 1 - P(A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

↳ Prove identities based on axioms/ corollaries

e.g. $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$

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→ Sampling methods (4 types)

w w/o ordering

w w/o replacement

Imp to identify the type.

→ **Conditional Probability**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

└ Given two events A and B calculate $P(A|B)$

→ **Total Probability**

$B_1 \; B_2 \; \dots \; B_n$ (partitions)

$$P(A) = \underbrace{P(A|B_1)}\,P(B_1) + \dots + P(A|B_n)\,P(B_n)$$

P(Event)

(*) Realize that you have to use total Probability

→ **Baye's Rule**

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$$

find $P(B|A)$ but $P(A|B)$ is easy.

→ Independence

$\underline{A \perp\!\!\!\perp B}$ if $P(A \cap B) = P(A)\,P(B)$.

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

→ Random Variables

Discrete RVs X →

PMF of X

$$\underbrace{P(X = x_i) \quad \forall \; x_i \in S_X}$$

$$\sum_{x_i \in S_X} P(X = x_i) = 1$$

$$E[g(X)] = \sum_{x_i \in S_X} g(x_i)\, P(X = x_i)$$

$$E[X] = \sum_{x_i \in S_X} x_i \, P(X = x_i)$$

$E[X^2]$

$$\boxed{VAR[X] = E[X^2] - (E[X])^2}$$

Qn. PMF X calculate mean, variance.

(*) Bernoulli (p) R.V

$$X = \begin{cases} 1 & \text{w.p} \quad p. \\ 0 & \text{w.p} \quad 1-p \end{cases}$$

$$E[X] = p = \left(1 \cdot p + 0 \cdot (1-p)\right)$$

$$E[X^2] = p = \left(1^2 \cdot p + 0^2 (1-p)\right)$$

$$VAR[X] = p(1-p) \qquad \left(\underset{p}{E[X^2]} \underset{-}{-} \underset{p^2}{(E[X])^2}\right)$$

Geometric (p) R.V

No. of trials until 1^{st} sucess in a series of Bernoulli(p) (independent) trials.

$$P(X=k) = (1-p)^{k-1} \cdot p \qquad k = 1, 2, \ldots$$

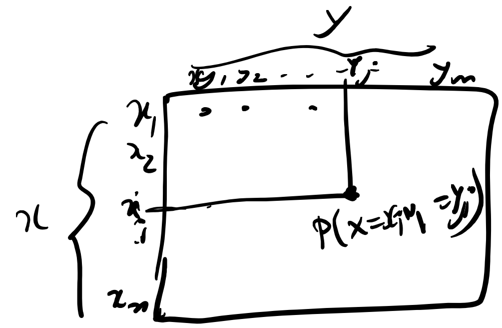
$$\left\{ \begin{array}{l} E[X] = \dfrac{1}{p} \\[2ex] VAR[X] = \dfrac{1-p}{p^2} \end{array} \right\}$$

(Recall differentiation step).

PMF of 2RVs.

$$P(X = x_i, Y = y_j) \quad \forall (x_i, y_j) \in S_{XY}$$

range of (X, Y)



$$\sum_i \sum_j P(X = x_i \;\; Y = y_j) = 1$$

marginalization.

$$\sum_{j=1}^{|S_Y|} P(X = x_i, Y = y_j) = P(X = x_i)$$

$$\underbrace{\sum_{i=1}^{|S_X|} P(X = x_i, Y = y_j)} = \underline{P(Y = y_j)}$$

$$E[g(X, Y)] = \sum_{i,j} g(x_i, y_j) \, P(X = x_i \;\; Y = y_j)$$

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

X_i's need not be independent.

$\circledast$ $X \perp\!\!\!\perp Y$

$$E[XY] = E[X]\,E[Y]$$

$\circledast$ if $E[XY] = E[X]E[Y]$

then

X, Y are uncorrelated.

Independence $\Rightarrow$ Uncorrelation. ✓

$\not\Leftarrow$

$\circledast$ $$VAR(X+Y) = VAR(X) + VAR(Y)$$

X, Y are uncorrelated

$$VAR(X_1 + X_2 + \cdots + X_n) = VAR(X_1) + \cdots + VAR(X_n)$$

X_i's pairwise uncorrelated.

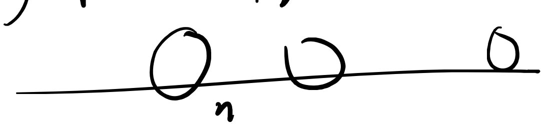
Binomial R.V (n, p)

n independent Bernoulli(p) trials.

Binomial R.V X : # sucesses.

$$P(X = K) = \binom{n}{K} p^K (1-p)^{n-K}$$

$(n-K)$



$$\sum_{K=0}^{n} K \underbrace{\binom{n}{K} p^K (1-p)^{n-K}}_{\text{difficult.}} \qquad X$$

$$Y_i = \begin{cases} 1 & i\text{th trial is success} \rightarrow \text{w.p } p \\ 0 & \text{otherwise} \end{cases}$$

$$X = Y_1 + Y_2 + \cdots + Y_n$$

H	H	T	T	H
1	1	0	0	1
Y_1	Y_2	Y_3	Y_4	Y_5

$$\underline{X} : Y_1 + Y_2 + Y_3 + Y_4 + Y_5 = 3$$

$$E[X] = E[Y_1 + \cdots + Y_n] = E[Y_1] + E[Y_2] + \cdots + E[Y_n]$$
$$= p + p + \cdots p$$
$$= np$$

$$VAR[X] = np(1-p) = VAR(Y_1 + \cdots + Y_n)$$
$$= VAR(Y_1) + \cdots + VAR(Y_n)$$

(as ~~each~~ trials are independent)

Q1. $P(M = k) = (1-p)^{k-1} \cdot p$ Geometric R.V

satisfies the following

$$P[M \geq k+j \mid M \geq j+1] = \underline{P[M \geq k]}$$ (Memoryless Property)

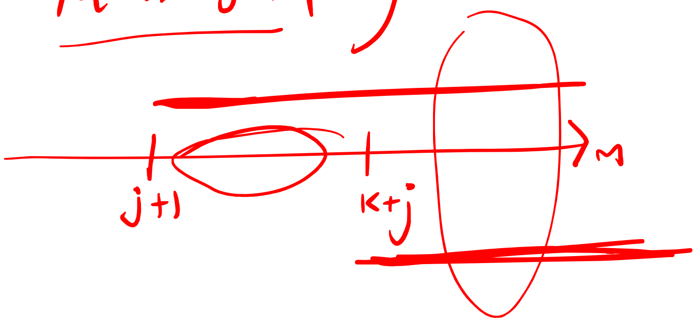
$k, j \geq 1$

$$= \frac{P(M \geq k+j \cap M \geq j+1)}{P(M \geq j+1)}$$

$k, j \geq 1$

$\left.\begin{array}{l} M \geq k+j \\ M \geq j+1 \end{array}\right\} \underline{M \geq k+j}$

$$= \boxed{\frac{P(M \geq k+j)}{P(M \geq j+1)}}$$



$P(M = k) = (1-p)^{k-1} \cdot p$

$P(M \geq r) = ?$

$P(M=r) + P(M=r+1) + P(M=r+2) + \dots$

$$= \sum_{k=r}^{\infty} (1-p)^{k-1} \cdot p$$

$$= p \sum_{k=r}^{\infty} (1-p)^{k-1}$$

$|r| < 1$

$a + ar + ar^2 + \dots$

$= \frac{a}{1-r}$

$$= p \cdot \left[\frac{(1-p)^{r-1}}{1-(1-p)}\right]$$

$$= \frac{p \cdot (1-p)^{r-1}}{p}$$

$$\boxed{P(M \geq r) = (1-p)^{r-1}}$$

$$P(M \geq k+j \mid M \geq j+1) = \frac{P(M \geq k+j)}{P(M \geq j+1)}$$

$$= \frac{(1-p)^{k+j-1}}{(1-p)^{j}}$$

$$= (1-p)^{k-1} = P(M \geq k)$$

$$P(\underline{M \geq k+j} \mid \underline{M \geq j+1}) = P(M \geq k) \quad (*)$$

e.g $P(\text{bulb will last for 15yrs} \mid \text{bulb lasted for 5yrs})$

$= P(\text{bulb will last for 10yrs})$

Q2. $\underline{20 \quad \text{m.c.qs}}$ 4 choices

1 correct answer. $\rightarrow$

You answer all questions Randomly.

X : $\underline{\# \text{ correct answers.}}$

i) PMF of X

each question Bernoulli$\left(\frac{1}{4}\right)$

X : Binomial$\left(20, \frac{1}{4}\right)$

$$\underline{P(X=k)} = \binom{20}{k}\left(\frac{1}{4}\right)^k\left(1-\frac{1}{4}\right)^{n-k}$$

$k=0,\dots,20$

ii) $E[X] = 20 \cdot \frac{1}{4} = 5 \qquad (n \cdot p)$

iii) $\underbrace{P(X>16)} = P(X=17) + P(X=18) + P(X=19) + P(X=20)$

$= \underbrace{2.98 \times 10^{-8}}$

Q3.

$\boxed{\text{5 red, 5 blue marbles}}$

choose 2 marbles randomly (without replacement one at a time)

win \$1.1 same color

loose \$1 diff color

(*) $E[\text{earning}]$? $\quad W = \begin{cases} 1.1 & \text{if } \underline{\text{same color}} \text{ w.p. } \frac{4}{9} \\ -1 & \text{if } \underline{\text{diff. color}} \text{ w.p. } \frac{5}{9} \end{cases}$

$P_{sc} = P(\text{Same color}) = P(B_2|B_1)\cdot \underbrace{P(B_1)} + P(R_2|R_1)\underbrace{P(R_1)}$

$= \frac{4}{9} \cdot \frac{5}{10} + \frac{4}{9}\,\frac{5}{10}$

$$= \frac{4}{9}$$

$$P_{dc} = P(\text{diff color}) = \frac{5}{9}$$

$$E[\text{earning}] = 1.1 \times \frac{4}{9} - 1 \times \frac{5}{9}$$

$$= \frac{-1}{15}$$

ii) $VAR(\text{earning})$

$$E[(\text{earning})^2] = 1.1^2 \cdot \frac{4}{9} + (-1)^2 \cdot \frac{5}{9}$$

$$= \frac{82}{75}$$

$$VAR[\text{earning}] = E[(\text{earning})^2] - (E[\text{earning}])^2$$

$$= \frac{82}{75} - \left(\frac{-1}{15}\right)^2$$

$$= \frac{49}{45}$$