EE 131A
Probability
Instructor: Lara Dolecek

Discussion Set 5 Solutions
February 5, 2021
TAs: Lev Tauz, Debarnab Mitra

Chapter 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *Gaussian RV.* If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

   (a) $P[X > 5]$
   Solution:
   We find

$$\begin{aligned} P[X > 5] &= P[\frac{X - 10}{6} > \frac{5 - 10}{6}] \\ &= P[Z > -0.833] = 1 - P[Z < -0.833] = Q(-0.833). \end{aligned}$$

   (b) $P[4 < X < 16]$
   Solution:

$$P[4 < X < 16] = P[\frac{4 - 10}{6} < \frac{X - 10}{6} < \frac{16 - 10}{6}] = P[-1 < Z < 1] = \Phi(1) - \Phi(-1).$$

   (c) $P[X < 8]$
   Solution:

$$P[X < 8] = P[\frac{X - 10}{6} < \frac{8 - 10}{6}] = P[Z < -0.333] = \Phi(-0.333).$$

2. Let X be an exponential random variable with parameter $\lambda > 0$. Find the expectation and variance of X.
   Solution: The PDF of X is given by

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \lambda e^{-\lambda x} & \text{for } x \geq 0. \end{cases}$$

Thus,

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_0^{\infty} \lambda x e^{-\lambda x} dx \\
&= \lambda(x \int e^{-\lambda x} dx - \int [\frac{d}{dx} x \int e^{-\lambda x} dx] dx) \\
&= \lambda(\Big[- x \frac{e^{-\lambda x}}{\lambda} \Big]_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx) \\
&= \lambda(\Big[\frac{-1}{\lambda^2} e^{-\lambda x} \Big]_0^{\infty}) \\
&= \frac{1}{\lambda}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\
&= \int_0^{\infty} \lambda x^2 e^{-\lambda x} dx \\
&= \lambda(x^2 \int e^{-\lambda x} dx - \int [\frac{d}{dx} x^2 \int e^{-\lambda x} dx] dx) \\
&= \lambda(\Big[- x^2 \frac{e^{-\lambda x}}{\lambda} \Big]_0^{\infty} + \frac{2}{\lambda} \int_0^{\infty} x e^{-\lambda x} dx) \quad (\text{since } \int_0^{\infty} x e^{-\lambda x} dx = \frac{1}{\lambda^2} \text{ from before}) \\
&= \lambda(\frac{2}{\lambda^3}) \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

Thus,

$$VAR[X] = E[X^2] - (E[X])^2 = \frac{1}{\lambda^2}$$

.

3. *Function of RV, $Y = g(X)$ where X is discrete and Y is continuous.*
   Assume $h << 1$ for all parts.
   Let X be a Bernoulli Random Variable with parameter p which is an input to a binary communication system. The output Y of the system is a Gaussian random variable with variance one and mean "0" when the input is "0" and mean "1" when the input is "1". In other words $Y \sim \mathcal{N}(X, 1)$.
   Note:
   The input output relation of the communication system can be modelled as $Y = X + N$ where X is the input to the channel, Y is the output of the channel and N is the noise introduced by the channel. Here, X is Bernouli(p), N is $\mathcal{N}(0, 1)$ and X and N are independent. Thus when X is 0, Y is $\mathcal{N}(0, 1)$ and when X is 1, Y is $\mathcal{N}(1, 1)$.

(a) Find $P[\text{input is } 1|y < Y < y + h]$ and $P[\text{input is } 0|y < Y < y + h]$.
Solution:
Assuming $h << 1$, then the conditional probability can be computed as follows:

$$\begin{aligned} P[\textit{input is 1}|y < Y < y+h] &= \frac{P[\textit{input is 1}, y < Y < y+h]}{P[y < Y < y+h]} \\ &= \frac{p\int_y^{y+h} f_1(t)\mathrm{d}t}{p\int_y^{y+h} f_1(t)\mathrm{d}t + (1-p)\int_y^{y+h} f_0(t)\mathrm{d}t} \\ &\simeq \frac{f_1(y)ph}{f_1(y)ph + f_0(y)(1-p)h} \\ &\simeq \frac{f_1(y)p}{f_1(y)p + f_0(y)(1-p)} \end{aligned}$$

(b) The receiver uses the following decision rule:
If $P[\text{input is } 1|y < Y < y+h] > P[\text{input is } 0|y < Y < y+h]$, decide input was 1; otherwise, decide input was 0. Show that this decision rule leads to the following threshold rule:
If $Y > T$, decide input was 1; otherwise, decide input was 0.
Solution:
The receiver uses the following decision rule:

$$P[\textit{input is 1}|y < Y < y+h] > P[\textit{input is 0}|y < Y < y+h]$$

$$\Longrightarrow f_1(y)p > f_0(y)(1-p)$$

$$\Longrightarrow \frac{p}{\sqrt{2\pi}}e^{-(y-1)^2/2} > \frac{1-p}{\sqrt{2\pi}}e^{-y^2/2}$$

$$\Longrightarrow e^{-(y-1)^2/2+y^2/2} > \frac{1-p}{p}$$

$$\Longrightarrow \frac{1}{2}(2y-1) > \frac{1-p}{p}$$

$$\Longrightarrow y > 1/2 + \ln\frac{1-p}{p}$$

Therefore, if $Y > T = 1/2 + \ln\frac{1-p}{p}$,decide input was 1; otherwise, decide input was 0.

(c) What is the probability of error for the above decision rule?
Solution:

$$
\begin{aligned}
P\{error\} &= P\{error|x=0\}(1-p) + P\{error|x=1\}p \\
&= P\{Y > T|x=0\}(1-p) + P\{y < T|x=1\}p \\
&= (1-p)\int_T^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \mathrm{d}y + p\int_{-\infty}^{T} \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2} \mathrm{d}y \\
&= (1-p)Q(T) + pQ(1-T)
\end{aligned}
$$

4. *Max of iid. uniform.* Problem 4.174, page 231 of ALG.
The random variable X is uniformly distributed in the interval $[0, a]$. Suppose a is unknown, so we estimate a by the maximum value observed in n independent repetitions of the experiment; that is, we estimate a by $Y = \max\{X_1, X_2, \ldots, X_n\}$.

   (a) Find $P[Y \leq y]$.
   Solution:
   The random variable Y is given by $Y = \max\{X_1, X_2, ..., X_n\}$, then we can compute the cdf of Y as follows:

$$
\begin{aligned}
P[Y \leq y] &= P[\max\{X_1, X_2, ..., X_n\} \leq y] \\
&= P[X_1 \leq y, X_2 \leq y, ..., X_n \leq y] \\
&= P[X_1 \leq y]P[X_2 \leq y]...P[X_n \leq y] \\
&= P[X \leq y]^n \\
&= (\frac{y}{a})^n
\end{aligned}
$$

   (b) Find the mean and variance of Y, and explain why Y is a good estimate for a when N is large.
   Solution:
   Given cdf function in (a), we first compute the pdf function of Y as follows:

$$
f_Y(y) = \frac{\mathrm{d}F_Y(y)}{\mathrm{d}y} = \frac{\mathrm{d}(\frac{y}{a})^n}{\mathrm{d}y} = \frac{ny^{n-1}}{a^n}
$$

   then the expectation and variance of Y is given by:

$$
E(Y) = \int_0^a y f_Y(y)\mathrm{d}y = \int_0^a y\frac{ny^{n-1}}{a^n}\mathrm{d}y = \frac{n}{n+1}\frac{y^{n+1}}{a^n}\ |_0^a = \frac{n}{n+1}a
$$

$$
E(Y^2) = \int_0^a y^2 f_Y(y)\mathrm{d}y = \int_0^a y^2\frac{ny^{n-1}}{a^n}\mathrm{d}y = \frac{n}{n+2}\frac{y^{n+2}}{a^n}\ |_0^a = \frac{n}{n+2}a^2
$$

$$
Var(Y) = E(Y^2) - E(Y)^2 = \frac{n}{n+2}a^2 - (\frac{n}{n+1}a)^2 = \frac{n}{(n+1)^2(n+2)}a^2
$$

   As $n \to \infty$, $E[Y] \to a$ and $Var(Y) \to 0$. Thus the estimate Y tends to a.

5. *Bonus:* A stick of length 1 is split at a point U that is uniformly distributed over $(0,1)$. Determine the expected length of the piece that contains the point p, $0 \leq p \leq 1$.

Solution: Let $L_p(U)$ denote the length of the substick that contains the point p, and note that

$$L_p(U) = \begin{cases} 1-U & U < p, \\ U & U > p. \end{cases} \tag{1}$$

Hence we have

$$\begin{aligned} E\left[L_p(U)\right] &= \int_0^1 L_p(u)du \\ &= \int_0^p (1-u)du + \int_p^1 udu \\ &= \frac{1}{2} - \frac{(1-p)^2}{2} + \frac{1}{2} - \frac{p^2}{2} \\ &= \frac{1}{2} + p(1-p). \end{aligned} \tag{2}$$