EE 131A
Probability and Statistics
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Homework 2 Solution
Monday, January 11, 2021
Due: Wednesday, January 20, 2021
before class begins
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Please upload your homework to Gradescope by January 20, 3:59 pm.
Please submit a single PDF directly on Gradescope
You may type your homework or scan your handwritten version. Make sure all the work is discernible.

Reading: Chapters 2.4-2.5 & 3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Roll two fair dice independently. In terms of the possible outcomes, define the events:

$$A = \{\text{First die is 1, 2 or 3}\}$$
$$B = \{\text{First die is 2, 3 or 6}\}$$
$$C = \{\text{Sum of outcomes is 9}\}$$

Are A, B, and C mutually independent? Hint: Three events A, B, and C are independent if all the four following constraints hold:

$$P(A \cap B) = P(A)P(B),$$
$$P(A \cap C) = P(A)P(C),$$
$$P(B \cap C) = P(B)P(C),$$
$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

Solution: Sample space is $S = \{(i, j) | 1 \le i, j \le 6\}$. And,

$$A = \{(i, j) : i = 1, 2, 3 \text{ and } 1 \le j \le 6\},$$
$$B = \{(i, j) : i = 2, 3, 6 \text{ and } 1 \le j \le 6\},$$
$$C = \{(i, j) : i + j = 9\} = \{(3, 6), (4, 5), (5, 4), (6, 3)\}.$$

So, $|S| = 36$, $|A| = 18$, $|B| = 18$, and $|C| = 4$. Since the two fair dice rolls are independent, $P(A) = \frac{18}{36} = \frac{1}{2}$, $P(B) = \frac{18}{36} = \frac{1}{2}$, and $P(C) = \frac{4}{36} = \frac{1}{9}$.

$$A \cap B = \{(i, j) | i = 2, 3 \text{ and } 1 \le j \le 6\},$$
$$A \cap C = \{(3, 6)\},$$
$$B \cap C = \{(3, 6), (6, 3)\},$$
$$A \cap B \cap C = \{(3, 6)\}.$$

Hence, $P(A \cap B) = \frac{12}{36} = \frac{1}{3}$, $P(A \cap C) = \frac{1}{36}$, $P(B \cap C) = \frac{2}{36} = \frac{1}{18}$, $P(A \cap B \cap C) = \frac{1}{36}$. Three events A, B, and C are independent if all the four following constraints hold:

$$P(A \cap B) = P(A)P(B), \tag{1}$$
$$P(A \cap C) = P(A)P(C), \tag{2}$$
$$P(B \cap C) = P(B)P(C), \tag{3}$$
$$P(A \cap B \cap C) = P(A)P(B)P(C). \tag{4}$$

Observe that (1) and (2) do not hold. Therefore, A, B,and C are **not** independent.

2. Assume there are 5 jars numbered 1 to 5. The i^{th} jar contains i black balls, $6 - i$ red balls, and 5 green balls. A jar is selected uniformly at random and a ball is selected uniformly at random from that jar. Let the events B, R, and G represent the events that a black, red, or green ball is chosen, respectively. Let J_k represent the event that the k^{th} jar is chosen.

   (a) What is $P(B|J_k)$?
   Solution: By the problem definition, $P(B|J_k) = \frac{k}{11}$.

   (b) What is $P(G)$, $P(B)$, and $P(R)$?
   Solution: Regardless of which jar is chosen, the green balls always make up 5 of the 11 available balls. Hence, $P(G) = \frac{5}{11}$.
   By symmetry, $P(B) = P(R)$. Therefore,

   $$1 = P(B) + P(R) + P(G) = 2 \cdot P(B) + \frac{5}{11}$$
   $$\implies P(B) = \frac{3}{11}$$

   Hence, $P(B) = P(R) = \frac{3}{11}$.

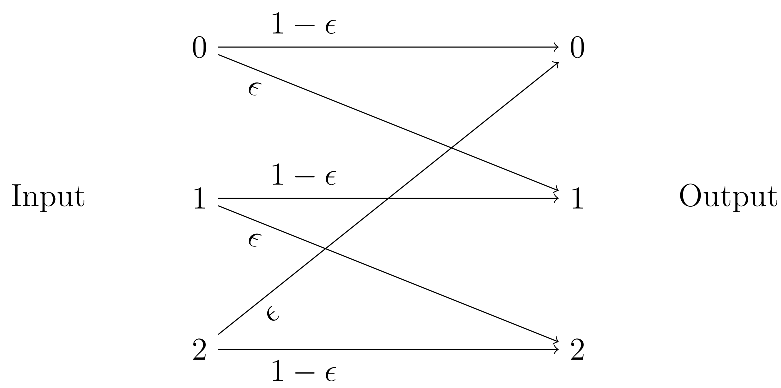
   (c) Given that the selected ball is black, what is the probability that the ball came from the kth jar, i.e. $P(J_k|B)$?
   Solution: By Bayes rule, we have

   $$P(J_k|B) = \frac{P(B|J_k)P(J_k)}{P(B)}$$

   By using the values determined in previous parts, we get

   $$P(J_k|B) = \frac{\frac{k}{11} \cdot \frac{1}{5}}{\frac{3}{11}} = \frac{k}{15}$$

3. A ternary communication channel is shown in the figure. Assume that input symbols $0, 1$, and 2 are chosen for transmission with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively.



(a) Calculate the probability of each output.

Solution:

$$\begin{aligned} P(\text{Output } = 0) &= P(\text{Output } = 0|\text{Input } = 0)P(\text{Input } = 0) \\ &+ P(\text{Output } = 0|\text{Input } = 1)P(\text{Input } = 1) \\ &+ P(\text{Output } = 0|\text{Input } = 2)P(\text{Input } = 2) \\ &= \frac{1}{4}(1-\epsilon) + 0 \cdot \frac{1}{2} + \frac{1}{4}\epsilon = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P(\text{Output } = 1) &= P(\text{Output } = 1|\text{Input } = 0)P(\text{Input } = 0) \\ &+ P(\text{Output } = 1|\text{Input } = 1)P(\text{Input } = 1) \\ &+ P(\text{Output } = 1|\text{Input } = 2)P(\text{Input } = 2) \\ &= \frac{1}{4}\epsilon + \frac{1}{2}(1-\epsilon) + 0 \cdot \frac{1}{4} = \frac{2-\epsilon}{4} \end{aligned}$$

$$\begin{aligned} P(\text{Output } = 2) &= P(\text{Output } = 2|\text{Input } = 0)P(\text{Input } = 0) \\ &+ P(\text{Output } = 2|\text{Input } = 1)P(\text{Input } = 1) \\ &+ P(\text{Output } = 2|\text{Input } = 2)P(\text{Input } = 2) \\ &= 0 \cdot \frac{1}{4} + \frac{1}{2}\epsilon + \frac{1}{4}(1-\epsilon) = \frac{1}{4} + \frac{1}{4}\epsilon = \frac{1+\epsilon}{4} \end{aligned}$$

(b) Given that the output was 1, what is the probability that the input was 0? 1? 2?

Solution:

We want to figure out $P(\text{Input } = k|\text{Output } = 1)$. By Bayes rule, we get

$$P(\text{Input } = k|\text{Output } = 1) = \frac{P(\text{Output } = 1|\text{Input } = k)P(\text{Input } = k)}{P(\text{Output } = 1)}.$$

All the necessary terms were calculated in part (a) which gives us the result

$$\begin{aligned}
P(\text{Input } = 0|\text{Output } = 1) &= \frac{\frac{1}{4}\epsilon}{\frac{2-\epsilon}{4}} = \frac{\epsilon}{2-\epsilon} \\
P(\text{Input } = 1|\text{Output } = 1) &= \frac{\frac{1}{2}(1-\epsilon)}{\frac{2-\epsilon}{4}} = \frac{2-2\epsilon}{2-\epsilon} \\
P(\text{Input } = 2|\text{Output } = 1) &= 0
\end{aligned}$$

4. A family has 5 natural children and has adopted 2 girls. Each natural child has equal probability of being a girl or a boy, independent of the other children. Find the PMF of the number of girls out of the 7 children.

   Solution: First, let N be the number of natural children that are girls. Let us first consider how we would get the pmf of N. Note that N can only take values in $\{0, 1, 2, 3, 4, 5\}$ since there are only 5 natural children. Now, consider the probability that $N = k$. This means that we want to figure out the probability that k out of the 5 children were born as girls and that $5-k$ were born as boys. There are $\binom{5}{k}$ ways to select k out of 5 children to be girls and that the probability that those k children are girls and that the other $5-k$ children are boys is $(P(\text{Child is girl}))^k(P(\text{Child is boy}))^{5-k} = \frac{1}{2^k}\frac{1}{2^{5-k}} = \frac{1}{2^5}$.

   Combining this together, we get the pmf of N as

$$P(N = k) = \begin{cases} \binom{5}{k}\frac{1}{2^5} & \text{, if } 0 \leq k \leq 5 \\ 0 & \text{else} \end{cases}$$

   This distribution is known as a binomial distribution.

   Since there are already 2 definite girls, we just need to shift this distribution. Thus, the pmf for G which is the number of girls out of the 7 children is

$$P(G = g) = \begin{cases} \binom{5}{g-2}\frac{1}{2^5} & \text{, if } 2 \leq g \leq 7 \\ 0 & \text{else} \end{cases}$$

5. Throw a pair of six-sided dice. Let X_1 be the number of dots on the resulting face of the first die and let X_2 be the number of dots on the resulting face of the second die. Let $Z = X_1 + X_2$ be the sum of the two dice rolls.

   (a) What is the pmf of Z?

   Solution:

$$P(Z = z) = \begin{cases} \frac{z-1}{36} & z \in \{2, 3, 4, 5, 6, 7\} \\ \frac{13-z}{36} & z \in \{8, 9, 10, 11, 12\} \\ 0 & \text{otherwise} \end{cases}$$

(b) What is $\mathbb{E}[Z]$? $Var(Z)$? **Solution:** By linearity of expectation, we get

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}[X_1 + X_2] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] \\ &= 3.5 + 3.5 = 7\end{aligned}$$

Note that X_1 and X_2 are independent, thus we get

$$\begin{aligned}Var(Z) &= Var(X_1 + X_2) \\ &= Var(X_1) + Var(X_2) \\ &= \frac{35}{12} + \frac{35}{12} = \frac{70}{12}\end{aligned}$$

(c) Given that $Z = 10$, what is the probability that $X_1 = k$ for $k \in \{1, 2, 3, 4, 5, 6\}$?

Solution:

By Bayes rule,

$$P(X_1 = k|Z = 10) = \frac{P(Z = 10|X_1 = k)P(X_1 = k)}{P(Z = 10)}.$$

From part (a), we know $P(Z = 10) = \frac{3}{36}$. Additionally, we know $P(X_1 = k) = \frac{1}{6}$. Due to $Z = X_1 + X_2$ and the independence of X_1 and X_2, we can write $P(Z = 10|X_1 = k) = P(X_2 = 10 - X_1|X_1 = k) = P(X_2 = 10 - k)$. Note that

$$P(X_2 = 10 - k) = \begin{cases} \frac{1}{6} & k \in \{4, 5, 6, 7, 8, 9\} \\ 0 & \text{otherwise} \end{cases}.$$

Combining these results together, we get

$$P(X_1 = k|Z = 10) = \begin{cases} \frac{1}{3} & k \in \{4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$

6. Assume that we flip a biased coin with probability of heads being p until a 2nd head is seen. Let X be the number of flips up until and including the flip that has the 2nd head. What is the pmf and expectation of X?

Solution:

Suppose $X = n$. Then the n^{th} flip must be a head. Of the first $(n-1)$ flips, there must be exactly 1 head. Since there are $(n-1)$ ways of getting exactly 1 head in $(n-1)$ flips, the probability that $X = n$ is

$$P(X = n) = \begin{cases} (n-1)p^2(1-p)^{n-2} & , n = 2, 3, 4, 5, \ldots \\ 0 & else \end{cases}$$

Now, we find the expectation of X.

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{n=2}^{\infty} n \cdot P(X = n) \\
&= \sum_{n=2}^{\infty} n(n-1)p^2(1-p)^{n-2} \\
&= p^2 \sum_{n=2}^{\infty} n(n-1)(1-p)^{n-2} \\
&= p^2 \frac{d^2}{dp^2}(\sum_{n=2}^{\infty}(1-p)^n) && \text{Differentiation Trick} \\
&= p^2 \frac{d^2}{dp^2}(\frac{(1-p)^2}{1-(1-p)}) && \text{Geometric Sum Formula} \\
&= p^2 \frac{d^2}{dp^2}(\frac{1-2p+p^2}{p}) \\
&= p^2 \frac{d^2}{dp^2}(\frac{1}{p} - 2 + p) \\
&= p^2(\frac{2}{p^3}) = \frac{2}{p}
\end{aligned}
$$

Another way we could have arrived at this expectation is by using the linearity of expectation. Observe that the number of flips until the first head arrives is a geometric random variable regardless of when you start counting. As such, the number of flips from the first head to the second head is also a geometric random variable. Hence, X can be written as $X = Y_1 + Y_2$ where Y_1 and Y_2 are geometric random variables with parameter p.

Hence,

$$\mathbb{E}[X] = \mathbb{E}[Y_1] + \mathbb{E}[Y_2] = \frac{1}{p} + \frac{1}{p} = \frac{2}{p}$$