ECE 131A
Probability
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Discussion Set 7 Solutions
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Chapters 5.1-5.5 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *Modem transmission and two RVs.* Problem 5.12, page 289 of ALG.
A modem transmits a two-dimensional signal (X, Y) given by:

$$X = r\cos(2\pi\Theta/8) \quad \text{and} \quad Y = r\sin(2\pi\Theta/8)$$

where Θ is a discrete uniform random variable in the set $\{0, 1, 2, \ldots, 7\}$.

   (a) Show the mapping from S_Θ to S_{XY}, the range of the pair (X, Y). **Solution:**
   The set $S_\Theta = \{0, 1, 2, 3, 4, 5, 6, 7\}$ maps to the set

$$\begin{aligned} S_{XY} =& \{(r,0), (\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}), (0,r), (-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}), \\ & (-r,0), (-\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}), (0,-r), (\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}})\} \end{aligned}$$

   in the corresponding order.

   (b) Find the joint pmf of X and Y.
   Solution:
   The joint pmf is given by:

$$p_{XY}(x,y) = \begin{cases} \frac{1}{8}, & (x,y) \in S_{XY} \\ 0, & otherwise \end{cases}$$

   (c) Find the marginal pmf of X and of Y.
   Solution:
   The marginal pmf is given by:

$$p_X(x) = \begin{cases} \frac{1}{8}, & x = r, -r \\ \frac{1}{4}, & x = \frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}, 0 \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{1}{8}, & y = r, -r \\ \frac{1}{4}, & y = \frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}, 0 \end{cases}$$

   (d) Find the probability of the following events: $A = \{X = 0\}, B = \{Y \leq r/\sqrt{2}\}, C = \{X \geq r/\sqrt{2}, Y \geq r/\sqrt{2}\}, D = \{X < -r/\sqrt{2}\}$.
   Solution:
   $P(A) = p_x(0) = \frac{1}{4}$

$P(B) = 1 - p_y(r) = \frac{7}{8}$

$P(C) = p_{XY}(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}) = \frac{1}{8}$

$P(D) = p_X(-r) = \frac{1}{8}$

(e) Determine if X and Y are independent random variables.
Solution:
$p_X(r) = p_Y(r) = \frac{1}{8}$, but $p_{XY}(r, r) = 0$. This implies that X and Y are not independent.

(f) Repeat part (e) if even values of Θ are twice as likely as odd values.
Solution:

In this case, we have the marginal pmf's as the following:

$$p_X(x) = \begin{cases} \frac{1}{6}, & x = r, -r, \frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}} \\ \\ \frac{1}{3}, & x = 0 \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{1}{6}, & y = r, -r, \frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}} \\ \\ \frac{1}{3}, & y = 0 \end{cases}$$

$p_X(r) = p_Y(r) = \frac{1}{6}$, but $p_{XY}(r, r) = 0$. This implies that X and Y are not independent.

2. A Bernoulli trail with P[Success] $= p$ is repeated until two successes have occurred. Let X be the trial on which the first success occurs and Y be the trial on which the 2^{nd} success occur.

(a) Find the joint PMF of X and Y.
Solution:
For $y > x$ we have

$$P(X = x, Y = y) = (1-p)^{x-1} p (1-p)^{y-x-1} p$$

Thus

$$P(X = x, Y = y) = \begin{cases} (1-p)^{y-2} p^2, & \text{if} \quad y > x \\ \\ 0, & \text{otherwise} \end{cases}$$

(b) The marginal probability of X and Y.
Solution:

$$
\begin{aligned}
P[X = x] &= \sum_y P(X = x, Y = y) \\
&= \sum_{y=x+1}^{\infty} p^2(1-p)^{y-2} \\
&= p(1-p)^{x-1}
\end{aligned}
$$

which is a geometric distribution as expected.

$$
\begin{aligned}
P[Y = y] &= \sum_x P(X = x, Y = y) \\
&= \sum_{x=0}^{y-1} p^2(1-p)^{y-2} \ \text{ if } \ y > 1 \\
&= \begin{cases} (y-1)p^2(1-p)^{y-2}, & \text{if } \ y > 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

3. Let $F_Z(z)$ and $F_Y(y)$ be valid one-dimensional CDFs. Show that $F_{Z,Y}(z, y) = F_Z(z)F_Y(y)$ satisfies the properties of a two-dimensional CDF.
Solution:
The desired properties are on page 243 of ALG.

   (a) Since $F_Z(z)$ and $F_Y(y)$ are valid CDF's, if $z_1 \leq z_2$ and $y_1 \leq y_2$, then $F_Z(z_1) \leq F_Z(z_2)$ and $F_Y(y_1) \leq F_Y(y_2)$. Thus, $F_Z(z_1)F_Y(y_1) \leq F_Z(z_2)F_Y(y_2)$, or, $F_{Z,Y}(z_1, y_1) \leq F_{Z,Y}(z_2, y_2)$.

   (b) $F_{Z,Y}(z, -\infty) = F_Z(z)F_Y(-\infty) = F_Z(z)(0) = 0$ (and similarly switching Z and Y). $F_{Z,Y}(\infty, \infty) = F_Z(\infty)F_Y(\infty) = 1 \times 1 = 1$.

   (c) $F_Z(z) = F_{Z,Y}(z, \infty) = F_Z(z)F_Y(\infty) = F_Z(z) \times 1$, and the same with Y instead of Z.

   (d) $\lim_{z \to a^+} F_{Z,Y}(z, y) = \lim_{z \to a^+} F_Z(z)F_Y(y) = F_Y(y) \lim_{z \to a^+} F_Z(z) = F_Y(y)F_Z(a) = F_{Z,Y}(a, y)$. We used the continuity property for the one-dimensional CDF. The other way is the similar.

   (e) This property follows from breaking up $F_{Z,Y}(z, y)$ into $F_Z(z)$ and $F_Y(y)$; then we get the product of $P[z_1 < Z \leq z_2]$ and $P[y_1 < Y \leq y_2]$, which is indeed a rectangle.

4. Problem 5.18, page 290 of ALG.

   A dart is equally likely to land at any points (X_1, X_2) inside a circular target of unit radius. Let R and Θ be the radius and angle of the point (X_1, X_2) where the angle is relative to the positive-real axis.

(a) Find the joint cdf of R and Θ.

Solution:

First note that $0 \le R \le 1$ and $0 \le \Theta \le 2\pi$.
Assume that $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

$$\begin{aligned} P(R \le r, \Theta \le \theta) &= \frac{\text{Area of pie slice of radius r and angle } \theta}{\text{Area of unit circle}} \\ &= \frac{\pi r^2 \frac{\theta}{2\pi}}{\pi 1^2} \\ &= r^2 \frac{\theta}{2\pi} \end{aligned}$$

Thus, the cdf is

$$F_{R\Theta}(r,\theta) = P(R \le r, \Theta \le \theta) = \begin{cases} r^2 \frac{\theta}{2\pi} & 0 \le r \le 1, 0 \le \theta \le 2\pi \\ \frac{\theta}{2\pi} & r > 1, 0 \le \theta \le 2\pi \\ r^2 & 0 \le r \le 1, \theta > 2\pi \\ 1 & r > 0, \theta > 2\pi \\ 0 & else \end{cases}$$

(b) Find the marginal cdf of R and Θ.

Solution:
For $0 \le r \le 1$,

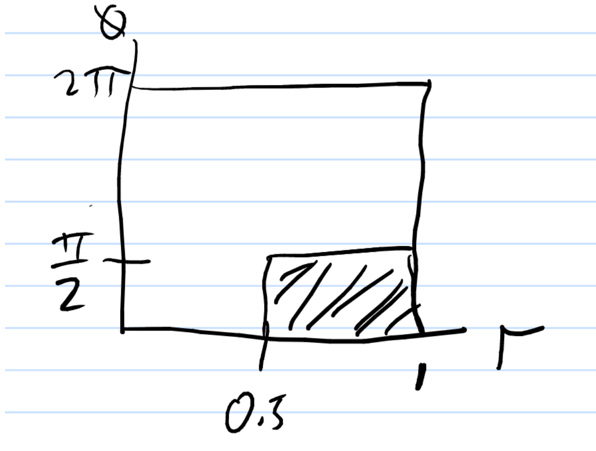
$$P(R \le r) = \lim_{\theta \to \infty} P(R \le r, \Theta \le \theta) = r^2$$

For $0 \le \theta \le 2\pi$,

$$P(\Theta \le \theta) = \lim_{r \to \infty} P(R \le r, \Theta \le \theta) = \frac{\theta}{2\pi}$$

(c) Use the joint cdf to find the probability that the dart lands in the first quadrant of the real plane and that the radius is greater than 0.5.

Solution:
From the problem, we want to calculate $P(R > 0.5, 0 \le \Theta \le \frac{\pi}{2})$. Visually, it is represented by the shaded area in the plot:



Thus,

$$P(R > 0.5, 0 \le \Theta \le \frac{\pi}{2}) = F_{R\Theta}(1, \frac{\pi}{2}) - F_{R\Theta}(1, 0) - F_{R\Theta}(0.5, \frac{\pi}{2}) + F_{R\Theta}(0.5, 0)$$
$$= \frac{1}{4} - 0 - \frac{1}{16} + 0 = \frac{3}{16}$$

5. *Computing joint pdf.* Problem 5.25, page 291 of ALG.
   The amplitudes of two signals X and Y have joint pdf:

$$f_{X,Y}(x, y) = e^{-\frac{x}{2}} y e^{-y^2} \quad \text{for} \quad x > 0, y > 0.$$

   (a) Find the joint cdf.
   Solution:
   The joint cdf is given by:

$$F_{XY}(x, y) = \int_{x=0}^{x} \int_{y=0}^{y} e^{-x/2} y e^{-y^2} dy dx = (1 - e^{-x/2})(1 - e^{-y^2})$$

   (b) Find $P[X^{\frac{1}{2}} > Y]$.
   Solution:
   $P(X^{1/2} > Y) =$

$$\int_{x=0}^{\infty} \int_{y=0}^{x^{1/2}} f_{XY}(x, y) dy dx = \int_{x=0}^{\infty} \frac{1}{2} e^{-x/2}(1 - e^{-x}) dx = -e^{-x/2} + \frac{1}{3} e^{-3x/2} \Big|_{x=0}^{\infty} = \frac{2}{3}$$

   (c) Find the marginal pdfs.
   Solution:
   The marginal pdf's are given as the following:

   $F_X(x) = \lim_{y \to \infty} F_{XY}(x, y) = 1 - e^{-x/2}$, then we have $f_X(x) = \frac{1}{2}e^{-x/2}$.

$F_Y(y) = \lim_{x\to\infty} F_{XY}(x, y) = 1 - e^{-y^2}$,

then we have $f_Y(y) = 2ye^{-y^2}$.