Discussion 7

Generating RVs

X CDF $F_X(x)$

$\hookrightarrow U \sim \text{Uniform}(0,1)$

$\tilde{X} = F_X^{-1}(U)$

$U_1 \; U_2 \; U_3 \; \ldots \; U_n$

$\downarrow F_X^{-1}(.)$

$\underbrace{F_X^{-1}(U_1) \; F_X^{-1}(U_2) \; \ldots \; F_X^{-1}(U_n)}$

$\downarrow$ distributed according to $F_X(x)$

2 RVs

S outcome $\in S$

$\hookrightarrow \underline{(X, Y)}$

S_{XY} : range of the pair (X,Y)

1) pair of discrete RVs

Joint PMF of (X, Y)

$$P_{XY}(x_i, y_k) = P(\underline{X = x_i} \overset{\cap}{,} \underline{Y = y_k})$$

Properties

i)
$$\sum\sum_{(x_i, y_k) \in S_{XY}} P_{XY}(x_i, y_k) = 1$$

ii) Marginalization

$$\sum_{\substack{x_i \in S_X \\ (x_i, y_k) \in S_{XY}}} P_{XY}(x_i, y_k) = \underset{\text{Marginal PMF of } Y}{P_Y(y_k)}$$

$$\sum_{\substack{y_k \in S_Y \\ (x_i, y_k) \in S_{XY}}} P_{XY}(x_i, y_k) = \underset{\text{Marginal PMF of } X}{P_X(x_i)}$$

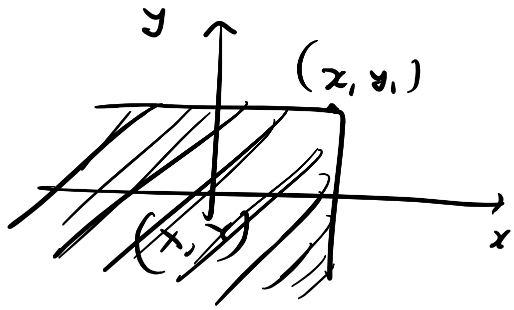
are X and Y independent ?

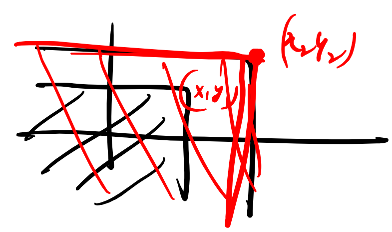
$$P_{XY}(x_i, y_k) = P_X(x_i)\, P_Y(y_k) \quad \forall (x_i, y_k) \in S_{XY}$$

$\longrightarrow$ X and Y are independent

Joint CDF

$$F_{XY}(x_1, y_1) = P(X \le x_1, Y \le y_1)$$





i) $x_1 \le x_2$ $\quad y_1 \le y_2$

$$F_{XY}(x_1, y_1) \le F_{XY}(x_2, y_2)$$

ii) $F_{XY}(+\infty, +\infty) = 1$

$F_{XY}(-\infty, -\infty) = 0 \longrightarrow P(X \le -\infty, Y \le -\infty)$

$F_{XY}(-\infty, y) = 0 \qquad P(X \le -\infty, Y \le y)$

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$F_{XY}(x, -\infty) = 0$

iii) Marginalization

$$F_{XY}(+\infty, y) = \underline{F_Y(y)}$$

$P(X \leq \infty, Y \leq y) = P(Y \leq y)$

$$F_{XY}(x, +\infty) = F_X(x)$$

Marginal CDF of X

Marginal CDFs

$(a,b) \to (a+h, b)$

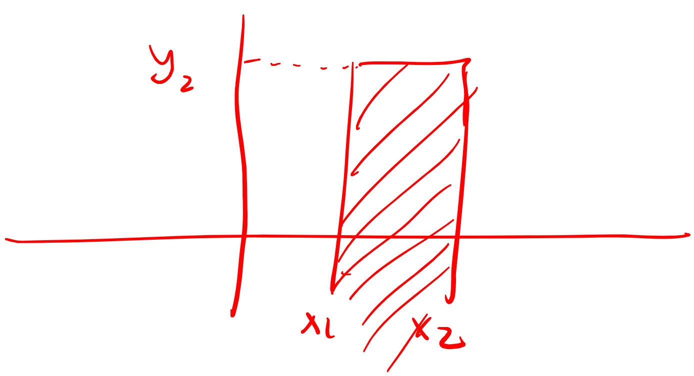
iv)
$$F_{XY}(a, b) = \lim_{\substack{h \to 0 \\ h > 0}} F_{XY}(a+h, b)$$

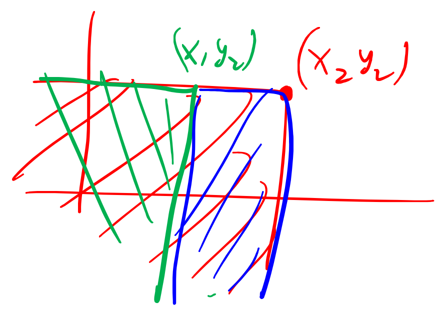
$(a, b+h)$

(a,b)

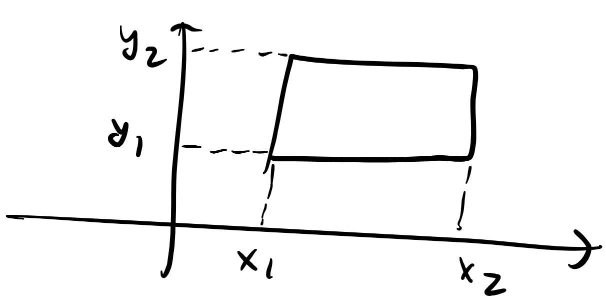
$$F_{XY}(a, b) = \lim_{\substack{h \to 0 \\ h > 0}} F_{XY}(a, b+h)$$

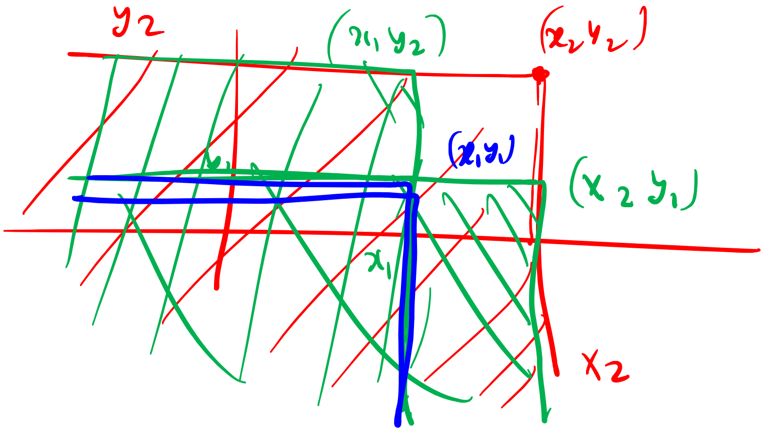
$$P(x_1 < X \leq x_2, \; Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2)$$





$$P(x_1 < X \leq x_2, \; y_1 < Y \leq y_2) = F_{XY}(x_2 y_2) - F_{XY}(x_1 y_2) - F_{XY}(x_2, y_1) + \underline{F_{XY}(x_1 y_1)}$$

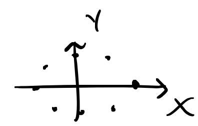




Q1. θ Uniform in $\underbrace{\{0, 1, ----, 7\}}$

$$\underline{X} = r\cos\left(\frac{2\pi\theta}{8}\right) \qquad \underline{Y} = r\sin\left(\frac{2\pi\theta}{8}\right)$$

$$S_\theta = \{0, 1, --- 7\}$$



a) $$\underline{S_{XY}} = \left\{ \underline{(r,0)},\ \underline{\left(\frac{r}{\sqrt2}, \frac{r}{\sqrt2}\right)},\ \underline{(0,r)},\ \underline{\left(-\frac{r}{\sqrt2}, \frac{r}{\sqrt2}\right)},\ \underline{(-r,0)},\ \underline{\left(-\frac{r}{\sqrt2}, -\frac{r}{\sqrt2}\right)},\ \underline{(0,-r)},\ \underline{\left(\frac{r}{\sqrt2}, -\frac{r}{\sqrt2}\right)} \right\}$$

b) Joint PMF of X and Y ?

$$\underline{P(X = r,\ Y = 0) = P(\theta = 0) = \frac{1}{8}}$$

$$P_{XY}(x, y) = \begin{cases} 1/8 & (x,y) \in S_{XY} \\ 0 & \text{otherwise.} \end{cases}$$

c) Marginal PMF of X and Y.

$$S_X = \left\{ r,\ \underline{\frac{r}{\sqrt2}},\ \underline{0},\ -\underline{\frac{r}{\sqrt2}},\ -r \right\}$$

$$P(X=r) = P(X=r, Y=0) = \frac{1}{8}$$

$$P\left(X=\frac{r}{\sqrt{2}}\right) = P\left(X=\frac{r}{\sqrt{2}}, Y=\frac{r}{\sqrt{2}}\right) + P\left(X=\frac{r}{\sqrt{2}}, Y=\frac{-r}{\sqrt{2}}\right)$$

$$= \frac{1}{8} + \frac{1}{8}$$

$$= \frac{1}{4}.$$

$$P(X=0) = \frac{1}{4} \qquad P(X=-r) = \frac{1}{8}.$$

$$P(X=-r/\sqrt{2}) = \frac{1}{4}$$

$$\boxed{P(Y=y) \qquad S_Y =}$$

d) $C = \left\{ X \geq \frac{r}{\sqrt{2}}, Y \geq \frac{r}{\sqrt{2}} \right\}$

$$P(C) = \underline{P\left(X=\frac{r}{\sqrt{2}}, Y=\frac{r}{\sqrt{2}}\right)}$$

$$= \frac{1}{8}$$

$B = \left\{ Y \leq r/\sqrt{2} \right\}$ $\qquad P(B) = 1 - P(\underline{Y=r}) \to \frac{1}{8}$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

e) Are X and Y independent?

$$P(X=r, Y=0) = \frac{1}{8}$$

$$P(X=r)\, P(Y=0) \to \frac{1}{24}$$

$\frac{1}{3}$

$$P(Y=0 \mid X=r) = \frac{P(X=r, Y=0)}{P(X=r)}$$

$$= 1 \neq \underline{P(Y=0)}$$

$P(A|B) = P(A)$

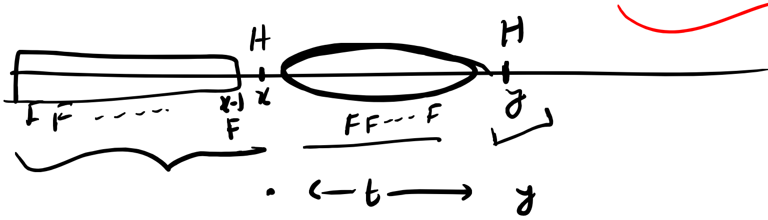
2. $P[\text{Sucess}] = p$

X : trial no. of 1st sucess. $S_X = \{1, 2, \ldots \infty\}$

Y : trial no. of 2nd sucess. $S_Y = \{2, 3, \ldots \infty\}$

a) Joint PMF of X and Y.

$$P(X=x, Y=y) = (1-p)^{x-1}\, p \;\big|\; (1-p)^{y-1-x} \cdot p \qquad y > x$$



$x = 10$

$y = 2$

$x + t + 1 = y$

$t = y - x - 1$

$$\boxed{P(X=x, Y=y) = \begin{cases} p^2 (1-p)^{y-2} & y > x \\ 0 & \text{otherwise} \end{cases}}$$

b) Marginal PMF of X and Y (via marginalization)

$$P(X=x) = \sum_y P(X=x, Y=y)$$

$$P(X=x) = \sum_{y=x+1}^{\infty} p^2 (1-p)^{y-2}$$

e.g. $P(X=5)$

$$= p^2 \left[(1-p)^{x-1} + (1-p)^{x} + \cdots\cdots \right]$$

$$= \frac{p^2 (1-p)^{x-1}}{1-(1-p)}$$

$a + ar + \cdots \infty$

$= \dfrac{a}{1-r}$

$$\boxed{P(X=x) = p(1-p)^{x-1} \qquad x \geq 1}$$

(Geometric)

$$P(Y=y) = \sum_{x} P(X=x, Y=y)$$

$$= \sum_{x=1}^{y-1} p^2 (1-p)^{y-2} \qquad y > 1$$

$P(Y=10)$

$$P(Y=2) = p^2(1-p)^{y-2} \underbrace{\sum_{x=1}^{y-1} 1}$$

$y=3$

1 2

$y=5$

1 2 3 4

$$\boxed{P(Y=y) = \begin{cases} (y-1)p^2(1-p)^{y-2} & y > 1 \\ 0 & \text{otherwise} \end{cases}}$$