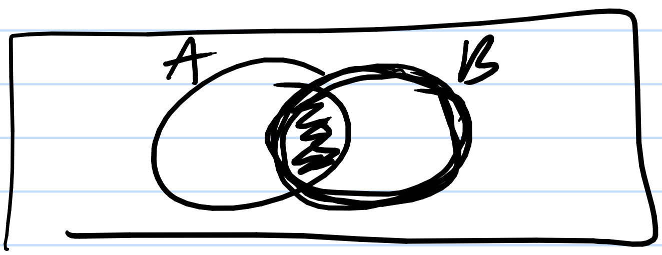
Discussion 2

Lecture 3 Recap

- Conditional Probability

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$



$P(B)=0$

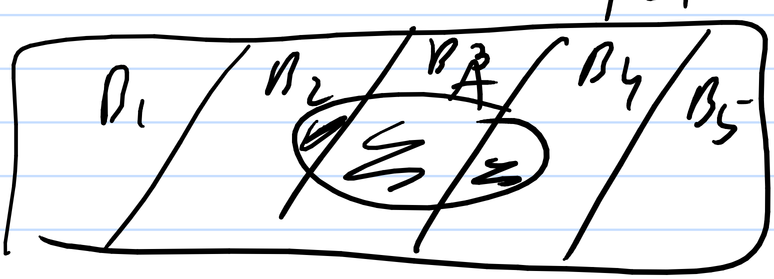
$\Rightarrow P(A|B)=0$

- Total Probability law

B_1 ---- B_n $\quad B_i \cap B_j = \emptyset \quad i \neq j$

$\cup B_i = S$

then $$P(A) = \sum P(A\cap B_i) = \sum_{i=1}^{n} P(A|B_i)\cdot P(B_i)$$



- Bayes Rule

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$P(\text{Study Hard?} \mid \text{Got an A in class})$$

$$= \frac{P(\text{Got A class} \mid \text{Given Study?}) \cdot P(\text{Study?})}{P(\text{Got an A in Class})}$$

- Independence

$$A \perp\!\!\!\perp B \implies P(A|B) = P(A)$$

$$\Updownarrow$$

$$P(B|A) = P(B)$$

$$\frac{P(A|B) \cdot P(B)}{P(A)} = \frac{\cancel{P(A)} \cdot P(B)}{\cancel{P(A)}}$$

Lecture 4 recap

- Random Variables

$f(w) = $ # of Heads in outcome

$$S = \{HH, HT, TH, TT\}$$

$$S_X = \{0, 1, 2\}$$

$X = $ # of Heads

$$S_Y = \{0, 1, 2\}$$

$Y = $ # of Tails

Probability Mass Function (pmf)

$$P(X=k) = \sum_{w:\, f(w)=k} P(w)$$

$X = $ # of Heads in 2 tosses

$$P(X=1) = P(HT) + P(TH)$$

- Expectations

$g(X)$ - some function of r.v. X

Ex: $g(X) = X$ $\quad g(X) = X^2$

$$E[g(X)] = \sum g(k) \cdot P(X=k)$$

$$E[X] = \sum k \cdot P(X=k) = m_X$$

$$Var(X) = E[(X - m_X)^2]$$

$$= E[X^2] - m_X^2$$

$$= E[X^2] - (E[X])^2$$

- Linearity of expectation

$$E[a \cdot g(X) + b \cdot h(X)]$$

$$= a\,E[g(X)] + b \cdot E[h(X)]$$

$$E[X+Y] = E[X] + E[Y]$$

- Common R.V. Examples
  - Uniform Discrete R.V.

$$S_X = \{1 \ \cdots \ L\}$$

$$P(X=k) = \frac{1}{L} \cdot \mathbb{1}_{\{k \in \{1 \cdots L\}\}}$$

$$= \begin{cases} \frac{1}{L} & k \in \{1 \cdots L\} \\ 0 & \text{else} \end{cases}$$

  - Bernoulli - parameter p

$$S_X = \{0, 1\}$$

$$P(X=1) = p \qquad P(X=0) = 1-p$$

$$E[X] = 0 \cdot (1-p) + 1 \cdot p = p$$

  - Geometric R.V - parameter p

$$S_X = \{1 \ \cdots \ \}$$

$$P(X=k) = (1-p)^{k-1} \cdot p$$

H	1
TH	2
TTH	3
TTTH	4

$$P(X=k) = \sum_{\omega : f(\omega)=k} P(\omega)$$

$$= P(\overbrace{T \cdots T}^{k-1} H)$$

$$= (1-p)^{k-1} \cdot p$$

$$E[X] = \frac{1}{p} = \sum k \cdot P(X=k)$$

$$= \sum_{k=1}^{\infty} k \cdot ((1-p)^{k-1} \cdot p$$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$= p \sum_{k=1}^{\infty} \frac{d}{dp}\left(-(1-p)^{k}\right)$$

$$= p \frac{d}{dp}\left(-\sum_{k=1}^{\infty} (1-p)^{k}\right)$$

$$= p \frac{d}{dp}\left(-\frac{(1-p)}{1-(1-p)}\right)$$

$$= p \frac{d}{dp}\left(-\frac{1}{p}+1\right)$$

$$= p \cdot \frac{1}{p^2} = \frac{1}{p}$$

Problem 1

G - Geometric R.V. with param p

$E[G] = \frac{1}{p}$

$\underline{Var(G) = \frac{1-p}{p^2}}$

$Var(G) = E[G^2] - (\overset{\frac{1}{p}}{E[G]})^2$

$E[G^2] = \sum k^2 \cdot P(G=k) = \sum_{k=1}^{\infty} k^2 \cdot (1-p)^{k-1} \cdot p$

$E[G^2] = E[G(G-1) + G]$

$= \underbrace{E[G \cdot (G-1)]} + \underbrace{E[G]}_{\frac{1}{p}}$

$$E[G(G-1)] = \sum_{k=1}^{\infty} (k)(k-1) \cdot P(G=k)$$

$$= \sum_{k=1}^{\infty} (k)(k-1)(1-p)^{k-1} p$$

$$= (p)(1-p) \sum_{k=1}^{\infty} k \cdot (k-1)(1-p)^{k-2}$$

$$= (p)(1-p) \sum_{k=1}^{\infty} \frac{d}{dp}\left(-k(1-p)^{k-1}\right)$$

$$= (p)(1-p) \sum_{k=1}^{\infty} \frac{d^2}{dp^2}\left((1-p)^{k}\right)$$

$$= (p)(1-p) \frac{d^2}{dp^2}\left(\sum_{k=1}^{\infty} (1-p)^{k}\right)$$

$$= (p)(1-p) \frac{d^2}{dp^2}\left(\frac{(1-p)}{1-(1-p)}\right)$$

$$= (p)(1-p) \frac{d^2}{dp^2}\left(\frac{1}{p} - 1\right)$$

$$= (p)(1-p) \cdot \frac{2}{p^3} = \frac{2-2p}{p^2} = E[G(G-1)]$$

$$\mathrm{Var}(G) = E[G^2] - (\underbrace{E[G]}_{\frac{1}{p}})^2$$

$$= E(G(G-1)) + E[G] - \frac{1}{p^2}$$

$$= \frac{2-2p}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1-p}{p^2}$$

Problem 2

90 students - split evenly in 3 class

Joe and Jane who want to stay together

$A = \{ J \text{ and } J \text{ stay in class together} \}$

$P(A) = ?$

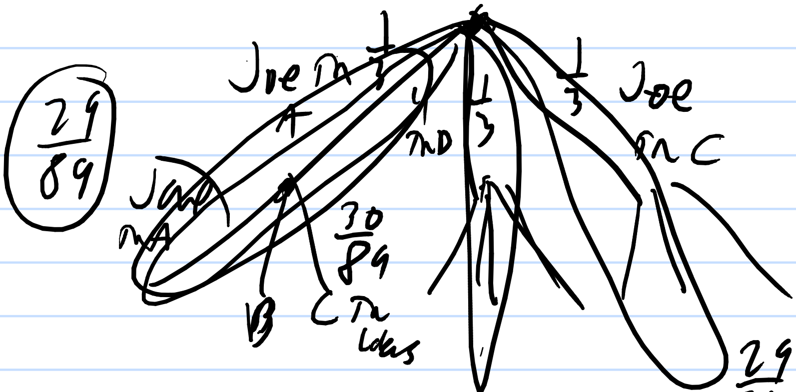
$P(A) = P(A \cap \text{Joe is in class A})$
$+ P(A \cap \text{Joe is in class B})$
$+ P(A \cap \text{Joe is in class C})$

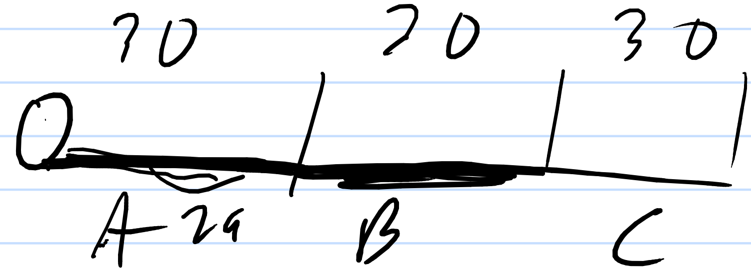
$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$

P(Both are in class A) = $\dfrac{\text{desired partitions}}{\text{Total partitions}}$

$$= \frac{\binom{88}{28}}{\binom{90}{30}}$$

$$P(A) = \frac{3\binom{88}{28}}{\binom{90}{30}} = \frac{29}{89}$$





$$P(A \cap \text{Joe is in } A) = \underbrace{P(A \mid \text{Joe is in } A)}_{\frac{29}{89}} \cdot \underbrace{P(\text{Joe is in } A)}_{\frac{1}{3}}$$

$$\frac{1}{3}\cdot\frac{29}{89} + \frac{1}{3}\cdot\frac{29}{89} + \frac{1}{3}\cdot\frac{29}{89} = \frac{29}{89}$$

$$\begin{aligned} P(A) = {} & P(A \cap \underline{\text{Bob is in class } A}) \\ & + P(A \cap \underline{\text{Bob is in class } B}) \\ & + P(A \cap \underline{\text{Bob is in class } C}) \end{aligned}$$

Problem 3

6-sides

Alice, Bob, and Claire each toss a die.

A – { Alice and Bob have same result }

B – { Alice and Claire " " }

C – { Bob and Claire " " }

A, B, C are pairwise independent but not mutually independent.

$A \perp\!\!\!\perp B \Rightarrow \cdot\, P(A|B) = P(A)$

$A \perp\!\!\!\perp C \qquad \cdot\, P(A \cap B) = P(A) \cdot P(B)$

$B \perp\!\!\!\perp C \qquad = P(A|B) \cdot P(B)$

$A \perp\!\!\!\perp B \perp\!\!\!\perp C$ Not true

$$P(A \cap B \cap C) = P(A) \cdot P(B) - P(C)$$

$$= P(B \cap C / A) \cdot P(A)$$

$$= P(C / A, B) \cdot P(B/A) \cdot P(A)$$

only if mutually independent $= P(C) \cdot P(B) \cdot P(A)$

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A) = P(\{1,1\}, \{2,2\} \cdots \{6,6\})$$

$$= 6 \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{6} = P(B) = P(C)$$

$$P(A \cap B) = P(\text{All same result})$$

$$= P((1,1,1), (2,2,2) \cdots)$$

$$= 6 \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

$$\begin{array}{ccc} P(A \cap B) & = & P(A) \cdot P(B) \\ \| & & \| \\ \frac{1}{36} & = & \frac{1}{6} \cdot \frac{1}{6} \end{array} \Rightarrow A \perp\!\!\!\perp B$$

$$P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$$

$$\| \qquad\qquad\qquad \|$$

$$P(A \cap B) = \frac{1}{6^2} \neq \left(\frac{1}{6}\right)^3$$

$$A \cap B \cap C = A \cap B$$

$$C \in (A \cap B)$$