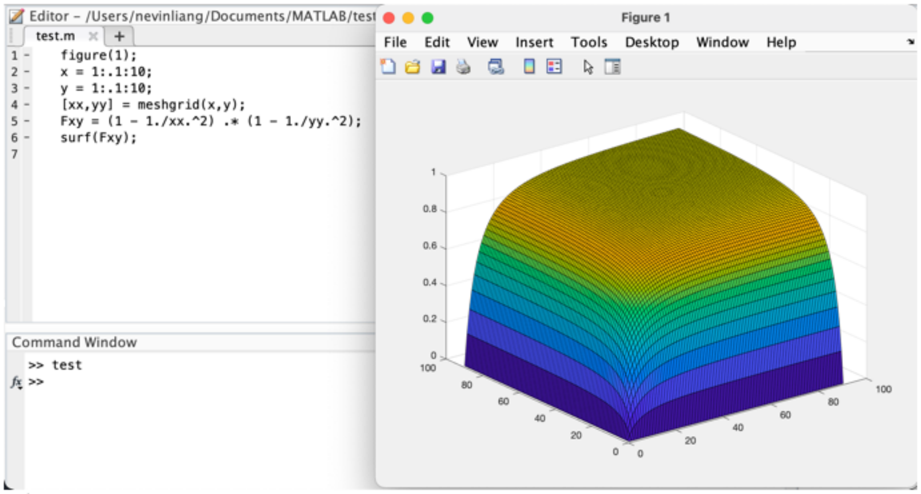
EE 131A : HW 6

1. (a)



(b) $F_x(x) = \lim_{y\to\infty} F_{xy}(x,y)$

$$= \lim_{y\to\infty} \begin{cases} \left(1-\frac{1}{x^2}\right)\left(1-\frac{1}{y^2}\right) & x>1, y>1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1-\frac{1}{x^2} & x>1 \\ 0 & \text{else.} \end{cases}$$

$$F_Y(y) = \begin{cases} 1-\frac{1}{y^2} & y>1 \\ 0 & \text{else.} \end{cases}$$

(c) $F_{X,Y}(x,y) = P(X \le x) \,\&\, P(Y \le y)$

$X<3, Y\le 5 \;:\; F_{X,Y}(3,5) = \frac{8}{\cancel{9}_{3}} \cdot \frac{\cancel{24}^{8}}{25} = \boxed{\frac{64}{75}}$

$X>4, Y>3 : F_x(4) + F_Y(3) - F_{XY}(4,3)\Big|_{\text{inverse}}$

$$= 1 - \left[\frac{15}{16} + \frac{8}{9} - \frac{5}{6}\right] = \boxed{\frac{1}{144}}$$

2. Verify:

$$\int_{0^+}^{\infty}\int_{0^+}^{\infty} y e^{-y(1+x)}\, dy\; dx$$

$$= \int_{0^+}^{\infty} -\frac{y(x+1)+1}{(x+1)^2}\cdot e^{-y(1+x)}\Big|_{0^+}^{\infty} dx$$

$$\lim_{k\to\infty} \frac{k(x+1)+1}{(x+1)^2}\cdot e^{-k(1+x)} = \frac{\cancel{x+1}}{(x+1)^2\cdot\cancel{(x+1)}\cdot e^{k(x+1)}} = 0$$

$$= \int_{0^+}^{\infty} -0+\frac{1}{(x+1)^2}\cdot e^{0}\; dx = \frac{-1}{x+1}\Big|_{0^+}^{\infty} = -0+1 = +1 \checkmark$$

$$f_X(x) = \int_{0^+}^{\infty} f_{X,Y}(x,y')\, dy' = \boxed{\frac{1}{(x+1)^2}} \text{ for } x>0$$

$$f_Y(y) = \int_{0^+}^{\infty} f_{X,Y}(x',y)\, dx' = \int_{0^+}^{\infty} y\cdot e^{-y(1+x')}\, dx'$$

$$= \frac{y}{-y}\cdot e^{-y(1+x')}$$

$$= -e^{-y(1+x')}\Big|_{0^+}^{\infty} = 0+\boxed{e^{-y}}$$

for $y>0$

3. Find f_x & f_y first

a) $f_x(x) = \int_1^5 f_{XY}(x,y)\,dy = \int_1^5 \frac{x}{5}+\frac{y}{20}\,dy = \frac{xy}{5}+\frac{y^2}{40}\Big|_1^5 = \frac{4x}{5}+\frac{3}{5}$ for $0\le x\le 1$

$f_Y(y) = \int_0^1 \frac{x}{5}+\frac{y}{20}\,dx = \frac{x^2}{10}+\frac{xy}{20}\Big|_0^1 = \frac{1}{10}+\frac{y}{20}$ for $1\le y\le 5$

$f_{XY}(x,y) \ne f_x(x)\cdot f_y(y)$ so $\boxed{\text{not } \perp\!\!\!\perp}$.

b) $\text{cov}(X,Y) = \mathbb{E}[(X-m_x)\cdot(y-m_y)] = \mathbb{E}[X\cdot Y] - \mathbb{E}[X]\cdot\mathbb{E}[y]$

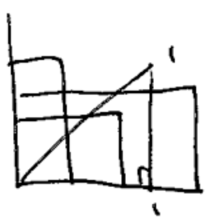
$\mathbb{E}[X] = \iint x\cdot f_{XY}(x,y)\,dy\,dx = \int_1^5\int_0^1 x\cdot\left(\frac{x}{5}+\frac{y}{20}\right)dx\,dy = \frac{4}{15}+\frac{3}{10} = \frac{17}{30}$

$\mathbb{E}[Y] = \iint y\cdot\left(\frac{x}{5}+\frac{y}{20}\right)dx\,dy \quad \frac{6}{5}+\frac{31}{15} = \frac{49}{15}$

$\mathbb{E}[XY] = \iint xy\left(\frac{x}{5}+\frac{y}{20}\right)dx\,dy = \int_1^5 y\int_0^1 \frac{x^2}{5}\,dx\,dy + \int_1^5 y^2\int_0^1 \frac{x}{20}\,dx\,dy = \frac{11}{6}$

$\frac{11}{6} - \frac{17}{30}\cdot\frac{49}{15} = \boxed{\frac{-4}{225}}$

4. (a) $\int_0^1\int_0^x k\,dy\,dx = 1 \quad \int_0^1 kx\,dx = \frac{k}{2}x^2\Big|_0^1 = 1 \rightarrow k=2 \rightarrow f_{XY}(x,y) = \begin{cases} 2 & 0\le y\le x\le 1 \\ 0 & \text{else} \end{cases}$



$F_{XY}(x,y) = \begin{cases} 0 & x<0,\ y<0 \\ x^2 & 0\le x\le 1,\ x\le y \\ 2xy-y^2 & 0<x\le 1 \quad x>y \\ 2y-y^2 & x>1 \quad 0\le y\le 1 \\ 1 & x>1 \quad y>1 \end{cases}$

(b) $F_X(x) = F_{XY}(x,\infty) = \begin{cases} x^2 & 0\le x\le 1 \\ 1 & x>1 \\ 0 & \text{otherwise} \end{cases} \qquad F_Y(y) = F_{XY}(\infty,y) = \begin{cases} 2y-y^2 & 0\le y\le 1 \\ 1 & y>1 \\ 0 & \text{otherwise} \end{cases}$

$f_x(x) = \begin{cases} 2x & 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_y(y) = \begin{cases} 2-2y & 0\le y\le 1 \\ 0 & \text{otherwise} \end{cases}$

(c) $P(A) = P(X\le\frac{1}{2},\ Y\le\frac{3}{4}) = \boxed{\frac{1}{4}}$

$P(B) = F_{XY}(\frac{3}{4},\frac{3}{4}) - F_{XY}(\frac{3}{4},\frac{1}{4}) - F_{XY}(\frac{1}{4},\frac{3}{4}) + F_{XY}(\frac{1}{4},\frac{1}{4})$

$= \boxed{\frac{1}{4}}$

5.

(a)

$$f_X(x) = \int_0^{\sqrt{x}} 6xy \, dy = 3x^2$$

$$= \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

(b)

$$f_Y(y) = \int_{y^2}^{1} 6xy \, d\cancel{x} = 3y(1-y^4)$$

$$= \begin{cases} 3y(1-y^4) & 0 \le y \le 1 \\ 0 & \text{else.} \end{cases}$$

(c)

$$\mathbb{E}[X \mid Y=y] = \int_{y^2}^{1} \frac{2x^2}{1-y^4} \, dx$$

$$= \boxed{\frac{2-2y^6}{3-3y^4}}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{6xy}{3y(1-y^4)}$$

$$= \begin{cases} \frac{2x}{1-y^4} & y^2 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

(d)

$$P[A] = \int_{1/2}^{1} 3x^2 dx = \boxed{\frac{7}{8}}$$

$$f_{X|A}(x) = \frac{P(X=x,\ x \ge 1/2)}{7/8} = \boxed{\frac{8}{7} \cdot \begin{cases} 3x^2 & \text{if } \frac{1}{2} \le x \le 1 \\ 0 & \text{else.} \end{cases}}$$

$$\mathbb{E}[X|A] = \int_{1/2}^{1} x \cdot f_{X|A}(x) \, dx = \int_{1/2}^{1} \frac{24}{7} x^3 dx = \boxed{\frac{45}{56}}$$

6. $f_{X,Y}(x,y) = 8xy \qquad 0 \le y \le x \le 1$

$$f_X(x) = \int_0^x 8xy\,dy = 4x^2 \cdot x = 4x^3$$

$$f_Y(y) = \int_y^1 8xy\,dx = 8y \cdot \tfrac{1}{2}x^2\Big|_y^1 = 4y(1-y^2)$$

$$f_{XY}(x,y) \neq f_X(x) \cdot f_Y(y) \text{ so NOT } \perp\!\!\!\perp \;\square$$

$$f_{X+Y}(z) = \cancel{\int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx = \int_{-\infty}^{\infty} 8xy\,dx = \int_{-\infty}^{\infty} 8 \cdot x \cdot (z-x)}\,dx$$

$$= \cancel{8z\,\tfrac{1}{2}x^2\Big|_{-\infty}^{\infty} = 8z}.$$

$$\cancel{F_Z(z)} = P(Z \le z) = \cancel{\int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx \qquad = \cancel{\int_0^1\int_y^{z-y} f_{xy}(x,y)\,dx\,dy}$$

$0 \le z \le 1$: $\frac{z}{2} \le x \le z$

$1 \le z \le 2$: $\frac{z}{2} \le x \le 1$

$$= \cancel{\int_0^1\int_y^{z-y} 8xy\,dx\,dy = \int_0^1 4yx^2\Big|_y^{z-y}}\,dy$$

$0 \le z \le 1$: $\displaystyle\int_{z/2}^{z} 8x(z-x)\,dx = \boxed{\tfrac{2}{3}z^3}$

$$= \cancel{\int_0^1 4y\cdot((z-y)^2 - y^2)\,dy = \int_0^1 4y(z^2 - 2zy)}\,dy$$

$$= \cancel{2z^2 y^2 - 8z \cdot \tfrac{1}{3}y^3}\Big|_0^1$$

$$\boxed{= \cancel{2z^2 - \tfrac{8}{3}z}}$$

$1 \le z \le 2$: $\displaystyle\int_{z/2}^{1} 8xz - 8x^2\,dx = \boxed{4z - \tfrac{8}{3} - \tfrac{2}{3}z^3}$

$$f_Z(z) = \begin{cases} \frac{2}{3}z^3 & 0 \le z \le 1 \\ -\frac{2}{3}z^3 + 4z - \frac{8}{3} & 1 \le z \le 2 \\ 0 & \text{else.} \end{cases}$$

7.

a) $F_x(x) = 1 - e^{-\lambda x}$

$F_x^{-1}(x) = y$

$x = 1 - e^{-\lambda y} \Rightarrow 1 - x = e^{-\lambda y} \quad y = \frac{\ln(1-x)}{-\lambda}$

b) mean of y = 2.0297

$\mathbb{E}[x] = \frac{1}{\lambda} = 2$

pretty close off by $\frac{0.02}{2} = 1\%$

plots & code

Editor – /Users/nevinliang/Documents/MATLAB/te

test.m +

```
x = rand(5000, 1);
lambda = 0.5;
y = -log(1-x)/lambda;
hold on;
histogram(y)
x=0:.01:10;
func = @(x) lambda * exp(-lambda*x);
plot(x, func(x));
hold off;
mean(y)
```

Command Window