ECE 131A
Probability
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Discussion Set 9 Solutions
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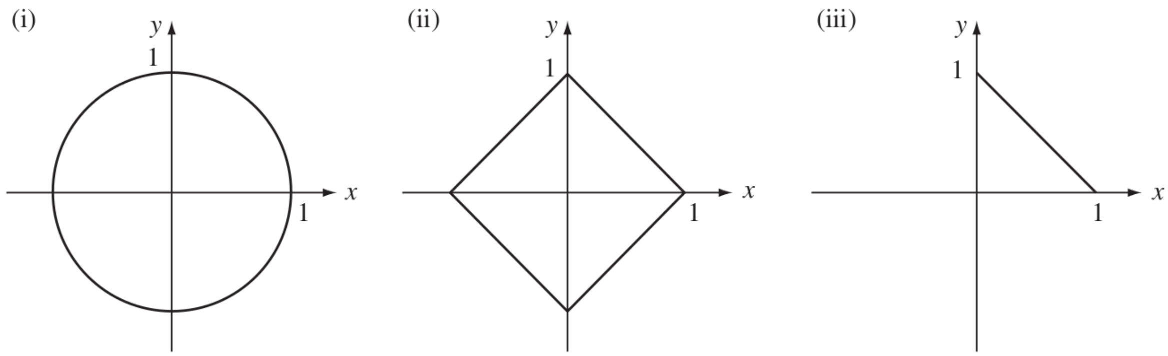
Chapters 5.1-5.10 and 6.4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *(Problem 5.28 and Problem 5.81 of ALG)* The random vector (X, Y) is uniformly distributed (i.e., $f(x, y) = k$) in the regions shown in the following figures and zero elsewhere.



(a) Find the value of k for each case.

Solution:

Since the distributions are uniform, k is equal to $\frac{1}{\text{Area of Section}}$.

(i) $k = \frac{1}{\pi}$ since radius is 1.

(ii) $k = \frac{1}{2}$ since we can just add the four triangles which have an area of 0.5 each.

(iii) $k = 2$

(b) Find the marginal pdf for X and the marginal pdf for Y in each case.

Solution:

(i)

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \begin{cases} \frac{2\sqrt{1-y^2}}{\pi} & -1 \leq x \leq 1 \\ 0 & else \end{cases}$$

By symmetry,

$$f_Y(y) = \begin{cases} \frac{2\sqrt{1-y^2}}{\pi} & -1 \leq y \leq 1 \\ 0 & else \end{cases}$$

(ii)

$$f_X(x) = \int_{-(1-|x|)}^{1-|x|} \frac{1}{2} dy = \begin{cases} 1 - |x| & -1 \leq x \leq 1 \\ 0 & else \end{cases}$$

By symmetry,

$$f_Y(y) = \begin{cases} 1-|y| & -1 \le y \le 1 \\ 0 & else \end{cases}$$

(iii)

$$f_X(x) = \int_0^{1-x} 2dy = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & else \end{cases}$$

By symmetry,

$$f_Y(y) = \int_0^{1-y} 2dx = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & else \end{cases}$$

(c) Find $P(X > 0, Y > 0)$ for each case.
Solution:

(i) $P(X > 0, Y > 0) = \frac{1}{4}$
(ii) $P(X > 0, Y > 0) = \frac{1}{4}$.
(iii) $P(X > 0, Y > 0) = 1$

(d) Find $f_{Y|X}(y|x)$ for each case. **Solution:**

(i)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{1}{2\sqrt{1-x^2}} & -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ 0 & else \end{cases}$$

(ii)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{1}{2(1-|x|)} & -1 \le x \le 1, -(1-|x|) \le y \le 1-|x| \\ 0 & else \end{cases}$$

(iii)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{1}{1-x} & 0 \le x \le 1, 0 \le y \le 1-x \\ 0 & else \end{cases}$$

Note that all of the conditional distributions are uniform.

(e) Find $\mathbb{E}[Y|X = x]$ and $\mathbb{E}[Y]$ for each case.
Solution:

(i)

$$\mathbb{E}[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{y}{2\sqrt{1-x^2}} dy = 0$$

Since this is true regardless of X, then $\mathbb{E}[Y] = 0$.

(ii)

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_{-(1-|x|)}^{1-|x|} \frac{y}{2(1-|x|)}dy = 0$$

Since this is true regardless of X, then $\mathbb{E}[Y] = 0$.

(iii)

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_{0}^{1-x} \frac{y}{1-x}dy = \frac{1-x}{2}$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} \mathbb{E}[Y|X=x] f_X(x) = \int_0^1 \frac{1-x}{2} 2(1-x)dx = \int_0^1 (1-x)^2 = \frac{1}{3}$$

2. *(Problem 5.23 and Problem 5.84 of ALG)*Let the number of uses logged onto a system be the RV N and the time until the next user logs off is the RV T. A joint probability is provided in the following:

$$P(N=n, T \leq t) = (1-p)p^{n-1}(1-e^{-n\lambda t}) \text{ for } n = 1,2,3,4,\ldots \quad t > 0$$

where $0 \leq p \leq 1$ and $\lambda > 0$ are parameters.

(a) Find the marginal pmf of N. **Solution:**

$$P(N=n) = \lim_{t\to\infty} P(N=n, T\leq t) = (1-p)p^{n-1}$$

for $n = 1,2,3,4,\ldots$.

(b) Find the marginal cdf of T. **Solution:**

$$\begin{aligned}
P(T \leq t) &= \sum_{n=1}^{\infty} P(N=n, T\leq t) \\
&= \sum_{n=1}^{\infty} (1-p)p^{n-1}(1-e^{-n\lambda t}) \\
&= \sum_{n=1}^{\infty} (1-p)p^{n-1} - (1-p)\sum_{n=1}^{\infty} p^{n-1}e^{-n\lambda t} \\
&= 1 - (1-p)e^{-\lambda t}\sum_{n=1}^{\infty} p^{n-1}e^{-(n-1)\lambda t} \\
&= 1 - (1-p)e^{-\lambda t}\sum_{n=1}^{\infty} (pe^{-\lambda t})^{n-1} \\
&= 1 - \frac{(1-p)e^{-\lambda t}}{1-pe^{-\lambda t}} = \frac{1-e^{-\lambda t}}{1-pe^{-\lambda t}}
\end{aligned}$$

(c) Find the conditional pdf $f_{T|N}(t|N=n)$.

Solution:

First, we find the conditional cdf.

$$
\begin{aligned}
F_{T|N}(t|N=n) &= \frac{P(N=n, T \le t)}{P(N=n)} \\
&= \frac{(1-p)p^{n-1}(1-e^{-n\lambda t})}{(1-p)p^{n-1}} = 1 - e^{-n\lambda t}
\end{aligned}
$$

Thus,

$$
f_{T|N}(t|n) = \begin{cases} n\lambda e^{-n\lambda t} & t \ge 0 \\ 0 & else \end{cases}
$$

which is an exponenetial RV with parameter $n\lambda$.

(d) Find $\mathbb{E}[T|N=n]$.

Solution:

Since the previous equation showed that it is the exponential RV, the expectation can be quickly derived as $\frac{1}{n\lambda}$.

3. Let X be a zero-mean, unit variance Gaussian RV and A be a Bernoulli random variable with parameter 0.5. Define a new random variable Y such that Y is X when $A=0$ and $-X$ when $A=1$. Is Y a Gaussian RV? Are X and Y uncorrelated? Are X and Y independent? Are X and Y jointly gaussian RVs?

Solution:

Is Y a Gaussian RV? Yes. We can see this by the pdf of Y.

$$
\begin{aligned}
f_Y(y) &= f_Y(x|A=0)p(A=0) + f_Y(-x|A=1)p(A=1) \\
&= \frac{1}{\sqrt{2\pi}}e^{-x^2}0.5 + \frac{1}{\sqrt{2\pi}}e^{-x^2}0.5 \\
&= \frac{1}{\sqrt{2\pi}}e^{-x^2}
\end{aligned}
$$

Are X and Y uncorrelated? Yes. First, note that $Cov(X,Y) = \mathbb{E}[XY]$ since X and Y are zero-mean RVs. Thus,

$$
\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[XY|A=0]P(A=0) + \mathbb{E}[XY|A=1]P(A=1) \\
&= \mathbb{E}[X^2]0.5 + \mathbb{E}[-X^2]0.5 = 0
\end{aligned}
$$

Are X and Y independent? No. Clearly, if X is fixed, Y can only be one of two values. So knowing X, reduces the sample space of Y meaning they can't be independent.

Are X and Y jointly gaussian RVs? No. If they were jointly Gaussian, uncorrelatedness would imply independence which we do not have here.

4. *(Problem 5.111 of ALG)*
   Let X and Y be jointly Gaussian random variables with PDF

$$f_{X,Y}(x,y) = \frac{\exp\left\{-\frac{1}{2}[x^2+4y^2-3xy+3y-2x+1]\right\}}{2\pi c} \quad \text{for all } x,\ y.$$

   (a) Find $E[X]$, $E[Y]$,$VAR[X]$,$VAR[Y]$, and $COV[X,Y]$ by pattern matching the above expression with the expression for jointly Gaussian random variables. Additionally, determine c.
   Solution:

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{\exp\left\{-\frac{1}{2}[x^2+4y^2-3xy+3y-2x+1]\right\}}{2\pi c} \\ &= \frac{\exp\left\{-\frac{1}{2}[(x-1)^2-3(x-1)y+4y^2]\right\}}{2\pi c} \end{aligned}$$

   Recall that the formula for the joint pdf of X and Y (equation (5.61a) in the book) is:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2}+\frac{(y-m_Y)^2}{\sigma_Y^2}-\frac{2\rho(x-m_X)(y-m_Y)}{\sigma_X\sigma_Y}\right)\right].$$

   Therefore, by comparing to the two formulas for we can conclude that:
   $m_1 = 1$, $m_2 = 0$, $\sigma_X^2 = \frac{16}{7}$, $\sigma_Y^2 = \frac{4}{7}$, and $\rho_{X,Y} = \frac{3}{4}$.
   Thus, $COV(X,Y) = \rho_{X,Y}\sigma_X\sigma_Y = \frac{6}{7}$ and $c = \sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2} = \frac{2}{\sqrt{7}}$.

   (b) Confirm the value of $\mathbb{E}[Y]$ by determining the marginal pdf of Y.
   Solution:
   The marginal PDF of Y, $f_Y(y)$ is derived as follows:

$$\begin{aligned} f_Y(y) &= \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_{x=-\infty}^{\infty} \frac{\exp\left\{-\frac{1}{2}[(x-1)^2-3(x-1)y+4y^2]\right\}}{2\pi c} \end{aligned} \tag{1}$$

   To perform this integral, we need to complete a square inside the argument of the exponential.

$$\begin{aligned} &-\frac{1}{2}[(x-1)^2-3(x-1)y+4y^2] \\ &= -\frac{1}{2}[((x-1)^2-3(x-1)y+\frac{9}{4}y^2)-\frac{9}{4}y^2+4y^2] \\ &= -\frac{1}{2}[((x-1)-\frac{3}{2}y)^2-\frac{9}{4}y^2+4y^2] \\ &= -\frac{1}{2}[((x-1)-\frac{3}{2}y)^2+\frac{7}{4}y^2] \end{aligned}$$

Substituting this exponential argument in the integral of $f_Y(y)$ gives us:

$$\begin{aligned}
f_Y(y) &= \int_{x=-\infty}^{\infty} \frac{1}{2\pi c} \exp\left[-\frac{1}{2}[((x-1)-\frac{3}{2}y)^2 + \frac{7}{4}y^2]\right] dx \\
&= \frac{1}{\sqrt{2\pi a}} \exp\left[-\frac{1}{2}(\frac{y}{\frac{2}{\sqrt{7}}})^2\right] \int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi b}} \exp\left[-\frac{1}{2}[((x-1)-\frac{3}{2}y)^2]\right] dx \\
&= \frac{1}{\sqrt{2\pi a}} \exp\left[-\frac{1}{2}(\frac{y}{\frac{2}{\sqrt{7}}})^2\right] \int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi b}} \exp\left[-\frac{1}{2}[(x-1-\frac{3}{2}y)^2]\right] dx
\end{aligned}$$

where$a$ and b a constants such that $ab = c$.
Observe that the term in the integral looks like a Gaussian pdf with mean $(1+\frac{3}{2}y)^2$ and variance 1. Integrating the pdf of any RV results in 1. We now need to assign the correct value to b so as to ensure it is a pdf of a Gaussian RV. For this example, $b = 1$ and $a = \frac{2}{\sqrt{7}}$.
Thus,

$$f_Y(y) = \frac{1}{\sqrt{2\pi\frac{2}{\sqrt{7}}}} \exp\left[-\frac{1}{2}(\frac{y}{\frac{2}{\sqrt{7}}})^2\right]$$

which proves that Y is a Gaussian random variable with mean 0 and variance $\frac{4}{7}$. Thus,we have confirmed $\mathbb{E}[Y] = 0$.

(c) Find $\mathbb{E}[X|Y]$.

Solution:
We will determine $\mathbb{E}[X|Y]$ by finding the conditional pdf first.
The conditional PDF $f_{X|Y}(x|y)$ is derived as follows:

$$\begin{aligned}
f_{X|Y}(x|y) &= f_{X,Y}(x,y)/f_Y(y) \\
&= \frac{\frac{\exp\left\{-\frac{1}{2}[(x-1)^2-3(x-1)y+4y^2]\right\}}{2\pi\frac{2}{\sqrt{7}}}}{\frac{1}{\sqrt{2\pi\frac{2}{\sqrt{7}}}}\exp\left[-\frac{1}{2}(\frac{y}{\frac{2}{\sqrt{7}}})^2\right]} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[(x-1)^2 - 3(x-1)y + 4y^2] + \frac{1}{2}(\frac{y}{\frac{2}{\sqrt{7}}})^2\right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[(x-1)^2 - 3(x-1)y + \frac{9}{4}y^2) - \frac{9}{4}y^2 + 4y^2 - \frac{7}{4}y^2]\right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[((x-1)-\frac{3}{2}y)^2 + \frac{7}{4}y^2 - \frac{7}{4}y^2]\right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[((x-1-\frac{3}{2}y)^2]\right\}
\end{aligned}$$

This proves that $f_{X|Y}(x|y)$ corresponds to another Gaussian random variable with mean $1 + \frac{3}{2}Y$ and variance 1. Thus, $\mathbb{E}[X|Y] = 1 + \frac{3}{2}Y$.