EE 131A	Homework 4 Solutions
Probability and Statistics	Monday, February 1, 2021
Instructor: Lara Dolecek	Due: Monday, February 8, 2021
	before class begins
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Please upload your homework to Gradescope by February 8, 3:59 pm.
Please submit a single PDF directly on Gradescope
You may type your homework or scan your handwritten version. Make sure all the work is discernible.

Reading: 2.6, 3 & 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. The PDF of a continuous random variable X is given by

$$f_X(x) = \begin{cases} c & 0 < x \le 2 \\ 2c & 4 < x \le 6 \\ c & 7 < x \le 9 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constrant.

(a) Find the numerical value of c.
Solution: For a valid PDF, $f_X(x)$ must satisfy $\int_{-\infty}^{\infty} f_X(x) = 1$. Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x) &= \int_0^2 c dx + \int_4^6 2c dx + \int_7^9 c dx \\ &= 2c + 4c + 2c = 1\end{aligned}$$

therefore, the numerical value of c is given by: $c = \frac{1}{8}$.

(b) Compute $P(1 < X \le 8)$.
Solution:

$$P(1 < X \le 8) = \int_1^8 f_X(x) dx = 6c = \frac{3}{4}$$

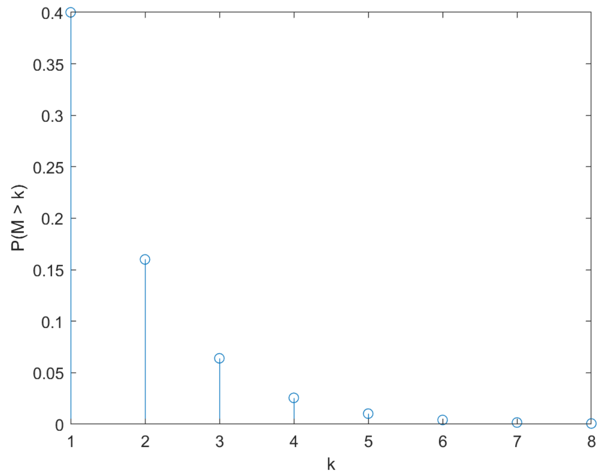
2. Let M be a geometric random variable with parameter p and let X be an exponential random variable with parameter λ.

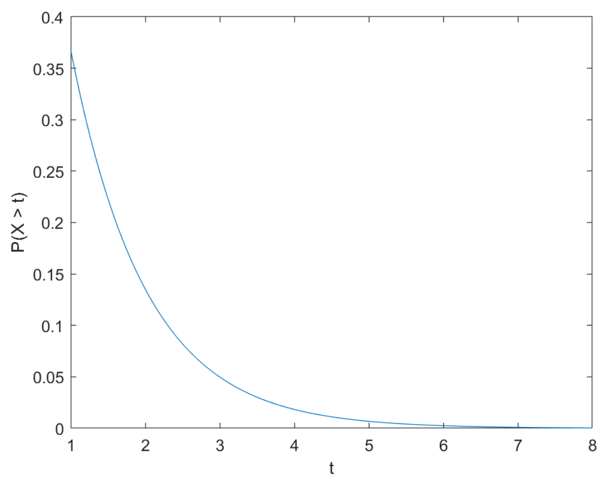
(a) Compute the tail probabilities $P(M > k)$ and $P(X > t)$ for the geometric and the exponential random variable where k is a positive integer and t is a non-negative real number.
Solution: The tail probabilities are given by

$$P(M > k) = 1 - P(M \le k) = 1 - \sum_{j=1}^{k} (1-p)^{j-1} p = (1-p)^k$$

$$P(X > t) = 1 - P(X \le t) = 1 - F_X(t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$$

(b) Plot $P(M > k)$ as a function of k and $P(X > t)$ as a function of t. Use $p = 0.6$ and $\lambda = 1$. Compare the two plots.
Solution: $P(M > k)$ and $P(X > t)$ are shown below:





From the above plots, we can see that both the tail probabilities exponentially decay down to zero.

(c) A continuous random variable X is said to satisfy the memoryless property if for all $t, h \geq 0$, $P(X > t+h|X > t) = P(X > h)$. Prove that the exponential random variable satisfies the memoryless property.
Recall that in Discussion 3, you proved that the geometric random variable satisfies the memoryless property.

Solution:

$$\begin{aligned} P(X > t+h | X > t) &= \frac{P(\{X > t+h\} \cap \{X > t\})}{P(X > t)} \\ &= \frac{P(X > t+h)}{P(X > t)} \quad \text{for } t > 0 \\ &= \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} \\ &= e^{-\lambda h} = P(X > h). \end{aligned}$$

Thus the exponential random variable satisfies the memoryless property.

3. Consider a random variable X with PDF given by $f_X(x) = be^{-a|x|}$.

   (a) If $P(|X| \leq 1) = 1 - e^{-2}$, find a and b.
   Solution:

$$P(|X| \leq 1) = \int_{-1}^{1} f_X(x)\mathrm{d}x = \int_{-1}^{1} be^{-a|x|}\mathrm{d}x = 2\int_{0}^{1} be^{-ax}\mathrm{d}x$$

$$2b\frac{e^{-ax}}{-a}\Big|_0^1 = \frac{2b}{a} - \frac{2b}{a}e^{-a} \longrightarrow \frac{2b}{a} = 1, a = 2 \longrightarrow b = 1.$$

   Note that the third equality in the above equation is due to the fact that $e^{-a|x|}$ is symmetric around 0.

   (b) Find the CDF of X.
   Solution: The CDF of X is given as follows,

$$F_X(x') = P(X \leq x') = \int_{-\infty}^{x'} e^{-2|x|}\mathrm{d}x.$$

   We split the solution to two cases $x' \leq 0$ and $x' > 0$:

   - $x' \leq 0$

$$F_X(x') = \int_{-\infty}^{x'} e^{2x}\mathrm{d}x = \frac{e^{2x}}{2}\Big|_{-\infty}^{x'} = 0.5e^{2x'}.$$

   - $x' > 0$

$$F_X(x') = \int_{-\infty}^{0} e^{2x}\mathrm{d}x + \int_{0}^{x'} e^{-2x}\mathrm{d}x = \frac{e^{2x}}{2}\Big|_{-\infty}^{0} - \frac{e^{-2x}}{2}\Big|_{0}^{x'} = 1 - 0.5e^{-2x'}.$$

   As a result,

$$F_X(x) = \begin{cases} 0.5e^{2x} & x \leq 0 \\ 1 - 0.5e^{-2x} & x > 0. \end{cases}$$

4. Let $X = U^n$ where n is a positive integer. Find the CDF and PDF of X when

(a) U is a uniform random variable in $[0,1]$
Solution:
The CDF is shown as the following:

$$F_X(x) = P[X \leq x] = P[U^n \leq x] = P[U \leq x^{\frac{1}{n}}]$$

Where $0 \leq x^{\frac{1}{n}} \leq 1$.
Since $P[U \leq k] = k$ for $0 \leq k \leq 1$, then we have:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^{\frac{1}{n}}, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Therefore we have the PDF as:
$f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{n} x^{\frac{1}{n}-1}$ for $0 \leq x \leq 1$, and $f_X(x) = 0$ elsewhere.

(b) U is a uniform random variable in $[-1,1]$
Solution:
For this problem, we have to consider when n is odd or even.

When n is odd:

$$\begin{aligned} F_X(x) &= P[X \leq x] = P[U^n \leq x] = P[U \leq x^{\frac{1}{n}}] \\ &= \begin{cases} 0, & x < -1 \\ 0.5 \times (x^{\frac{1}{n}} + 1), & -1 \leq x \leq 1 \\ 1, & x > 1 \end{cases} \end{aligned}$$

$f_X(x) = \frac{d}{dx} F_X(x) = 0.5 \times \frac{1}{n} x^{\frac{1}{n}-1}$ for $-1 \leq x \leq 1$, and $f_X(x) = 0$ elsewhere.

When n is even:

$$\begin{aligned} F_X(x) &= P[X \leq x] = P[U^n \leq x] = P[-x^{\frac{1}{n}} \leq U \leq x^{\frac{1}{n}}] \\ &= \begin{cases} 0, & x < 0 \\ x^{\frac{1}{n}}, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases} \end{aligned}$$

$f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{n} x^{\frac{1}{n}-1}$ for $0 \leq x \leq 1$, and $f_X(x) = 0$ elsewhere.

(c) U is an exponential random variable with parameter 1.
Solution:
We have $F_U(u) = 1 - e^{-u}$, $u \geq 0$. Now,

$$F_X(x) = P[X \leq x] = P[U^n \leq x] = P[U \leq x^{\frac{1}{n}}]$$

We have,

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x^{\frac{1}{n}}}, & 0 \leq x \leq \infty \end{cases}$$

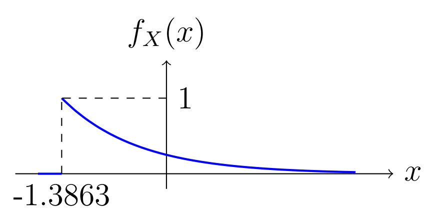
$f_X(x) = \frac{d}{dx}F_X(x) = \frac{1}{n}x^{-(1-\frac{1}{n})}e^{-x^{\frac{1}{n}}}$ for $0 \leq x \leq \infty$, and $f_X(x) = 0$ elsewhere.

5. Find and plot the PDF of $X = -\ln(4-4U)$, where U is a continuous random variable, uniformly distributed on the $[0,1]$ interval.
Solution:
First, we get the CDF. For $x < -\ln(4)$, $F_X(x) = 0$. For $x \geq -\ln(4)$,

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(-\ln(4-4U) \leq x) = P(4-4U \geq e^{-x}) \\ &= P(4U \leq 4 - e^{-x}) = P\left(U \leq 1 - \frac{e^{-x}}{4}\right) \\ &= 1 - \frac{e^{-x}}{4}. \end{aligned}$$

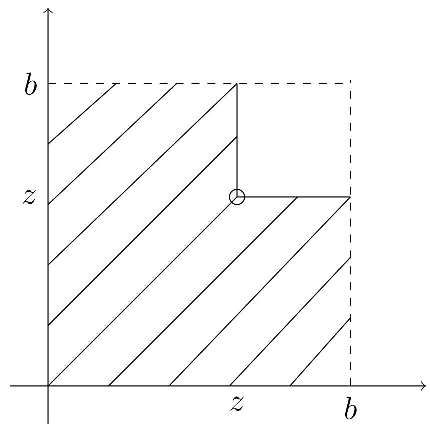
Now we get the PDF. For $x < -\ln(4)$, $f_X(x) = 0$, and for $x \geq -\ln(4)$,

$$\begin{aligned} f_X(x) &= \frac{d}{dx}F_X(x) \\ &= \frac{e^{-x}}{4}, \; x \geq -\ln(4) = -1.3863. \end{aligned}$$



6. A point is selected at random inside a square defined by $\{(x,y) : 0 \leq x \leq b, 0 \leq y \leq b\}$. Assume the point is equally likely to fall anywhere in the square. Let the random variable Z be given by the minimum of the two coordinates of the point.

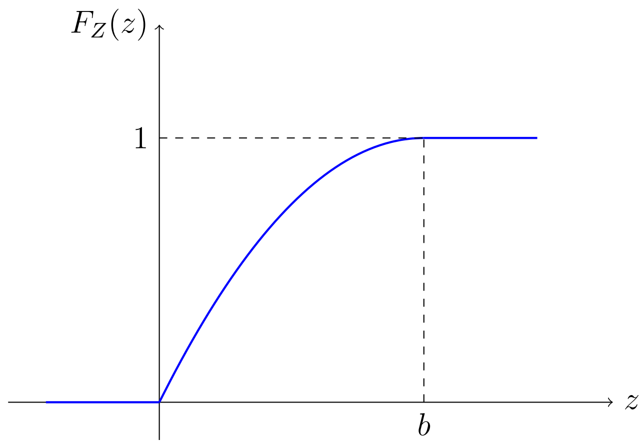
   (a) Find the region in the square corresponding to the event $\{Z \leq z\}$.
   Solution:
   The region is shown below



(b) Find and plot the CDF of Z.
Solutions:
For $0 \leq z \leq b$, the CDF of Z is defined by:

$$F_Z(z) = P(Z \leq z) = 1 - \left(\frac{b-z}{b}\right)^2$$
$$= \frac{2z}{b} - \frac{z^2}{b^2} \quad (0 \leq z \leq b\ )$$

The CDF is plotted below



(c) Use the CDF to find $P[Z > 0], P[Z > b], P[Z \leq b/2], P[Z > b/4]$.
Solution:

$$P[Z > 0] = 1 - P[Z \leq 0] = 1 - 0 = 1$$

$$P[Z > b] = 1 - P[Z \leq b] = 1 - 1 = 0$$

$$P[Z \leq b/2] = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P[Z > b/4] = 1 - P[Z \leq b/4] = 1 - \frac{7}{16} = \frac{9}{16}$$

(d) Find and plot the PDF of Z.
Solution:
The PDF of Z is given by:

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{2}{b} - \frac{2z}{b^2} \quad (0 \leq z \leq b)$$

and the PDF is plotted below