Discussion 4

CDF

$$F_X(x) = P(X \le x)$$

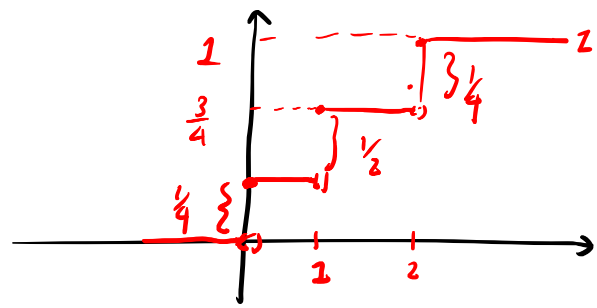
e.g. Toss a coin 2 times

X: No. of heads.

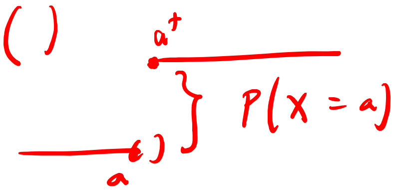
$P(X=0) = \frac{1}{4}$

$P(X=1) = \frac{1}{2}$

$P(X=2) = \frac{1}{4}$



$F_X(\ )$



CDF properties

① $0 \le F_X(x) \le 1$ because $F_X(x) = \underbrace{P(X \le x)}_{0 \le \ \le 1}$

② $\lim_{x \to \infty} F_X(x) = 1$

$\quad \to P(X \le \infty)$

③ $\lim_{x \to -\infty} F_X(x) = 0$

$\quad \to P(X \le -\infty)$

④ Non-decreasing.

$a < b$

$F_X(a) \le F_X(b)$

⑤ Right continuous

$F_X(a) = F_X(a^+)$

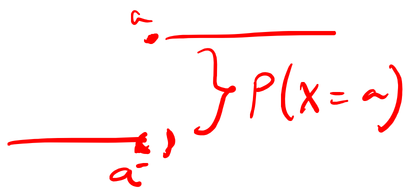
$$\underbrace{P(a < X \le b)} = \underbrace{P(X \le b)} - \underbrace{P(X \le a)}$$

$$= F_X(b) - F_X(a)$$

$$P(a \le X \le b) = \underbrace{P(X \le b)} - \underbrace{P(X < a)}$$

$$= F_X(b) - F_X(a^-)$$

$$F_X(a^-) \ne F_X(a)$$



$X \le a$

e.g. $P(X > a) = 1 - P(X \le a)$

$$= 1 - F_X(a)$$

Continuous R.Vs

- CDF does not have jumps.
- $P(X = a) = \underline{0}$

PDF

$$\underline{f_X(x)} = \frac{d}{dx} F_X(x) \quad \text{where the derivative exists.}$$

$$\left. \underline{F_X(x)} = \int_{-\infty}^{x} f_X(x)\, dx \right\}$$

Properties.

1) $f_X(x) \ge 0$ CDF is nondecreasing

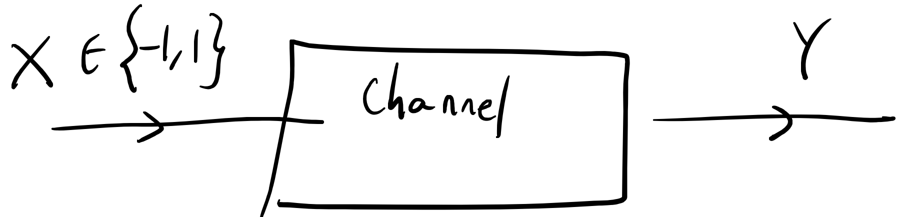
2) $\int_{-\infty}^{\infty} f_X(x) = 1$ $F_X(\infty) = 1$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

$$\left.\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx \\ E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \end{aligned}\right\}$$

$$\underline{VAR[X]} = E[X^2] - (E[X])^2$$

Q1. PMF of 2 RVs



$$Y = \begin{cases} X & \text{w.p} \quad 1 - p - P_e \\ -X & \text{w.p} \quad p \\ 0 & \text{w.p} \quad P_e \end{cases}$$

a) S_{XY} (Range of (X, Y))

$$S_{XY} = \left\{ \begin{matrix} (-1,1), & (-1,-1), & (-1,0), \\ (1,1), & (1,-1), & (1,0) \end{matrix} \right\}$$

b) Calculate the Probability of all pairs of (X, Y)

$P(X=-1) = \frac{1}{4}$ $\quad$ $P(X=1) = \frac{3}{4}$

$P(A\cap B)=P(A)\cdot P(B|A)$

$P(X=-1)\cdot P(Y=1|X=-1)$

$$P(X=-1,\ Y=1) = \frac{1}{4}\cdot P \qquad \frac{1}{4}\cdot\frac{1}{6}$$

$P(X=1)\cdot P(Y=1|X=1)$

$$P(X=1,\ Y=1) = \frac{3}{4}\cdot(1-P-P_e)$$

$$P(X=-1,\ Y=0) = \frac{1}{4}\cdot P_e.$$

c) $$\underbrace{P(X\neq Y)} = P(X=-1,\ Y=1) + P(X=-1,\ Y=0) + P(X=1,\ Y=-1) + P(X=1,\ Y=0)$$

$$= 1 - \underbrace{P(X=1,\ Y=1)} - \underbrace{P(X=-1,\ Y=-1)}$$

$$= 1 - \frac{3}{4}\cdot(1-P-P_e) - \frac{1}{4}(1-P-P_e)$$

$$= 1 - (1-P-P_e)$$

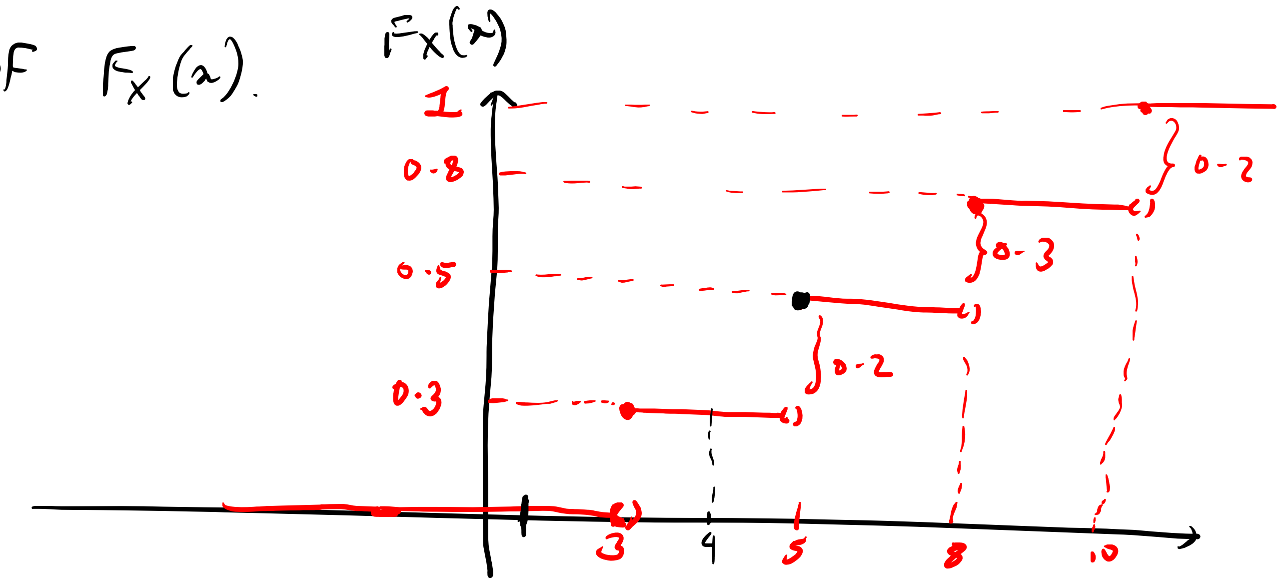
$$= \underline{P+P_e}$$

$$P(Y=0) = P_e. \leftarrow P(X=-1,\ Y=0) + P(X=1,\ Y=0)$$

Q2.

$$P_X(x) = \begin{cases} 0.3 & \text{---- } x=3 \\ 0.2 & \text{---- } x=5 \\ 0.3 & \text{---- } x=8 \\ 0.2 & \text{---- } x=10 \\ 0 & \text{---- otherwise} \end{cases}$$

CDF $F_X(x)$.



$$\underbrace{P(2 < X \le 5)} = P(X \le 5) - P(X \le 2)$$

$$= F_X(5) - \underbrace{F_X(2)}$$

$$= 0.5 - 0$$

$$= 0.5$$

$$P(X > 4) = 1 - \underbrace{P(X \le 4)}$$

$$= 1 - F_X(4)$$

$$= 1 - 0.3$$

$$= 0.7.$$

Q3.

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases} \qquad \lambda > 0$$

Q. Is it a valid CDF ?

check all CDF. properties.

$x < 0$ $\quad 0$

$x \geq 0$ $\quad 1 - e^{-\lambda x}$

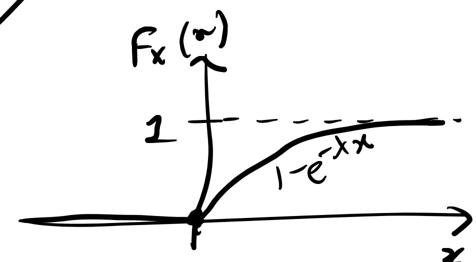
$\lambda > 0$

1) $0 \leq F_X(x) \leq 1$ ✓

2) $\lim_{x \to \infty} F_X(x) \overset{?}{=} 1$ ✓ $\qquad \underbrace{\lim_{x \to \infty} \left(1 - \underbrace{\cancel{e^{-\lambda x}}}_{0}\right)}_{1}$

3) $\lim_{x \to -\infty} F_X(x) = 0$ ✓

4) Non-decreasing ? ✓



$x \uparrow \quad e^{-\lambda x} \downarrow \quad 1 - e^{-\lambda x} \uparrow \quad x = 0$

5) Right continuous ✓ $\qquad$ $F_X(x)$ is continuous

YES VALID.

ii) $f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$ $\quad \lambda > 0$

Q. Is is a valid PDF ?

check properties

i) $f_X(x) \geq 0$ → ✓

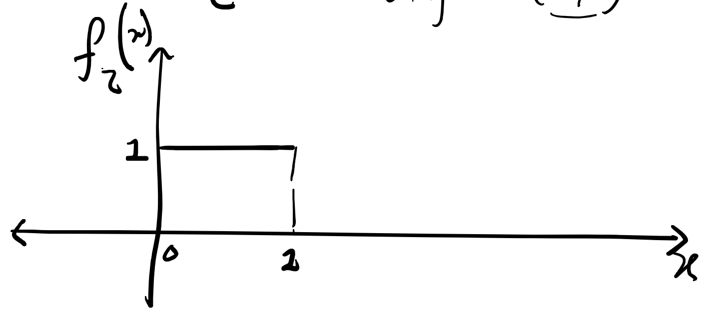
ii) $\int_{-\infty}^{\infty} f_X(x) = 1$ ✓

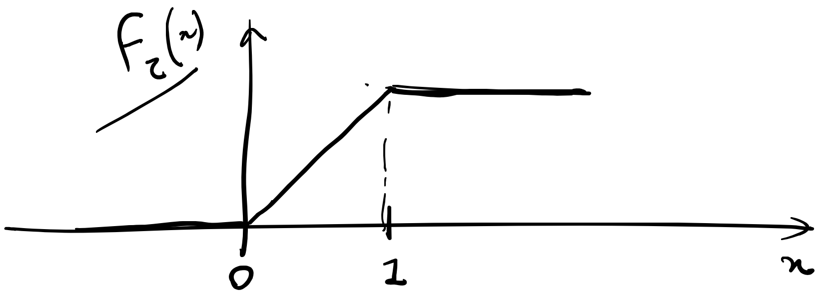
$$\int_0^{\infty} \lambda e^{-\lambda x} = \lambda \left(\frac{-e^{-\lambda x}}{\lambda} \right)_0^{\infty} = 1$$

YES VALID PDF.

Q.4. $Z \sim$ Uniform $(0,1)$

$\frac{1}{b-a}$





$$X = \frac{1}{\sqrt{1 - Z}}$$

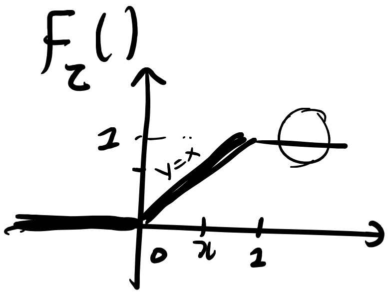
CDF of X ?

$$F_X(x) = P(X \le x) \qquad \text{CDF} \quad \underbrace{P(Z \le \,?)}_{F_Z(?)}$$

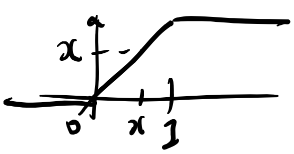
$$= P\left(\frac{1}{\sqrt{1-Z}} \le x\right)$$

$$= P\left(\frac{1}{x^2} \le 1-Z\right)$$

$$= P\left(Z \le 1-\frac{1}{x^2}\right) = F_Z\left(1-\frac{1}{x^2}\right)$$



$$= \begin{cases} 0 & 1-\frac{1}{x^2} < 0 \\ 1-\frac{1}{x^2} & 0 < 1-\frac{1}{x^2} < 1 \\ 1 & 1-\frac{1}{x^2} > 1 \end{cases}$$



$$F_X(x) = \begin{cases} 0 & 1-\frac{1}{x^2} < 0 \quad x<1 \\ 1-\frac{1}{x^2} & 0 \le 1-\frac{1}{x^2} \le 1 \quad x \ge 1 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x<1 \\ 1-\frac{1}{x^2} & x \ge 1 \end{cases}$$

$$P(5 < X < 7) = \underbrace{F_X(7) - F_X(5)} \qquad 7 \quad 5$$

$$= \left(1-\frac{1}{7^2}\right) - \left(1-\frac{1}{5^2}\right)$$