EE 131A	Homework 3 Solutions
Probability and Statistics	Wednesday, January 20, 2021
Instructor: Lara Dolecek	Due: Monday, February 1, 2021
	before class begins
TA: Lev Tauz	levtauz@ucla.edu
Debarnab Mitra	debarnabucla@ucla.edu

Please upload your homework to Gradescope by February 1, 3:59 pm.
Please submit a single PDF directly on Gradescope
You may type your homework or scan your handwritten version. Make sure all the work is discernible.

Reading: 2.6, 3 & 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the probability mass function of X, the total dollar value of all sales.
Solution:
There are 9 possible outcomes, as summarized in Table 1. Summing all possible ways to get the various values of X we find

$$\begin{aligned}
P[X = 0] &= 0.28 \\
P[X = 500] &= 0.21 + 0.06 = 0.27 \\
P[X = 1000] &= 0.21 + 0.045 + 0.06 = 0.315 \\
P[X = 1500] &= 0.045 + 0.045 = 0.09 \\
P[X = 2000] &= 0.045.
\end{aligned}$$

Sale from Customer 1	Sales from Customer 2	X	Probability
0	0	0	$(1-0.3)(1-0.6) = 0.28$
0	500	500	$(1-0.3)(0.6)(0.5) = 0.21$
0	1000	1000	$(1-0.3)(0.6)(0.5) = 0.21$
500	0	500	$(0.3)(0.5)(1-0.6) = 0.06$
500	500	1000	$(0.3)(0.5)(0.6)(0.5) = 0.045$
500	1000	1500	$(0.3)(0.5)(0.6)(0.5) = 0.045$
1000	0	1000	$(0.3)(0.5)(1-0.6) = 0.06$
1000	500	1500	$(0.3)(0.5)(0.6)(0.5) = 0.045$
1000	1000	2000	$(0.3)(0.5)(0.6)(0.5) = 0.045$

Table 1: Encyclopedia Sales.

2. Suppose that a die is rolled twice. Let X and Y denote the maximum and minimum value to appear in the two rolls respectively. What are the possible values that X and Y can take? Compute the pmf of X and Y. What is the expectation of $X+Y$?
   Solution: X and Y both take the values {1,2,3,4,5,6}. They have the following pmf:

$$
\begin{aligned}
P(X=6) &= \frac{11}{36} = P(Y=1) \\
P(X=5) &= \frac{9}{36} = P(Y=2) \\
P(X=4) &= \frac{7}{36} = P(Y=3) \\
P(X=3) &= \frac{5}{36} = P(Y=4) \\
P(X=2) &= \frac{3}{36} = P(Y=5) \\
P(X=1) &= \frac{1}{36} = P(Y=6)
\end{aligned}
$$

   Let A be the number obtained on the first die roll and B be the number obtained on the second die roll. Then $X+Y=A+B$. Thus $E[X+Y]=E[A+B]=E[A]+E[B]=3.5+3.5=7$. (Recall from lecture that a discrete random variable taking values uniformly in $\{1,2,\dots L\}$ has expectation $\frac{L+1}{2}$. Here A and B are uniform in $\{1,2,3,4,5,6\}$).

3. Two coins are simultaneously tossed until one of them comes up a head and the other a tail. The first coin comes up a head with probability p and the second with probability q. All tosses are assumed independent.

   (a) Find the PMF, the expected value, and the variance of the number of tosses.
   Solution: Let X be the number of tosses until one of them comes up as a head and the other as a tail. This experiment can be seen as a series of Beroulli trials with the success determined by the outcomes $\{HT, TH\}$. Thus X is a Geometric RV with probability of success

$$P(\{HT, TH\}) = p(1-q) + q(1-p).$$

   Thus the PMF of X is given by

$$P(X=k) = (1-p(1-q)-q(1-p))^{k-1}\left(p(1-q)+q(1-p)\right), \quad k=1,2,\dots$$

   Therefore

$$
\begin{aligned}
E[X] &= \frac{1}{p(1-q)+q(1-p)} \\
VAR(X) &= \frac{1-p(1-q)-q(1-p)}{(p(1-q)+q(1-p))^2}.
\end{aligned}
$$

(b) What is the probability that the last toss of the first coin is a head?
Solutions: The probability that the last toss of the first coin is a head is

$$P(HT|\{HT,TH\}) = \frac{P(HT)}{P(\{HT,TH\})} = \frac{p(1-q)}{p(1-q)+q(1-p)}$$

4. Suppose X is a Binomial random variable with parameters $n = 4$, and p.

(a) Express $E\left[sin\left(\pi X/2\right)\right]$ in terms of p.
Solution:
The PMF of X is given by:

$$p_X(x) = \binom{4}{x} p^x(1-p)^{4-x}, \; x = 0, 1, 2, \ldots, n.$$

As a result, we conclude that:

$$\begin{aligned} E\left[sin\left(\frac{\pi X}{2}\right)\right] &= \sum_{x=0}^{4} sin\left(\frac{\pi x}{2}\right)\binom{4}{x} p^x(1-p)^{4-x} \\ &= 0 + sin\left(\frac{\pi}{2}\right)\binom{4}{1} p(1-p)^3 + 0 + sin\left(\frac{3\pi}{2}\right)\binom{4}{3} p^3(1-p) + 0 \\ &= 4p(1-p)^3 - 4p^3(1-p) = 4p(1-p)(1-2p). \end{aligned}$$

(b) Express $E\left[cos\left(\pi X/2\right)\right]$ in terms of p.
Solution:
Similar to part (a), we can see that:

$$\begin{aligned} E\left[cos\left(\frac{\pi X}{2}\right)\right] &= \sum_{x=0}^{4} cos\left(\frac{\pi x}{2}\right)\binom{4}{x} p^x(1-p)^{4-x} \\ &= cos\left(0\right)\binom{4}{0}(1-p)^4 + 0 + cos\left(\pi\right)\binom{4}{2} p^2(1-p)^2 \\ &+ 0 + cos\left(2\pi\right)\binom{4}{4} p^4 = (1-p)^2(1-2p-5p^2) + p^4. \end{aligned}$$

5. A modem transmits over a binary error-prone channel. To improve reliability, it transmits each bit, i.e "0" or "1", five times. We call each such group of five bits a "codeword." The channel changes an input bit to its complement with probability $p = 1/10$ and it does so independently of its treatment of other input bits. The modem receiver takes a majority vote of the five received bits to estimate the input bit. Find the probability that the receiver makes the wrong decision.
Solution:
A wrong decision will be made by the receiver if at least 3 of the 5 received bits are in error. Thus the probability of error can be written as

$$\begin{aligned} P(\text{decoding error}) &= P(3 \text{ or more bits in error}) \\ &= \binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p) + \binom{5}{5} p^5 \end{aligned}$$

Substituting $p = \frac{1}{10}$, we get $P(\text{decoding error}) = 0.00856$.

6. Consider a random variable X with pdf given by

$$f_X(x) = \begin{cases} cx(1-x^2) & 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

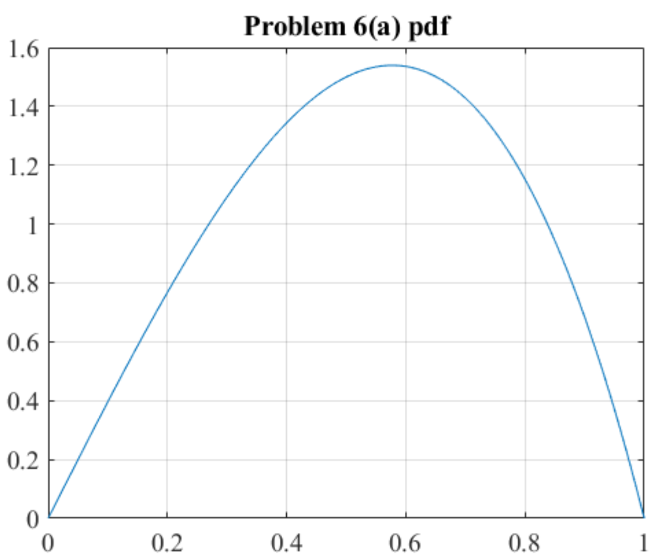
   (a) What is c? Plot the pdf of X using MATLAB.

   (b) Plot the cdf of X using MATLAB.

   (c) Find $P(0.25 < X < 0.3)$.

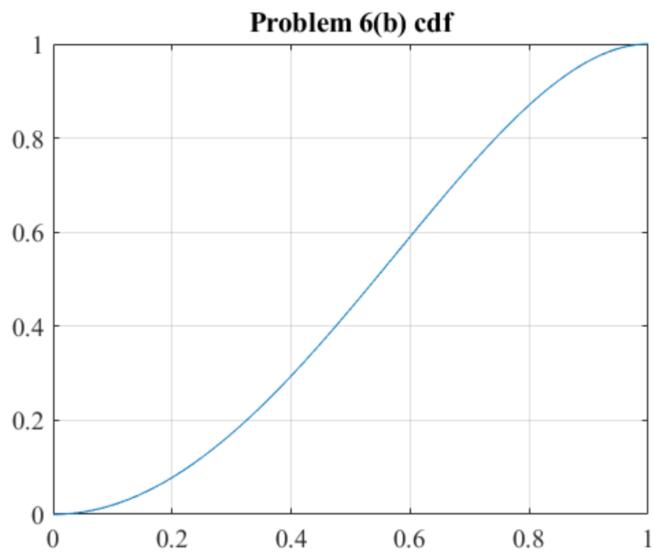
   Solution:
   (a) Using $\int_{-\infty}^{\infty} f_X(x)dx = 1$, we get

$$\begin{aligned} 1 &= c\int_0^1 x(1-x^2)dx \\ \implies 1 &= c\Big[\frac{x^2}{2} - \frac{x^4}{4}\Big]_0^1 \\ \implies c &= 4 \end{aligned}$$



(b)

$$
\begin{aligned}
F_X(x) &= \int_0^x f_X(x)dx \\
&= 4\int_0^x (x - x^3)dx \\
&= 4\Big[\frac{x^2}{2} - \frac{x^4}{4}\Big]
\end{aligned}
$$



(c) $P(0.25 < X < 0.3) = F_X(0.3) - F_X(0.25) = 0.0508$