EE 131A
Probability and Statistics
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Homework 6 solution
Wednesday, February 17, 2021
Due: Monday, March 3, 2021
before class begins
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Please upload your homework to Gradescope by March 1, 3:59 pm.
Please submit a single PDF directly on Gradescope
You may type your homework or scan your handwritten version. Make sure all the work is discernible.

Reading: 5.1-5.10 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. The pair (X, Y) has joint cdf given by:

$$F_{X,Y}(x,y) = \begin{cases} (1-\frac{1}{x^2})(1-\frac{1}{y^2}) & \text{for } x > 1, y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Plot the joint cdf as a 3D plot using MATLAB.
Solution:
The joint cdf is sketched in Figure 1:

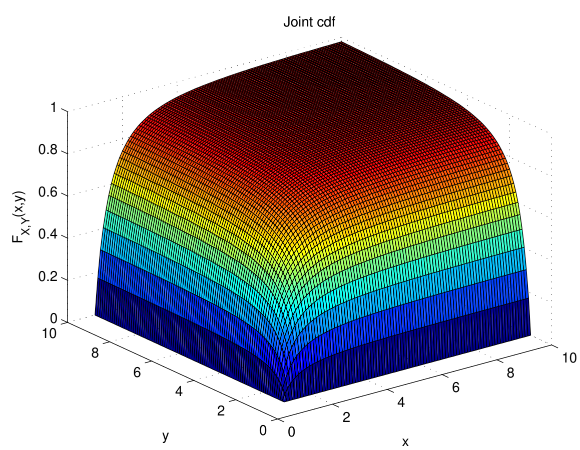

Figure 1: Joint cdf in Problem 3

(b) Find the marginal cdf of X and of Y.
Solution:
The marginal cdf of X and Y can be computed as follows:

$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = 1 - \frac{1}{x^2},\ x > 1$$

$$F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y) = 1 - \frac{1}{y^2},\ y > 1$$

(c) Find the probability of the following events: $\{X < 3, Y \leq 5\}, \{X > 4, Y > 3\}$.
Solution:
The probabilities of the following events are given by:

$$P\{X < 3, Y \leq 5\} = F_{X,Y}(3,5) = (1 - \frac{1}{3^2})(1 - \frac{1}{5^2}) = \frac{64}{75}$$

$$\begin{aligned} P\{X > 4, Y > 3\} &= 1 - F_{X,Y}(4,\infty) - F_{X,Y}(\infty,3) + F_{X,Y}(4,3) \\ &= 1 - \frac{15}{16} - \frac{8}{9} + \frac{5}{6} = \frac{1}{144} \end{aligned}$$

2. Let X and Y have the joint pdf:

$$f_{X,Y}(x,y) = ye^{-y(1+x)} \quad \text{for } x > 0, y > 0.$$

Verify that $f_{X,Y}(x,y)$ is a valid pdf. Find the marginal pdf of X and of Y.
Solution:
The joint pdf is given by:

$$f_{X,Y}(x,y) = ye^{-y(1+x)},\ x > 0,\ y > 0$$

$f_{X,Y}(x,y)$ satisfies the following properties:

- $f_{X,Y}(x,y) \geq 0$ for all $x > 0, y > 0$.
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = \int_0^\infty\int_0^\infty ye^{-y(1+x)}dxdy = \int_0^\infty e^{-y}dy = 1.$

Thus it is a valid pdf. The marginal pdfs of X and Y can be found as follows:

$$\begin{aligned} f_X(x) &= \int_0^\infty ye^{-y(1+x)}\mathrm{d}y = -\frac{1}{1+x}e^{-y(1+x)}y\ |_0^\infty + \frac{1}{1+x}\int_0^\infty e^{-y(1+x)}\mathrm{d}y \\ &= \frac{1}{(1+x)^2},\ x > 0 \end{aligned}$$

$$f_Y(y) = \int_0^\infty ye^{-y(1+x)}\mathrm{d}x = ye^{-y}\int_0^\infty e^{-xy}\mathrm{d}x = e^{-y},\ y > 0$$

3. The joint pdf for two random variables X and Y is given below:

$$f_{XY}(x,y) = \begin{cases} \frac{x}{5} + \frac{y}{20}, & 0 \leq x \leq 1, 1 \leq y \leq 5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are these two random variables independent?
Solution: We first find the marginal pdfs:

$$f_X(x) = \begin{cases} \int_1^5 f_{XY}(x,y)\mathrm{d}y = \int_1^5 (\frac{x}{5} + \frac{y}{20})\mathrm{d}y = \frac{4}{5}x + \frac{3}{5}, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^1 f_{XY}(x,y)\mathrm{d}x = \int_0^1 (\frac{x}{5} + \frac{y}{20})\mathrm{d}x = \frac{1}{20}y + \frac{1}{10}, & 1 \leq y \leq 5, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_{XY}(x,y) = f_X(x)f_Y(y)$ does not hold $\forall x, y$, these two random variables are not independent.

(b) Find the covariance for X and Y.

Solution:

$$
\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\mathrm{d}y\mathrm{d}x = \int_0^1\int_1^5 xy(\frac{x}{5}+\frac{y}{20})\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \frac{x^2}{5}\int_1^5 y\mathrm{d}y\mathrm{d}x + \int_0^1 \frac{x}{20}\int_1^5 y^2\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 12\frac{x^2}{5}\mathrm{d}x + \int_0^1 \frac{124}{3}\frac{x}{20}\mathrm{d}x = \frac{12}{15}+\frac{124}{120} = \frac{220}{120}. \\
E[X] &= \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x = \int_0^1 x(\frac{4}{5}x+\frac{3}{5})\mathrm{d}x \\
&= \frac{4}{5}\int_0^1 x^2\mathrm{d}x + \frac{3}{5}\int_0^1 x\mathrm{d}x = \frac{4}{15}+\frac{3}{10} = \frac{17}{30}. \\
E[Y] &= \int_{-\infty}^{\infty} y f_Y(x)\mathrm{d}y = \int_1^5 y(\frac{1}{20}y+\frac{1}{10})\mathrm{d}y \\
&= \frac{1}{20}\int_1^5 y^2\mathrm{d}y + \frac{1}{10}\int_1^5 y\mathrm{d}y = \frac{124}{60}+\frac{12}{10} = \frac{98}{30}. \\
COV[X,Y] &= E[XY]-E[X]E[Y] = \frac{220}{120} - \frac{17}{30}\frac{98}{30} = -\frac{4}{225}.
\end{aligned}
$$

4. A point (X, Y) is selected at random inside a triangle defined by $\{(x,y) : 0 \le y \le x \le 1\}$. Assume the point is equally likely to fall anywhere in the triangle.

(a) Find the joint cdf and joint pdf of X and Y.

Solution:

Note the area of triangle is 1/2, then the joint pdf is

$$
f_{XY}(x,y) = \begin{cases} 2 & \text{for } 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Therefore the joint cdf of X and Y can be found under 4 different regions.

i. If $0 \le y \le x \le 1$, the joint cdf of X and Y is given by:

$$
F_{XY}(x,y) = \int_0^y\int_u^x 2dvdu = 2xy - y^2
$$

ii. If $0 \le x \le y, x \le 1$, the joint cdf of X and Y is given by:

$$
F_{XY}(x,y) = \int_0^x\int_0^v 2dudv = x^2
$$

iii. If $0 \le y \le 1 \le x$, the joint cdf of X and Y is given by:

$$
F_{XY}(x,y) = \int_0^y\int_u^1 2dvdu = 2y - y^2
$$

iv. If $x \geq 1, y \geq 1$, the joint cdf of X and Y is given by:

$$F_{XY}(x, y) = 1$$

v. $F_{XY}(x, y) = 0$ for $x < 0, y < 0$.

(b) Find the marginal cdf and marginal pdf of X and Y.
Solution:
The marginal cdf of X and Y can be found by taking limit of joint cdf:

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x, y) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$$F_Y(y) = \lim_{x\to\infty} F_{XY}(x, y) = \begin{cases} 0 & \text{if } y \leq 0 \\ 2y - y^2 & \text{if } 0 < y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

The marginal pdf of X and Y can be found by taking derivative of marginal cdf:

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 2x & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \frac{d}{dx} F_Y(y) = \begin{cases} 2 - 2y & \text{for } 0 \leq y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find the probabilities of the following events in terms of the joint cdf: $A = \{X \leq 1/2, Y \leq 3/4\}$; $B = \{1/4 < X \leq 3/4, 1/4 < Y \leq 3/4\}$.
Solution:
The probabilities of A and B are given by:

$$P(A) = F_{XY}(\frac{1}{2}, \frac{3}{4}) = (\frac{1}{2})^2 = \frac{1}{4}$$

$$\begin{aligned} P(B) &= F_{XY}(\frac{3}{4}, \frac{3}{4}) - F_{XY}(\frac{1}{4}, \frac{3}{4}) - F_{XY}(\frac{3}{4}, \frac{1}{4}) + F_{XY}(\frac{1}{4}, \frac{1}{4}) \\ &= (\frac{3}{4})^2 - (\frac{1}{4})^2 - [2\frac{1}{4}\frac{3}{4} - (\frac{1}{4})^2] + (\frac{1}{4})^2 \\ &= \frac{1}{4} \end{aligned}$$

(d) Find $P[Y < X^2]$.
Solution:

$$P[Y < X^2] = \int_0^1 \int_0^{x^2} 2dydx = \frac{2}{3}$$

5. Let X and Y be two jointly continuous random variables with joint pdf

$$f_{XY}(x,y) = \begin{cases} 6xy, & 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}, \\ 0, & \text{otherwise}, \end{cases}$$

(a) Find $f_X(x)$.
Solution:
For $0 \leq x \leq 1$,

$$\begin{aligned} f_X(x) &= \int_{\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^{\sqrt{x}} 6xydy \\ &= 3x^2 \end{aligned}$$

Thus

$$f_X(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find the conditional pdf of X given $Y = y$, $f_{X|Y}(x|y)$.
Solution:
For $0 \leq y \leq 1$,

$$\begin{aligned} f_Y(y) &= \int_{\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_{y^2}^{1} 6xydy \\ &= 3y(1-y^4) \end{aligned}$$

Thus

$$f_Y(y) = \begin{cases} 3y(1-y^4), & 0 \leq y \leq 1 \\ 0, & \text{otherwise}, \end{cases}$$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{2x}{1-y^4} & y^2 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(c) Find $E[X|Y = y]$, for $0 \leq y < 1$. What is $E[X|Y]$?
Solution:
For $0 \leq y < 1$

$$\begin{aligned} E[X|Y = y] &= \int_{\infty}^{\infty} x f_{X|Y}(x|y)dx \\ &= \int_{y^2}^{1} x\frac{2x}{1-y^4}dx \\ &= \frac{2(1-y^6)}{3(1-y^4)} \end{aligned}$$

$E[X|Y] = \frac{2(1-Y^6)}{3(1-Y^4)}$

(d) Let A be the event $\{X \geq \frac{1}{2}\}$. Find $P[A]$, $f_{X|A}(x)$, and $E[X|A]$.
Solution:

$$P(A) = \int_A f_X(x)dx = \int_{\frac{1}{2}}^1 3x^2 dx = \frac{7}{8}$$

$$f_{X|A}(x|A) = \begin{cases} \frac{f_X(x)}{P[A]} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|A}(x|A) = \begin{cases} \frac{24x^2}{7} & \text{if } \frac{1}{2} \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$E[X|A] = \int_{\infty}^{\infty} x f_{X|A}(x|A)dx = \int_{\frac{1}{2}}^1 x\frac{24x^2}{7}dx = \left[\frac{6x^4}{7}\right]_{\frac{1}{2}}^1 = \frac{45}{56}$$

6. The random variables X and Y have the joint pdf

$$f_{X,Y}(x,y) = 8xy \quad \text{for } 0 \leq y \leq x \leq 1.$$

Are X and Y independent? Find the pdf of $Z = X + Y$.
Solution:

$$f_X(x) = \int_0^x 8xydy = 4x^3$$

$$f_Y(y) = \int_y^1 8xydx = 4y(1-y^2)$$

.

Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

$$\begin{aligned} F_Z(z) &= P[Z \leq z] \\ &= P[X + Y \leq z] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)dydx \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \int_{-\infty}^{\infty}\left[\frac{d}{dz}\int_{-\infty}^{z-x} f(x,y)dy\right]dx \\ &= \int_{-\infty}^{\infty} f(x, z-x)dx \end{aligned}$$

Now, $f(x, z-x) = 8x(z-x)$ for $0 \leq z-x \leq x \leq 1$.

Thus, when $0 \leq z \leq 1$, $f(x, z-x)$ is non-zero when $\frac{z}{2} \leq x \leq z$. Hence

$$f_Z(z) = \int_{\frac{z}{2}}^{z} 8x(z-x)dx = \frac{2z^3}{3}$$

When, $1 \leq z \leq 2$, $f(x, z-x)$ is non-zero when $\frac{z}{2} \leq x \leq 1$. Hence

$$f_Z(z) = \int_{\frac{z}{2}}^{1} 8x(z-x)dx = 4z - \frac{8}{3} - \frac{2z^3}{3}$$