Discussion E.5

Lecture 7 Review

- New RVs.
  - Exponential RV. $Exp(\lambda)$

$$f(x) = \begin{cases} 0 & x<0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda} \quad Var[X] = \frac{1}{\lambda^2}$$

$$P(X > a+b \mid X > b) = P(X > a)$$

  - Gaussian RV. $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$$E[X] = \mu \quad Var[X] = \sigma^2$$

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X}{n} \sim N(\overset{E[X]}{\mu}, \sigma^2) \quad (CLT)$$

$$Q(a) = P(X > a) \quad X \sim N(0,1)$$

$$Y \sim N(\mu, \sigma^2)$$

$$P(Y > a) = P\left(X > \frac{a-\mu}{\sigma}\right) = Q\left(\frac{a-\mu}{\sigma}\right)$$

$$\Phi(a) = P(X \leq a) = 1 - Q(a)$$

  - Chi-squared

$$Y = \sum_{i=1}^{k} X_i^2 \quad X_i \sim N(0,1)$$

$$f(x) = \frac{y^{\frac{k-2}{2}} \cdot e^{-\frac{y}{2}}}{2^{k/2} \cdot \Gamma(k/2)}$$

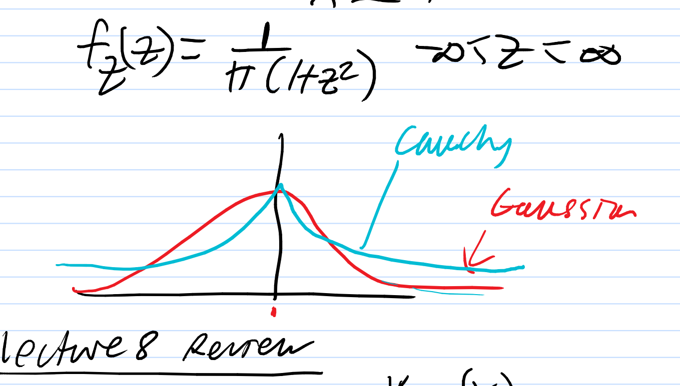
$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \leftarrow \text{Gamma Function}$$

$$\Gamma(n) = (n-1)!$$

  - Cauchy RV.

$$Z = \frac{X}{Y} \quad X, Y \sim N(0,1) \quad X \perp Y$$

$$f_Z(z) = \frac{1}{\pi(1+z^2)} \quad -\infty < z < \infty$$



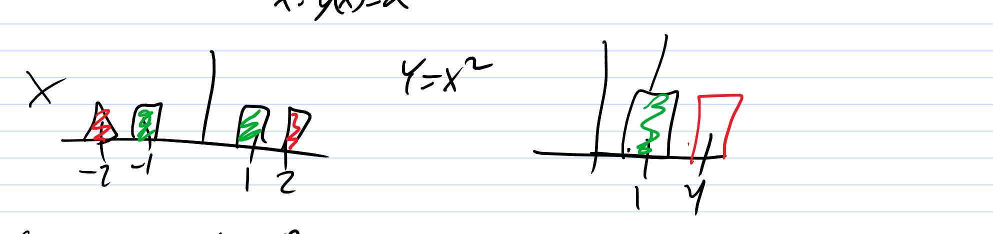
Lecture 8 Review

Function of RV. $Y = g(X)$

		Y: Discrete	Y: Continuous
X	Discrete	Lec	Now
	Continuous	Lec	Lec

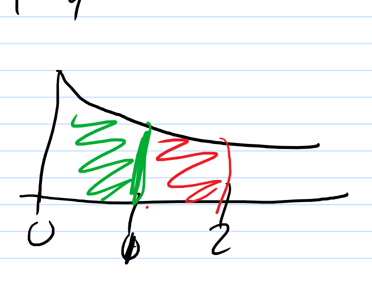
Discrete ⇒ Discrete

$$P(Y=k) = \sum_{x:\, g(x)=k} P(X=x)$$



Continuous to Discrete

$$P(Y=k) = \int_{x:\, g(x)=k} f(x)\,dx$$



Continuous ⇒ Continuous

$$Y = g(X)$$

$$P(Y \leq y) = P(g(X) \leq y) = P(X \leq z : z \in g^{-1}(y))$$

Problem 1

$$X \sim N(10, 36)$$

a) $P(X > 5) = P\left(\frac{X-\mu}{\sigma} > \frac{5-\mu}{\sigma}\right)$

$Y = \frac{X-\mu}{\sigma} = N(0,1)$ $= P\left(Y > \frac{5-10}{6}\right) = Q\left(\frac{5-10}{6}\right) = Q\left(\frac{-5}{6}\right)$

b) $P(4 < X < 16)$

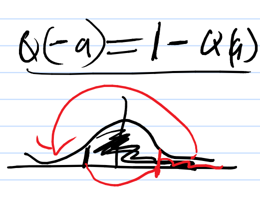
$$= P\left(\frac{4-10}{6} < \frac{X-10}{6} < \frac{16-10}{6}\right)$$

$$= P\left(-1 < \frac{X-10}{6} < 1\right)$$

$$= Q(-1) - Q(1)$$

c) $P(X < 8) = P\left(\frac{X-\mu}{\sigma} < \frac{8-10}{6}\right) = 1 - Q\left(-\frac{2}{6}\right)$

$$Q(-a) = 1 - Q(a)$$



Problem 2

$$X \sim Exp(\lambda) \quad E[X] = \frac{1}{\lambda} \quad Var(X) = \frac{1}{\lambda^2}$$

$$f(x) = \begin{cases} 0 & x<0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$\boxed{\text{Integration by parts}} \quad \int u \frac{dv}{dx} dx = u \cdot v - \int v \frac{du}{dx} dx$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty \underbrace{x}_{u} \underbrace{e^{-\lambda x}}_{\frac{dv}{dx}} dx$$

$$= \lambda \left(x \cdot \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-\lambda x}}{-\lambda} \right)$$

$$= \lambda \left(x \cdot \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty - \frac{e^{-\lambda x}}{\lambda^2} \Big|_0^\infty \right)$$

$$= \lambda \left(\underbrace{\frac{\infty \cdot e^{-\lambda\infty}}{-\lambda}}_{0} - \underbrace{\frac{0 \cdot e^{-\lambda 0}}{-\lambda}}_{0} - \left(\underbrace{\frac{e^{-\lambda\infty}}{\lambda^2}}_{0} - \frac{e^{-\lambda 0}}{\lambda^2} \right) \right)$$

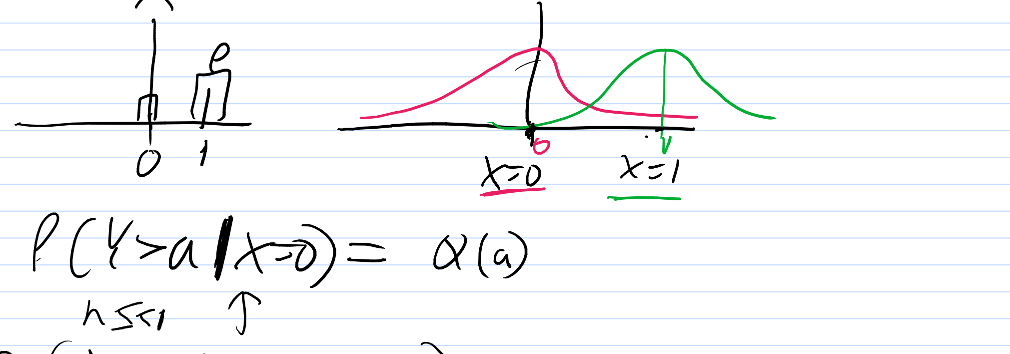
$$\left. \lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = 0 \right\} \text{Calculus Review}$$

$$= \lambda \left(\frac{1}{\lambda^2} \right) = \frac{1}{\lambda}$$

$$Var(X) = E[X^2] - (E[X])^2$$

Problem 3

$Y = g(X)$ X discrete, Y continuous



$$P(Y > a \mid X=0) = Q(a)$$

$h \ll 1$

$$P(X=1 \mid y < Y < y+h)$$

$$= \frac{P(X=1,\, y < Y < y+h)}{P(y < Y < y+h)}$$

$$= \frac{P(X=1) \cdot P(y < Y < y+h \mid X=1)}{P(X=1,\, y < Y < y+h) + P(X=0,\, y < Y < y+h)}$$

$$= \boxed{\frac{p \int_y^{y+h} f_1(x)\,dx}{p \int_y^{y+h} f_1(x)\,dx + (1-p) \int_y^{y+h} f_0(x)\,dx}}$$

$f_1(x)$ = pdf of Y when X=1

$f_0(x)$ = pdf of Y when X=0

$$\approx \frac{p \cdot f_1(y) h}{p \cdot f_1(y) h + (1-p) f_0(y) \cdot h}$$

$$\approx \frac{p f_1(y)}{p \cdot f_1(y) + (1-p) f_0(y)}$$

b) $P(X=1 \mid y < Y < y+h) > P(X=0 \mid y < Y < y+h)$

$\Rightarrow$ $Y > T$ for some T

if true output 1

else output 0

$$\frac{f_1(y) \cdot p}{\cancel{p\, f_1(y) + (1-p) f_0(y)}} > \frac{f_0(y)(1-p)}{\cancel{p\, f_1(y) + (1-p) f_0(y)}}$$

$$f_1(y)\, p > f_0(y)(1-p)$$

$$\frac{p}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} > e^{-\frac{y^2}{2}} \cdot \frac{(1-p)}{\sqrt{2\pi}}$$

$$e^{-\frac{(y-1)^2}{2} + \frac{y^2}{2}} > \frac{(1-p)}{p}$$

$$e^{\frac{1}{2}(2y-1)} > \frac{1-p}{p}$$

$$\frac{1}{2}(2y-1) > \ln\left(\frac{1-p}{p}\right)$$

$$\boxed{y > \frac{1}{2} + \ln\left(\frac{1-p}{p}\right)}$$