EE 131A
Probability
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Discussion Set 3 Solutions
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Chapter 2.1-2.3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. *Memoryless property of Geometric RV.*
   A discrete random variable M is said to satisfy the memoryless property if $P[M \geq k+j|M \geq j+1] = P[M \geq k]$ for all $j, k \geq 1$. Show that the geometric random variable satisfies the memoryless property.
   Solution:
   Let M be a geometric random variable. Then $P[M = k] = (1-p)^{k-1}p \quad k = 1, 2, ...$
   The probability that $M \leq k$ can be written in closed form:
   $P[M > k] = 1 - P[M \leq k] = 1 - \sum_{j=1}^{k}(1-p)^{j-1}p = (1-p)^k$

$$
\begin{aligned}
P[M \geq k+j|M > j] &= \frac{P[M \geq k+j, M > j]}{P[M > j]} = \frac{P[M \geq k+j]}{P[M > j]} \text{ for } k \geq 1 \\
&= \frac{(1-p)^{k+j-1}}{(1-p)^j} \\
&= (1-p)^{k-1} \\
&= P[M \geq k]
\end{aligned}
$$

   Therefore, geometric random variable satisfies the memoryless property.

2. You take an exam that contains 20 multiple-choice questions. Each question has 4 possible options and only one correct answer. You have no idea about any of the questions, so you choose answers randomly. Your score X on the exam is the total number of correct answers.

   (a) Find the PMF of X.
   Solution: For each question, you have a success probability of $\frac{1}{4}$ (the probability of getting each question right). Thus X is the sum of 20 independent Bernoulli($\frac{1}{4}$) trails and hence X is a Binomial($20, \frac{1}{4}$) RV. Thus the PMF of X is given by

$$
P(X = k) = \begin{cases} \binom{20}{k}(\frac{1}{4})^k(\frac{3}{4})^{20-k} & k = 0, 1, \ldots, 20 \\ 0 & \text{otherwise.} \end{cases}
$$

   (b) What is your expected score on the test?
   Solution: Since X is a Binomial($20, \frac{1}{4}$) RV, $E[X] = 20 \cdot \frac{1}{4} = 5$. Thus your expected score on the test is 5.

   (c) What is $P(X > 16)$?
   Hint: *You may approximate* 32551 *as* 2^{15}.

Solution:

$$
\begin{aligned}
P(X > 16) &= P(X = 17) + P(X = 18) + P(X = 19) + P(X = 20) \\
&= \binom{20}{17}\left(\frac{1}{4}\right)^{17}\left(\frac{3}{4}\right)^{3} + \binom{20}{18}\left(\frac{1}{4}\right)^{18}\left(\frac{3}{4}\right)^{2} \\
&+ \binom{20}{19}\left(\frac{1}{4}\right)^{19}\left(\frac{3}{4}\right)^{1} + \binom{20}{20}\left(\frac{1}{4}\right)^{20}\left(\frac{3}{4}\right)^{0} \\
&= \frac{\binom{20}{17}\cdot 3^3 + \binom{20}{18}\cdot 3^2 + \binom{20}{19}\cdot 3 + \binom{20}{20}}{4^{20}} \\
&= \frac{1140\cdot 27 + 190\cdot 9 + 20\cdot 3 + 1}{2^{40}} \\
&= \frac{32551}{2^{40}} \\
&\approx \frac{2^{15}}{2^{40}} \qquad\qquad (\text{since } 2^{15} = 32768) \\
&= 2.98\times 10^{-8} \qquad (\text{using } 2 \approx 10^{0.301}).
\end{aligned}
$$

3. *A Game with Marbles.* A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly, one at a time without replacement. If they are the same color, then you win \$1.10; if they are different colors, then you lose \$1.00. Calculate the expected value of the amount you win and the variance of the amount you win.
Solution:
Define W to be the random variable, and it represents the winnings obtained when one plays the proposed game. The expected value of W is then given by

$$E[W] = 1.1P_{sc} - 1.0P_{dc}$$

where the notation"sc" means that the two drawn marbles are of the same color and the notation"dc" means that the two drawn marbles are of different colors. Now to calculate each of these probabilities, we introduce the four possible events that can happen when we draw to marbles: RR, BB, RB, and BR. As an example the notation RB denotes the event that we first draw a red marble and then second draw a black marble. With this notation we see that P_{sc} is given by

$$
\begin{aligned}
P_{sc} &= P[RR] + P[BB] \\
&= \frac{5}{10}\left(\frac{4}{9}\right) + \frac{5}{10}\left(\frac{4}{9}\right) = \frac{4}{9}.
\end{aligned}
$$

while P_{dc} is given by

$$
\begin{aligned}
P_{dc} &= P[RB] + P[BR] \\
&= \frac{5}{10}\left(\frac{5}{9}\right) + \frac{5}{10}\left(\frac{5}{9}\right) = \frac{5}{9}.
\end{aligned}
$$

With these two results the expected profit is then given by

$$1.1\left(\frac{4}{9}\right) - 1.0\left(\frac{5}{9}\right) = -\frac{1}{15}.$$

The variance of the amount one wins can be computed by the standard expression for variance in term of expectations. Specifically we have

$$Var[W] = E[W^2] - E[W]^2.$$

Now using the results from Part (a) above we see that

$$E[W^2] = \frac{4}{9}1.1^2 + \frac{5}{9}(-1.0)^2 = \frac{82}{75}.$$

so that

$$Var[W] = \frac{82}{75} - \left(\frac{1}{15}\right)^2 = \frac{49}{45}.$$

4. *(Problem 3.9 and 3.26 of ALG)* A coin is tossed n times. Let the random variable Y be the difference between the number of heads and the number of tails in the n tosses of a coin. Assume $P[\text{heads}] = p$.

   (a) Describe the sample space of S.

   (b) Find the probability of the event $\{Y = 0\}$.

   (c) Find the probabilities for the other values of Y.

   (d) Find $E[Y]$ and $VAR[Y]$. In a large number of repetitions of this random experiment, what is the meaning of $E[Y]$?

   Solution:

   (a) Let the random variable X be the number of heads, then $S_X = \{0, 1, ..., n\}$. Then the random variable Y, which is difference between the number of heads and the number of tails in the n tosses of a coin, is given by $Y = X - (n - X) = 2X - n$. Therefore the sample space S_Y is given by
   $S_Y = \{-n, -n + 2, ...n - 2, n\}$.

   (b) $P[Y = 0] = P[2X - n = 0] = P[X = n/2] = \binom{n}{n/2}p^{n/2}(1 - p)^{n/2}$ for n is even.

   (c) $P[Y = k] = P[2X - n = k] = \binom{n}{(n+k)/2}p^{(n+k)/2}(1 - p)^{(n-k)/2}$ for $n + k$ is even.

   (d) Note $Y = 2X - n$ and X is the number of heads in n tosses of a coin, which is the binomial random variable. In the class, we already proved $E(X) = np$ and $Var(X) = np(1 - p)$. Therefore $E(Y)$ and $Var(Y)$ are given by
   $E(Y) = E(2X - n) = 2E(X) - n = 2np - n$
   $Var(Y) = Var(2X - n) = 4Var(X) = 4np(1 - p)$
   In a large number of repetitions of this random experiment, we expect the difference between the number of heads and the number of tails to be $E(Y)$.

5. *Bonus:* n people arrive at a restaurant and give their hats to a hat-check person. The hat-check person losses the receipts of who the hats belong to and returns the hats randomly. What is the expected number of people who get their own hat?
Solution:
Consider the following random variables:

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person gets own hat} \\ 0 & \text{otherwise.} \end{cases}$$

Also, let X be the random variable denoting the total number of people who get their own hat. We wish to compute $E(X)$.
Clearly,

$$X = \sum_{i=1}^{n} X_i$$

Now the probability that the i^{th} person gets his own hat is $\frac{1}{n}$ i.e., $P(X_i = 1) = \frac{1}{n}$. Thus $E(X_i) = \frac{1}{n}$. Hence we get

$$E(X) = E\big(\sum_{i=1}^{n} X_i\big)$$
$$= \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Thus the expected number of people who receive their own hats back is 1. It is important to note that linearity of expectation holds even when the corresponding random variables are not independent.