

# 10. Multi-objective least squares

- multi-objective least squares
- regularized data fitting
- control
- estimation and inversion

# Multi-objective least squares

we have several objectives

$$J_1 = \|A_1x - b_1\|^2, \quad \dots, \quad J_k = \|A_kx - b_k\|^2$$

- $A_i$  is an  $m_i \times n$  matrix,  $b_i$  is an  $m_i$ -vector
- we seek *one*  $x$  that makes all  $k$  objectives small
- usually there is a trade-off: no single  $x$  minimizes all objectives simultaneously

**Weighted least squares formulation:** find  $x$  that minimizes

$$\lambda_1 \|A_1x - b_1\|^2 + \dots + \lambda_k \|A_kx - b_k\|^2$$

- coefficients  $\lambda_1, \dots, \lambda_k$  are positive weights
- weights  $\lambda_i$  express relative importance of different objectives
- without loss of generality, we can choose  $\lambda_1 = 1$

# Solution of weighted least squares

- weighted least squares is equivalent to a standard least squares problem

$$\text{minimize} \quad \left\| \begin{bmatrix} \sqrt{\lambda_1} A_1 \\ \sqrt{\lambda_2} A_2 \\ \vdots \\ \sqrt{\lambda_k} A_k \end{bmatrix} x - \begin{bmatrix} \sqrt{\lambda_1} b_1 \\ \sqrt{\lambda_2} b_2 \\ \vdots \\ \sqrt{\lambda_k} b_k \end{bmatrix} \right\|^2$$

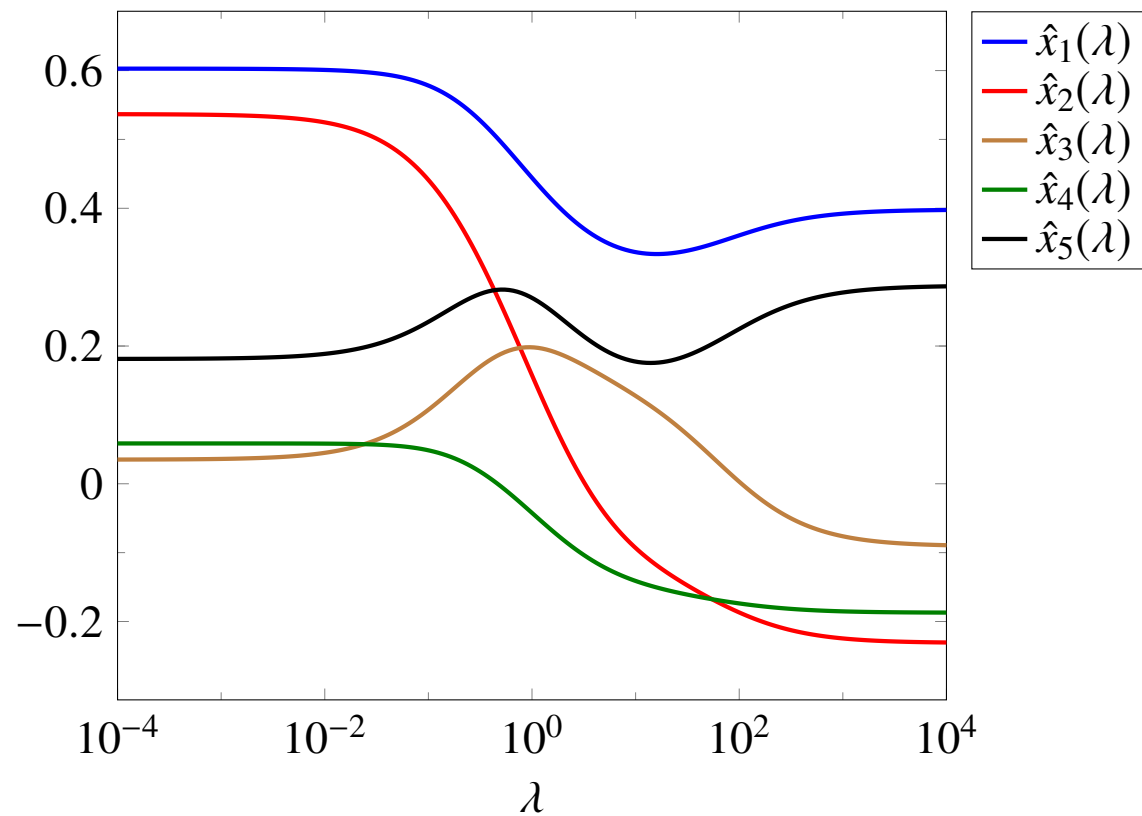
- solution is unique if the *stacked matrix* has linearly independent columns
- each matrix  $A_i$  may have linearly dependent columns (or be a wide matrix)
- if the stacked matrix has linearly independent columns, the solution is

$$\hat{x} = \left( \lambda_1 A_1^T A_1 + \cdots + \lambda_k A_k^T A_k \right)^{-1} \left( \lambda_1 A_1^T b_1 + \cdots + \lambda_k A_k^T b_k \right)$$

## Example with two objectives

$$\text{minimize} \quad \|A_1x - b_1\|^2 + \lambda \|A_2x - b_2\|^2$$

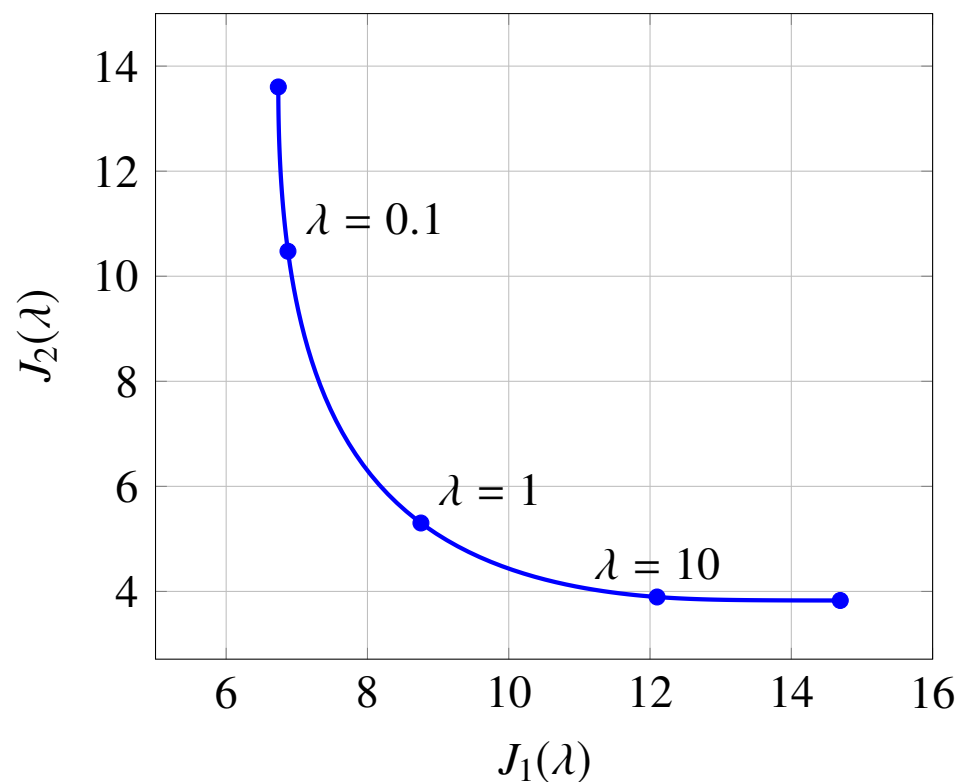
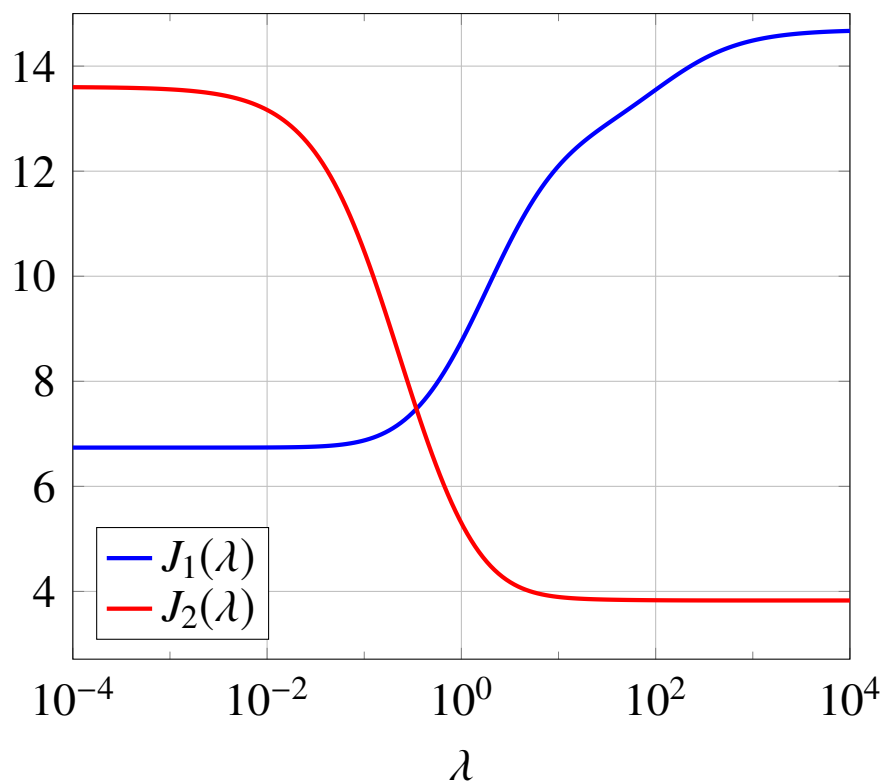
$A_1$  and  $A_2$  are  $10 \times 5$



plot shows weighted least squares solution  $\hat{x}(\lambda)$  as function of weight  $\lambda$

## Example with two objectives

$$\text{minimize } \|A_1x - b_1\|^2 + \lambda \|A_2x - b_2\|^2$$



- left figure shows  $J_1(\lambda) = \|A_1\hat{x}(\lambda) - b_1\|^2$  and  $J_2(\lambda) = \|A_2\hat{x}(\lambda) - b_2\|^2$
- right figure shows optimal trade-off curve of  $J_2(\lambda)$  versus  $J_1(\lambda)$

# Outline

- multi-objective least squares
- **regularized data fitting**
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# Motivation

- consider linear-in-parameters model

$$\hat{f}(x) = \theta_1 f_1(x) + \cdots + \theta_p f_p(x)$$

we assume  $f_1(x)$  is the constant function 1

- we fit the model  $\hat{f}(x)$  to examples  $(x^{(1)}, y^{(1)}), \dots, (x^{(N)}, y^{(N)})$
- large coefficient  $\theta_i$  makes model more sensitive to changes in  $f_i(x)$
- keeping  $\theta_2, \dots, \theta_p$  small helps avoid over-fitting
- this leads to two objectives:

$$J_1(\theta) = \sum_{k=1}^N (\hat{f}(x^{(k)}) - y^{(k)})^2, \quad J_2(\theta) = \sum_{j=2}^p \theta_j^2$$

primary objective  $J_1(\theta)$  is sum of squares of prediction errors

## Weighted least squares formulation

$$\text{minimize } J_1(\theta) + \lambda J_2(\theta) = \sum_{k=1}^N (\hat{f}(x^{(k)}) - y^{(k)})^2 + \lambda \sum_{j=2}^p \theta_j^2$$

- $\lambda$  is positive *regularization parameter*
- equivalent to least squares problem: minimize

$$\left\| \begin{bmatrix} A_1 \\ \sqrt{\lambda} A_2 \end{bmatrix} \theta - \begin{bmatrix} y^d \\ 0 \end{bmatrix} \right\|^2$$

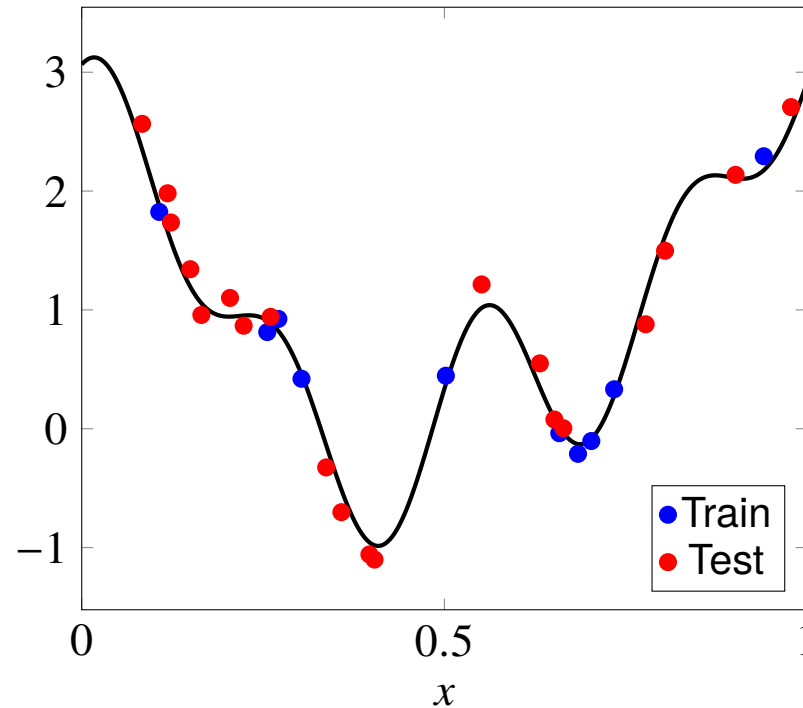
with  $y^d = (y^{(1)}, \dots, y^{(N)})$ ,

$$A_1 = \begin{bmatrix} 1 & f_2(x^{(1)}) & \cdots & f_p(x^{(1)}) \\ 1 & f_2(x^{(2)}) & \cdots & f_p(x^{(2)}) \\ \vdots & \vdots & & \vdots \\ 1 & f_2(x^{(N)}) & \cdots & f_p(x^{(N)}) \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

- stacked matrix has linearly independent columns (for positive  $\lambda$ )
- value of  $\lambda$  can be chosen by out-of-sample validation or cross-validation



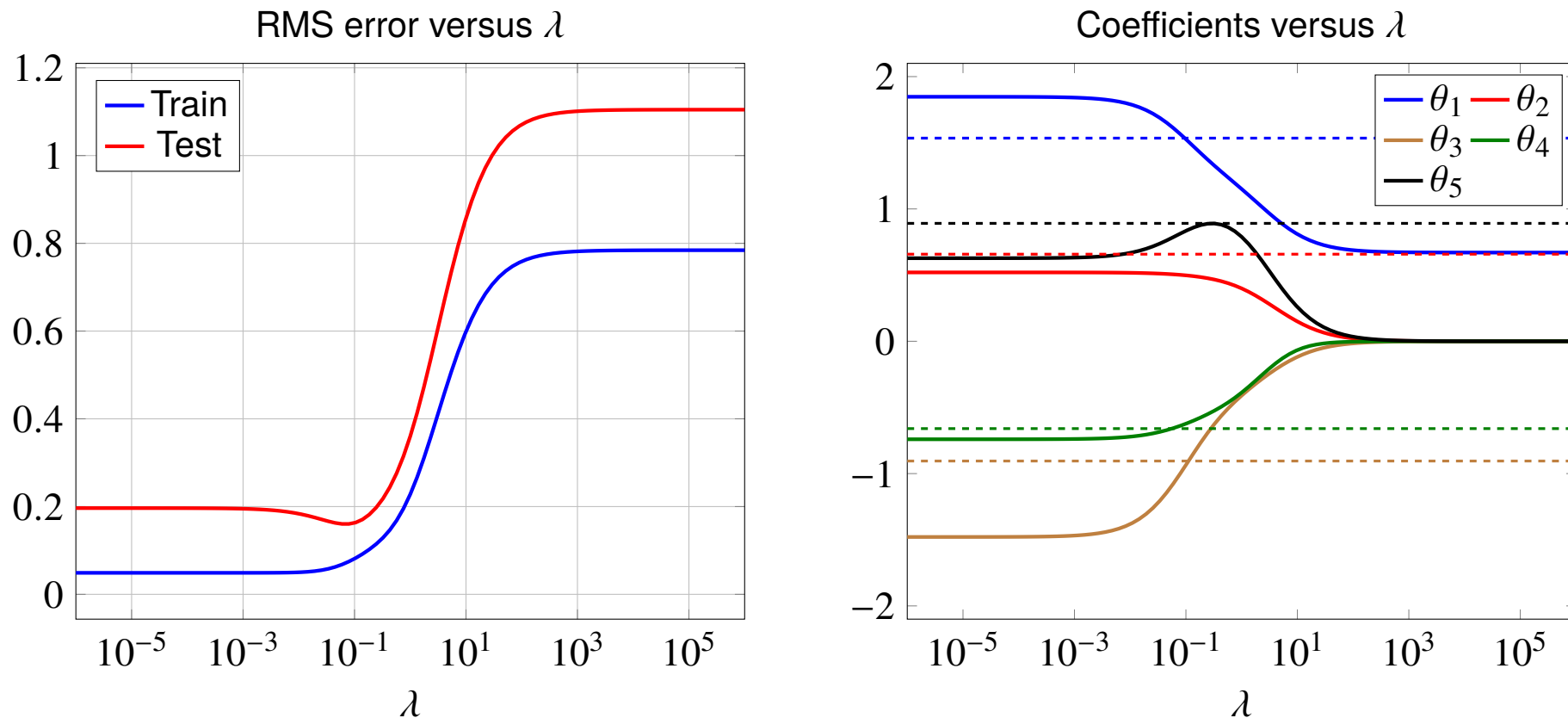
## Example



- solid line is signal used to generate synthetic (simulated) data
- 10 blue points are used as training set; 20 red points are used as test set
- we fit a model with five parameters  $\theta_1, \dots, \theta_5$ :

$$\hat{f}(x) = \theta_1 + \sum_{k=1}^4 \theta_{k+1} \sin(\omega_k x + \phi_k) \quad (\text{with given } \omega_k, \phi_k)$$

# Result of regularized least squares fit



- minimum test RMS error is for  $\lambda$  around 0.08
- increasing  $\lambda$  “shrinks” the coefficients  $\theta_2, \dots, \theta_5$
- dashed lines show coefficients used to generate the data
- for  $\lambda$  near 0.08, estimated coefficients are close to these “true” values

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# Control

$$y = Ax + b$$

- $x$  is  $n$ -vector of *actions* or *inputs*
- $y$  is  $m$ -vector of *results* or *outputs*
- relation between inputs and outputs is a known affine function

the goal is to choose inputs  $x$  to optimize different objectives on  $x$  and  $y$

# Optimal input design

## Linear dynamical system

$$y(t) = h_0 u(t) + h_1 u(t-1) + h_2 u(t-2) + \dots + h_t u(0)$$

- output  $y(t)$  and input  $u(t)$  are scalar
- we assume input  $u(t)$  is zero for  $t < 0$
- coefficients  $h_0, h_1, \dots$  are the *impulse response coefficients*
- output is convolution of input with impulse response

## Optimal input design

- optimization variable is the input sequence  $x = (u(0), u(1), \dots, u(N))$
- goal is to track a desired output using a small and slowly varying input

# Input design objectives

$$\text{minimize } J_t(x) + \lambda_v J_v(x) + \lambda_m J_m(x)$$

- primary objective: track desired output  $y_{\text{des}}$  over an interval  $[0, N]$ :

$$J_t(x) = \sum_{t=0}^N (y(t) - y_{\text{des}}(t))^2$$

- secondary objectives: use a small and slowly varying input signal:

$$J_m(x) = \sum_{t=0}^N u(t)^2, \quad J_v(x) = \sum_{t=0}^{N-1} (u(t+1) - u(t))^2$$

# Tracking error

$$\begin{aligned}
 J_t(x) &= \sum_{t=0}^N (y(t) - y_{\text{des}}(t))^2 \\
 &= \|A_t x - b_t\|^2
 \end{aligned}$$

with

$$A_t = \begin{bmatrix} h_0 & 0 & 0 & \cdots & 0 & 0 \\ h_1 & h_0 & 0 & \cdots & 0 & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 & 0 \\ h_N & h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix}, \quad b_t = \begin{bmatrix} y_{\text{des}}(0) \\ y_{\text{des}}(1) \\ y_{\text{des}}(2) \\ \vdots \\ y_{\text{des}}(N-1) \\ y_{\text{des}}(N) \end{bmatrix}$$

# Input variation and magnitude

## Input variation

$$J_v(x) = \sum_{t=0}^{N-1} (u(t+1) - u(t))^2 = \|Dx\|^2$$

with  $D$  the  $N \times (N+1)$  matrix

$$D = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

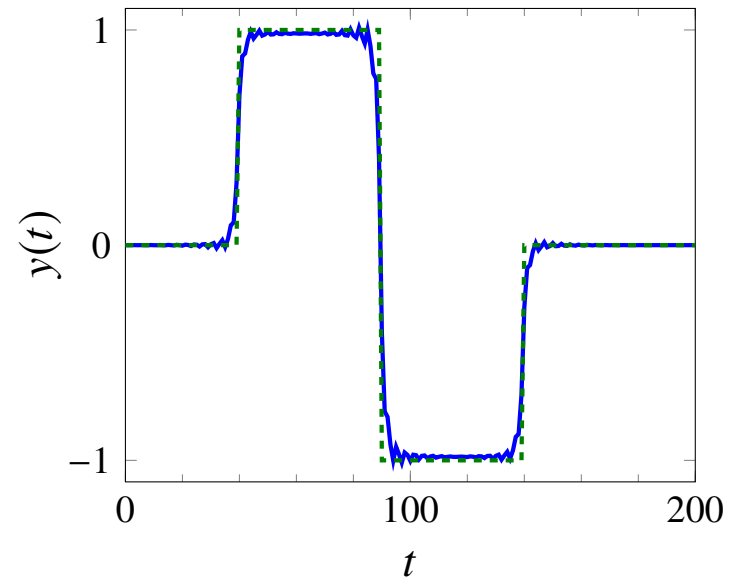
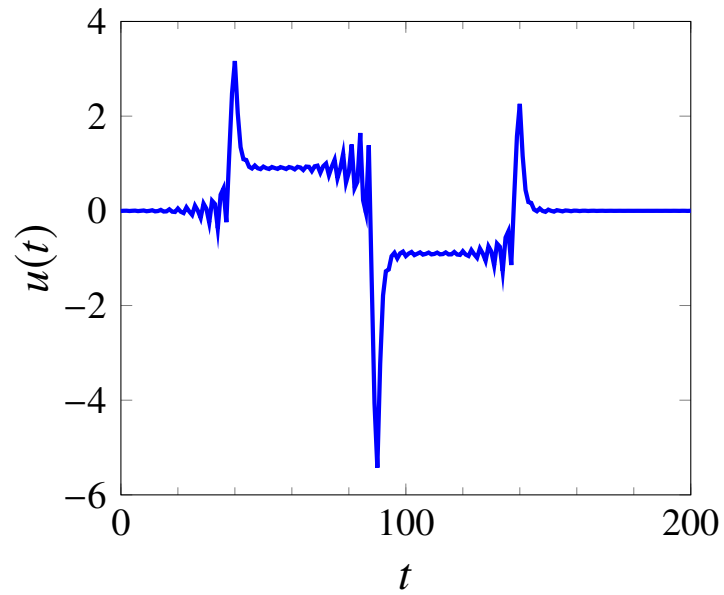
## Input magnitude

$$J_m(x) = \sum_{t=0}^N u(t)^2 = \|x\|^2$$

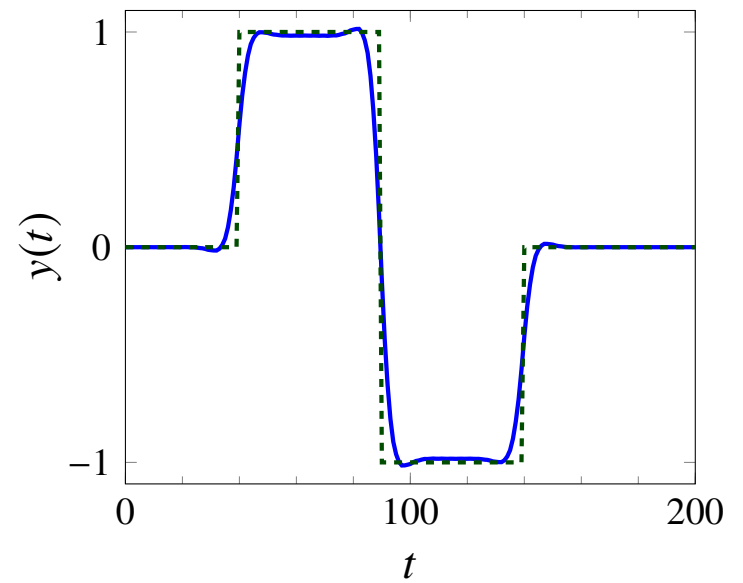
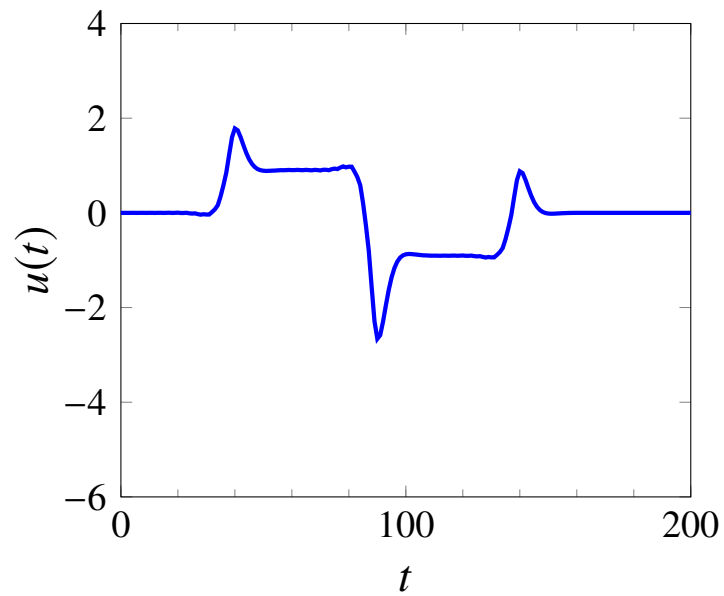


# Example

$\lambda_v = 0$ ,  
small  $\lambda_m$



larger  $\lambda_v$   
larger  $\lambda_m$



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# Estimation

## Linear measurement model

$$y = Ax_{\text{ex}} + v$$

- $n$ -vector  $x_{\text{ex}}$  contains parameters that we want to estimate
- $m$ -vector  $v$  is unknown measurement error or noise
- $m$ -vector  $y$  contains measurements
- $m \times n$  matrix  $A$  relates measurements and parameters

**Least squares estimate:** use as estimate of  $x_{\text{ex}}$  the solution  $\hat{x}$  of

$$\text{minimize } \|Ax - y\|^2$$

# Regularized estimation

add other terms to  $\|Ax - y\|^2$  to include information about parameters

## Example: Tikhonov regularization

$$\text{minimize } \|Ax - y\|^2 + \lambda \|x\|^2$$

- goal is to make  $\|Ax - y\|$  small with small  $x$
- equivalent to solving

$$(A^T A + \lambda I)x = A^T y$$

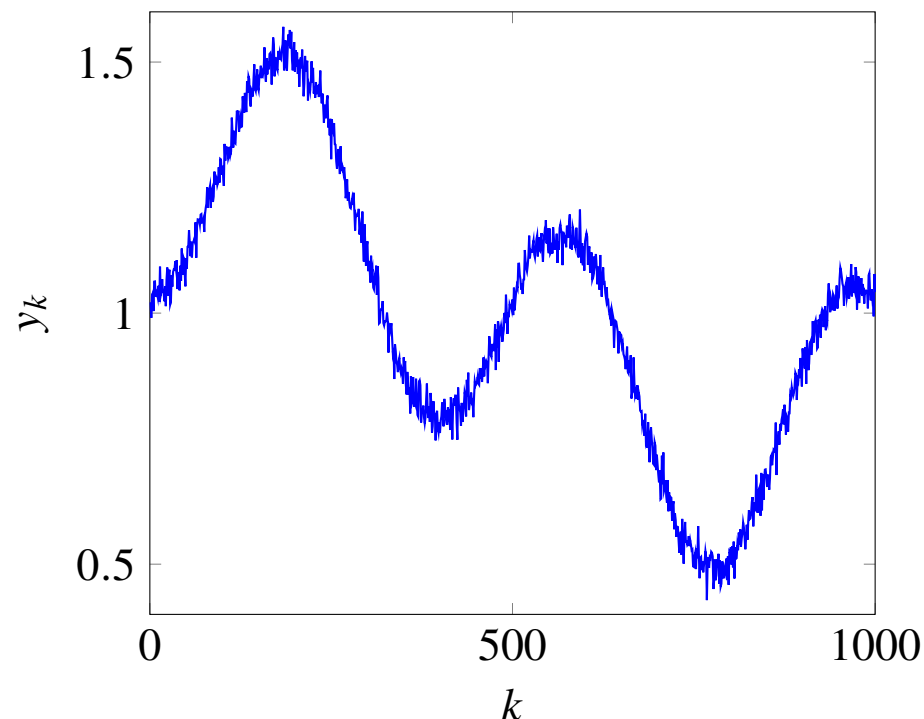
- solution is unique (if  $\lambda > 0$ ) even when  $A$  has linearly dependent columns

# Signal denoising

- observed signal  $y$  is  $n$ -vector

$$y = x_{\text{ex}} + v$$

- $x_{\text{ex}}$  is unknown signal
- $v$  is noise



**Least squares denoising:** find estimate  $\hat{x}$  by solving

$$\text{minimize} \quad \|x - y\|^2 + \lambda \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$$

goal is to find slowly varying signal  $\hat{x}$ , close to observed signal  $y$

## Matrix formulation

$$\text{minimize} \quad \left\| \begin{bmatrix} I \\ \sqrt{\lambda} D \end{bmatrix} x - \begin{bmatrix} y \\ 0 \end{bmatrix} \right\|^2$$

- $D$  is  $(n - 1) \times n$  finite difference matrix

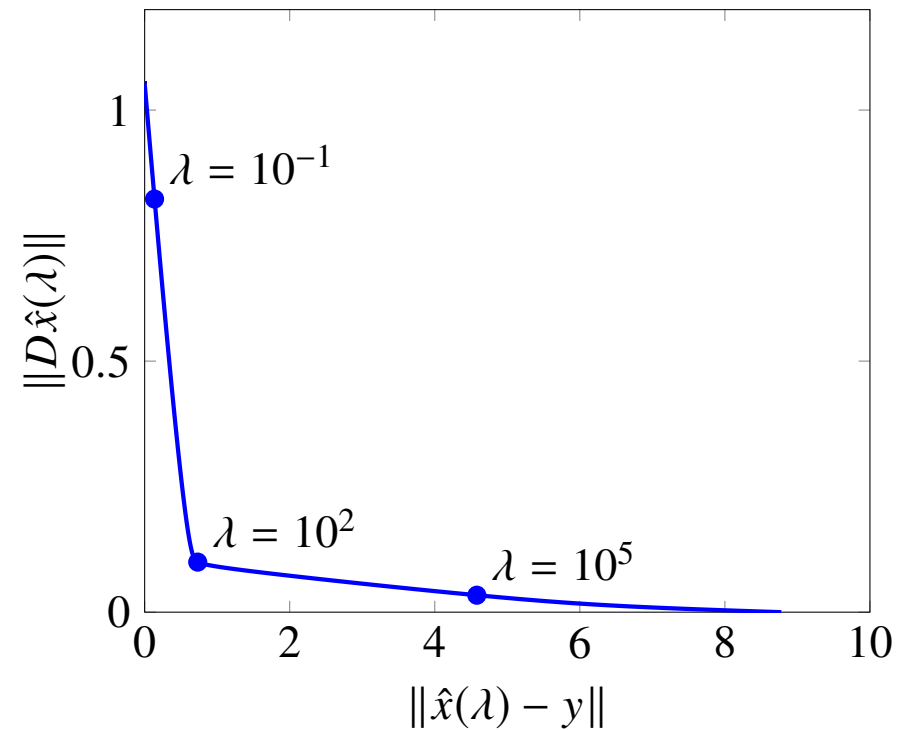
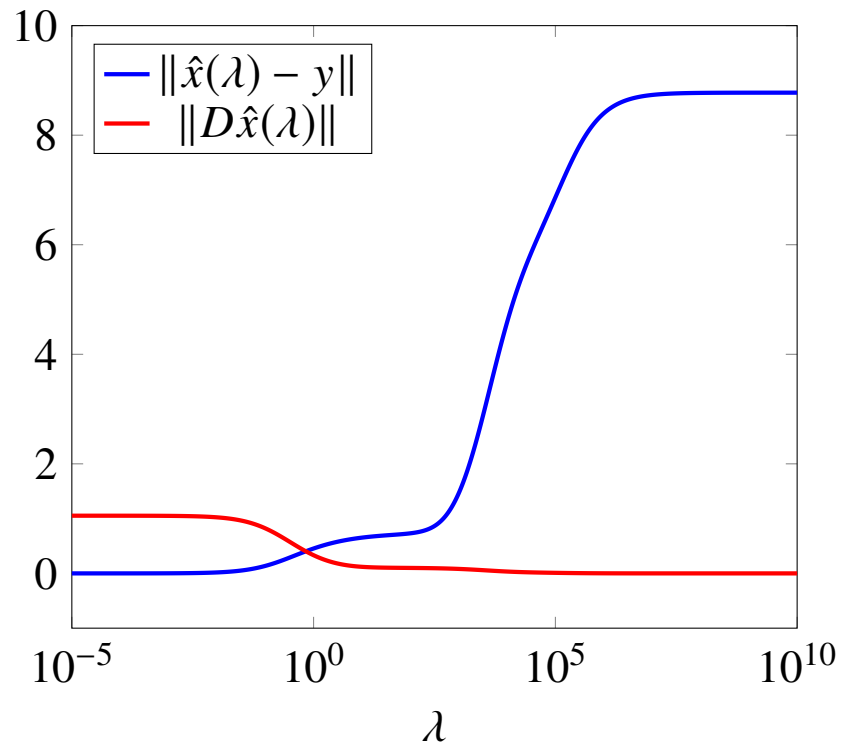
$$D = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

- equivalent to linear equation

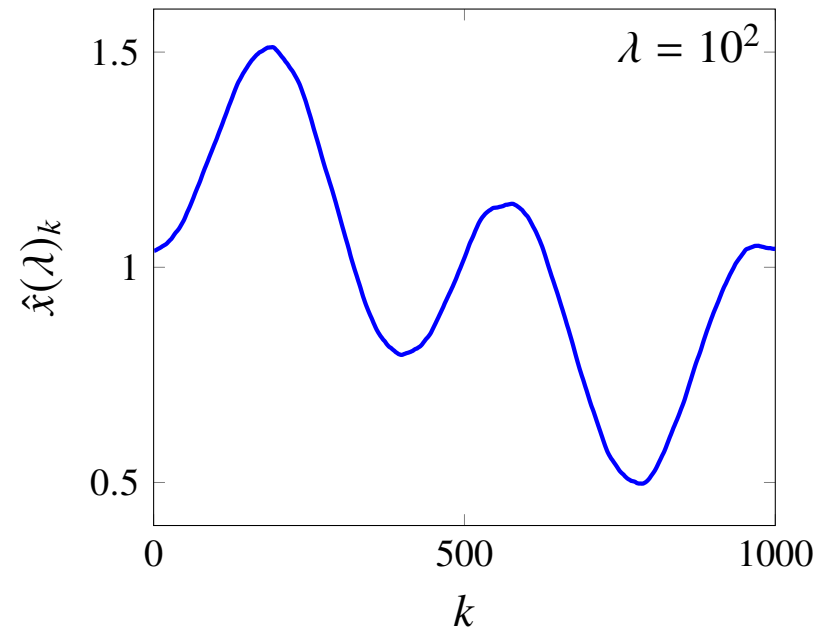
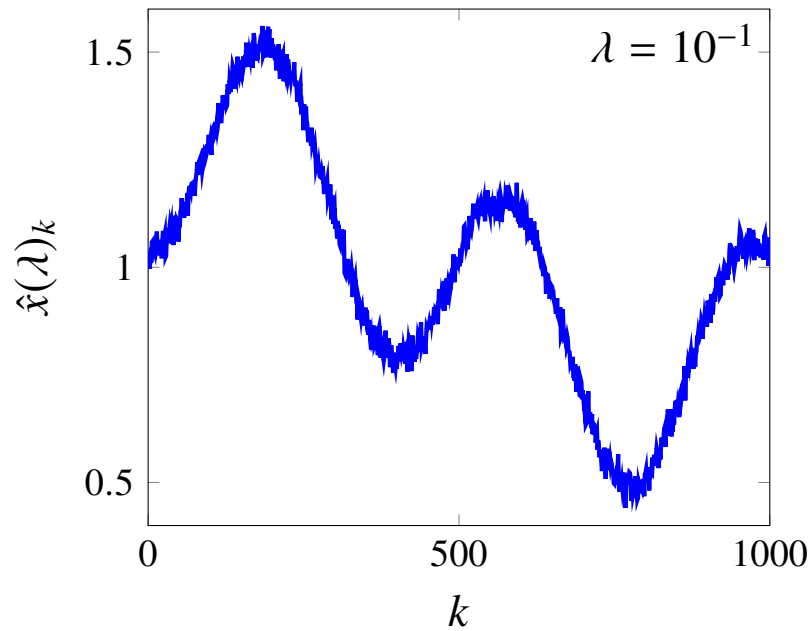
$$(I + \lambda D^T D)x = y$$

# Trade-off

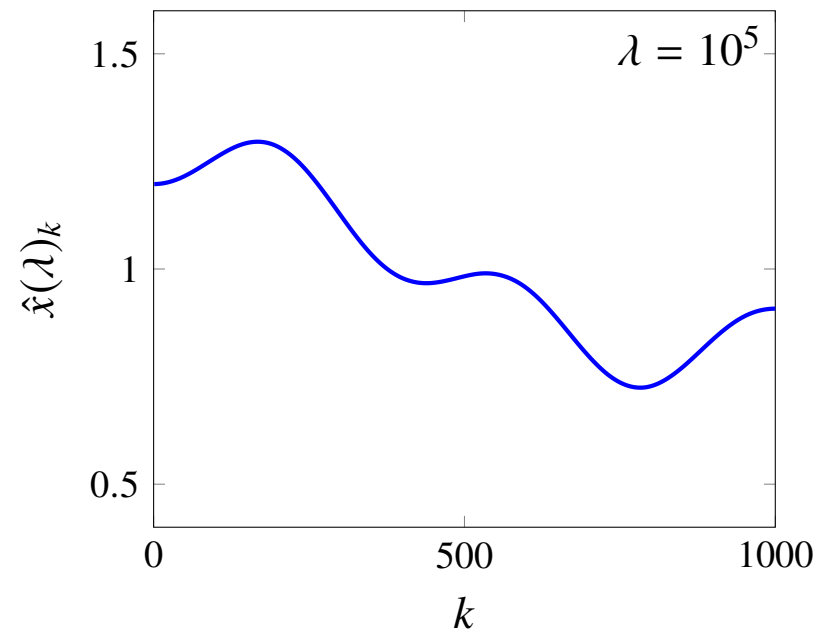
the two objectives  $\|\hat{x}(\lambda) - y\|$  and  $\|D\hat{x}(\lambda)\|$  for varying  $\lambda$



# Three solutions



- $\hat{x}(\lambda) \rightarrow y$  for  $\lambda \rightarrow 0$
- $\hat{x}(\lambda) \rightarrow \mathbf{avg}(y)\mathbf{1}$  for  $\lambda \rightarrow \infty$
- $\lambda \approx 10^2$  is good compromise





# Image deblurring

$$y = Ax_{\text{ex}} + v$$

- $x_{\text{ex}}$  is unknown image,  $y$  is observed image
- $A$  is (known) blurring matrix,  $v$  is (unknown) noise
- images are  $M \times N$ , stored as  $MN$ -vectors



blurred, noisy image  $y$



deblurred image  $\hat{x}$

## Least squares deblurring

$$\text{minimize } \|Ax - y\|^2 + \lambda(\|D_v x\|^2 + \|D_h x\|^2)$$

- 1st term is “*data fidelity*” term: ensures  $A\hat{x} \approx y$
- 2nd term penalizes differences between values at neighboring pixels

$$\|D_h x\|^2 + \|D_v x\|^2 = \sum_{i=1}^M \sum_{j=1}^{N-1} (X_{i,j+1} - X_{ij})^2 + \sum_{i=1}^{M-1} \sum_{j=1}^N (X_{i+1,j} - X_{ij})^2$$

if  $X$  is the  $M \times N$  image stored in the  $MN$ -vector  $x$

# Differencing operations in matrix notation

suppose  $x$  is the  $M \times N$  image  $X$ , stored column-wise as  $MN$ -vector

$$x = (X_{1:M,1}, X_{1:M,2}, \dots, X_{1:M,N})$$

- horizontal differencing:  $(N - 1) \times N$  block matrix with  $M \times M$  blocks

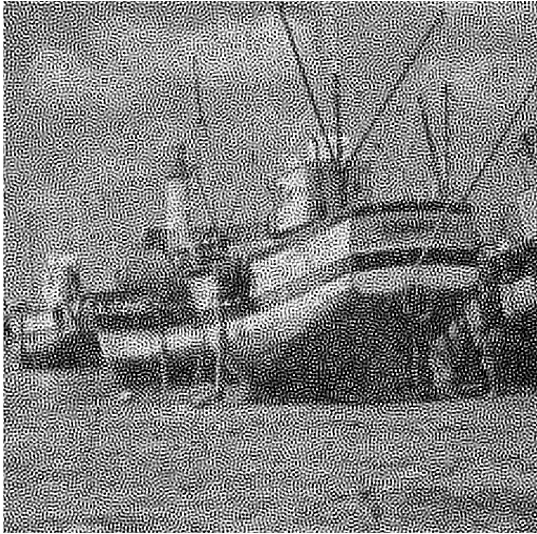
$$D_h = \begin{bmatrix} -I & I & 0 & \dots & 0 & 0 & 0 \\ 0 & -I & I & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -I & I \end{bmatrix}$$

- vertical differencing:  $N \times N$  block matrix with  $(M - 1) \times M$  blocks

$$D_v = \begin{bmatrix} D & 0 & \dots & 0 \\ 0 & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}$$

# Deblurred images

$$\lambda = 10^{-6}$$



$$\lambda = 10^{-4}$$



$$\lambda = 10^{-2}$$



$$\lambda = 1$$



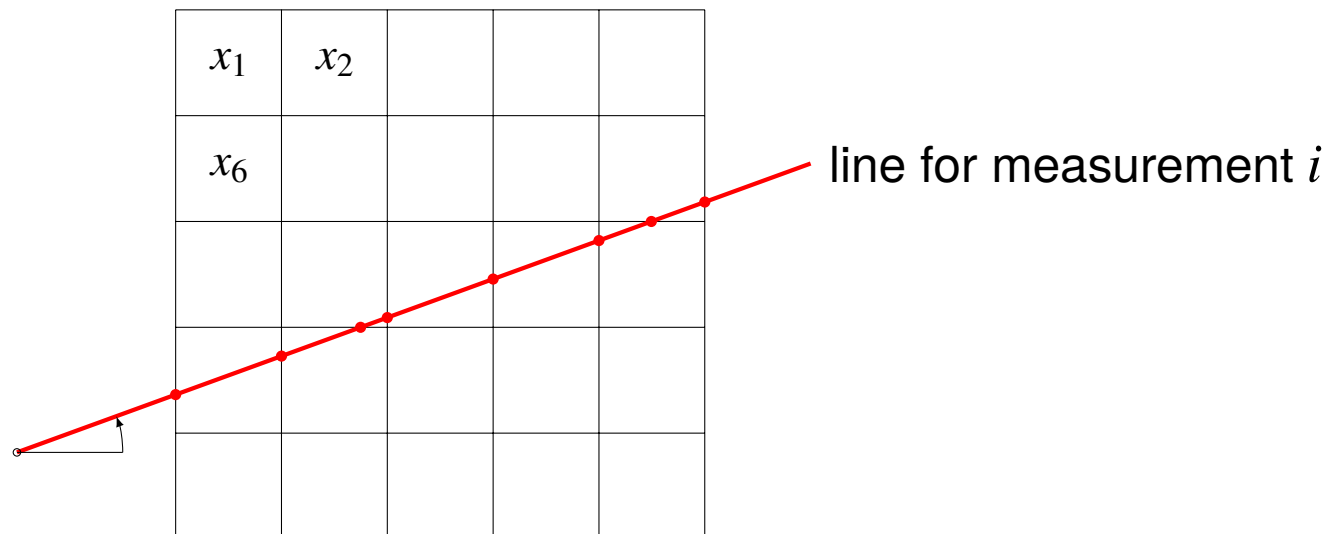
# Tomography

$$y = Ax_{\text{ex}} + v$$

- $x_{\text{ex}}$  represents values of some quantity in a region of interest of  $n$  voxels (pixels)
- $Ax$  represents measurements of the integral along lines through the region

$$(Ax)_i = \sum_{j=1}^n A_{ij}x_j$$

$A_{ij}$  is the length of the intersection of the line in measurement  $i$  with voxel  $j$

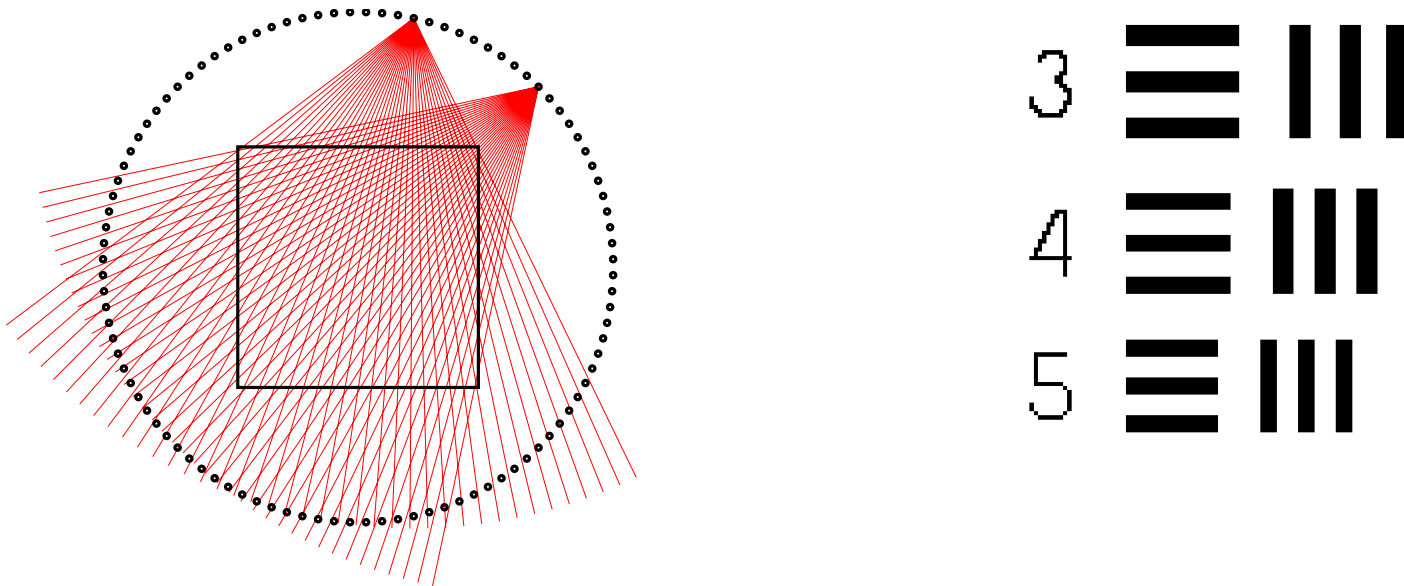


# Tomographic reconstruction

$$\text{minimize } \|Ax - y\|^2 + \lambda(\|D_v x\|^2 + \|D_h x\|^2)$$

$D_v$  and  $D_h$  are defined as in image deblurring example on page 10.23

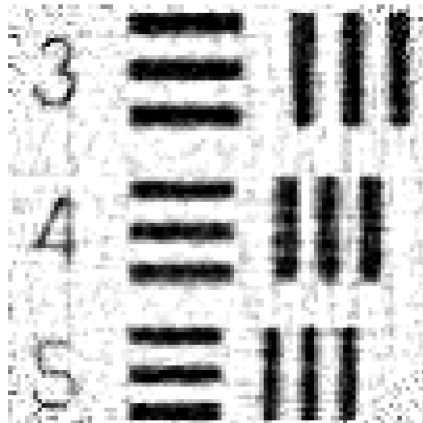
## Example



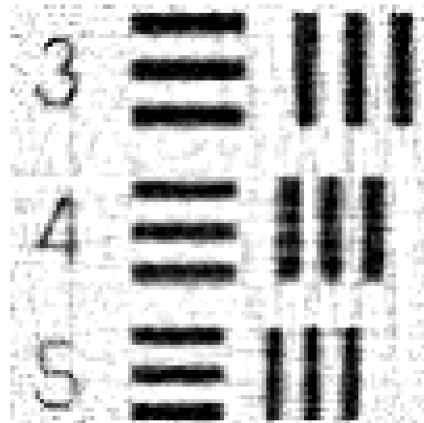
- left: 4000 lines (100 points, 40 lines per point)
- right: object placed in the square region at the center of the picture on the left
- region of interest is divided in 10000 pixels

# Regularized least squares reconstruction

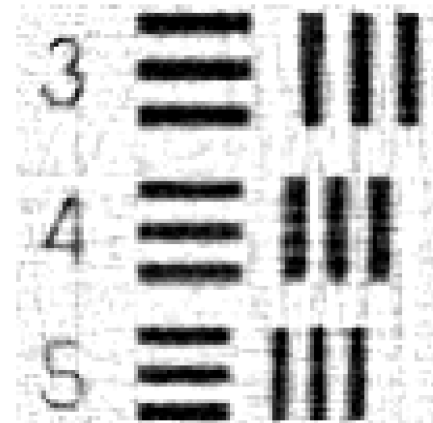
$$\lambda = 10^{-2}$$



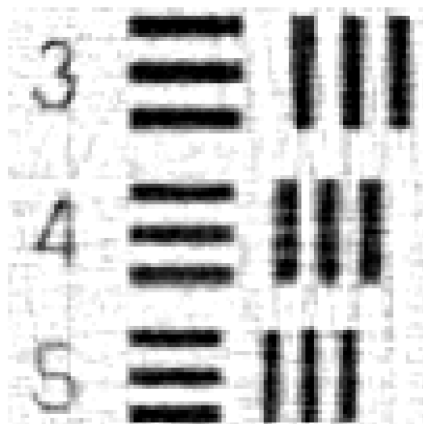
$$\lambda = 10^{-1}$$



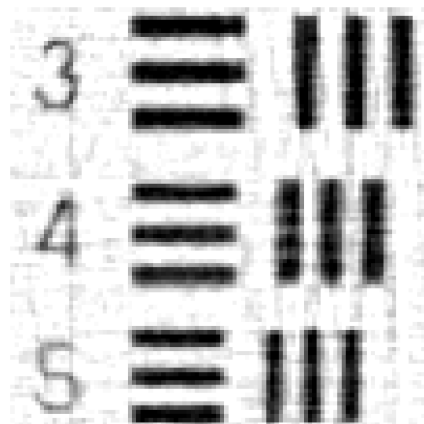
$$\lambda = 1$$



$$\lambda = 5$$



$$\lambda = 10$$



$$\lambda = 100$$

