PROBLEM 1

1. $f(s,t) = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} s^{i-1} t^{j-1}$

$$= c_{11} s^0 t^0 + c_{12} s^0 t^1 + c_{13} s^0 t^2$$
$$+ c_{21} s^1 t^0 + c_{22} s^1 t^1 + c_{23} s^1 t^2$$
$$+ c_{31} s^2 t^0 + c_{32} s^2 t^1 + c_{33} s^2 t^2$$

$$\begin{bmatrix} 1 & t_1 & t_1^2 & s_1 & s_1t_1 & s_1t_1^2 & s_1^2 & s_1^2t_1 & s_1^2t_1^2 \\ 1 & t_2 & t_2^2 & s_2 & s_2t_2 & s_2t_2^2 & s_2^2 & s_2^2t_2 & s_2^2t_2^2 \\ 1 & t_3 & t_3^2 & | & | & | & | & & \\ 1 & t_4 & t_4^2 & | & & | & & & | \\ 1 & t_5 & t_5^2 & | & | & | & | & & | \\ 1 & t_6 & t_6^2 & & | & | & | & & | \\ 1 & t_7 & t_7^2 & | & | & | & | & & | \\ 1 & t_8 & t_8^2 & | & & & & & \\ 1 & t_9 & t_9^2 & & & & & & \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix}$$

for row i, row $= [1 \;\; t_i \;\; t_i^2 \;\; s_i \;\; s_it_i \;\; s_it_i^2 \;\; s_i^2 \;\; s_i^2t_i \;\; s_i^2t_i^2]$ 9×1

2. $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 5 \\ 5 \\ 7 \\ 5 \\ 3 \\ 5 \\ 3 \end{bmatrix}$$

(s_i, t_i) given in problem.

for $i = 1:9$
    $A[i,:] = [\qquad\qquad]$
end;

$$A \backslash y = \cancel{\begin{bmatrix} 7 \\ -2 \\ 0 \\ -13 \\ 25 \\ -8.5 \\ 6 \\ -12 \\ 4 \end{bmatrix}} \begin{bmatrix} 7 \\ -13 \\ 6 \\ -2 \\ 25 \\ -12 \\ 0 \\ -8.5 \\ 4 \end{bmatrix}$$

$$\boxed{\cancel{\begin{bmatrix} 7 & -2 & 0 \\ -13 & 25 & -8.5 \\ 6 & -12 & 4 \end{bmatrix}}}$$

accidentally flipped s and t

$$\boxed{\begin{bmatrix} 7 & -13 & 6 \\ -2 & 25 & -12 \\ 0 & -8.5 & 4 \end{bmatrix}}$$

Problem 2:

$$C = \begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{bmatrix}$$

Where A is a diagonal matrix. Let the diagonals be elements $a_1, a_2, \dots a_n$.

$$Ab = \begin{bmatrix} a_1b_1 \\ a_2b_2 \\ \vdots \\ a_nb_n \end{bmatrix}$$

Since the diagonal of $A^k = a_1^k, a_2^k, \dots a_n^k$

$$A^kb = \begin{bmatrix} a_1^kb_1 \\ \vdots \\ a_n^kb_n \end{bmatrix}$$

forgot a column.

$$C = \begin{bmatrix} b_1 & a_1b_1 & a_1^2b_1 & a_1^3b_1 & \cdots & a_1^{n-1}b_1 \\ b_2 & a_2b_2 & a_2^2b_2 & a_2^3b_2 & \cdots & a_2^{n-1}b_2 \\ \vdots & & & \vdots & & \\ b_n & a_nb_n & a_n^2b_n & \cdots & & a_n^{n-1}b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

prove $CX = 0$ has only 1 solution $x=0$.

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} \begin{bmatrix} b_1x_1 \\ b_2x_2 \\ \vdots \\ b_nx_n \end{bmatrix} = 0$$

representation of $(n-1)^{th}$ degree polynomial with variable a, n roots $a_1 - a_n$ and coefficients $b_1x_1 - b_nx_n$.

n roots and $n-1$ degree $=$ 1 roots w/ multiplicity 2. aka $a_i = a_j$ for some $i \neq j$. This is a contradiction by the problem statement ($A_{ii} \neq A_{jj}$) for $i \neq j$. Thus, must be 0 polynomial and $b_ix_i = 0$. Since b has nonzero elements, $x_i = 0$ and $X = 0 \Rightarrow$ nonsingular $\square$.

Problem 3

1. $Q = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $R = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $= \begin{bmatrix} a & -a+b & -a-b+c \\ d & -d+e & -d-e+f \\ g & -g+h & -g-h+i \end{bmatrix}$

$\|q_i\| = 1$ $\quad q_i^T q_j = 0$ if $i \neq j$

after finishing this i realized norms of A is same as norms of R. this is because Q has orthonormal columns i think and is orthogonal? preserves norms through multiplication. oops. would've saved a lot of time.

$\|a_1\| = \|q_1\| = \boxed{1}$

$\|a_2\| = \sqrt{\cancel{a^2} - 2ab + \cancel{b^2} + \cancel{d^2} - 2de + \cancel{e^2} + g^2 - 2gh + \cancel{h^2}}$

$= \sqrt{1 + 1 - 2(ab + de + gh)}$ b/c $q_1^T q_2 = 0$

$= \boxed{\sqrt{2}}$

$\|a_3\| = \sqrt{(a+b-c)^2 + (d+e-f)^2 + (g+h-i)^2} = \sqrt{(a+b)^2 + (d+e)^2 + (g+h)^2 + c^2 + f^2 + i^2 \underbrace{- 2(a+b)c + 2(d+e)f + 2(g+h)i}_{\to 0 \text{ b/c}}}$

$= \sqrt{1 + a^2 + b^2 + d^2 + e^2 + g^2 + h^2 + \underbrace{(2ab + 2de + 2gh)}_{\to 0 \text{ b/c}}}$ $\to q_i^T q_j = 0$

$= \sqrt{1 + 1 + 1} = \boxed{\sqrt{3}}$

2. $\cos\theta = \dfrac{a \cdot b}{\|a\| \|b\|} \Rightarrow \theta = \cos^{-1}\left(\dfrac{a^T b}{\|a\| \|b\|}\right)$

$\theta_{12} = \cos^{-1}\left(\dfrac{a_1^T a_2}{1 \cdot \sqrt{2}}\right) = \cos^{-1}\left(\dfrac{-a^2 - d^2 - g^2 + \overset{0}{\cancel{ab + de + gh}}}{\sqrt{2}}\right) = \cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right) = 2.3562$

$\theta_{13} = \cos^{-1}\left(\dfrac{a_1^T a_3}{1 \cdot \sqrt{3}}\right) = \cos^{-1}\left(\dfrac{-a^2 - \cancel{ab} + \cancel{ac} - d^2 - \cancel{de} + \cancel{df} - g^2 + \cancel{gh} + \cancel{gi}}{\sqrt{3}}\right) = \cos^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = 2.1863$

$ac + df + gi = 0$ & $ab + de + gh = 0$

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$$\theta_{23} = \cos^{-1}\left(\frac{a_2^T a_3}{\|a_2\|\ \|a_3\|}\right) = \cos^{-1}\left(\frac{(a-b)\cdot(a+b-c) + (d-e)\cdot(d+e-f) + (g-h)(g+h-i)}{\sqrt{6}}\right)$$

$$= \cos^{-1}\left(\frac{a^2-b^2+d^2-e^2+g^2-h^2-\left(c(a-b)+f(d-e)+i(g-h)\right)}{\sqrt{6}}\right)$$

$ac+df+gi=0$ since $q_1^T q_3 = 0$

$bc+ef+ih=0$ since $q_2^T q_3 = 0$

$$= \cos^{-1}\left(\frac{1-1}{\sqrt{6}}\right) = \frac{\pi}{2} = \boxed{1.5708}$$

Problem 4.

1) we see that computing Ax_i is common for every row.

Let's precompute that.

$$Ax_1, Ax_2, Ax_3 \ldots Ax_n \qquad A = n \times n, \quad x = n \times 1$$

Normally it would take $2n^2$, but A is lower/upper triangle so each multiplication takes n^2 flops. Total: $\boxed{n^3}$ flops for all n of them:

$$\begin{bmatrix} A_{11}A & A_{12}A & \cdots & A_{1n}A \\ & \vdots & & \\ A_{n1}A & \cdots & & A_{nn}A \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$ becomes.

$$\begin{bmatrix} A_{11}(Ax_1) + A_{12}(Ax_2) + \ldots + A_{1n}(Ax_n) \\ \vdots \\ A_{n1}(Ax_1) + \cdots + A_{nn}(Ax_n) \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Calculating b_i: $A_{ij}(Ax_j)$ takes 1 flop. total: n^2 flops.

adding vectors together n-1 flops each: $n(n-1)$.

$2n^2 - n$

total all $b_i = n(2n^2-n) = 2n^3 - n^2$

$$2n^3 - n^2 + \boxed{n^3} = \boxed{3n^3 - n^2}$$

regular $n^2 \times n^2 = n^2 \times n^2 \times 1 = \boxed{n^4}$

not 2. b/c triangular

2)

Step 1: $A = PLU$ factor: $\frac{2}{3}n^3$ flops.

Step 2: $Ax_i = PLUx_i = P \cdot L \cdot Ux_i$

Matrix vector $\rightarrow n^2$

triangular

each i so

$n \times$

same n^2

0 b/c permutation

total $= n \times (n^2 + n^2 + 0) = 2n^3$ flops.

solving
$$\begin{bmatrix} A_{11}(Ax_1) & A_{12} & \cdots & A_{1n}(Ax_n) \\ & & \ddots & \\ A_{n1}(Ax_1) & & & A_{nn}(Ax_n) \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Same as previous part: $(n^2 + n(n-1))n = 2n^3 - n^2$

$2n^3 - n^2 + 2n^3 = \boxed{4n^3 - n^2}$ flops.

Same as before, several SET of linear eq
(not necessarily triangular)

$2 \cdot n^2 \cdot n^2 \cdot 1 = \boxed{2n^4 \text{ flops}}$