15. Problem condition

- condition of a mathematical problem
- matrix norm
- condition number

Sources of error in numerical computation

Example: evaluate a function $f : \mathbf{R} \to \mathbf{R}$ at a given x

sources of error in the result:

- x is not exactly known
  - measurement errors
  - errors in previous computations

  $\longrightarrow$ how sensitive is $f(x)$ to errors in x?

- the algorithm for computing $f(x)$ is not exact
  - discretization (*e.g.*, algorithm uses a table to look up function values)
  - truncation (*e.g.*, function is evaluated by truncating a Taylor series)
  - rounding error during the computation

  $\longrightarrow$ how large is the error introduced by the algorithm?

Condition (conditioning) of a problem

describes sensitivity of the solution to changes in the problem data

- **well-conditioned problem:**

  small changes in the data produce small changes in the solution

- **ill-conditioned (badly conditioned) problem:**

  small changes in the data can produce large changes in the solution

a rigorous definition depends on what “large error” means

- absolute or relative error, which norm is used, ...
- the informal definition is sufficient for our purposes

Example: function evaluation

here the problem is: given x, evaluate $y = f(x)$

- if x is changed to $x + \Delta x$, solution changes to

$$y + \Delta y = f(x + \Delta x)$$

- condition with respect to absolute error in x and y

$$|\Delta y| \approx |f'(x)||\Delta x|$$

problem is ill-conditioned with respect to absolute error if $|f'(x)|$ is very large

- condition with respect to relative errors in x and y

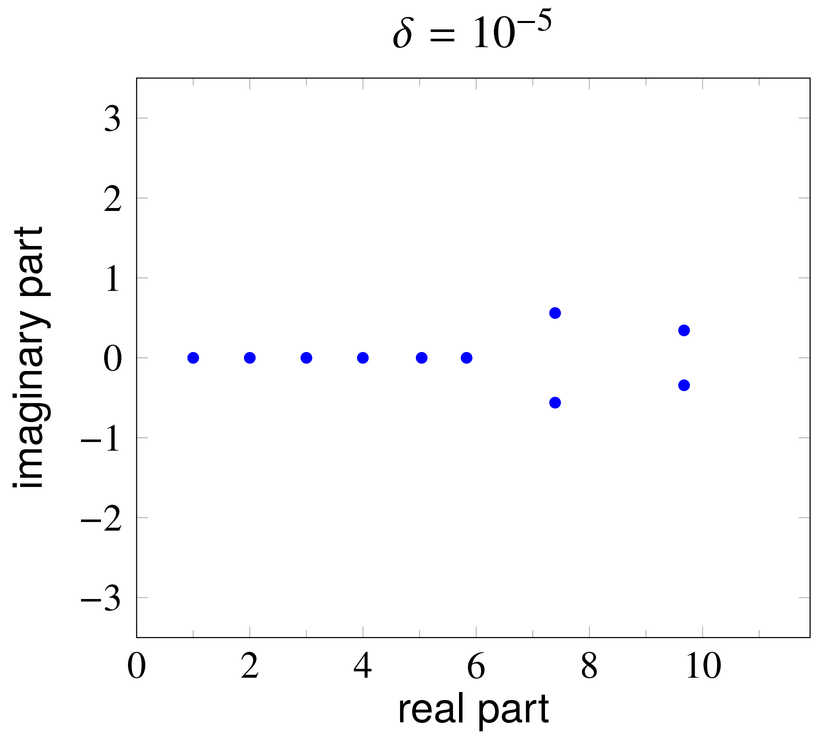
$$\frac{|\Delta y|}{|y|} \approx \frac{|f'(x)||x|}{|f(x)|} \frac{|\Delta x|}{|x|}$$

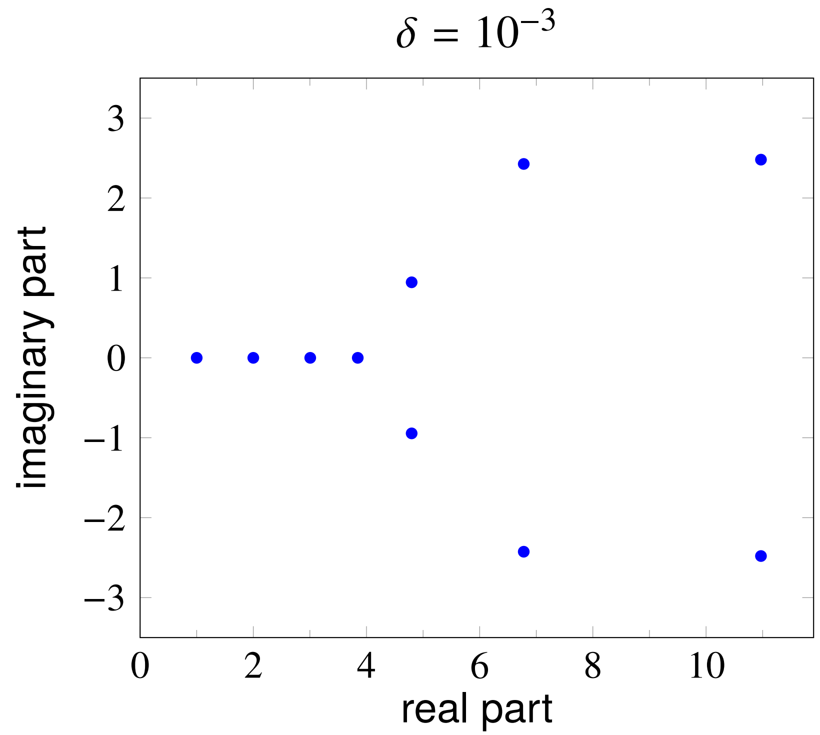
ill-conditioned with respect to relative error if $|f'(x)||x|/|f(x)|$ is very large

Roots of a polynomial

$$p(x) = (x-1)(x-2)\cdots(x-10) + \delta \cdot x^{10}$$

roots of p computed by MATLAB for two values of δ





roots can be very sensitive to errors in the coefficients

Condition of a set of linear equations

- assume A is nonsingular and $Ax = b$
- if we change b to $b + \Delta b$, the new solution is $x + \Delta x$ with

$$A(x + \Delta x) = b + \Delta b$$

- the change in x is

$$\Delta x = A^{-1}\Delta b$$

Condition

- the equations are *well-conditioned* if small Δb results in small Δx
- the equations are *ill-conditioned* if small Δb can result in large Δx

Example of ill-conditioned equations

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$$

- solution for $b = (1,1)$ is $x = (1,1)$
- change in x if we change b to $b + \Delta b$:

$$\Delta x = A^{-1}\Delta b = \begin{bmatrix} \Delta b_1 - 10^{10}(\Delta b_1 - \Delta b_2) \\ \Delta b_1 + 10^{10}(\Delta b_1 - \Delta b_2) \end{bmatrix}$$

small Δb can lead to extremely large Δx

Outline

- condition of a mathematical problem
- **matrix norm**
- condition number

Matrix norms

the **Frobenius norm** of an $m \times n$ matrix A is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}$$

- denoted $\|A\|$ in the textbook
- in MATLAB: `norm(A,'fro')`; in Julia: `norm(A)`

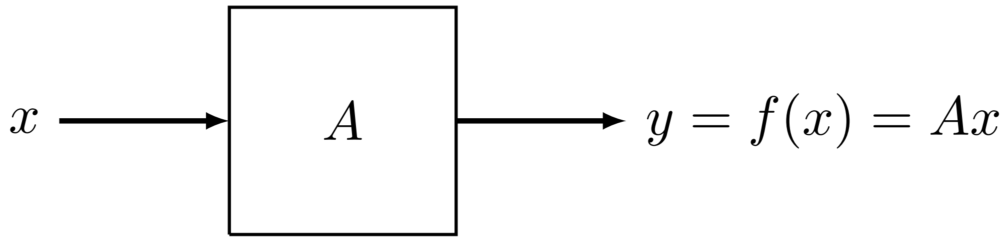
the **2-norm** or **spectral norm** is defined as

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

- the norms $\|Ax\|$ and $\|x\|$ are Euclidean norms of vectors
- no simple explicit expression, except for special A
- readily computed numerically (in MATLAB: `norm(A)`; in Julia: `opnorm(A)`)

Interpretation of 2-norm

the $m \times n$ matrix A defines a linear function $f(x) = Ax$



- $\|Ax\|/\|x\|$ gives the *amplification factor* or *gain* for input x
- the gain only depends on the direction of x
- the 2-norm of A is the maximum gain over all directions:

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

Computing the 2-norm of a matrix

Simple matrices: sometimes it is easy to maximize $\|Ax\|/\|x\|$

- zero matrix: $\|0\|_2 = 0$
- identity matrix: $\|I\|_2 = 1$
- diagonal matrix:

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}, \qquad \|A\|_2 = \max_{i=1,\ldots,n} |A_{ii}|$$

- matrix with orthonormal columns: $\|A\|_2 = 1$

General matrices: $\|A\|_2$ must be computed by numerical algorithms

Properties of the matrix norm

Properties satisfied by all matrix norms

- *nonnegative:* $\|A\|_2 \geq 0$ for all A
- *positive definiteness:* $\|A\|_2 = 0$ only if $A = 0$
- *homogeneity:* $\|\beta A\|_2 = |\beta|\|A\|_2$
- *triangle inequality:* $\|A + B\|_2 \leq \|A\|_2 + \|B\|_2$

Additional properties satisfied by the 2-norm

- $\|Ax\| \leq \|A\|_2\|x\|$ if the product Ax exists
- $\|AB\|_2 \leq \|A\|_2\|B\|_2$ if the product AB exists
- if A is nonsingular: $\|A\|_2\|A^{-1}\|_2 \geq 1$
- if A is nonsingular: $1/\|A^{-1}\|_2 = \min_{x \neq 0} \left(\|Ax\|_2/\|x\|\right)$
- $\|A^T\|_2 = \|A\|_2$

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Bound on absolute error

suppose A is nonsingular and define

$$x = A^{-1}b, \qquad \Delta x = A^{-1}\Delta b$$

Upper bound on $\|\Delta x\|$:

$$\|\Delta x\| \le \|A^{-1}\|_2 \, \|\Delta b\|$$

- follows from property 4 on page 15.11
- small $\|A^{-1}\|_2$ means that $\|\Delta x\|$ is small when $\|\Delta b\|$ is small
- large $\|A^{-1}\|_2$ means that $\|\Delta x\|$ can be large, even when $\|\Delta b\|$ is small
- for every A, there exists nonzero Δb such that $\|\Delta x\| = \|A^{-1}\|_2 \, \|\Delta b\|$

Bound on relative error

suppose in addition that $b \neq 0$; hence $x \neq 0$

Upper bound on $\|\Delta x\|/\|x\|$:

$$\frac{\|\Delta x\|}{\|x\|} \leq \|A\|_2 \|A^{-1}\|_2 \frac{\|\Delta b\|}{\|b\|} \tag{1}$$

- follows from $\|\Delta x\| \leq \|A^{-1}\|_2 \|\Delta b\|$ and $\|b\| \leq \|A\|_2 \|x\|$
- $\|A\|_2 \|A^{-1}\|_2$ small means $\|\Delta x\|/\|x\|$ is small when $\|\Delta b\|/\|b\|$ is small
- $\|A\|_2 \|A^{-1}\|_2$ large means $\|\Delta x\|/\|x\|$ can be much larger than $\|\Delta b\|/\|b\|$
- for every A, there exist nonzero b, Δb such that equality holds in (1)

Condition number

Definition: the condition number of a nonsingular matrix A is

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$$

Properties

- $\kappa(A) \geq 1$ for all A (last property on page page 15.11)
- A is a *well-conditioned* matrix if $\kappa(A)$ is small (close to 1):

  the relative error in x is not much larger than the relative error in b
- A is *badly conditioned* or *ill-conditioned* if $\kappa(A)$ is large:

  the relative error in x can be much larger than the relative error in b

Example

- A is blurring matrix, nonsingular with condition number $\approx 10^9$
- we apply A to image x

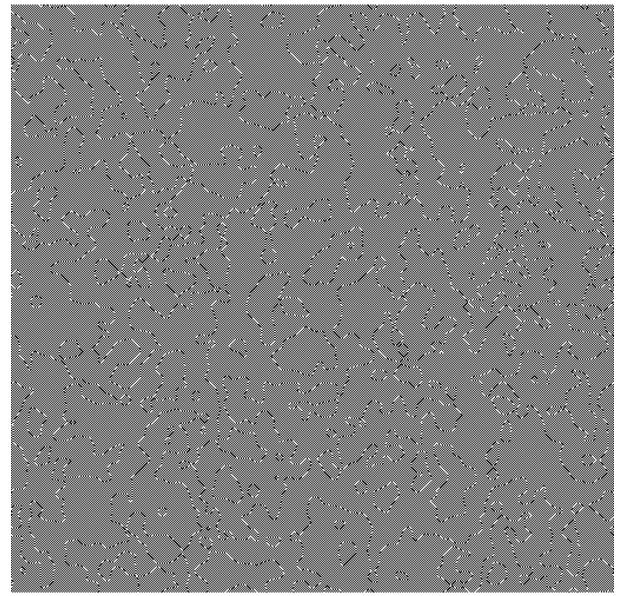
blurred image
$y_1 = Ax$

blurred and noisy image
$y_2 = Ax$ + small noise

Example

we solve $Ax = y$ for the two blurred images

$A^{-1}y_1$



$A^{-1}y_2$

- illustrates ill conditioning of A
- explains need for regularization in deblurring algorithms

Exercises

Exercise 1

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1+a & 1-a \end{bmatrix}, \qquad A^{-1} = \frac{1}{a} \begin{bmatrix} a-1 & 1 \\ a+1 & -1 \end{bmatrix}$$

a is small and nonzero ($a = 10^{-10}$ on page 15.7); show that $\kappa(A) \geq 1/|a|$

Exercise 2

suppose $A = UBV$ with U, V orthogonal, and B nonsingular; show that

$$\kappa(A) = \kappa(B)$$

Exercise 3

suppose $A = uv^T$ where u and v are vectors; show that $\|A\|_2 = \|u\|\|v\|$

Exercises

Exercise 4 (ex. A15.3)

- let u be a vector; show that

$$\|u\| = \max_{v \neq 0} \frac{v^T u}{\|v\|}$$

- let A be a matrix; show that

$$\|A\|_2 = \max_{y \neq 0,\, x \neq 0} \frac{y^T A x}{\|x\| \|y\|}$$

therefore $\|A\|_2 = \|A^T\|_2$