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ECE133A

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Homework 7 solutions

1. *Exercise A10.1 (a,b,c).* We use the following notation:

$$s(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 0.95 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

With this notation, the system equation is $s(t+1) = As(t) + bu(t)$. By applying this recursively, we can eliminate the intermediate states and express $s(N)$ as a linear function of $u(0)$, ..., $u(N-1)$:

$$\begin{aligned} s(1) &= Bu(0) \\ s(2) &= As(1) + bu(1) \\ &= Abu(0) + bu(1) \\ s(3) &= As(2) + bu(2) \\ &= A^2bu(0) + Abu(1) + bu(0) \\ &\vdots \\ s(N) &= A^{N-1}bu(0) + A^{N-2}bu(1) + \cdots + Abu(N-2) + bu(N-1). \end{aligned}$$

In matrix form,

$$s(N) = \begin{bmatrix} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-2) \\ u(N-1) \end{bmatrix}.$$

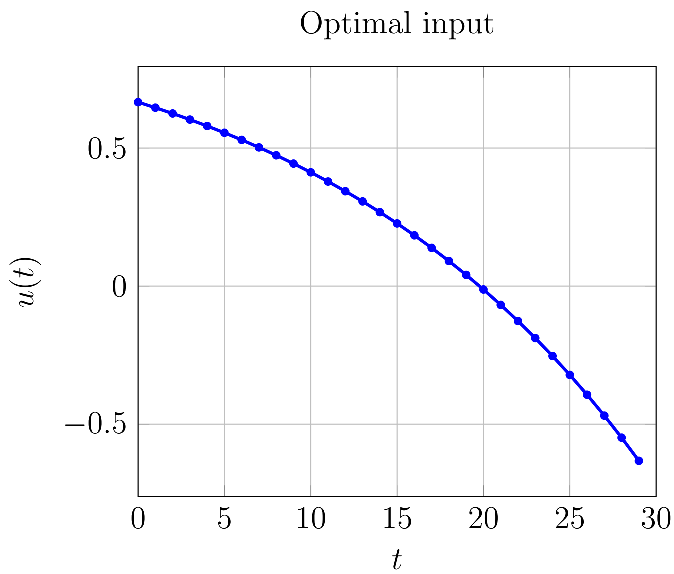
(a) The minimum energy optimal control problem

$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{N-1} u(t)^2 \\ \text{subject to} & s(N) = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \end{array}$$

is a least norm problem with

$$x = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-2) \\ u(N-1) \end{bmatrix}, \qquad C = \begin{bmatrix} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{bmatrix}, \qquad d = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

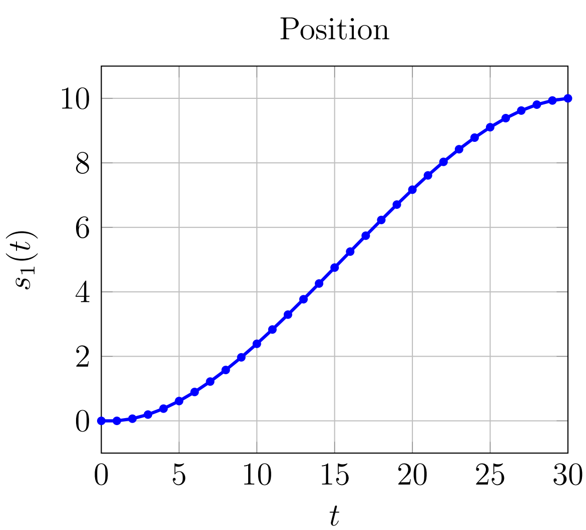
(b) The optimal input for $N = 30$ is shown in the figure below.

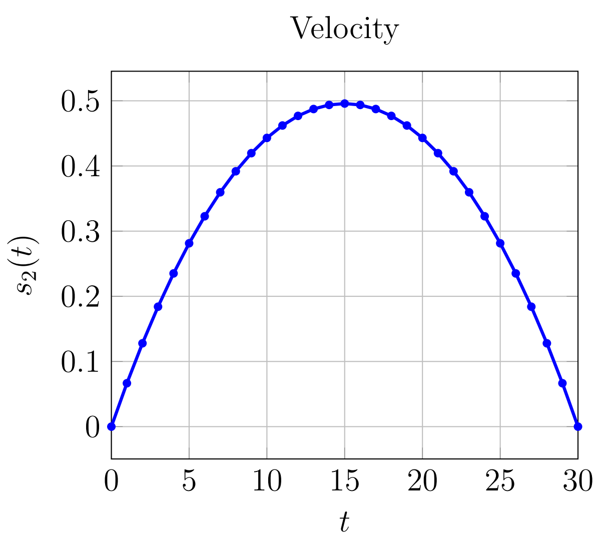


We use the following MATLAB code.

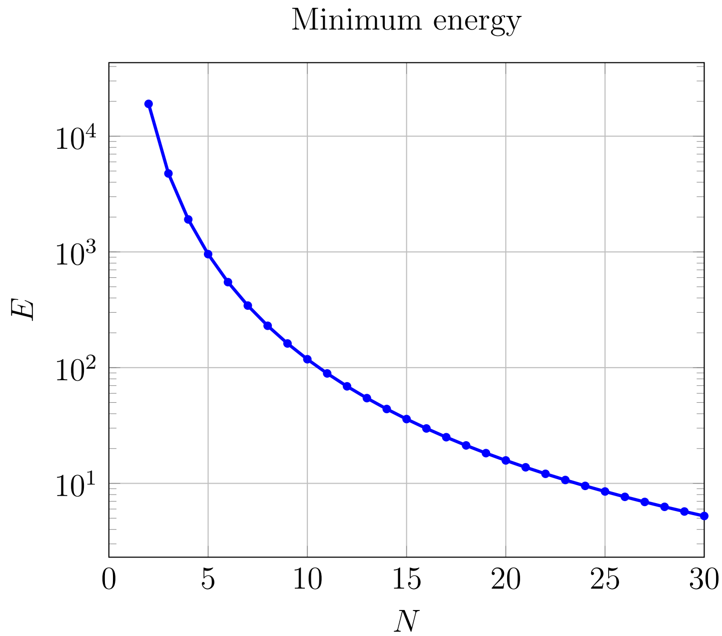
```
A = [ 1 1; 0 0.95 ];
b = [ 0; 0.1 ];
N = 30;
C = zeros(2, N);
C(:,N) = b;
for t = N-1:-1:1
    C(:,t) = A*C(:,t+1);
end;
u = C' * ( (C*C') \ [10;0] );
```

The resulting position and velocity are as follows.





(c) We repeat part (b) for different N, and calculate $E = \sum_{t=0}^{N-1} u(t)^2$ for each value of N.



2. *Exercise A10.14.*

(a) The optimality conditions are

$$\begin{bmatrix} A^TA & e_i \\ e_i^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

From the first equation, $x = -z(A^TA)^{-1}e_i$. Substituting this in the second equation gives

$$e_i^Tx = -ze_i^T(A^TA)^{-1}e_i = -1.$$

Therefore $z = 1/(e_i^T(A^TA)^{-1}e_i)$ and

$$x = -z(A^TA)^{-1}e_i = -\frac{1}{e_i^T(A^TA)^{-1}e_i}(A^TA)^{-1}e_i.$$

Note that $(A^TA)^{-1}e_i$ is the ith column of the matrix $(A^TA)^{-1}$, and $e_i^T(A^TA)^{-1}e_i$ is the ith diagonal entry of $(A^TA)^{-1}$.

(b) We first factorize $A = QR$ ($2mn^2$ flops). We then compute $H = (A^TA)^{-1} = R^{-1}R^{-T}$ by solving $R^TRH = I$.

- We first compute R^{-T} by solving $R^TX = I$, column by column, using forward substitution. The solution X is lower triangular, so we can start the forward substitution for column i at the diagonal entry, and the cost for column i is $(n-i+1)^2$. The total for $X = R^{-T}$ is $n^2 + (n-1)^2 + \cdots + 1 \approx n^3/3$ flops.

- We then compute H by solving $RH = X$, column by column, using back substitution. We know the result will be symmetric, so it is sufficient to compute the lower triangular part, and we can stop the back substitution for column i when we have computed the diagonal entry. The cost for column i is therefore $(n-i+1)^2$, and the total cost for the entire lower-triangular part of H is $n^3/3$ flops.

After computing H, we obtain

$$\hat{x}^{(i)} = -\frac{1}{e_i^T H e_i} H e_i = -\frac{1}{H_{ii}} H e_i,$$

by dividing He_i (which is column i of H) by the diagonal entry $e_i^T H e_i = H_{ii}$. This takes n^2 flops for the n vectors $\hat{x}^{(i)}$.

The total for computing all vectors $\hat{x}^{(i)}$ is therefore

$$2mn^2 + (2/3)n^3.$$

3. Exercise A11.5.

   (a) A and B are positive semidefinite.

   The angle between x and its projection $f(x)$ is always less than or equal to 90°. Therefore the inner product $x^T Ax$ is nonnegative. Similarly, the angle between x and $g(x)$ is less than or equal to 90°, and therefore the inner product $x^T Bx$ is always nonnegative.

   The result also follows algebraically:

$$x^T Ax = \frac{(a^T x)^2}{\|a\|^2} \geq 0 \quad \text{for all } x,$$

   and

$$x^T Bx = \|x\|^2 - \frac{(a^T x)^2}{\|a\|^2} \geq 0 \quad \text{for all } x$$

   because of the Cauchy–Schwarz inequality ($|a^T x| \leq \|a\| \, \|x\|$).

   (b) A and B are not positive definite. To see this, we note that $x^T Ax = 0$ if x is any vector orthogonal to a, and that $x^T Bx = 0$ for $x = a$. Hence $x^T Ax = 0$ is possible with nonzero x (if $n \geq 2$) and $x^T Bx = 0$ with nonzero x (if $a \neq 0$).

4. Exercise A11.8 (b,d).

   (b) $|a| < 1/\sqrt{2}$.

   We compute the Cholesky factorization

$$\begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{12} & R_{22} & 0 \\ R_{13} & R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix},$$

using the standard method. We start with the first row

$$R_{11} = 1, \qquad R_{12} = \frac{a}{R_{11}} = a, \qquad R_{13} = \frac{0}{R_{11}} = 0.$$

This gives the partial factorization

$$\begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & R_{22} & 0 \\ 0 & R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix}.$$

Next we compute the factorization of the 2×2 matrix

$$\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} - \begin{bmatrix} a \\ 0 \end{bmatrix} \begin{bmatrix} a & 0 \end{bmatrix} = \begin{bmatrix} 1-a^2 & a \\ a & 1 \end{bmatrix}.$$

The Cholesky factorization is

$$\begin{bmatrix} 1-a^2 & a \\ a & 1 \end{bmatrix} = \begin{bmatrix} R_{22} & 0 \\ R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} R_{22} & R_{23} \\ 0 & R_{33} \end{bmatrix}.$$

This requires $1 - a^2 > 0$. If this inequality holds,

$$R_{22} = \sqrt{1-a^2}, \qquad R_{23} = \frac{a}{R_{22}} = \frac{a}{\sqrt{1-a^2}}.$$

We now have the partial factorization

$$\begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & \sqrt{1-a^2} & 0 \\ 0 & a/\sqrt{1-a^2} & R_{33} \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & \sqrt{1-a^2} & a/\sqrt{1-a^2} \\ 0 & 0 & R_{33} \end{bmatrix}.$$

The last element is computed from

$$1 - \frac{a^2}{1-a^2} = \frac{1-2a^2}{1-a^2} = R_{33}^2.$$

Since R_{33} must be positive, we need $1 - 2a^2 > 0$.

To summarize, the matrix A can be factored as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & \sqrt{1-a^2} & 0 \\ 0 & a/\sqrt{1-a^2} & \sqrt{1-2a^2}/\sqrt{1-a^2} \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & \sqrt{1-a^2} & a/\sqrt{1-a^2} \\ 0 & 0 & \sqrt{1-2a^2}/\sqrt{1-a^2} \end{bmatrix}$$

if and only if $1-a^2 > 0$ and $1-2a^2 > 0$. In other words, if and only if $|a| < 1/\sqrt{2}$.

(d) $|a| < 1$. The Cholesky factorization

$$A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & \sqrt{1-a^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{1-a^2} \end{bmatrix}$$

exists if and only if $|a| < 1$.

5. *Exercise A11.26.*

(a) We determine R_{11}, R_{12}, R_{22} from the factorization

$$\begin{bmatrix} B & -C^T \\ C & D \end{bmatrix} = \begin{bmatrix} R_{11}^T & 0 \\ -R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^T R_{11} & R_{11}^T R_{12} \\ -R_{12}^T R_{11} & -R_{12}^T R_{12} + R_{22}^T R_{22} \end{bmatrix}.$$

Since B is positive definite it has a Cholesky factorization $B = R_{11}^T R_{11}$. From the 1,2 block, we determine $R_{12} = -R_{11}^{-T} C^T$. This also satisfies the equation for the 2,1 block, since $-R_{12}^T R_{11} = C R_{11}^{-1} R_{11} = C$. Finally, we determine R_{22} from $D + R_{12}^T R_{12} = R_{22}^T R_{22}$. The matrix on the left-hand side is the sum of a positive definite and a positive semidefinite matrix, so it is positive definite and has a Cholesky factorization $R_{22}^T R_{22}$.

(b) The main steps are the following.

- Cholesky factorization of B. $(1/3)n^3$ flops.
- Solve $R_{11}^T R_{12} = -C^T$ for R_{12}, column by column, by forward substitution. $n^2 m$ flops.
- Compute the symmetric matrix $E = D + R_{12}^T R_{12}$. $m^2 n$ flops.
- Cholesky factorization of E. $(1/3)m^3$ flops.

The total is the same as for a Cholesky factorization of the size $m + n$:

$$\frac{1}{3}n^3 + n^2 m + m^2 n + \frac{1}{3}m^3 = \frac{1}{3}(m+n)^3.$$