2. Norm, distance, angle

- norm
- distance
- k-means algorithm
- angle
- complex vectors

Euclidean norm

(Euclidean) norm of vector $a \in \mathbf{R}^n$:

$$\begin{aligned} \|a\| &= \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \\ &= \sqrt{a^T a} \end{aligned}$$

- if $n = 1$, $\|a\|$ reduces to absolute value $|a|$
- measures the magnitude of a
- sometimes written as $\|a\|_2$ to distinguish from other norms, *e.g.*,

$$\|a\|_1 = |a_1| + |a_2| + \cdots + |a_n|$$

Properties

Positive definiteness

$$\|a\| \geq 0 \quad \text{for all } a, \qquad \|a\| = 0 \quad \text{only if } a = 0$$

Homogeneity

$$\|\beta a\| = |\beta|\|a\| \quad \text{for all vectors } a \text{ and scalars } \beta$$

Triangle inequality (proved on page 2.7)

$$\|a + b\| \leq \|a\| + \|b\| \quad \text{for all vectors } a \text{ and } b \text{ of equal length}$$

Norm of block vector: if a, b are vectors,

$$\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{\|a\|^2 + \|b\|^2}$$

Cauchy–Schwarz inequality

$$|a^T b| \le \|a\| \|b\| \quad \text{for all } a, b \in \mathbf{R}^n$$

moreover, equality $|a^T b| = \|a\| \|b\|$ holds if:

- $a = 0$ or $b = 0$; in this case $a^T b = 0 = \|a\| \|b\|$
- $a \ne 0$ and $b \ne 0$, and $b = \gamma a$ for some $\gamma > 0$; in this case

$$0 < a^T b = \gamma \|a\|^2 = \|a\| \|b\|$$

- $a \ne 0$ and $b \ne 0$, and $b = -\gamma a$ for some $\gamma > 0$; in this case

$$0 > a^T b = -\gamma \|a\|^2 = -\|a\| \|b\|$$

Proof of Cauchy–Schwarz inequality

1. trivial if $a = 0$ or $b = 0$

2. assume $\|a\| = \|b\| = 1$; we show that $-1 \le a^T b \le 1$

$$\begin{aligned} 0 &\le \|a - b\|^2 \\ &= (a-b)^T(a-b) \\ &= \|a\|^2 - 2a^Tb + \|b\|^2 \\ &= 2(1 - a^Tb) \end{aligned}$$

with equality only if $a = b$

$$\begin{aligned} 0 &\le \|a + b\|^2 \\ &= (a+b)^T(a+b) \\ &= \|a\|^2 + 2a^Tb + \|b\|^2 \\ &= 2(1 + a^Tb) \end{aligned}$$

with equality only if $a = -b$

3. for general nonzero a, b, apply case 2 to the unit-norm vectors

$$\frac{1}{\|a\|}a, \quad \frac{1}{\|b\|}b$$

Average and RMS value

let a be a real n-vector

- the *average* of the elements of a is

$$\mathbf{avg}(a) = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\mathbf{1}^T a}{n}$$

- the *root-mean-square* value is the root of the average squared entry

$$\mathbf{rms}(a) = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} = \frac{\|a\|}{\sqrt{n}}$$

Exercise: show that $|\mathbf{avg}(a)| \leq \mathbf{rms}(a)$

Triangle inequality from Cauchy–Schwarz inequality

for vectors a, b of equal size

$$\begin{array}{rcll}
\|a+b\|^2 & = & (a+b)^T(a+b) & \\
& = & a^Ta + b^Ta + a^Tb + b^Tb & \\
& = & \|a\|^2 + 2a^Tb + \|b\|^2 & \\
& \leq & \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 & \text{(by Cauchy–Schwarz)} \\
& = & (\|a\| + \|b\|)^2 &
\end{array}$$

- taking squareroots gives the triangle inequality
- triangle inequality is an equality if and only if $a^Tb = \|a\|\|b\|$ (see page 2.4)
- also note from line 3 that $\|a+b\|^2 = \|a\|^2 + \|b\|^2$ if $a^Tb = 0$

Outline

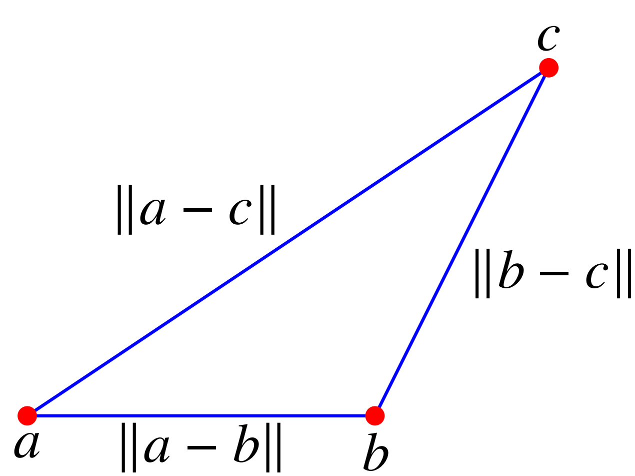
- norm
- **distance**
- k-means algorithm
- angle
- complex vectors

Distance

the (Euclidean) distance between vectors a and b is defined as $\|a - b\|$

- $\|a - b\| \geq 0$ for all a, b and $\|a - b\| = 0$ only if $a = b$
- triangle inequality

$$\|a - c\| \leq \|a - b\| + \|b - c\| \quad \text{for all } a, b, c$$



- RMS deviation between n-vectors a and b is $\mathbf{rms}(a - b) = \dfrac{\|a - b\|}{\sqrt{n}}$

Standard deviation

let a be a real n-vector

- the *de-meaned* vector is the vector of deviations from the average

$$a - \mathbf{avg}(a)\mathbf{1} = \begin{bmatrix} a_1 - \mathbf{avg}(a) \\ a_2 - \mathbf{avg}(a) \\ \vdots \\ a_n - \mathbf{avg}(a) \end{bmatrix} = \begin{bmatrix} a_1 - (\mathbf{1}^T a)/n \\ a_2 - (\mathbf{1}^T a)/n \\ \vdots \\ a_n - (\mathbf{1}^T a)/n \end{bmatrix}$$

- the *standard deviation* is the RMS deviation from the average

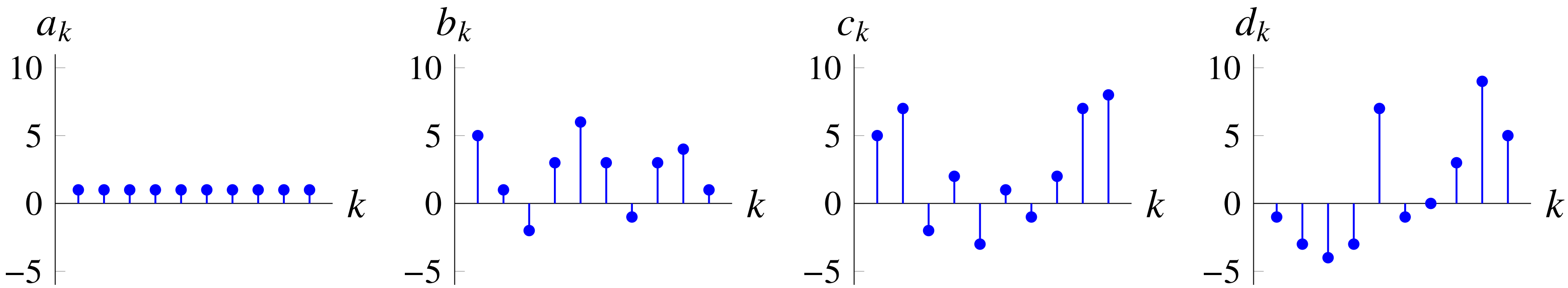
$$\mathbf{std}(a) = \mathbf{rms}(a - \mathbf{avg}(a)\mathbf{1}) = \frac{\left\| a - ((\mathbf{1}^T a)/n)\mathbf{1} \right\|}{\sqrt{n}}$$

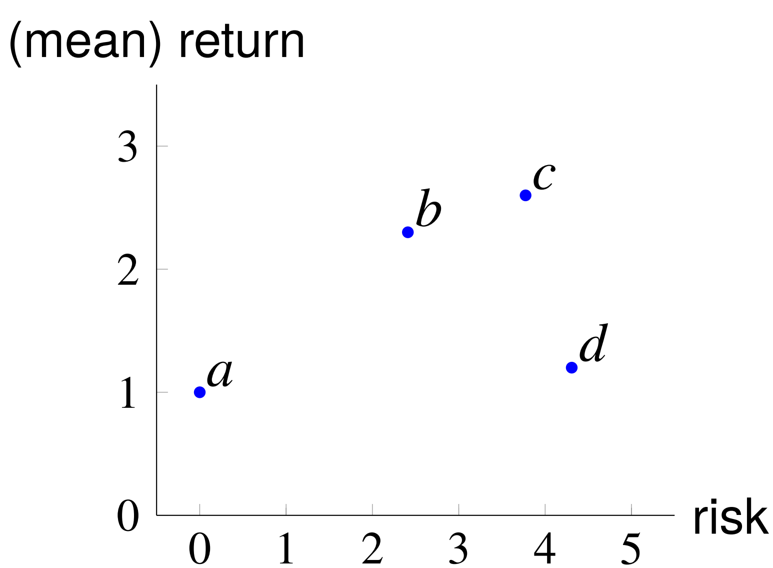
- the de-meaned vector in *standard units* is

$$\frac{1}{\mathbf{std}(a)}(a - \mathbf{avg}(a)\mathbf{1})$$

Mean return and risk of investment

- vectors represent time series of returns on an investment (as a percentage)
- average value is *(mean) return* of the investment
- standard deviation measures variation around the mean, *i.e.*, *risk*





Exercise

show that

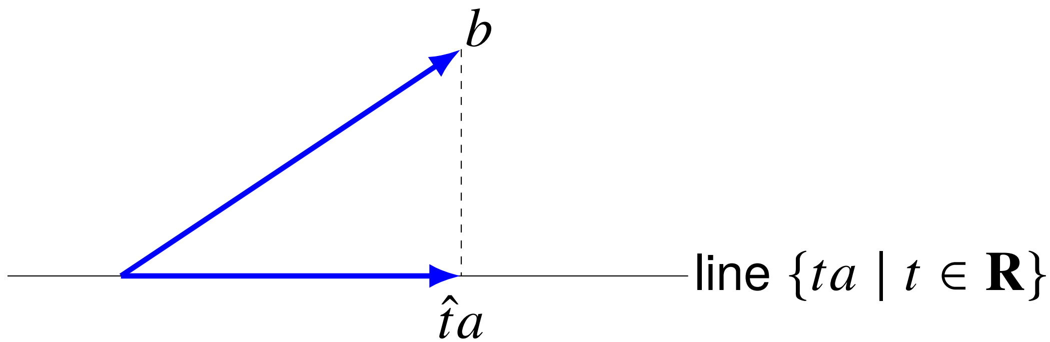
$$\mathbf{avg}(a)^2 + \mathbf{std}(a)^2 = \mathbf{rms}(a)^2$$

Solution

$$\begin{aligned}
\mathbf{std}(a)^2 &= \frac{\|a - \mathbf{avg}(a)\mathbf{1}\|^2}{n} \\
&= \frac{1}{n}\left(a - \frac{\mathbf{1}^T a}{n}\mathbf{1}\right)^T \left(a - \frac{\mathbf{1}^T a}{n}\mathbf{1}\right) \\
&= \frac{1}{n}\left(a^T a - \frac{(\mathbf{1}^T a)^2}{n} - \frac{(\mathbf{1}^T a)^2}{n} + \left(\frac{\mathbf{1}^T a}{n}\right)^2 n\right) \\
&= \frac{1}{n}\left(a^T a - \frac{(\mathbf{1}^T a)^2}{n}\right) \\
&= \mathbf{rms}(a)^2 - \mathbf{avg}(a)^2
\end{aligned}$$

Exercise: nearest scalar multiple

given two vectors $a, b \in \mathbf{R}^n$, with $a \neq 0$, find scalar multiple ta closest to b



Solution

- squared distance between ta and b is

$$\|ta - b\|^2 = (ta - b)^T (ta - b) = t^2 a^T a - 2ta^T b + b^T b$$

  a quadratic function of t with positive leading coefficient $a^T a$

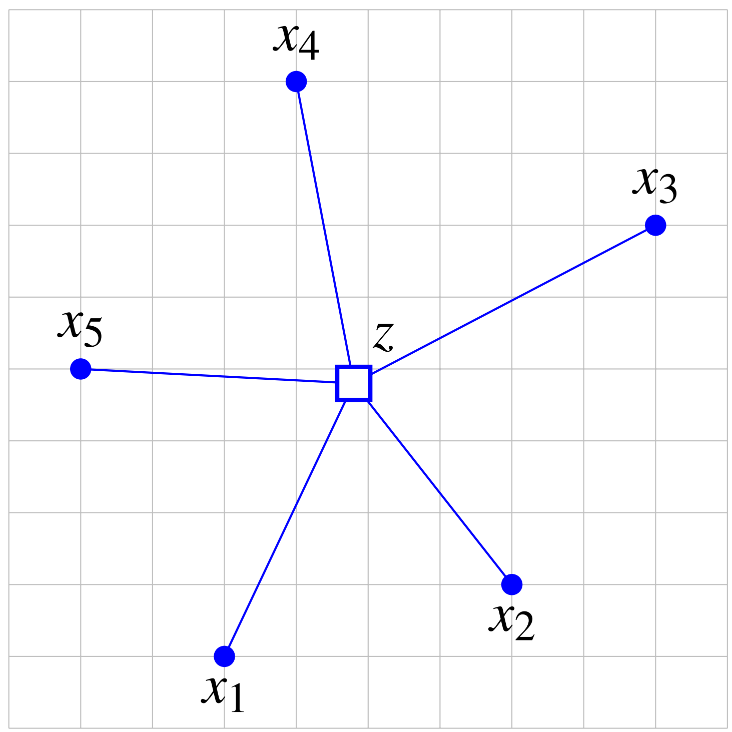
- derivative with respect to t is zero for

$$\hat{t} = \frac{a^T b}{a^T a} = \frac{a^T b}{\|a\|^2}$$

Exercise: average of collection of vectors

given N vectors $x_1, \ldots, x_N \in \mathbf{R}^n$, find the n-vector z that minimizes

$$\|z - x_1\|^2 + \|z - x_2\|^2 + \cdots + \|z - x_N\|^2$$



z is also known as the *centroid* of the points $x_1, \ldots, x_N$

Solution: sum of squared distances is

$$\begin{aligned}
&\|z - x_1\|^2 + \|z - x_2\|^2 + \cdots + \|z - x_N\|^2 \\
&\quad = \sum_{i=1}^{n} \left((z_i - (x_1)_i)^2 + (z_i - (x_2)_i)^2 + \cdots + (z_i - (x_N)_i)^2 \right) \\
&\quad = \sum_{i=1}^{n} \left(N z_i^2 - 2 z_i \left((x_1)_i + (x_2)_i + \cdots + (x_N)_i \right) + (x_1)_i^2 + \cdots + (x_N)_i^2 \right)
\end{aligned}$$

here $(x_j)_i$ is ith element of the vector x_j

- term i in the sum is minimized by

$$z_i = \frac{1}{N}((x_1)_i + (x_2)_i + \cdots + (x_N)_i)$$

- solution z is component-wise average of the points $x_1, \ldots, x_N$:

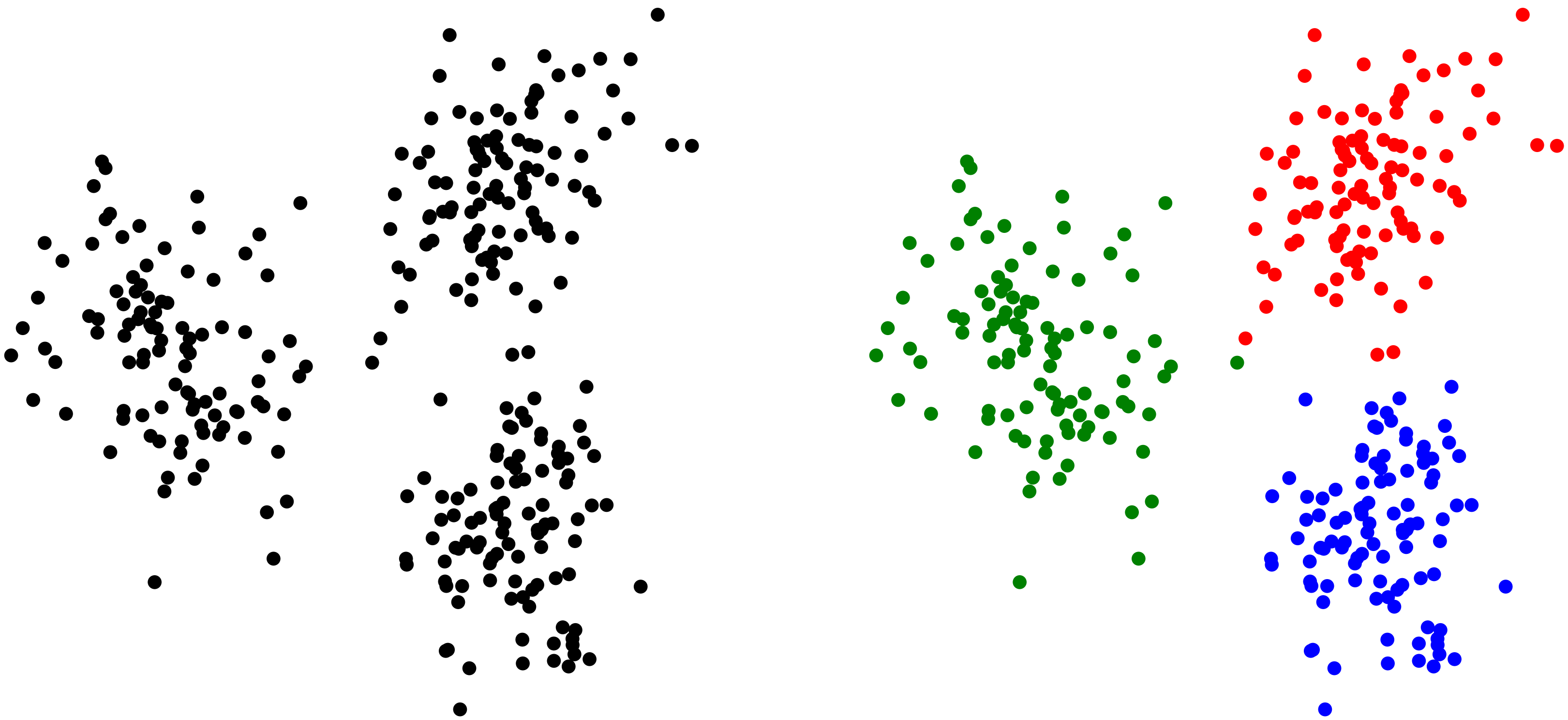
$$z = \frac{1}{N}(x_1 + x_2 + \cdots + x_N)$$

Outline

- norm
- distance
- **k-means algorithm**
- angle
- complex vectors

k-means clustering

a popular iterative algorithm for partitioning N vectors $x_1, \ldots, x_N$ in k clusters



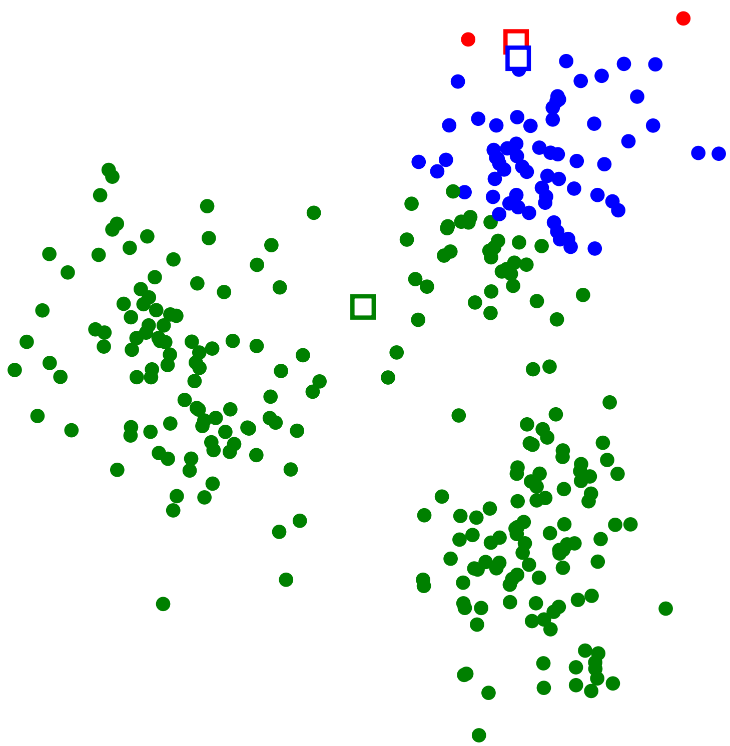
Algorithm

choose initial 'representatives' $z_1, \ldots, z_k$ for the k groups and repeat:

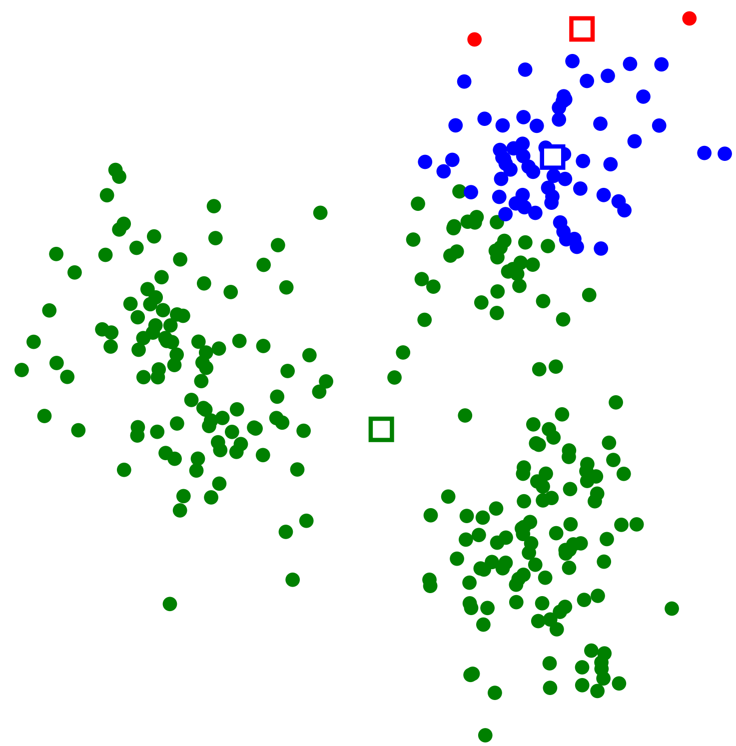
1. assign each vector x_i to the nearest group representative z_j
2. set the representative z_j to the mean of the vectors assigned to it

- initial representatives are often chosen randomly
- as a variation, choose a random initial partition and start with step 2
- solution depends on choice of initial representatives or partition
- can be shown to converge in a finite number of iterations
- in practice, often restarted a few times, with different starting points

Example: first iteration

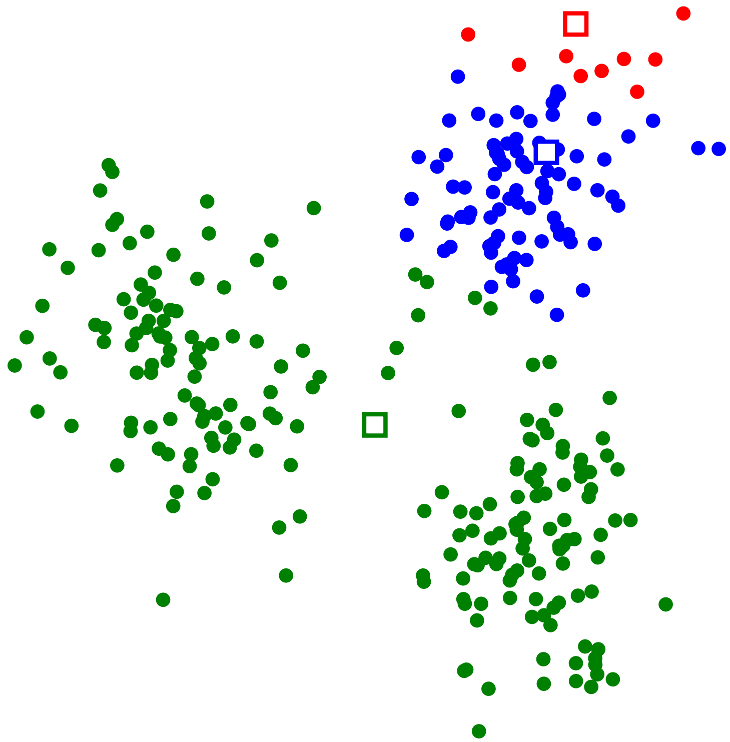


assignment to groups

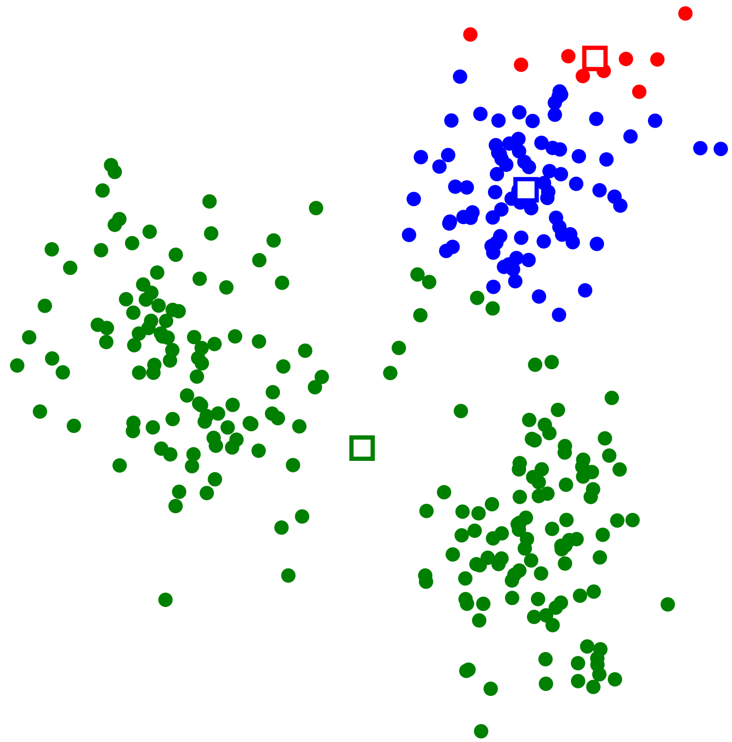


updated representatives

Iteration 2

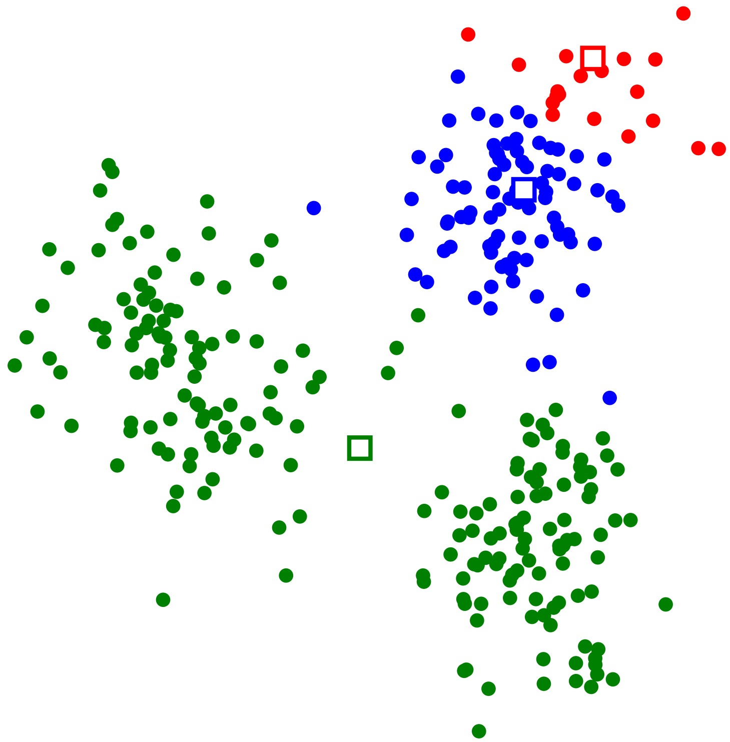


assignment to groups

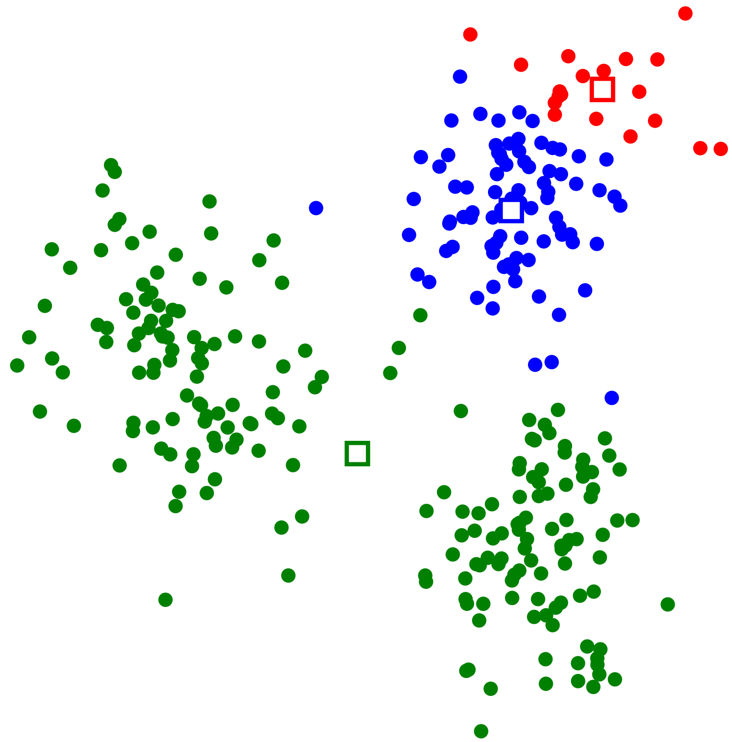


updated representatives

Iteration 3

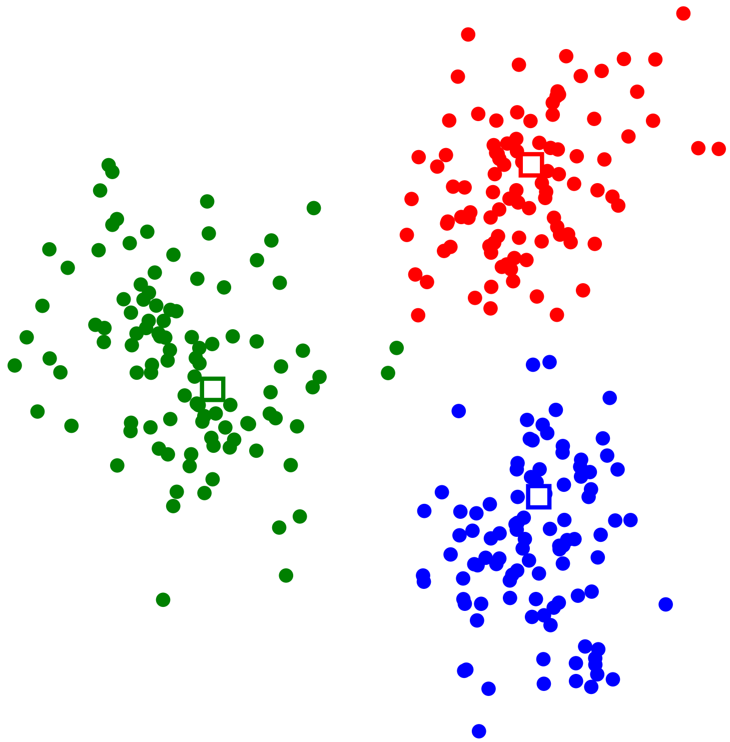


assignment to groups

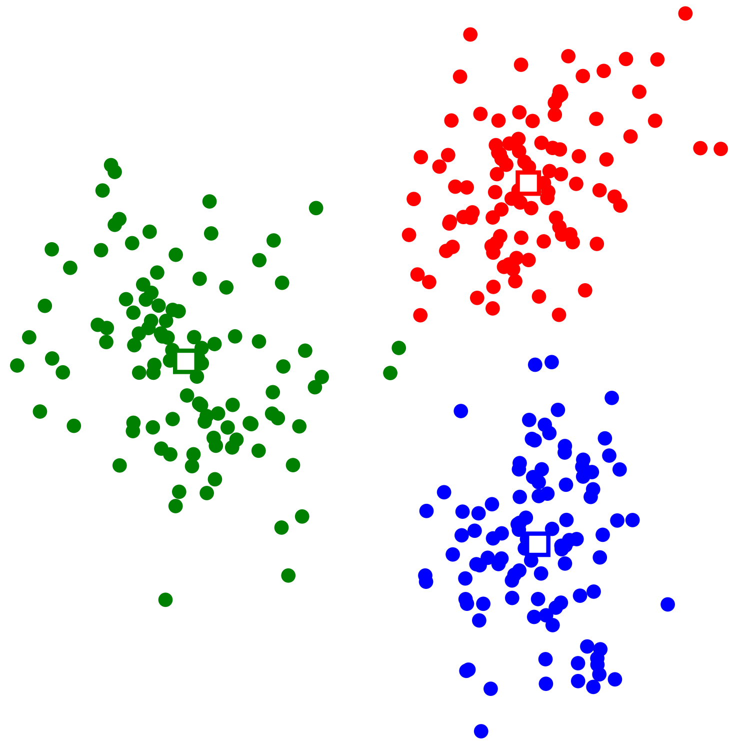


updated representatives

Iteration 11

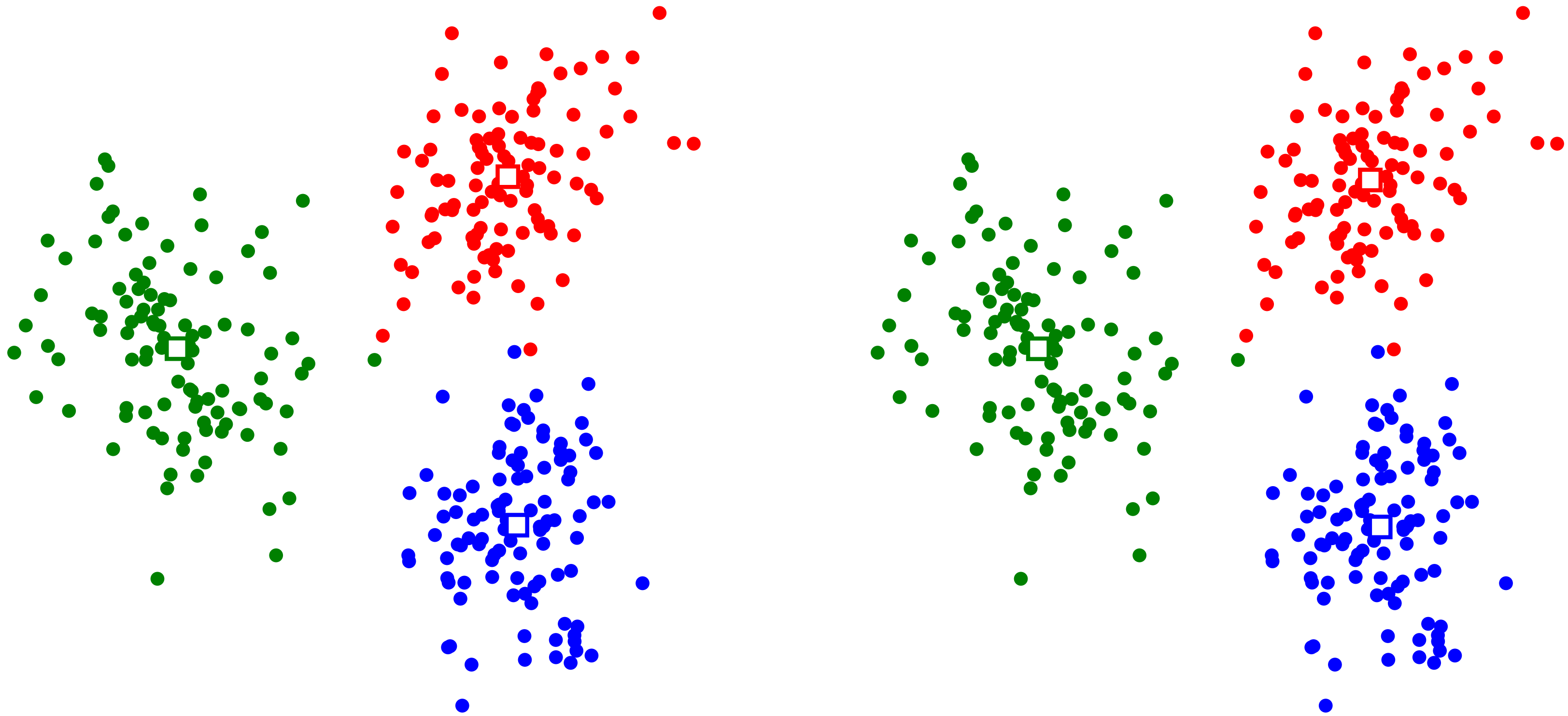


assignment to groups



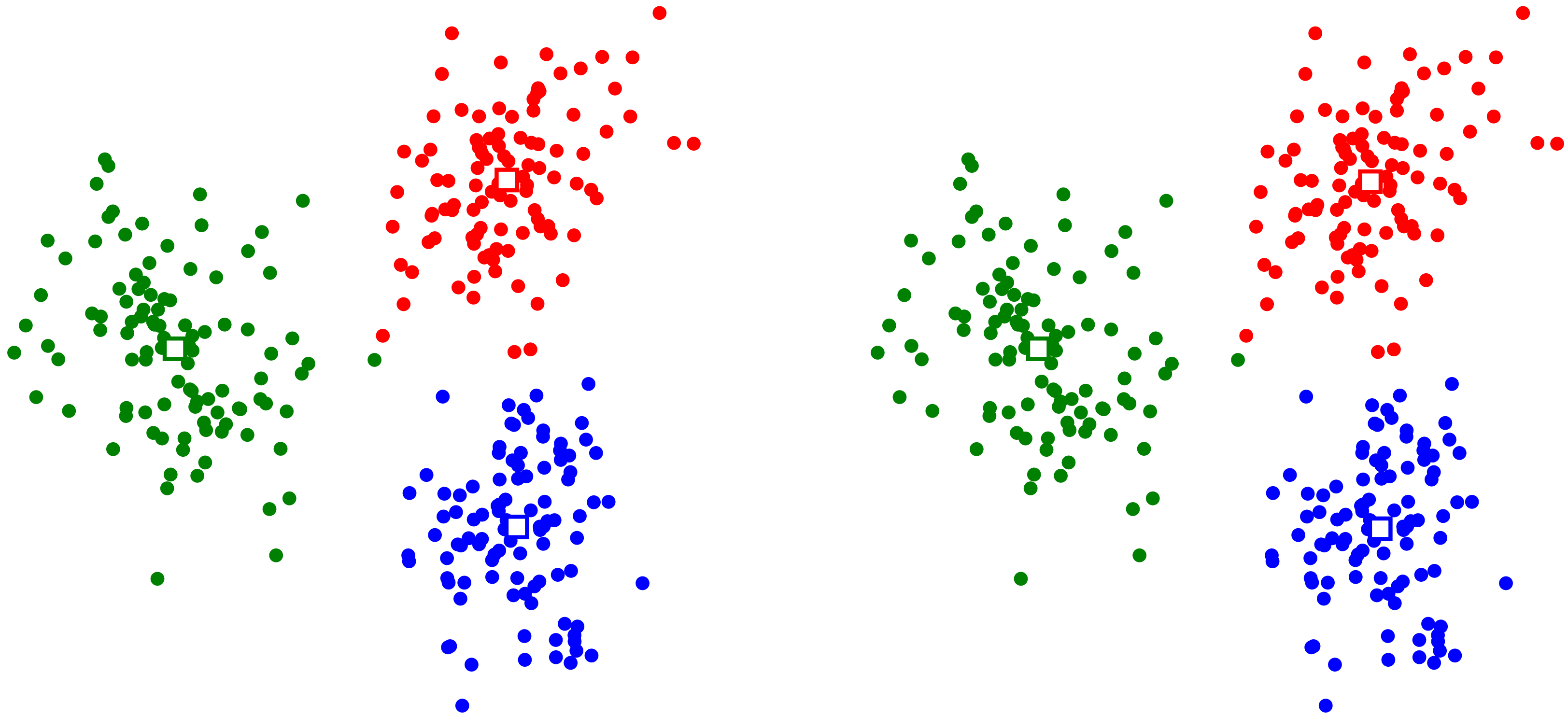
updated representatives

Iteration 12



assignment to groups updated representatives

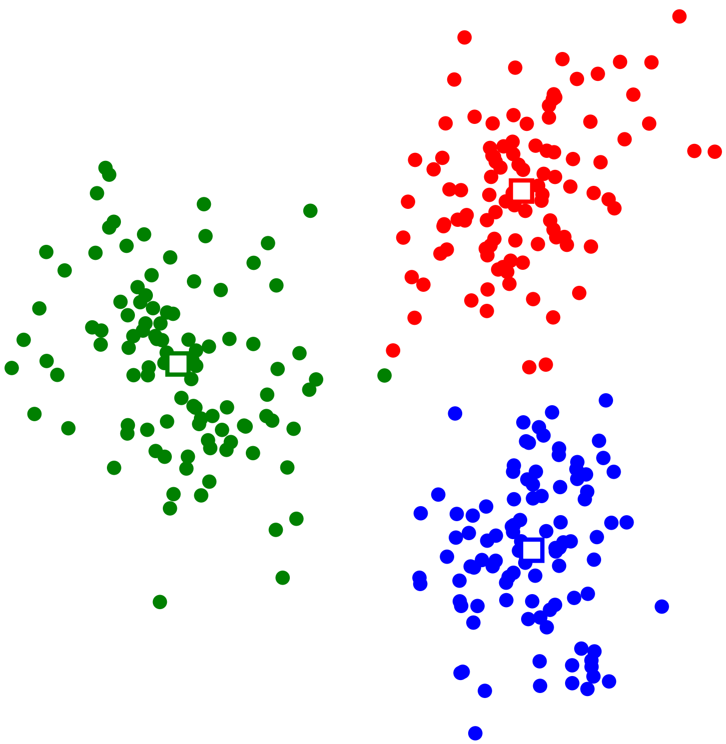
Iteration 13



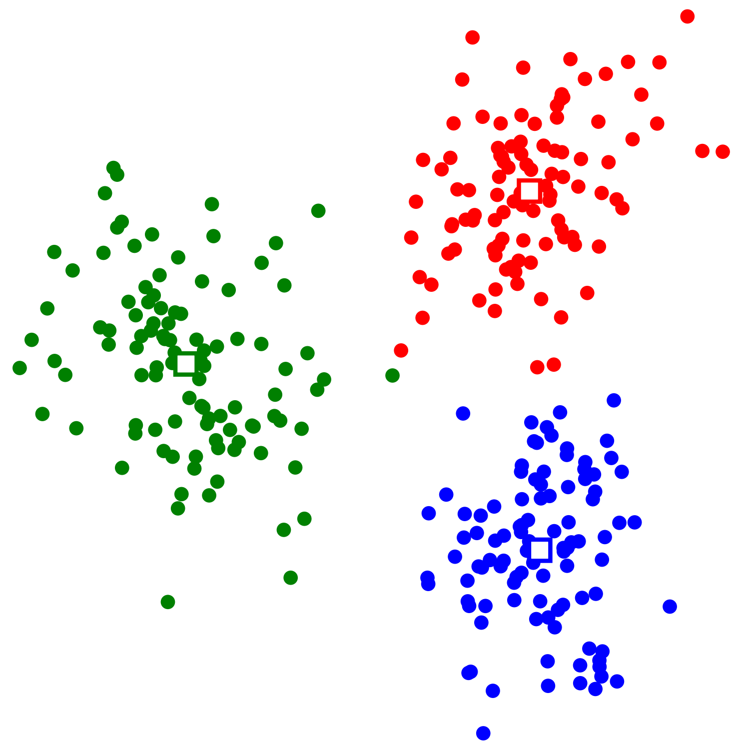
assignment to groups

updated representatives

Iteration 14



assignment to groups



updated representatives

Final clustering

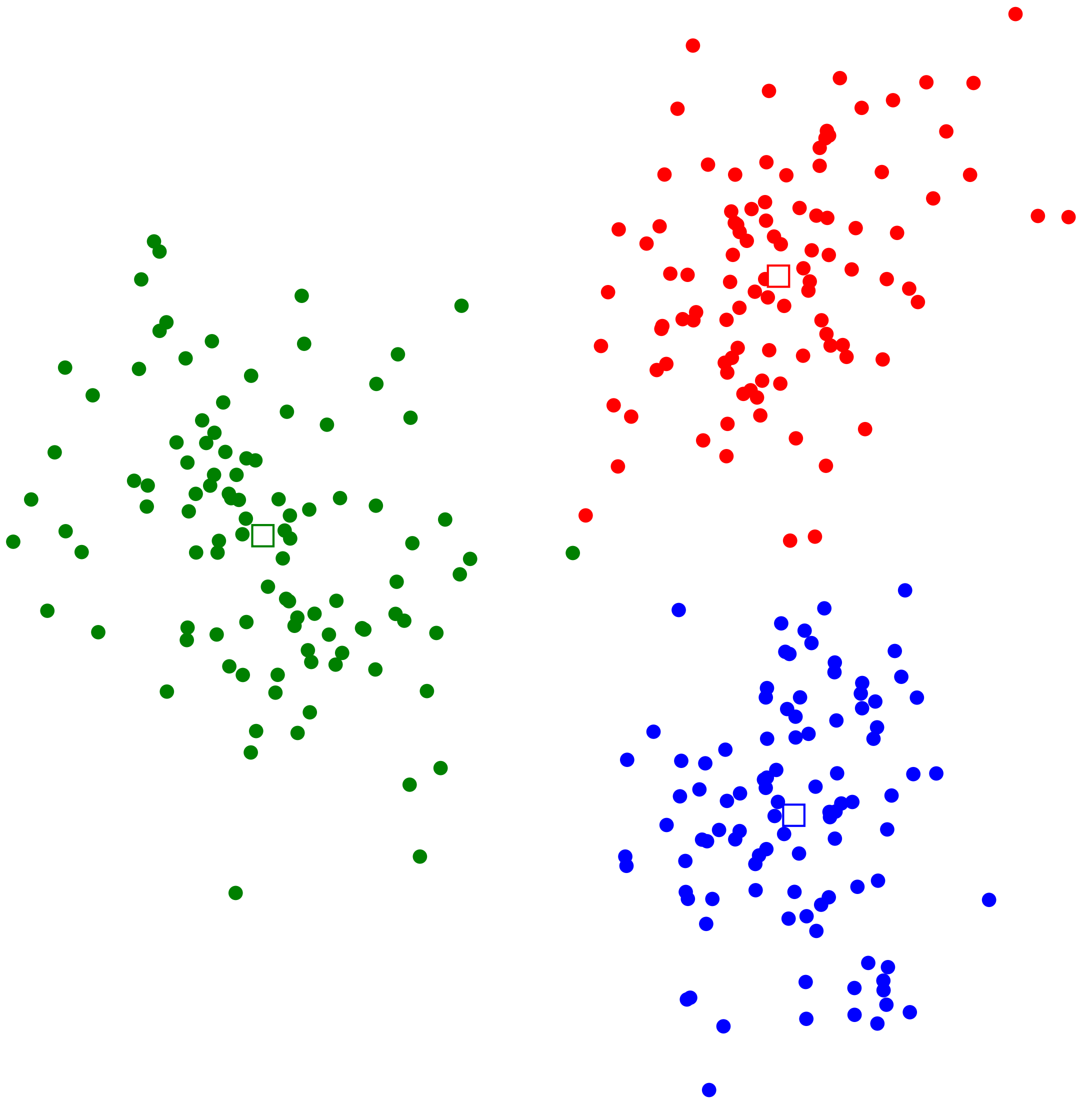
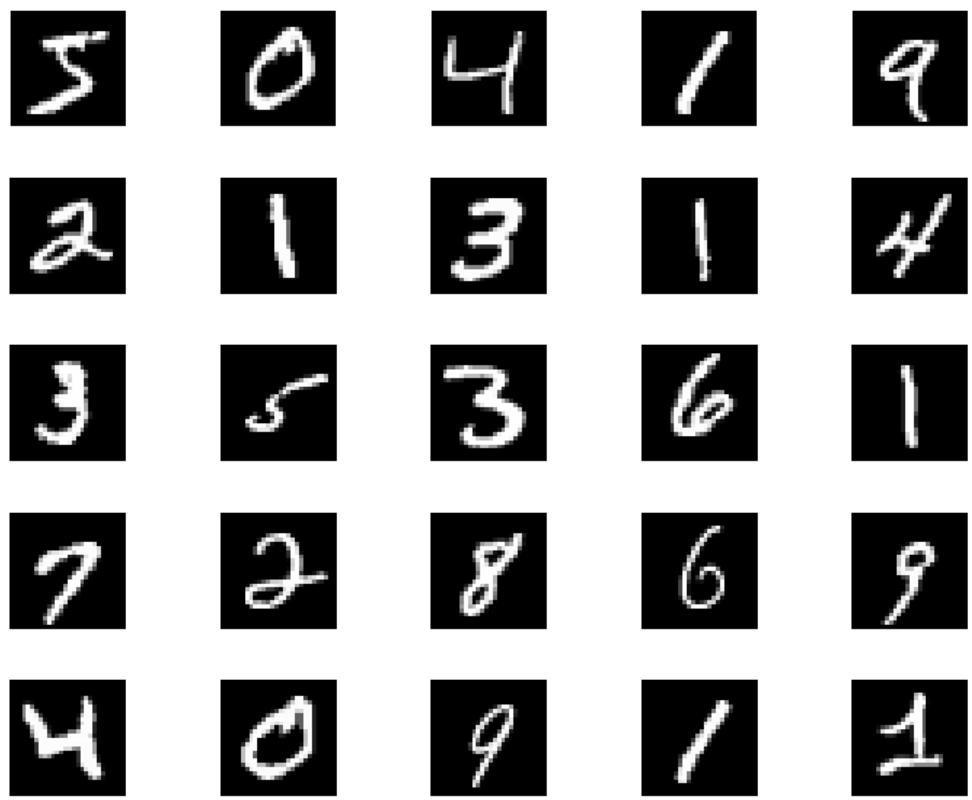


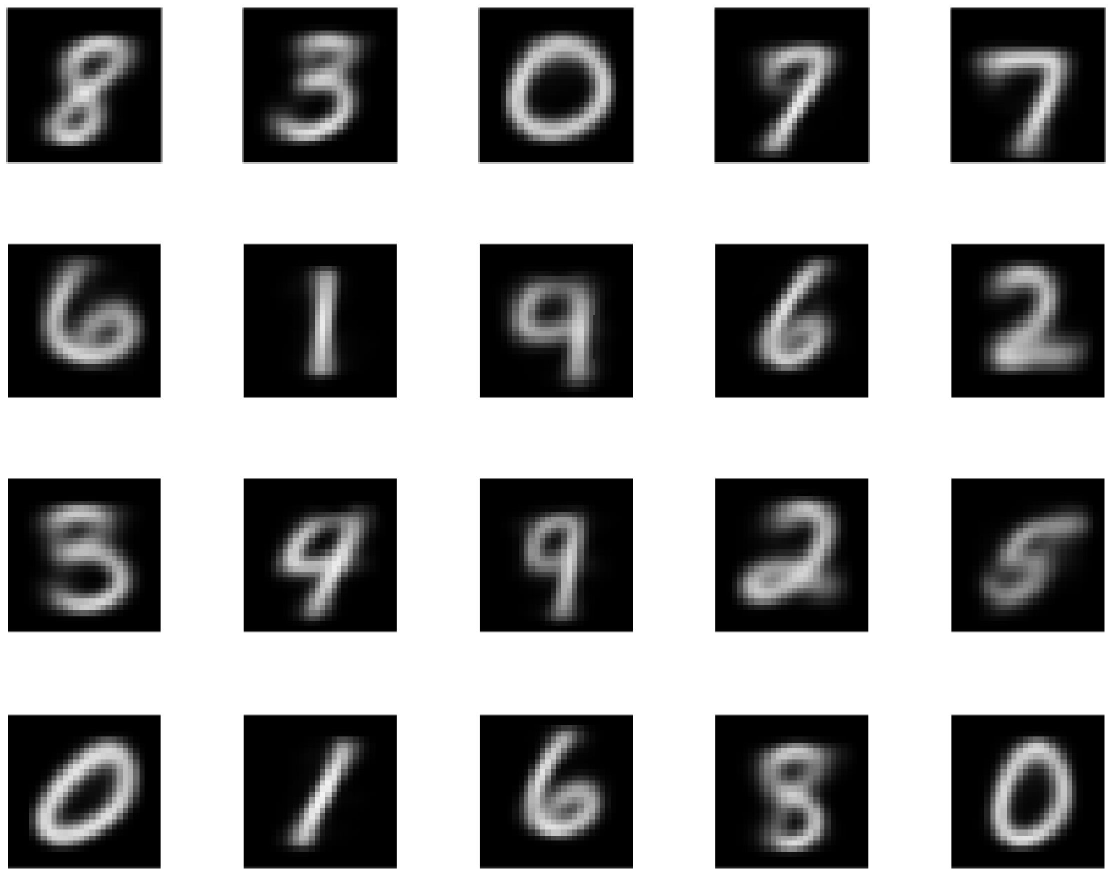
Image clustering

- MNIST dataset of handwritten digits
- $N = 60000$ grayscale images of size 28×28 (vectors x_i of size $28^2 = 784$)
- 25 examples:



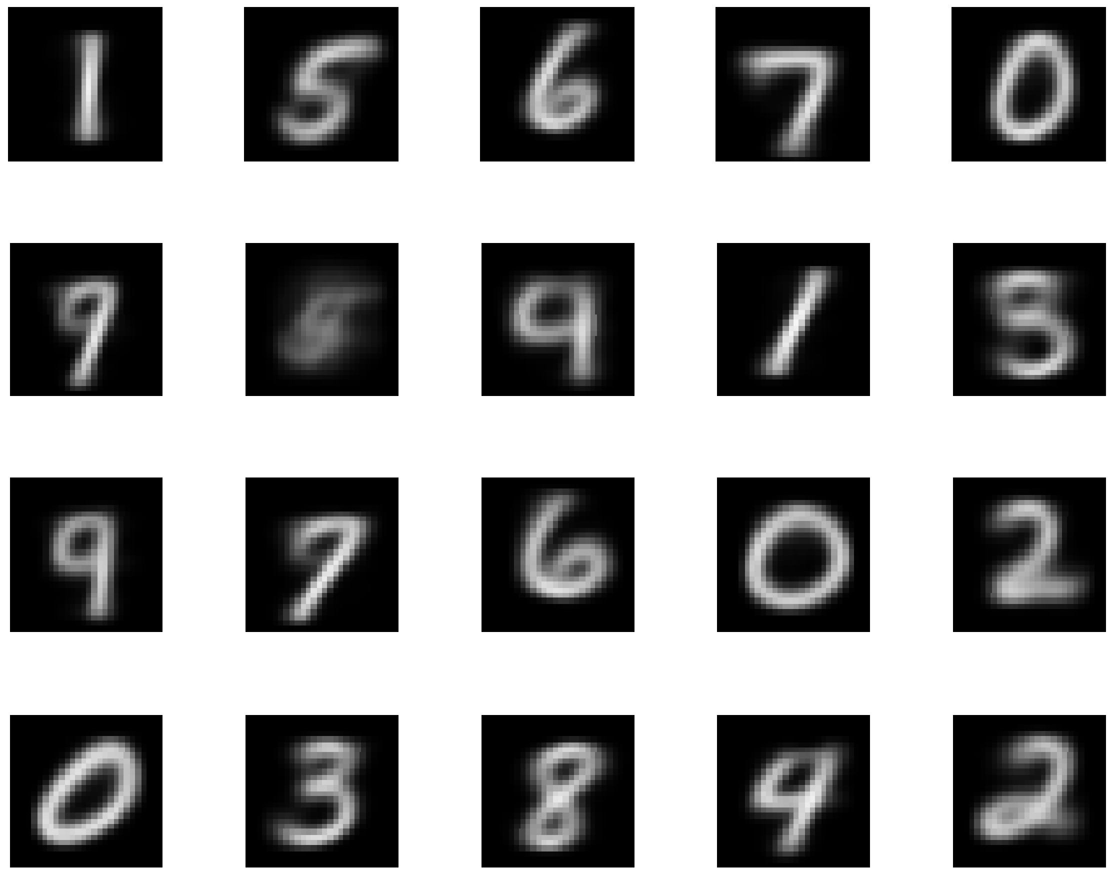
Group representatives ($k = 20$)

- k-means algorithm, with $k = 20$ and randomly chosen initial partition
- 20 group representatives



Group representatives ($k = 20$)

result for another initial partition



Document topic discovery

- $N = 500$ Wikipedia articles, from weekly most popular lists (9/2015–6/2016)
- dictionary of 4423 words
- each article represented by a word histogram vector of size 4423
- result of k-means algorithm with $k = 9$ and randomly chosen initial partition

Cluster 1

- largest coefficients in cluster representative z_1

word	fight	win	event	champion	fighter	...
coefficient	0.038	0.022	0.019	0.015	0.015	...

- documents in cluster 1 closest to representative

  "Floyd Mayweather, Jr", "Kimbo Slice", "Ronda Rousey", "José Aldo", "Joe Frazier", ...

Cluster 2

- largest coefficients in cluster representative z_2

word	holiday	celebrate	festival	celebration	calendar	...
coefficient	0.012	0.009	0.007	0.006	0.006	...

- documents in cluster 2 closest to representative

  "Halloween", "Guy Fawkes Night", "Diwali", "Hannukah", "Groundhog Day", ...

Cluster 3

- largest coefficients in cluster representative z_3

word	united	family	party	president	government	...
coefficient	0.004	0.003	0.003	0.003	0.003	...

- documents in cluster 3 closest to representative

  "Mahatma Gandhi", "Sigmund Freund", "Carly Fiorina", "Frederick Douglass", "Marco Rubio", ...

Cluster 4

- largest coefficients in cluster representative z_4

word	album	release	song	music	single	...
coefficient	0.031	0.016	0.015	0.014	0.011	...

- documents in cluster 4 closest to representative

  "David Bowie", "Kanye West", "Celine Dion", "Kesha", "Ariana Grande", ...

Cluster 5

- largest coefficients in cluster representative z_5

word	game	season	team	win	player	...
coefficient	0.023	0.020	0.018	0.017	0.014	...

- documents in cluster 5 closest to representative

  "Kobe Bryant", "Lamar Odom", "Johan Cruyff", "Yogi Berra", "José Mourinho", ...

Cluster 6

- largest coefficients in representative z_6

word	series	season	episode	character	film	...
coefficient	0.029	0.027	0.013	0.011	0.008	...

- documents in cluster 6 closest to cluster representative

  "The X-Files", "Game of Thrones", "House of Cards", "Daredevil", "Supergirl", ...

Cluster 7

- largest coefficients in representative z_7

word	match	win	championship	team	event	...
coefficient	0.065	0.018	0.016	0.015	0.015	...

- documents in cluster 7 closest to cluster representative

  "Wrestlemania 32", "Payback (2016)", "Survivor Series (2015)", "Royal Rumble (2016)", "Night of Champions (2015)", ...

Cluster 8

- largest coefficients in representative z_8

word	film	star	role	play	series	...
coefficient	0.036	0.014	0.014	0.010	0.009	...

- documents in cluster 8 closest to cluster representative

  "Ben Affleck", "Johnny Depp", "Maureen O'Hara", "Kate Beckinsale", "Leonardo DiCaprio", ...

Cluster 9

- largest coefficients in representative z_9

word	film	million	release	star	character	...
coefficient	0.061	0.019	0.013	0.010	0.006	...

- documents in cluster 9 closest to cluster representative

  "Star Wars: The Force Awakens", "Star Wars Episode I: The Phantom Menace", "The Martian (film)", "The Revenant (2015 film)", "The Hateful Eight", ...

Outline

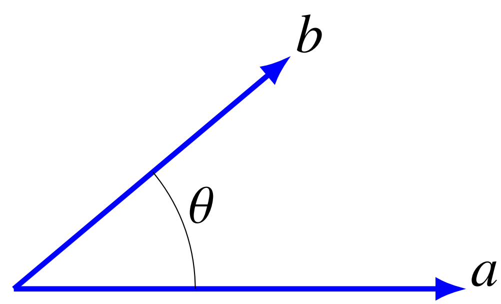
- norm
- distance
- k-means algorithm
- **angle**
- complex vectors

Angle between vectors

the angle between nonzero real vectors a, b is defined as

$$\arccos\left(\frac{a^T b}{\|a\|\,\|b\|}\right)$$

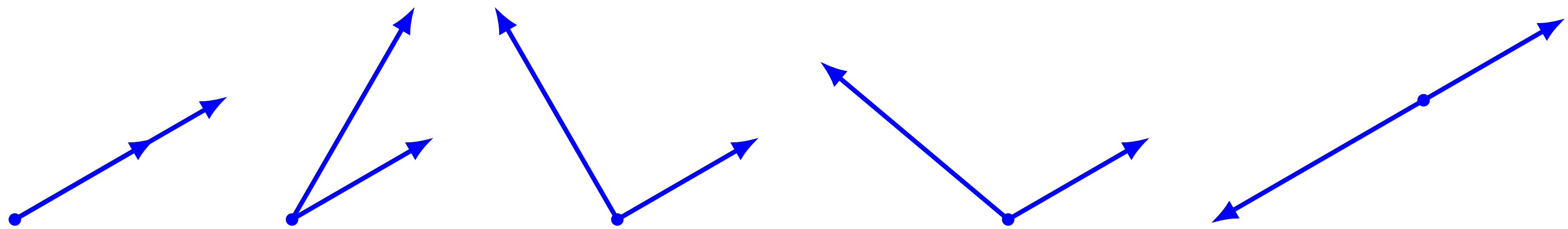
- this is the unique value of $\theta \in [0, \pi]$ that satisfies $a^T b = \|a\|\|b\| \cos\theta$



- Cauchy–Schwarz inequality guarantees that

$$-1 \le \frac{a^T b}{\|a\|\,\|b\|} \le 1$$

Terminology



$\theta = 0$	$a^T b = \|a\|\|b\|$	vectors are aligned or parallel
$0 \le \theta < \pi/2$	$a^T b > 0$	vectors make an acute angle
$\theta = \pi/2$	$a^T b = 0$	vectors are orthogonal ($a \perp b$)
$\pi/2 < \theta \le \pi$	$a^T b < 0$	vectors make an obtuse angle
$\theta = \pi$	$a^T b = -\|a\|\|b\|$	vectors are anti-aligned or opposed

Correlation coefficient

the *correlation coefficient* between non-constant vectors a, b is

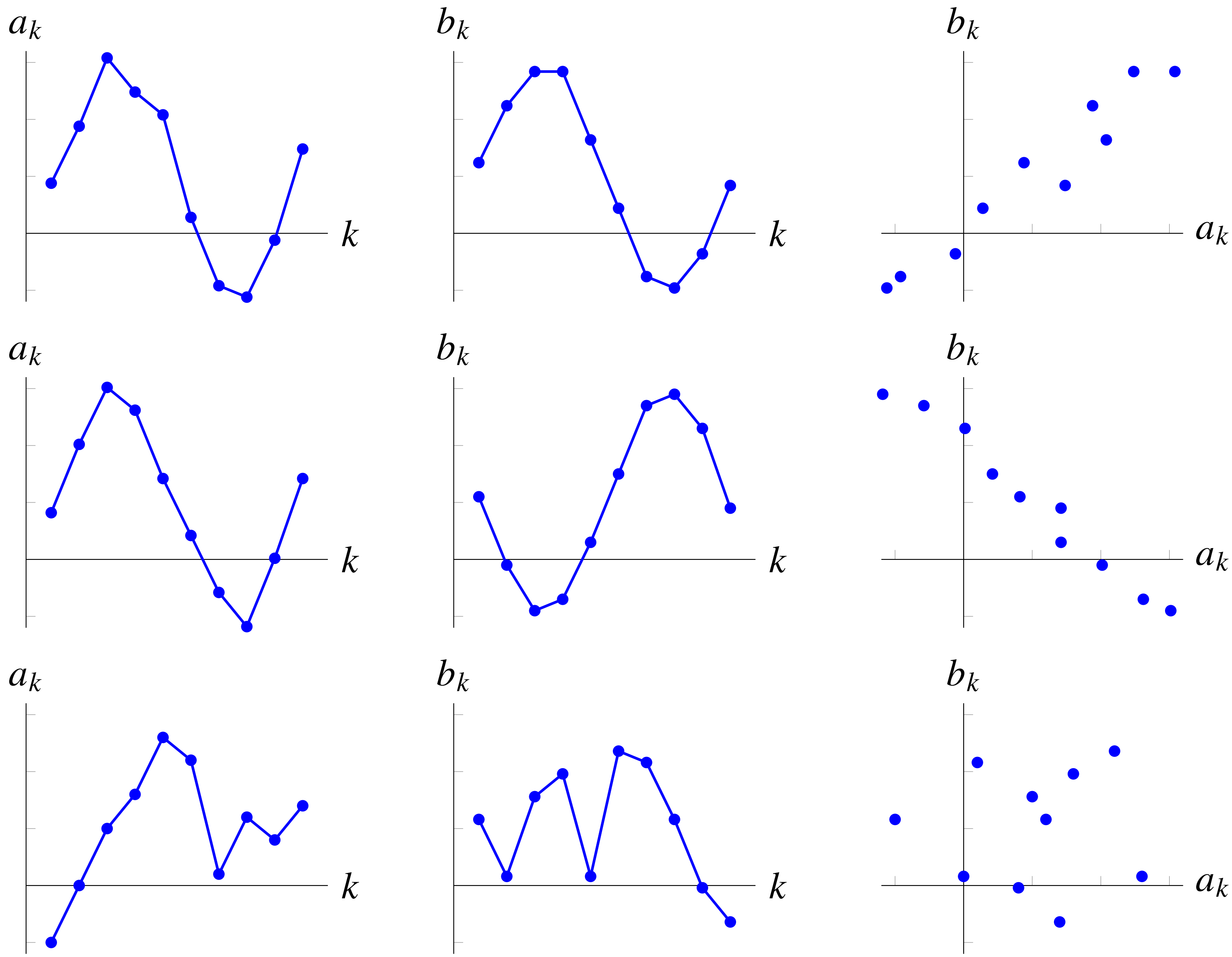
$$\rho_{ab} = \frac{\tilde{a}^T \tilde{b}}{\|\tilde{a}\| \, \|\tilde{b}\|}$$

where $\tilde{a} = a - \mathbf{avg}(a)\mathbf{1}$ and $\tilde{b} = b - \mathbf{avg}(b)\mathbf{1}$ are the de-meaned vectors

- only defined when a and b are not constant ($\tilde{a} \neq 0$ and $\tilde{b} \neq 0$)
- ρ_{ab} is the cosine of the angle between the de-meaned vectors
- a number between -1 and 1
- ρ_{ab} is the average product of the deviations from the mean in standard units

$$\rho_{ab} = \frac{1}{n} \sum_{i=1}^{n} \frac{(a_i - \mathbf{avg}(a))}{\mathbf{std}(a)} \frac{(b_i - \mathbf{avg}(b))}{\mathbf{std}(b)}$$

Examples



$\rho_{ab} = 0.968$

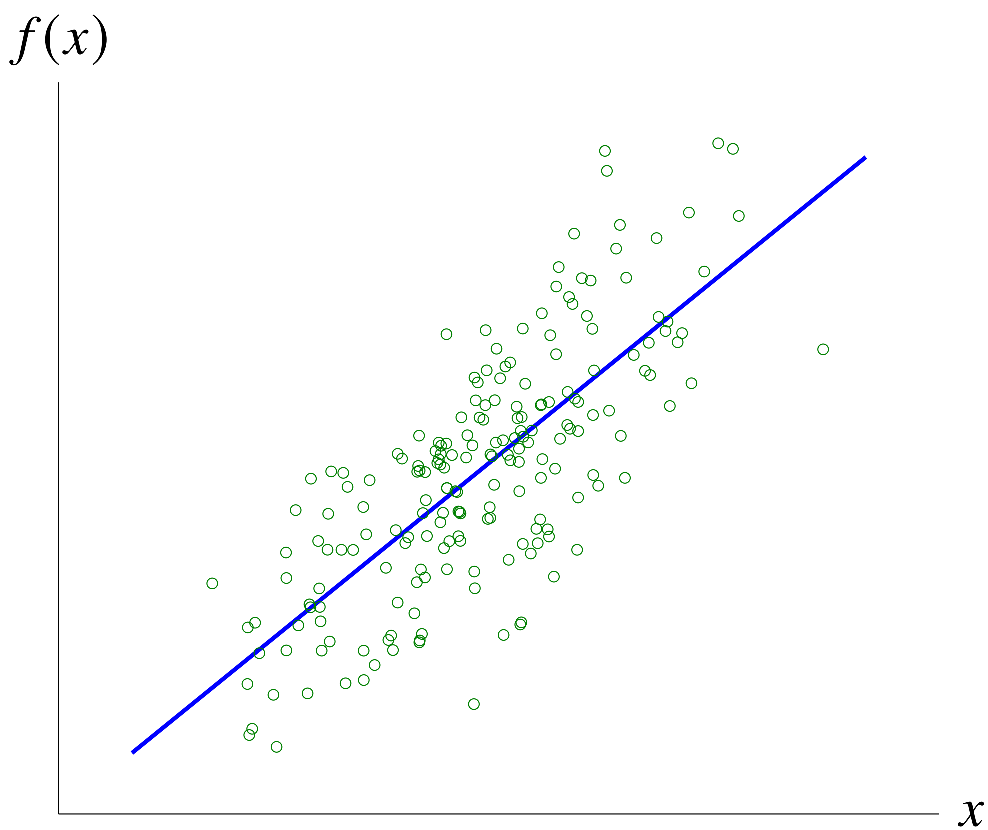
$\rho_{ab} = -0.988$

$\rho_{ab} = 0.004$

Regression line

- scatter plot shows two n-vectors a, b as n points (a_k, b_k)
- straight line shows affine function $f(x) = c_1 + c_2 x$ with

$$f(a_k) \approx b_k, \quad k = 1, \ldots, n$$



Least squares regression

use coefficients c_1, c_2 that minimize $J = \frac{1}{n}\sum_{k=1}^{n}(f(a_k) - b_k)^2$

- J is a quadratic function of c_1 and c_2:

$$\begin{aligned} J &= \frac{1}{n}\sum_{k=1}^{n}(c_1 + c_2 a_k - b_k)^2 \\ &= \left(nc_1^2 + 2n\,\mathbf{avg}(a)c_1c_2 + \|a\|^2c_2^2 - 2n\,\mathbf{avg}(b)c_1 - 2a^Tbc_2 + \|b\|^2\right)/n \end{aligned}$$

- to minimize J, set derivatives with respect to c_1, c_2 to zero:

$$c_1 + \mathbf{avg}(a)c_2 = \mathbf{avg}(b), \qquad n\,\mathbf{avg}(a)c_1 + \|a\|^2c_2 = a^Tb$$

- solution is

$$c_2 = \frac{a^Tb - n\,\mathbf{avg}(a)\,\mathbf{avg}(b)}{\|a\|^2 - n\,\mathbf{avg}(a)^2}, \qquad c_1 = \mathbf{avg}(b) - \mathbf{avg}(a)c_2$$

Interpretation

slope c_2 can be written in terms of correlation coefficient of a and b:

$$c_2 = \frac{(a - \mathbf{avg}(a)\mathbf{1})^T (b - \mathbf{avg}(b)\mathbf{1})}{\|a - \mathbf{avg}(a)\mathbf{1}\|^2} = \rho_{ab} \frac{\mathbf{std}(b)}{\mathbf{std}(a)}$$

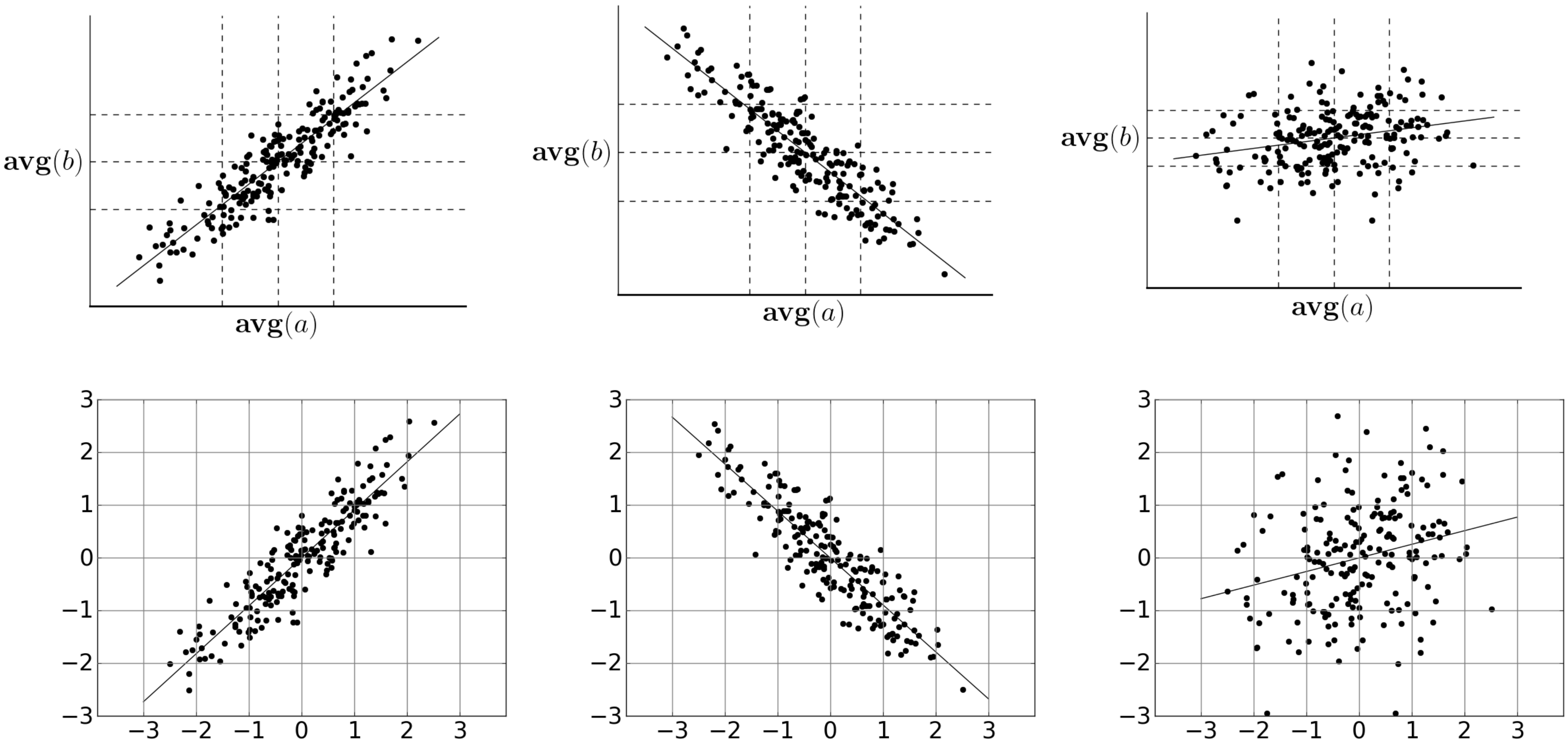
- hence, expression for regression line can be written as

$$f(x) = \mathbf{avg}(b) + \frac{\rho_{ab}\,\mathbf{std}(b)}{\mathbf{std}(a)}(x - \mathbf{avg}(a))$$

- correlation coefficient ρ_{ab} is the slope after converting to standard units:

$$\frac{f(x) - \mathbf{avg}(b)}{\mathbf{std}(b)} = \rho_{ab} \frac{x - \mathbf{avg}(a)}{\mathbf{std}(a)}$$

Examples



$\rho_{ab} = 0.91$ $\rho_{ab} = -0.89$ $\rho_{ab} = 0.25$

- dashed lines in top row show average ± standard deviation
- bottom row shows scatter plots of top row in standard units

Outline

- norm
- distance
- k-means algorithm
- angle
- **complex vectors**

Norm

norm of vector $a \in \mathbf{C}^n$:

$$\begin{aligned} \|a\| &= \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2} \\ &= \sqrt{a^H a} \end{aligned}$$

- positive definite:

$$\|a\| \geq 0 \quad \text{for all } a, \qquad \|a\| = 0 \quad \text{only if } a = 0$$

- homogeneous:

$$\|\beta a\| = |\beta| \|a\| \quad \text{for all vectors } a\text{, complex scalars } \beta$$

- triangle inequality:

$$\|a + b\| \leq \|a\| + \|b\| \quad \text{for all vectors } a, b \text{ of equal size}$$

Cauchy–Schwarz inequality for complex vectors

$$|a^H b| \leq \|a\| \|b\| \quad \text{for all } a, b \in \mathbf{C}^n$$

moreover, equality $|a^H b| = \|a\| \|b\|$ holds if:

- $a = 0$ or $b = 0$
- $a \neq 0$ and $b \neq 0$, and $b = \gamma a$ for some (complex) scalar γ

- exercise: generalize proof for real vectors on page 2.4
- we say a and b are *orthogonal* if $a^H b = 0$
- we will not need definition of angle, correlation coefficient, ... in $\mathbf{C}^n$