12.12

ALL MATLAB AT END

(a) $D(u)$ where $u = \begin{bmatrix} (p_1)_1 \\ (p_2)_1 \\ \vdots \\ (p_N)_1 \end{bmatrix}$

is $\sum_{\substack{\text{edges} \\ i,j}} \left((p_i)_1 - (p_j)_1\right)^2$ aka sum of squares of all x coordinates of points w/ edge in between.

$\sum_{\substack{\text{edges} \\ i,j}} \left((p_i)_2 - (p_j)_2\right)^2$ same this for Y coordinates.

~~Then~~, Now, $D(u)+D(v) = \underbrace{\sum_{\substack{\text{edges} \\ i,j}} \left((p_i)_1 - p(j)_1\right)^2 + \left((p_i)_2 - (p_j)_2\right)^2}$ b/c edges i,j are same for both sums.

↳ sum of squared distances

(b) $x = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-K} \\ v_1 \\ v_2 \\ \vdots \\ v_{N-K} \end{bmatrix}$ Let $M_{\text{INCIDENT}} =$ ~~N x L~~ matrix $L \times N$

$\begin{array}{cc} & \text{nodes} \\ \text{edges} & \begin{bmatrix} \; \end{bmatrix} \end{array}$

if M_{ij} is 1, then node j is attached to edge i. otherwise 0.

Let A_1^{m} be the first N-K colums of M_{INC}.

Let us make the 2^{nd} 1 in A_1's rows be -1 (if it exists).

$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}$ when $A \cdot x$, we get all edges btwn the nodes of class N-K, as well as the coordinates of points in that class, but whose other side of the edge is one of the K fixed points.

That is where b_k comes in. Let B_1 be the last K columns of M_{INC}

$b = \begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix} \begin{bmatrix} u_{N-K+1} \\ \vdots \\ u_N \\ v_{N-K+1} \\ \vdots \\ v_N \end{bmatrix}$ } length 2K

8.3) $y_i = \dfrac{e^{\alpha t_i + \beta}}{1 + e^{\alpha t_i + \beta}} \Rightarrow \dfrac{1}{y_i} = \dfrac{1}{e^{\alpha t_i + \beta}} + 1$

$$\frac{y_i}{1-y_i} = e^{\alpha t_i + \beta}$$

$$\ln\left(\frac{y_i}{1-y_i}\right) = \alpha t_i + \beta$$

Error: $\displaystyle\sum_{i=1}^{m} \left(\alpha t_i + \beta - \ln\left(\frac{y_i}{1-y_i}\right)\right)^2$

$\downarrow$

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_n \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \ln\left(\frac{y_1}{1-y_1}\right) \\ \ln\left(\frac{y_2}{1-y_2}\right) \\ \vdots \end{bmatrix}$$

8.8) (a) $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ where x is a m-vector, y is a n-vector

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} Ix + Ay \\ A^T x + 0y \end{bmatrix} = 0 \Rightarrow \begin{matrix} Ay + x = 0 \\ A^T x = 0 \end{matrix}$$

$$Ay = -x \Rightarrow x = -Ay \Rightarrow A^T(-Ay) = 0 \Rightarrow \cancel{A^Ty}\ A^TAy = 0$$

Since A is ~~nonsingular~~ has lin. indep columns, A^TA is nonsingular. thus, $y=0$ is the only solution to $A^TAy=0$. if $y=0$, $x=-Ay \Rightarrow x=0$. Thus, $\begin{bmatrix} x \\ y \end{bmatrix} = 0$ vector. thus, $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}$ is nonsingular

(b) $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \cancel{\begin{bmatrix} b \\ 0 \end{bmatrix}} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$

$x = n \times 1$ $\quad \bar{x} = b - Ax_{ls} = b - (m\times n)(n \times 1)$

$x_{ls} = n \times 1$ $\quad \bar{x} = m \times 1$

$\bar{y} = n \times 1$

$$\begin{bmatrix} I\bar{x} + A\bar{y} \\ A^T\bar{x} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$A^T(b - Ax_{ls}) = 0$

$I(b - Ax_{ls}) + A(x_{ls}) = b$

$\cancel{-Ax_{ls}} + \cancel{Ax_{ls}} = \cancel{0}$

~~$A^T \neq 0$ so $b - Ax_{ls} = 0 \Rightarrow$ solution to $\|Ax-b\|^2$~~

$A^Tb = A^TAx_{ls}$

$x_{ls} = (A^TA)^{-1}A^Tb = A^\dagger b$ = solution to LS problem.

$= A \backslash b$

8.12) $\|(Ay-b)\|^2 + (c^Ty-d)^2$

(a)

$$\frac{\partial}{\partial y_k}\left(\|Ay-b\|^2 + (c^Ty-d)^2\right) = 2\sum_{i=1}^{m} A_{ik}\left(\sum_{j=1}^{n} A_{ij}\cancel{x}y_j - b_i\right)$$

$$+ 2c_k\left(\sum_{j=1}^{n} c_j y_j - d\right)$$

$$= \left[2\,A^T(Ay-b)\right]_k + \left[2c^Ty - 2d\right]c_k = 0$$

$$\nabla f = A^T(Ay-b) + \cancel{2}(c^Ty - \cancel{2}d)c = 0$$

$$A^TAy + cc^Ty = A^Tb + dc$$

$$A^TA\left(\hat{x} + \frac{d-c^T\hat{x}}{1+c^T(A^TA)^{-1}c}(A^TA)^{-1}c\right) + cc^T\left(\hat{x} + \frac{d-c^T\hat{x}}{1+c^T(A^TA)^{-1}c}(A^TA)^{-1}c\right)$$

$$= A^TA\hat{x} + \frac{d-c^T\hat{x}}{1+c^T(A^TA)^{-1}c}c + cc^T\hat{x} + \frac{dcc^T(A^TA)^{-1}c - c^T\hat{x}cc^T(A^TA)^{-1}c}{1+c^T(A^TA)^{-1}c}$$

$$= A^TA\hat{x} + \frac{dc - \cancel{cc^T\hat{x}} + \cancel{cc^T\hat{x}} + \cancel{c^T\hat{x}cc^T(A^TA^{-1})c} + dcc^T(A^TA)^{-1}c - \cancel{c^T\hat{x}cc^T(A^TA)^{-1}c}}{1+c^T(A^TA)^{-1}c}$$

$$= A^TA\hat{x} + \frac{dc + dcc^T(A^TA)^{-1}c}{1+c^T(A^TA^{-1})c}$$

$$= A^TA\hat{x} + dc = A^TA(A^TA)^{-1}A^Tb + dc = A^Tb + dc \checkmark$$

(b)

$A = QR \quad 2mn^2$ flop

$\cancel{x = A^{\dagger}b = (A^TA)^{-1}A^Tb = R^{-1}Q^Tb}$ $\quad 2mn \quad n^2$

<<actually i need to pay attention to this because the problem asks for quadratic terms as well

$A = QR$

$y = \hat{x} + \dfrac{d - c^T\hat{x}}{1 + c^T(A^TA)^{-1}c}(A^TA)^{-1}c \qquad (A^TA)^{-1} = R^{-1}R^{-T}$

$z = R^{-T}c$

$= \hat{x} + \dfrac{d - c^T\hat{x}}{1 + z^Tz}(R^{-1}z) \to u$

$R^Tz = c$ for z $\quad n^2$ flops

$Ru = z$ for u $\quad n^2$ flops

multiplication $= n$ flops

total flops $= \boxed{2mn^2}$ flops since $m, n \gg 1$

forgot that i needed to include all quadratic terms as well:
2n^2 from figuring out z and u
2mn + n^2 from figuring out x (the crossed out stuff)

total = 2mn^2 + 3n^2 + 2mn

#6 : 8.13)

(a) we want

$$\min \sum_{i=1}^{k-1} (a_i^T y - b_i)^2 + (a_k^T y + z - b_k) + \sum_{i=k+1}^{m} (a_i^T y - b_i)^2$$

given a min value of y that minimizes the 1^{st} and 3^{rd} terms, we can find a min value of z AFTER since the minimum possible value of 2^{nd} term is 0 and for any y we can find a value of z that makes the term 0.

$\text{Min}\left(\sum_{1}^{k-1} + \sum_{k+1}^{m}\right)$ = occurs @ $y = \hat{y}_k$ and the min of $a_k^T y + z - b_k$ occurs at $z = b_k - a_k^T y$. Together these values of y, z minimize f.

(b) Let $\tilde{A} = \begin{bmatrix} A & e_k \end{bmatrix}$ and $\tilde{b} = b$ and $x = \begin{bmatrix} y \\ z \end{bmatrix}$

$$\|\tilde{A}x - \tilde{b}\| = \sum_{i=1}^{k-1}\left((a_i^T y + 0 \cdot z) - b_i\right)^2 + \left(a_k^T y + z - b_k\right)^2 + \sum_{i=k+1}^{m}\left((a_i^T y + 0z) - b_i\right)^2$$

$=$ ~~solution~~ of eq. $\boxed{19}$

Normal eq: $\tilde{A}^T\tilde{A}x = \tilde{A}^T\tilde{b} \Rightarrow$

$$\begin{bmatrix} A^T \\ e_k^T \end{bmatrix}\begin{bmatrix} A & e_k \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} A^T \\ e_k^T \end{bmatrix}\begin{bmatrix} b \end{bmatrix}$$

$\cancel{A^TAy +}$ $\begin{bmatrix} A^TA & a_k \\ a_k^T & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} =$ $\quad A^TAy + a_k z = A^T b$

$\quad a_k^T y + z = e_k^T b = b_k$

$$y = (A^TA)^{-1}(A^Tb - a_k z) = (A^TA)^{-1}A^Tb - (A^TA)^{-1} z a_k = \hat{x} - z(A^TA)^{-1}a_k$$

$$a_k^T\hat{x} - a_k^T z (A^TA)^{-1} a_k + z = b_k$$

$$z(1 - a_k^T(A^TA)^{-1}a_k) = b_k - a_k^T\hat{x}$$

$$z = \frac{b_k - a_k^T\hat{x}}{1 - a_k^T(A^TA)^{-1}a_k} \Rightarrow \hat{y}_k = \hat{x} - \frac{b_k - a_k^T\hat{x}}{1 - a_k^T(A^TA)^{-1}a_k}(A^TA)^{-1}a_k$$

(c) Step 1: $A = QR$ $\quad 2mn^2$ flops

2: solve for $\hat{x}$ $\quad 2mn$ flops

$$\hat{y}_k = \hat{x} - \frac{b_k - a_k^T \hat{x}}{1 - a_k^T (A^T A)^{-1} a_k} \boxed{(A^T A)^{-1} a_k} \rightarrow \text{same as } 8.12\ (c)$$

$z = R^{-T} a_k$

$\rightarrow z^T z$

$u = R^{-1} z$

$R^T z = a_k \quad \rightarrow n^2$ flops

$R u = z \quad \rightarrow n^2$ flops

$$\hat{y}_k = \hat{x} - \frac{b_k - a_k^T \hat{x}}{1 - z^T z} u = O(n) \text{ flops}$$

Total flops $= 2mn^2$ flops if $m, n \gg 1$.

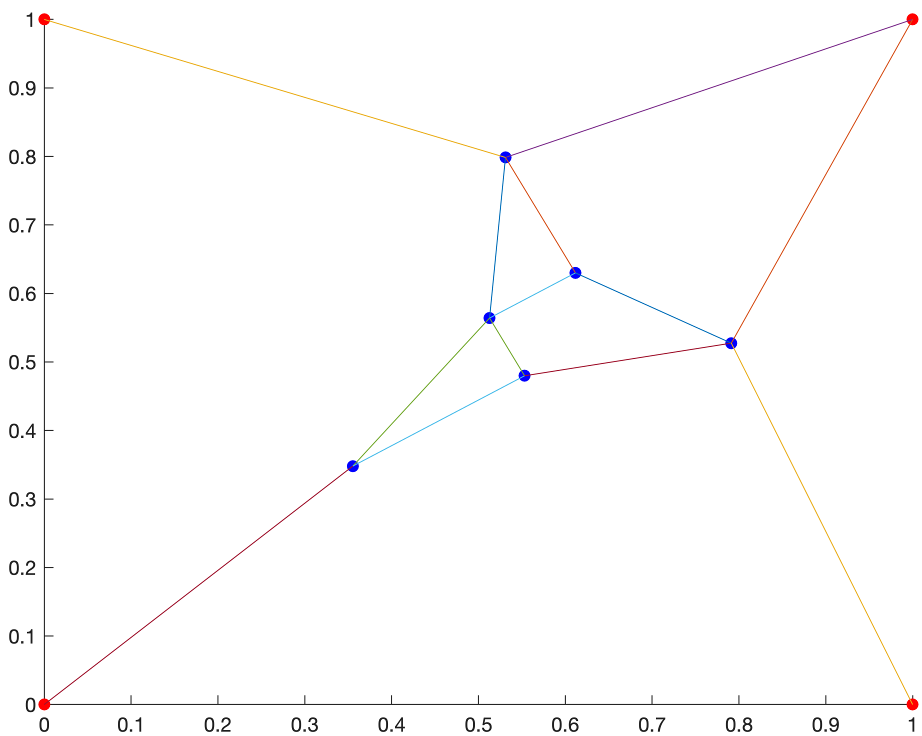
12.12)

(c)

```
figure(1);
% initialize
N = 10;
K = 4;
L = 13;
e = [[1,3];[1,4];[1,7];[2,3];[2,5];[2,8];[2,9];[3,4];[3,5];[4,6];
    [5,6];[6,9];[6,10]];
known = [[0,0];[0,1];[1,1];[1,0]];
uv = [known(:,1); known(:,2)];
% fill in incidence matrix
M = zeros(L, N);
for i = 1:13
    M(i,e(i,1)) = 1;
    M(i,e(i,2)) = -1;
end
% note that every row of M has a 1 and a -1. This is just for the
% distance formula to work.
A1 = M(:,1:(N-K));
% these are just the outgoing edges for the non-fixed points
A = [A1, zeros(size(A1)); zeros(size(A1)), A1];
B1 = abs(M(:,N-K+1:N));
% b is all the fixed points ends of the edges in question
b = [B1, zeros(size(B1)); zeros(size(B1)), B1] * uv;
% solve the linear regression equation
x = A \ b;
% plot
hold on;
all = [[x(1:N-K) x(N-K+1:end)]; known]
```

```
all = 10×2
    0.3553    0.3480
    0.5311    0.7985
    0.5128    0.5641
    0.5531    0.4799
    0.6117    0.6300
    0.7912    0.5275
         0         0
         0    1.0000
    1.0000    1.0000
    1.0000         0
```

```
scatter(all(1:N-K,1), all(1:N-K,2), 'filled', 'blue');
scatter(all(N-K+1:end,1), all(N-K+1:end,2), 'filled', 'red');
for edge = 1:13
    pA = all(e(edge,1),:);
    pB = all(e(edge,2),:);
    plot([pA(1) pB(1)], [pA(2) pB(2)]);
end
```



13.3)

(a)

```
figure(2);
hold on;
mooreslaw
scatter(T(:,1), T(:,2));
set(gca, 'yscale', 'log')

n = length(T);
A = [ones(n, 1) T(:,1)-1970];
y = log(T(:,2))/log(10);

% trying to minimize
w = A \ y;

theta1 = w(1)
```

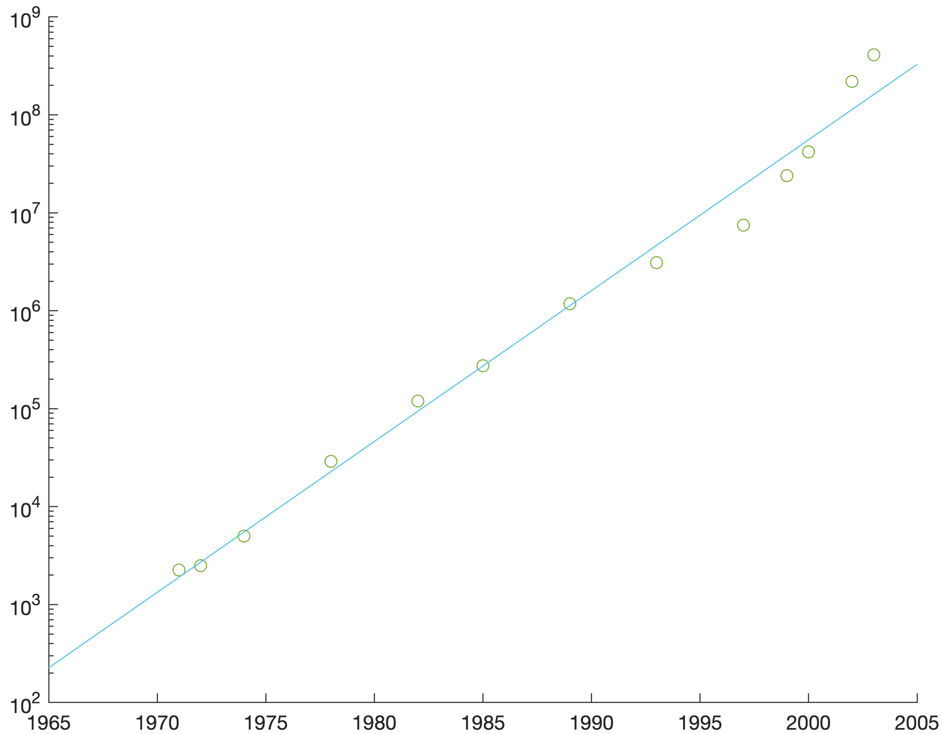
```
theta1 = 3.1256
```

```
theta2 = w(2)
```

```
theta2 = 0.1540
```

```
x = (1965:1:2005);
```

```
plot(x, 10.^((x-1970) * theta2 + theta1));
```



(b)

```
tran_2015 = 10.^((2015-1970) * theta2 + theta1)
```

```
tran_2015 = 1.1387e+10
```

```
acc_ratio = tran_2015 / (4 * 10^9)
```

```
acc_ratio = 2.8468
```

(c)

```
rate20 = log(2)/(log(10) * theta2)
```

```
rate20 = 1.9545
```

8.3)

```
figure(3);
hold on;
[t, y] = logistic_fit;
n = length(t);
scatter(t, y);
A = [ones(n, 1) t];
b = log(y ./ (1 - y));
w = A \ b;
```

```
alpha = w(2)
```

```
alpha = 1.8676
```

```
beta = w(1)
```

```
beta = -3.7397
```

```
x = (-0.5:0.01:4.5);
plot(x, (exp(alpha * x + beta) ./ (1 + exp(alpha * x + beta))));
```