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ECE133A

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Midterm solutions

Problem 1

Formulate the following problem as a set of linear equations: find a polynomial of two variables s, t,

$$f(s,t) = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} s^{i-1} t^{j-1}$$

that satisfies nine interpolation conditions

$$f(s_k, t_k) = y_k, \qquad k = 1, \ldots, 9.$$

The points (s_k, t_k) and values y_k are given. The unknowns are the coefficients c_{ij}.

1. Write the equations in matrix–vector form $Ax = b$. Clearly state what A, x, and b are.
2. Solve the problem for the following interpolation conditions. The points (s_k, t_k) are

$$\begin{array}{lll} (s_1,t_1) = (0,0), & (s_2,t_2) = (0,1), & (s_3,t_3) = (0,2), \\ (s_4,t_4) = (1,0), & (s_5,t_5) = (1,1), & (s_6,t_6) = (1,2), \\ (s_7,t_7) = (2,0), & (s_8,t_8) = (2,1), & (s_9,t_9) = (2,2), \end{array}$$

   and $y_1, \ldots, y_9$ are the nine digits of your UID.

   In your answer, write the coefficients in an array

$$\begin{array}{ccc} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{array}.$$

Solution.

1.

$$\begin{bmatrix} 1 & s_1 & s_1^2 & t_1 & s_1t_1 & s_1^2t_1 & t_1^2 & s_1t_1^2 & s_1^2t_1^2 \\ 1 & s_2 & s_2^2 & t_2 & s_2t_2 & s_2^2t_2 & t_2^2 & s_2t_2^2 & s_2^2t_2^2 \\ 1 & s_3 & s_3^2 & t_3 & s_3t_3 & s_3^2t_3 & t_3^2 & s_3t_3^2 & s_3^2t_3^2 \\ 1 & s_4 & s_4^2 & t_4 & s_4t_4 & s_4^2t_4 & t_4^2 & s_4t_4^2 & s_4^2t_4^2 \\ 1 & s_5 & s_5^2 & t_5 & s_5t_5 & s_5^2t_5 & t_5^2 & s_5t_5^2 & s_5^2t_5^2 \\ 1 & s_6 & s_6^2 & t_6 & s_6t_6 & s_6^2t_6 & t_6^2 & s_6t_6^2 & s_6^2t_6^2 \\ 1 & s_7 & s_7^2 & t_7 & s_7t_7 & s_7^2t_7 & t_7^2 & s_7t_7^2 & s_7^2t_7^2 \\ 1 & s_8 & s_8^2 & t_8 & s_8t_8 & s_8^2t_8 & t_8^2 & s_8t_8^2 & s_8^2t_8^2 \\ 1 & s_9 & s_9^2 & t_9 & s_9t_9 & s_9^2t_9 & t_9^2 & s_9t_9^2 & s_9^2t_9^2 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix}.$$

Problem 2

Suppose b is an n-vector with nonzero elements ($b_i \neq 0$ for all i) and A is a diagonal $n \times n$-matrix with distinct diagonal elements ($A_{ii} \neq A_{jj}$ for $i \neq j$). Prove that the $n \times n$ matrix

$$C = \left[\begin{array}{ccccc} b & Ab & A^2b & \cdots & A^{n-1}b \end{array} \right]$$

is nonsingular.

Solution. Denote the diagonal elements of A by $t_1, \ldots, t_n$. We show that the columns of C are linearly independent: $Cx = 0$ only if $x = 0$.

$$\begin{aligned} Cx &= x_1 b + x_2 Ab + x_3 A^2 b + \cdots + x_n A^{n-1} b \\ &= (x_1 I + x_2 A + x_3 A^2 + \cdots + x_n A^{n-1}) b \\ &= \left[\begin{array}{cccc} p(t_1) & 0 & \cdots & 0 \\ 0 & p(t_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(t_n) \end{array} \right] b \\ &= \left[\begin{array}{c} p(t_1) b_1 \\ p(t_2) b_2 \\ \vdots \\ p(t_n) b_n \end{array} \right] \end{aligned}$$

where $p(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$. Since the elements b_i are nonzero, $Cx = 0$ holds only if

$$p(t_1) = \cdots = p(t_n) = 0,$$

and since the diagonal elements t_i are distinct this is only possible if $x_1 = \cdots = x_n = 0$.

Problem 3

Let $A = \left[\begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \right]$ be a matrix with three columns and QR factorization $A = QR$ where

$$R = \left[\begin{array}{rrr} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array} \right].$$

1. What are the norms $\|a_1\|$, $\|a_2\|$, $\|a_3\|$ of the columns of A?
2. Denote by θ_{ij} the angle between columns a_i and a_j of A. What are θ_{12}, θ_{13}, θ_{23} (in degrees)?

Explain your answers.

Solution. We have

$$a_1 = q_1 R_{11}, \qquad a_2 = q_1 R_{12} + q_2 R_{22}, \qquad a_3 = q_1 R_{13} + q_2 R_{23} + q_3 R_{33}.$$

Since q_1, q_2, q_3 are orthonormal,

$$\|a_1\| = R_{11} = 1, \qquad \|a_2\| = (R_{12}^2 + R_{22}^2)^{1/2} = \sqrt{2}, \qquad \|a_3\| = (R_{13}^2 + R_{23}^2 + R_{33}^2)^{1/2} = \sqrt{3}.$$

The cosines of the angles are

$$\cos\theta_{12} = \frac{a_1^T a_2}{\|a_1\|\|a_2\|} = \frac{R_{11}R_{12}}{R_{11}\sqrt{R_{12}^2 + R_{22}^2}} = \frac{-1}{\sqrt{2}},$$

$$\cos\theta_{13} = \frac{a_1^T a_3}{\|a_1\|\|a_3\|} = \frac{R_{11}R_{13}}{R_{11}\sqrt{R_{13}^2 + R_{23}^2 + R_{33}^2}} = \frac{-1}{\sqrt{3}},$$

and

$$\cos\theta_{23} = \frac{a_2^T a_3}{\|a_2\|\|a_3\|} = \frac{R_{12}R_{13} + R_{22}R_{23}}{\sqrt{R_{12}^2 + R_{22}^2}\sqrt{R_{13}^2 + R_{23}^2 + R_{33}^2}} = 0.$$

Hence

$$\theta_{12} = 135^\circ, \qquad \theta_{13} = 125.3^\circ, \qquad \theta_{23} = 90^\circ.$$

Problem 4

Recall (from exercise A2.8) the definition of the Kronecker product $A \otimes B$ of two $n \times n$ matrices:

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}.$$

This is a matrix of size $n^2 \times n^2$. A useful property of the Kronecker product is

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

(You are not asked to prove this.) We consider the linear equation

$$(A \otimes A)x = b, \tag{1}$$

where A is an $n \times n$ matrix, b is an n^2-vector, and the variable x is an n^2-vector. By partitioning x and b in subvectors $b_1, \ldots, b_n$ and $x_1, \ldots, x_n$ of length n, we can write the equation as

$$\begin{bmatrix} A_{11}A & A_{12}A & \cdots & A_{1n}A \\ A_{21}A & A_{22}A & \cdots & A_{2n}A \\ \vdots & \vdots & & \vdots \\ A_{n1}A & A_{n2}A & \cdots & A_{nn}A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

1. Suppose A is upper or lower triangular and nonsingular. Then $A \otimes A$ is upper or lower triangular. Describe an efficient method for solving (1). What is the complexity (number of flops for large n)? Compare with the complexity of solving a general triangular set of linear equations of size $n^2 \times n^2$.

2. Suppose A is nonsingular. Explain how you can solve (1) using the LU factorization of A. Clearly state the different steps in the algorithm, the complexity of each step, and the overall complexity. Compare with the complexity of solving a general set of linear equations of size $n^2 \times n^2$.

Solution.

1. Consider lower triangular A.

$$\begin{bmatrix} A_{11}A & 0 & \cdots & 0 & 0 \\ A_{21}A & A_{22}A & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_{n-1,1}A & A_{n-1,2}A & \cdots & A_{n-1,n-1}A & 0 \\ A_{n1}A & A_{n2}A & \cdots & A_{n,n-1}A & A_{nn}A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}.$$

We first solve for $y_1 = Ax_1, \ldots, y_n = Ax_n$ from the equations

$$\begin{aligned} A_{11}y_1 &= b_1 \\ A_{21}y_1 + A_{22}y_2 &= b_2 \\ &\vdots \\ A_{n-1,1}y_1 + A_{n-1,2}y_2 + \cdots + A_{n-1,n-1}y_{n-1} &= b_{n-1} \\ A_{n1}y_1 + A_{n2}y_2 + \cdots + A_{n,n-1}y_{n-1} + A_{nn}y_n &= b_n. \end{aligned}$$

The system can be solved by forward substitution in

$$n(1 + 2 + 3 + \cdots + n - 1) = n^3 \text{ flops.}$$

Next we solve the n equations

$$Ax_1 = y_1, \qquad Ax_2 = y_2, \qquad \ldots, \qquad Ax_n = y_n,$$

each by forward substitution, at a total cost of n^3. The total for the two steps is $2n^3$, one order faster than the cost of n^4 for a general triangular system of this size.

As an alternative solution, we can note that the equation $(A \otimes A)x = b$ can be written as a matrix equation

$$AXA^T = B$$

where X is the $n \times n$ matrix with columns $x_1, \ldots, x_n$ and B is the $n \times n$ matrix with columns $b_1, \ldots, b_n$. In the method just described, we first solve the matrix equation $YA^T = B$ or, equivalently, $AY^T = B^T$, row by row. Then we solve the matrix equation $AX = Y$ for the columns of X.

2. Substituting $A = PLU$ in $(A \otimes A)x = b$ and using the property in the statement gives

$$(P \otimes P)(L \otimes L)(U \otimes U)x = b.$$

We can solve this in three steps.

- Solve $(P \otimes P)z = b$. Zero flops because $P \otimes P$ is a permutation matrix.
- Solve $(L \otimes L)y = z$ using the method in part 1. $2n^3$ flops.
- Solve $(U \otimes U)x = y$ using the method in part 1. $2n^3$ flops.

The total is $(2/3)n^3 + 4n^3 = (14/3)n^3$, as compared to $(2/3)n^6$ for a general system of this size.

As in part 1 we can interpret the problem as a matrix equation

$$AXA^T = B.$$

If we substitute the LU factorization of A, this becomes

$$PLUXU^TL^TP^T = B.$$

In the method described we first solve $PZP^T = B$ for Z. The solution $Z = P^TBP$ is a symmetric reordering of the matrix B. In step 2 we solve $LYL^T = Z$ for Y, using the method of part 1. In step 3 we solve $UXU^T = Y$ for X.

Other possibilities exist, with the same complexity. For example, we can first solve the matrix equation $PLUW = B$ column by column for the matrix W. Then we solve $XU^TL^TP^T = W$ for X or, equivalently, $PLUX^T = W^T$ for X^T.