5.6) (b)

$$LX + XL^T = B$$

$$\begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ \vdots & \vdots & & \ddots \\ L_{n1} & L_{n2} & & \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & \\ X_{21} & X_{22} & \\ X_{31} & \vdots & \cdots \\ \vdots & \vdots & \end{bmatrix} + \begin{bmatrix} X_{11} & \\ X_{21} & \\ \vdots & \cdots \\ \vdots & \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & L_{31} & \cdots \\ 0 & L_{22} & L_{32} & \cdots \\ 0 & 0 & L_{33} & \\ \vdots & & \ddots & \end{bmatrix} = B$$

we solve for B column by column.

Assume that ~~nth~~ ~~xth~~ ~~ath~~ i^{th} column of A is A_i

$$B_1 = (LX + XL^T)_1 = (LX)_1 + (XL^T)_1 = L \cdot X_1 + L_{11} X_1$$ ← in terms of only X_i

$$B_2 = (LX + XL^T)_2 = (LX)_2 + (XL^T)_2 = LX_2 + (L_{22} X_2 + L_{12} X_1)$$ ← note that we already know X_i

$$B_j = \underbrace{LX_j + L_{jj} X_j}_{(L + L_{jj} I) X_j} + \underbrace{\sum_{k=1}^{j-1} L_{kj} X_k}$$ — all the previous X_k have been calculated.

$(L + L_{jj} I) X_j$ → Equation of the form $AX = B$ where $L + L_{jj} I$ is lower triangular. So, forward sub works. ($L_{jj} \neq 0$)

forward sub is n^2 flops

n flops since loop along diagonal.

this is j-1 multiplications and j-1 +/- b/c we include B_j (−). each is an n-vector so total flops =

$n \cdot 2 \cdot (j-1)$. j ranges from 1−n

so total: $n^3 - n^2$

$$n(n + n^2) + n^3 - n^2 = \boxed{2n^3}$$

A 6.3) a) to prove that $I-S$ is nonsingular, it suffices to show that its columns are linearly $\perp\!\!\!\perp$.

Specifically, $(I-S)x=0 \iff x=0$ for an $n\times 1$ vector x. $\boxed{\text{DONE}}$

$$x^T(I-S)x = x^T\cdot 0 = 0 \rightarrow x^T\cdot I\cdot x - x^T\cdot S\cdot x = 0 \rightarrow x^Tx = x^T\cdot S\cdot x.$$

Let us take a closer look @ $x^T\cdot S\cdot x$. $(1\times n)(n\times n)(n\times 1) = (1\times 1)$ so this product is essentially a scalar.

Thus, $x^TSx = (x^TSx)^T$ since $(1\times 1)^T = (\text{itself})$

$$= x^TS^Tx$$

$$= x^T(-S)x = -x^TSx$$

Since $A=-A$, matrix A must $=0$. $x^TSx=0$ for all x. ($n\times 1$ vectors).

$x^Tx = x^TSx \Rightarrow 0 = x^Tx \Rightarrow x=0$. Thus, $I-S$ is nonsingular. $\square$

b) $I-S$ is nonsingular. Thus, there exists matrix A s.t.

$$(I-S)A = I \quad \& \quad A(I-S) = I$$

where the 2 A's are the same and $A=(I-S)^{-1}$

Now, $IA - SA = I \rightarrow SA = A-I$ & $AI - AS = I \rightarrow AS = A-I$

Therefore $AS = SA$.

Also, $A(I+S) = I + 2AS$ & $(I+S)A = I + 2SA$.

$AS = SA$ so $A(I+S) = (I+S)A \rightarrow (I-S)^{-1}(I+S) = (I+S)(I-S)^{-1}$ $\square$

(c) If matrix A is square and $A^TA = I$, then it has orthonormal columns and is orthogonal.

$A = (I+S)(I-S)^{-1}$

$A^T = \left[(I+S)(I-S)^{-1}\right]^T = \left[(I-S)^{-1}(I+S)\right]^T = (I+S)^T\left((I-S)^{-1}\right)^T$

$= (I^T + S^T)(I^T - S^T)^{-1} = (I-S)(I+S)^{-1}$ using part (b) and $(A^{-1})^T = (A^T)^{-1}$

$A^TA = (I-S)(I+S)^{-1}(I+S)(I-S)^{-1}$

$= (I-S)(I-S)^{-1} = I$. since $A^TA = I$, $A = (I+S)(I-S)^{-1}$ is orthogonal. $\square$

A 6.9)

a) S rotates a matrix's columns by 1. S^{k-1} rotates it k^{-1} times, so that the new 1^{st} column was column k.

$$WS^{k-1} = \begin{bmatrix} 1 & 1 & \\ w^{-(k-1)} & w^{-k} & \\ w^{-2(k-1)} & w^{-2k} & \cdots \\ \vdots & \vdots & \ddots \\ w^{-(n-1)(k-1)} & w^{-(n-1)k} & \end{bmatrix}$$

diag(We_k) is the k^{th} column of W written in an $n \times n$ diagonal matrix form.

$$\begin{bmatrix} 1 & & & 0 \\ & w^{-(k-1)} & & \\ & & w^{-2(k-1)} & \\ 0 & & \ddots & w^{-(n-1)(k-1)} \end{bmatrix}$$

we multiply that by W

$$\begin{bmatrix} 1 & 1 & 1 & & 1 \\ 1 & w^{-1} & w^{-2} & & w^{-(n-1)} \\ 1 & w^{-2} & w^{-4} & & \vdots \\ \vdots & \vdots & \vdots & \cdots & \\ 1 & w^{-(n-1)} & w^{-2(n-1)} & & w^{-(n-1)(n-1)} \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & & 1 & 1 & \\ w^{-(k-1)} & & w^{-(k-1)} & w^{-(n-k+1)} & \\ w^{-2(k-1)} & \cdots & w^{-2(k-1)} & w^{-2(n-k+1)} & \cdots \\ \vdots & & & \vdots & \\ w^{-(n-1)(k-1)} & & w^{-(n-1)(k-1)} & w^{-(n-1)(n-k+1)} & \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 \\ w^{-n} \\ w^{-2n} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

b/c $w = e^{2\pi i/n}$

so $w^{-n} = 1$

Thus, $WS^{k-1} = \text{diag}(We_k)W$. they are both just the $k-1^{th}$ rotation of W.

b) $a = a_1 e_1 + a_2 \cdot e_2 + a_3 \cdot e_3 \ldots a_n \cdot e_n$

where a_i are scalars and e_i are unit vectors

$$T(a) = \begin{bmatrix} a & Sa & S^2 a & \ldots & S^{n-1} a \end{bmatrix}$$

$$\frac{1}{n} W^H \operatorname{diag}(Wa) W = \frac{1}{n} W^H \operatorname{diag}\left(W\left(a_1 e_1 + a_2 e_2 + \ldots a_n e_n\right)\right) W$$

$$= \sum_{i=1}^{n} \frac{1}{n} W^H \operatorname{diag}(W a_i e_i) W \quad \text{b/c diag is linear}$$

$$= \sum_{i=1}^{n} \frac{a_i}{n} W^H \operatorname{diag}(W e_i) W \quad \text{b/c diag is linear and } a_i \text{ is scalar}$$

$$= \sum_{i=1}^{n} a_i \cdot S^{i-1} = T(a) \quad \square$$

(c) $T(a)x = \frac{1}{n} W^H \operatorname{diag}(Wa) W x$

$$= W^{-1} \cdot \operatorname{diag}(Wa) \cdot Wx$$

$Wa \rightarrow n \log n$ → diagonal then multiply = dot product

$Wx \rightarrow n \log n$ → $O(n)$ time

$W^{-1}(\quad) \rightarrow n \log n$

total = $\boxed{3n \log n + n}$ $\Rightarrow \approx n \log n$ complexity.

(d) $Ax = b$

$x = A \backslash b$

$A = T(a)$

$T(a) x = b$

$x = T(a)^{-1} \cdot b$

$= \left(\frac{W^H}{n} \operatorname{diag}(Wa) \cdot W\right)^{-1} \cdot b$

inverse of diag matrix is 1/each term.

$= W^{-1} \operatorname{diag}(Wa)^{-1} Wb = W^{-1} \left(Wb \; ./ \; Wa\right)$

Nevin's Terminal

```
~/Desktop/ee133a                                                              took 20s
❯ micro matlab4.m

~/Desktop/ee133a
❯ matlab

                                     < M A T L A B (R) >
                           Copyright 1984-2020 The MathWorks, Inc.
                       R2020b Update 5 (9.9.0.1592791) 64-bit (maci64)
                                      February 4, 2021


For online documentation, see https://www.mathworks.com/support
For product information, visit www.mathworks.com.

>> matlab4

ans =

    0.0141


ans =

    0.2221

>>
```

matlab (MATLAB) ⌘1 | matlab (MATLAB) ⌘2

Nevin's Terminal

```
n = 500;
a = randn(n, 1);
b = randn(n, 1);
A = toeplitz(a, [a(1), flipud(a(2:n))']);

t = 100;
time = zeros(1, t);
for i = 1:t
    tic;
    x = ifft(fft(b) ./ fft(a));
    time(i) = toc;
end
sum(time)

time = zeros(1, t);
for i = 1:t
    tic;
    x = A \ b;
    time(i) = toc;
end
sum(time)
```

```
matlab4.m (22,1) | ft:objective-c | unix | utf-8          Alt-g: bindings, Ctrl-g: help
```

matlab (MATLAB) ⌘1 | micro (micro) ⌘2

6.13) (a) A is orthogonal iff $AA^T = I$ and A is square.

$\boxed{\text{DONE}}$

A is $2n \times 2n$ so square.

$$A \cdot A^T = \begin{bmatrix} aa^T & I-aa^T \\ I-aa^T & aa^T \end{bmatrix} \begin{bmatrix} aa^T & I-aa^T \\ I-aa^T & aa^T \end{bmatrix}$$

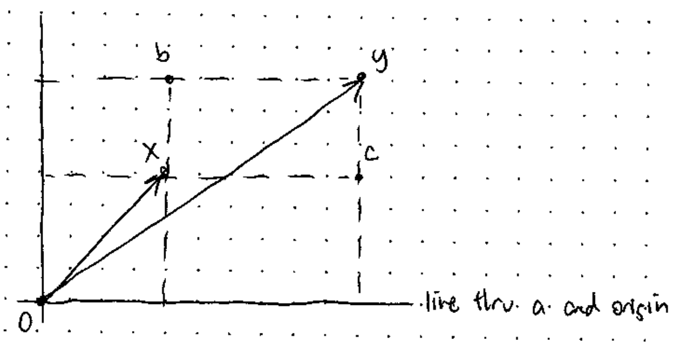
$$= \begin{bmatrix} aa^Taa^T + I - 2aa^T + aa^Taa^T & 2aa^T - 2aa^Taa^T \\ aa^T - aa^Taa^T + aa^T - aa^Taa^T & I + 2aa^Taa^T - 2aa^T \end{bmatrix}$$

Now, $aa^Taa^T = a(a^Ta)a^T$. Since $\|a\| = 1$, $a^Ta = 1$.

Thus, $aa^Taa^T = aa^T$ and $aa^Taa^T - aa^T = 0$

$$AA^T = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I \quad \square \quad AA^T \text{ is orthogonal.}$$

(b)



$aa^T x + (I - aa^T)y = b$

$(I-aa^T)y$: projection of y onto H

$aa^T x$: proj of x onto a

b has a-component of x and H-component of y.

& $aa^T y + (I - aa^T)x = c$

c has a-component of y

H-component of x

6.1b) By Gram-Schmidt,

$$q_i = a_i - (q_1^T a_i) q_1 - (q_2^T a_i) q_2 - \dots - (q_{i-1}^T a_i) q_{i-1}$$

(let q_i and a_i denote the i^{th} column vector of q and a).

Since q_i is a linearly combination of a_i and $q_1 \to q_{i-1}$ and all vectors satisfy the propery that $V_{ij} = 0$ for $i > j+1$, q_i must satisfy this propery as well. Example:

$$q_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

a_3 is of the form $\begin{bmatrix} X \\ X \\ X \\ X \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ q_1 is of the form $\begin{bmatrix} X \\ X \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$ q_2 is of the form $\begin{bmatrix} X \\ X \\ X \\ 0 \\ 0 \\ \vdots \end{bmatrix}$

Thus, the last $n-4$ rows have to be 0 in any linear combination of these vectors.

7.5). $x = (I + A^{-1} + A^{-2} + A^{-3})b$

The first thing we note is that $(n\times n) \times (n \times n)$ is $O(2n^3)$ which is slow.

It is much better to first multiply by a vector $(n \times 1)$ and then multiply by a matrix.

$n\times n \times n \times 1 = O(2n^2)$ which is MUCH better.

Using LU factorization we can get $A = PLU$ in $\frac{2}{3}n^3$ flops

$PLUx_1 = b$ for x_1 takes $2n^2$

$PLUx_2 = x_1$ " x_2 takes $2n^2$

$PLUx_3 = x_2$ " x_3 " $2n^2$

$x_1 = A^{-1}b \quad x_2 = A^{-1}x_1 = A^{-2}b \quad x_3 = A^{-3}b$

$X = b + x_1 + x_2 + x_3$ takes $3n$ flops.

Total $= \frac{2}{3}n^3 + 6n^2 + 3n$ flops approx.

A7.30) (a) $\tilde{A}$ is singular depending on if the only solution to

DONE

$$\tilde{A}x = 0 \quad \text{is} \quad x = 0$$

$$\tilde{A}x = 0$$

$$(A + (c - a_i)e_i^T)\, x = 0$$

$$Ax + (c - a_i)e_i^T x = 0$$

$$x + A^{-1}(c - a_i)e_i^T x = 0 \quad \Rightarrow \quad e_i^T x = [0\,0\,0 \ldots 0\,1\,0\,0 \ldots 0]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \end{bmatrix} = x_i$$

$$x + A^{-1}(c - a_i) \cdot x_i = 0$$

$$x + x_i \cdot A^{-1} \cdot c - x_i \cdot A^{-1} \cdot a_i = 0 \quad \Rightarrow \quad A^{-1} \cdot A = I$$

$$A^{-1} \cdot a_i = e_i$$

$$x + x_i A^{-1} c - x_i \cdot e_i = 0 \quad \Rightarrow \quad x = x_i e_i - x_i A^{-1} c$$

since the i^{th} element of $A^{-1}c = 0$, then for any value of x_i this equation has a solution for x. (the i^{th} values match; $x_i = x_i \cdot 1 - x_i \cdot 0$ ✓)

Since non-zero x works, $\tilde{A}$ is singular. ▫

(b) since $\tilde{A}$ square we can prove either side of invertibility

$C = n \times 1$

$\tilde{A}, A, A^{-1}, \tilde{A}^{-1} = n \times n$

$$\tilde{A}\tilde{A}^{-1} = \tilde{A}^{-1}\tilde{A} = I \quad \text{IF } \tilde{A} \text{ is nonsingular}$$

$$\tilde{A}\tilde{A}^{-1} = (A + (c - a_i)e_i^T)\left(A^{-1} - \frac{1}{(A^{-1}c)_i}(A^{-1}c - e_i)\, e_i^T A^{-1}\right)$$

$$= AA^{-1} - \frac{1}{(A^{-1}c)_i} A(A^{-1}c - e_i)\, e_i^T A^{-1} + (c - a_i)e_i^T A^{-1} - (c - a_i)e_i^T \frac{1}{(A^{-1}c)_i}(A^{-1}c - e_i)\, e_i^T A^{-1}$$

$$= I - \frac{1}{(A^{-1}c)_i}(c - \overset{a_i}{\cancel{Ae_i}})\, e_i^T A^{-1} + (c - A_i)e_i^T A^{-1} - \frac{1}{(A^{-1}c)_i} e_i^T (A^{-1}c - e_i)\, (c - a_i)\, e_i^T A^{-1}$$

$$= I - \left(\frac{1}{(A^{-1}c)_i}\vec{1} - \vec{1} + \frac{1}{(A^{-1}c)_i} e_i^T (A^{-1}c - e_i)\right)(c - a_i) \cdot e_i^T A^{-1}$$

$(A^{-1}c)_i = e_i^T(A^{-1}c)$ so we have

DONE

$$= I - \left(\frac{1}{e_i^T(A^{-1}c)} \cdot 1 - 1 + \frac{1}{e_i^T(A^{-1}c)} e_i^T(A^{-1}c - e_i) \right) (c - a_i) e_i^T A^{-1}$$

$$= I - \left(\frac{1}{e_i^T(A^{-1}c)} \cdot 1 - \cancel{1} + \cancel{1} - \frac{1}{e_i^T(A^{-1}c)} \underbrace{e_i^T e_i}_{1} \right) (c - a_i) e_i^T A^{-1}$$

$$= I - (0)(c-a) e_i^T A^{-1} = I \quad \square$$

(c) $$\tilde{A}^{-1} = A^{-1} - \frac{1}{(A^{-1}c)_i} (A^{-1}c - e_i) e_i^T A^{-1}$$

$A = PLU$ takes $\frac{2}{3}n^3$ flops.

Let's first solve $Ax = b \Rightarrow x = A^{-1}b$. simple propagated substitution $2n^2$ flops.

Now, for $\tilde{A}y = b \Rightarrow y = \tilde{A}^{-1}b = A^{-1}b - \frac{1}{(A^{-1}c)_i}(A^{-1}c - e_i) e_i^T A^{-1} b$

$$= x - \frac{e_i^T(A^{-1}b)}{(A^{-1}c)_i} (A^{-1}c - e_i)$$

$w = A^{-1}c$
solvable in
$2n^2$ flops same
as $x = A^{-1}b$.

$$= x - \frac{x_i}{w_i}(A^{-1}c - e_i) = x - \frac{x_i}{w_i}(w - e_i)$$

$\frac{x_i}{w_i} = 1$ flop

$w - e_i = n$ flops

multiplying these two
scalar $\times$ vector
is n flops.

$x - (\quad) = n$ flops.

Total $= \frac{2}{3}n^3 + 2n^2 + 2n^2 + n + n + 1$
$+ n$
$= \boxed{\frac{2}{3}n^3 + 4n^2 + 3n + 1}$ flops

NOTE: $w - e_i = 1$ flop if
we code it like $w(i) = w(i) - 1$.

$\frac{2}{3}n^3 + 4n^2 + 2n + 2$ also works.