Homework 3 T8.8, A4.16, A4.13, A4.15, A4.9, T8.11

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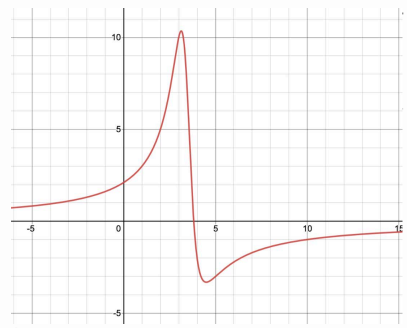
Apr 22, 2021

1. T8.8

$$f(t) = \frac{c_1 + c_2 t + c_3 t^2}{1 + d_1 t + d_2 t^2} \qquad \text{interpolation : } f(t_i) = y_i$$



$$(t, f(t)) = (1,3), (2,5), (3,10), (4,-2), (5,-3)$$

$$c_1 + c_2 + c_3 = 3(1 + d_1 + d_2) \Rightarrow c_1 + c_2 + c_3 - 3d_1 - 3d_2 = 3$$

$$c_1 + 2c_2 + 4c_3 = 5(1 + 2d_1 + 4d_2) \Rightarrow c_1 + 2c_2 + 4c_3 - 10d_1 - 20d_2 = 5$$

$$c_1 + 3c_2 + 9c_3 - 30d_1 - 90d_2 = 10$$

$$c_1 + 4c_2 + 16c_3 + 8d_1 + 32d_2 = \cancel{4} -2$$

$$c_1 + 5c_2 + 25c_3 + 15d_1 + 75d_2 = -3$$

$$A \cdot \theta = b \Rightarrow \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ d_1 \\ d_2 \end{bmatrix} = b$$

$$\begin{bmatrix} 1 & 1 & 1 & -3 & -3 \\ 1 & 2 & 4 & -10 & -20 \\ 1 & 3 & 9 & -30 & -90 \\ 1 & 4 & 16 & 8 & 32 \\ 1 & 5 & 25 & 15 & 75 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 10 \\ -2 \\ -3 \end{bmatrix}$$

$$\theta = A \backslash b = \begin{bmatrix} 2.1092 \\ -0.5499 \\ -0.0008 \\ -0.5616 \\ 0.0811 \end{bmatrix}$$

2. A4.16

$$m = 4 \quad n = 8 \Rightarrow p(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3 + x_5 t^4 + x_6 t^5 + x_7 t^6 + x_8 t^7$$

$$\boxed{-2 = x_1} \qquad p'(t) = x_2 + 2x_3 t + 3x_4 t^2 + 4x_5 t^3 + 5x_6 t^4 + 6x_7 t^5 + 7x_8 t^6$$

$$2 = x_1 + 0.3 x_2 + 0.3^2 x_3 + \cdots + 0.3^7 x_8$$

$$-1 = x_1 + 0.6 x_2 + 0.6^2 x_3 + \cdots + 0.6^7 x_8$$

$$1 = x_1 + x_2 + x_3 + \cdots + x_8$$

$$0 = x_2$$

$$0 = x_2 + 2x_3 \cdot 0.3 + 3x_4 \cdot 0.3^2 + \cdots + 7x_8 \cdot 0.6^{3\,2}$$

$$0 = x_2 + 2x_3 \cdot 0.6 + 3x_4 \cdot 0.6^2 + \cdots 7x_8 \cdot 0.6^6$$

$$0 = x_2 + 2x_3 + 3x_4 + \cdots 7x_8$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & .3 & .3^2 & .3^3 & \cdots & & & .3^7 \\ 1 & .6 & .6^2 & \cdots & & & & .6^7 \\ 1 & 1 & 1 & \cdots & & & & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 \cdot .3 & 3 \cdot .3^2 & \cdots & & & 7 \cdot .3^6 \\ 0 & 1 & 2 \cdot .6 & 3 \cdot .6^2 & \cdots & & & 7 \cdot .6^6 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = A \backslash b = \begin{bmatrix} -2 \\ 0 \\ 134.7 \\ -75.3 \\ -1859.6 \\ 4962 \\ -4696.6 \\ 1537.8 \end{bmatrix}$$

plot ⇒

(b) ~~no it is not non-singular.~~

~~first of all, if n=2m~~

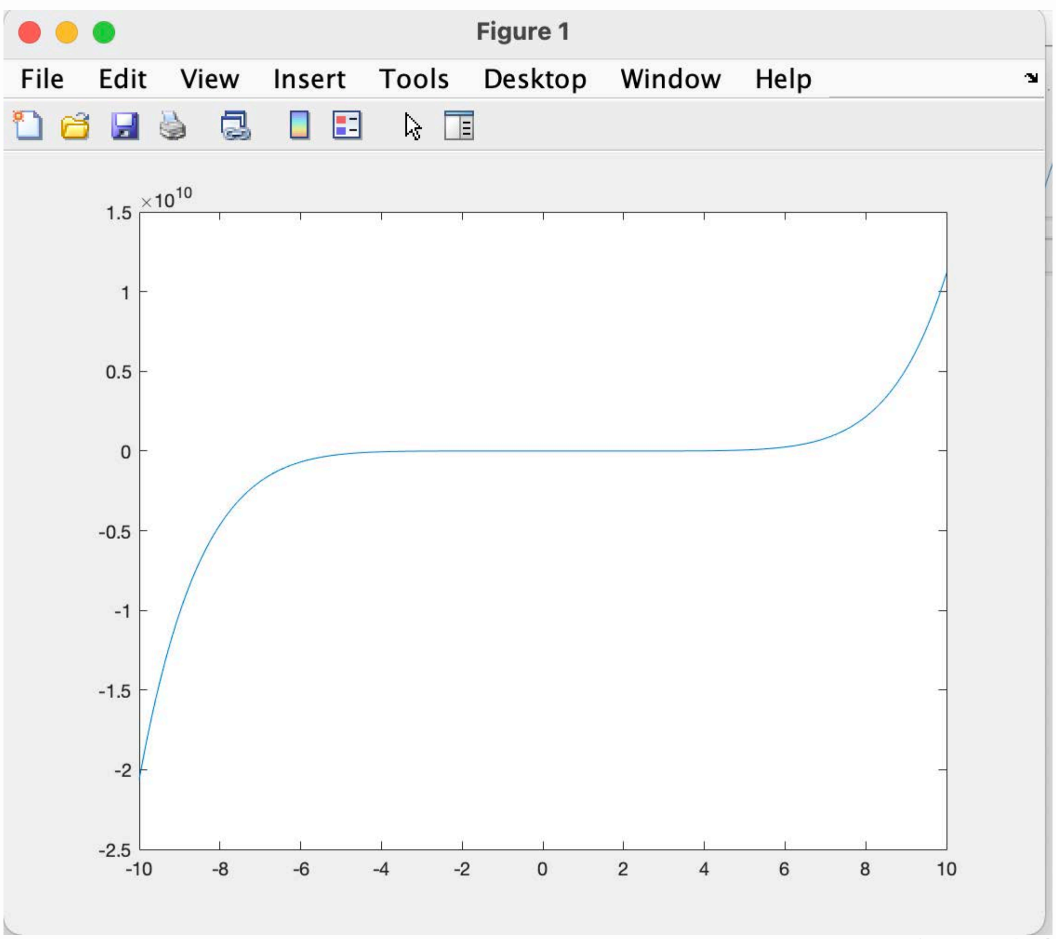
~~there are n columns and m rows. if the matrix is tall, it cannot be nonsingular~~

~~also,~~

$n=2m$. My matrix for A is $2m \times n$. if $n=2m$, this is now a square matrix. Now, we have to prove columns are linearly independent.

we know the m points t_i are distinct and so $\boxed{\text{yes, nonsingular}}$.

(a) PLOT FOR A



$\boxed{2.2}$

3. T8.11

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$$\rho^2 = \|x - a\|^2 = \left\| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \right\|^2$$

$$= (x_1 - a_x)^2 + (x_2 - a_y)^2 + (x_3 - a_z)^2$$

$$\rho_1^2 = (x_1 + 10)^2 + (x_2 - 10)^2 + (x_3 - 10)^2$$

$$\rho_2^2 = (x_1 \quad)^2 + (x_2 - 10)^2 + (x_3 \quad)^2$$

$$\rho_3^2 = (x_1 + 10)^2 + (x_2 - 10)^2 + (x_3 \quad)^2$$

$$\rho_4^2 = (x_1 + 20)^2 + (x_2 + 10)^2 + (x_3 + 10)^2$$

$$\rho_2^2 - \rho_3^2 = x_1^2 - (x_1 + 10)^2 = -20x_1 - 100$$

$$\rho_1^2 - \rho_3^2 = (x_3 - 10)^2 - x_3^2 = -20x_3 + 100$$

$$\rho_4^2 - \rho_2^2 = (x_1 + 20)^2 - x_1^2 + (x_2 + 10)^2 - (x_2 - 10)^2 + (x_3 + 10)^2 - x_3^2$$

$$= 40x_1 + 400 + 40x_2 + 20x_3 + 100$$

$$-20x_1 = 100 + \rho_2^2 - \rho_3^2 \quad , \quad -20x_3 = \rho_1^2 - \rho_3^2 - 100$$

$$40x_1 + 40x_2 + 20x_3 = \rho_4^2 - \rho_2^2 - 500$$

$$\begin{bmatrix} -20 & 0 & 0 \\ 0 & 0 & -20 \\ 40 & 40 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 100 + \rho_2^2 - \rho_3^2 \\ \rho_1^2 - \rho_3^2 - 100 \\ \rho_4^2 - \rho_2^2 - 500 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cancel{1.2094} \\ \cancel{-0.2} \\ \cancel{-0.5035} \end{bmatrix} \begin{bmatrix} 0.6047 \\ 0.4047 \\ -0.5035 \end{bmatrix}$$

4. A4.13

(a)

$$A = a I - \vec{1} \cdot \vec{1}^T$$

$$= \begin{bmatrix} a-1 & -1 & -1 & -1 \\ -1 & a-1 & -1 & -1 \\ -1 & -1 & a-1 & \ddots \\ -1 & -1 & \ddots & \ddots \end{bmatrix}$$

$x_1(a-1) + x_2(-1) + x_3 \ldots = 0$

$x_1(-1) + x_2(a-1) + x_3 \ldots = 0$

$x_1(-1) + x_2(-1) + x_3(a-1) \ldots = 0$

$\vdots$

if A is singular the columns can't be linearly independent.

must have a solution that isn't $x_i = 0$.

$x_1 a = \sum x_i \qquad x_3 a = \sum x_i$

$x_2 a = \sum x_i \qquad \ldots$

two solutions for a:

if $a = 0$, x_i can be anything as long as $\sum x_i = 0$.

if $a = n$, x_i can be anything as long as $x_i = x_j$ for all $i, j < n$.

$\boxed{a = 0, n}$

(b) $A = aI - 1 \cdot 1^T$

Let $1 \cdot 1^T = B$

We want to find values of x, y s.t.

$(aI - 1 \cdot 1^T)^{-1} = x \cdot I + y \cdot B \implies (aI - B)^{-1} = xI + yB$

$\implies (aI - B)(xI + yB) = I \implies axI - xB + ayB - yB^2 = I$

$\implies (ax-1) I + (ay - x) B - yB^2 = 0$

$ax - 1 + (ay - x) - ny = 0 \qquad ax - 1 = 0 \implies x = 1/a$

$0 + (ay - x) - ny = 0 \qquad \cancel{ay+x}\ ay - x - ny = 0 \implies y = \frac{1}{a(a-n)}$

$$A^{-1} = \frac{1}{a} I + \frac{1}{a(a-n)} (1 \cdot 1^T) \checkmark$$

#5) 5.15

(a) $(A+B)^{-1}(A+B) = I$

$(A+B)^{-1}A + (A+B)^{-1}B = I$

$A(A+B)^{-1}A + A(A+B)^{-1}B = A$

$C + A(A+B)^{-1}A = A$

$C = A - A(A+B)^{-1}A$ ▫

(b) $A - A = 0$

$\cancel{A - A(A+B)(A+B)^{-1} = 0}$ b/c $(A+B)(A+B)^{-1} = I$

$A - (A+B)(A+B)^{-1}A = 0$ ←

$A - A(A+B)^{-1}A - B(A+B)^{-1}A = 0$

$B(A+B)^{-1}A = A - A(A+B)^{-1}A = C$ ▫

(c) $C \cdot C^{-1} = \cancel{B(A+B)^{-1}A \cdot (A^{-1} + B^{-1})}$

$= \cancel{B(A+B)^{-1}A \cdot A^{-1} + B(A+B)^{-1}A \cdot B^{-1}}$

$= \cancel{B(A+B)^{-1} + B(A+B)^{-1}A \cdot B^{-1}}$

from (b) $C = B(A+B)^{-1}A$ ... ①

$B - C = B - B(A+B)^{-1}A$

$= B(A+B)^{-1}(A+B) - B(A+B)^{-1}A = B(A+B)^{-1} \cdot B$

$B - B + C = (A+B)(A+B)^{-1}B - B(A+B)^{-1}B = A(A+B)^{-1}B$ ②

$C \cdot C^{-1} = C(A^{-1} + B^{-1}) =$ ① $\cdot A^{-1} +$ ② $\cdot B^{-1} = B(A+B)^{-1} + A(A+B)^{-1}$

$= I$ ▫ ✓

5. A 4.9

$X = AB \quad Y = B^{\dagger} A^{\dagger}$

$B^{\dagger} = (B^T B)^{-1} \cdot B^T$

$A^{\dagger} = (A^T A)^{-1} \cdot A^T$

(a) $YX = (B^T B)^{-1} \cdot B^T \cdot (A^T A)^{-1} \cdot A^T \cdot A \cdot B$

$= (B^T B)^{-1} \cdot B^T \cdot \cancel{(A^T A)^{-1}} \cdot \cancel{(A^T A)} \cdot B$

$= (B^T B)^{-1} (B^T \cdot B) = I$ = symmetric

(b) $XY = \cancel{AB \cdot (B^T B)^{-1} B^T} = AB \cdot (B^T B)^{-1} \cdot B^T \cdot (A^T A)^{-1} A^T$

$= \cancel{AB \cdot B^{\dagger}} A^{\dagger}$

$= \cancel{(Y^T X^T)^T = (A \cdot (A^T A)^{-1} B (B^T B)^{-1} B^T A^T}\,)$

$B (B^T B)^{-1} B^T = X$

$B \cancel{(B^T B)^{-1} B^T B} = X \cdot B$

$B = X B \Rightarrow X = I$ or $B = \emptyset$ but not possible b/c B has lin. ll rows

$XY = A \cdot I \cdot (A^T A)^{-1} A^T = I$ by same reasoning and I is symmetric.

(c) $YXY = Y \cdot I = Y$ dimensions also match and can easily be verified.

(d) $XYX = X \cdot I = X$ ✓