PROBLEM #1

#1 $\epsilon_M = 10^{-16}$

when $x = -3 \cdot e(-16)$

b/c $3\epsilon_M = 3.33 \cdot 10^{-16}$ ← closer to $3 \cdot 10^{-16}$

$2\epsilon_M = 2.22 \cdot 10^{-16}$

$$\frac{\exp(x)-1}{x} = \frac{(1+x)-1}{x} = \frac{(1-3e-16)-1}{-3e-16} = \frac{1-3\epsilon_M-1}{-3e-16} = \frac{-3\epsilon_M}{-3e-16} = 1.1102$$

when $x = 3 \cdot e-16$

$$\frac{\exp(x)-1}{x} = \frac{(1+x)-1}{x} = \frac{(1+3e-16)-1}{+3e-16} = \frac{1+2\epsilon_M-1}{3e-16} = \frac{2\epsilon_M}{3e^{-16}} = 0.7401$$

#2

$$g(x) = \frac{e^x - e^{-2x}}{x} = \frac{e^{-2x}(e^{3x} - e^0)}{x} = \frac{e^{-2x}(e^{3x}-1)}{x}$$ ← cides diff

#2

$$= \frac{e^{-2x}(e^x-1)(e^{2x}+e^x+1)}{x}$$

expm1

```
x = 1e-16;
exp(-2*x)*expm1(x)*(exp(2*x)+exp(x)+1)/x
x = 1e-15;
exp(-2*x)*expm1(x)*(exp(2*x)+exp(x)+1)/x
x = 1e-14;
exp(-2*x)*expm1(x)*(exp(2*x)+exp(x)+1)/x
x=1e-13;
exp(-2*x)*expm1(x)*(exp(2*x)+exp(x)+1)/x
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ans = 3.0000
ans = 3.0000
ans = 3.0000
ans = 3.0000
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PROBLEM 2

1. $A+A^T$ is PSD $\Rightarrow$ for all y, $y^T(A+A^T)y \geq 0$

So, $y^TAy + y^TA^Ty \geq 0$. Note that $(y^TAy)^T = y^TA^Ty$. Also, y^TAy is a scalar because $(1\times m)(m\times m)(m\times 1) = 1\times 1$.

$$\text{So, } y^TAy = y^TA^Ty.$$

$$y^TAy + y^TA^Ty = 2y^TAy \geq 0 \rightarrow y^TAy \geq 0 \text{ for all } y.$$

$$\boxed{x^TAx \geq 0 \text{ for all } x}$$

accidentally used y instead of x.

2. To prove $I+A$ is nonsingular we can prove that ~~$\Rightarrow$~~ $I+A$ is posi definite

because $Mx = 0 \rightarrow x^TMx = 0 \rightarrow x = 0$

$y^T(I+A)y = y^Ty + y^TAy$. Now, $y^TAy \geq 0$ for all y from (1).

$y^Ty = \|y\|^2 > 0$ if $y \neq 0$

Thus, $y^Ty + y^TAy > 0$ if $y \neq 0 \rightarrow$ posi definite $(I+A) \rightarrow$ nonsingular.

3.

4.

$I - S^T S$ is PSD $\Rightarrow$ $y^T(I - S^T S)y \geq 0$

$y^T y - y^T S^T S y \geq 0$

$y^T y \geq y^T S^T S y$

$\|y\|^2 \geq \|Sy\|^2$

$\dfrac{\|Sy\|^2}{\|y\|^2} \leq 1$ for all $y \neq 0$

$\|S\|_2 = \max\limits_{y \neq 0} \dfrac{\|Sy\|}{\|y\|} < 1$ $\square$

$\dfrac{\|Sy\|}{\|y\|} \leq 1$ b/c $\|Sy\|, \|y\| > 0$

3. 1) $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = $ P.D. prove A_{11}, A_{22} are P.D.

Let $V = \begin{bmatrix} X \\ 0 \end{bmatrix}$ ← N vector ← N vector

$$V^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} V = \begin{bmatrix} X^T & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} X^T A_{11} & X^T A_{12} \end{bmatrix} \begin{bmatrix} X \\ 0 \end{bmatrix}$$

$$= X^T A_{11} X > 0$$

Note: $V^T A V > 0$ for All $V \neq 0$. Thus, it is true for a subset of All V. Thus, true for a set of vectors V that can be written as $\begin{bmatrix} X \\ 0 \end{bmatrix}$ where $\vec{X}, \vec{0}$ are N dimensional.

$\vec{X}$ can be ANY vector, so the proof holds.

Let $w = \begin{bmatrix} 0 \\ y \end{bmatrix}$ (Same as before thingy) $\Rightarrow$ $w^T A w = y^T A_{22} y > 0$

so A_{22} P.D.

2). $B_{11} = A_{11}^{-1}$. A_{11} is P.D. For ANY vector x, let $y = A_{11}^{-1} x$

$A_{11} y = x$. $x^T A_{11}^{-1} x = y^T A_{11}^T A_{11}^{-1} A_{11} y = y^T A_{11}^T y$

$= y^T A_{11} y > 0$. So, A_{11}^{-1} is P.D. and B_{11} is P.D.

Since $y^T A_{11}^T y$ is scalar.

$B_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$

COME BACK LATER

(3) $B_{11} = A_{11}^{-1}$. $A_{11} = R_1^T R_1 \rightarrow A_{11}^{-1} = R_1^{-1} (R_1^T)^{-1} = R_1^{-1} (R_1^{-1})^T$

$= \cancel{((R_1^{-1})^T)^T ((R_1^{-1})^T)}$

$B_{11} = R_1^{-1} (R_1^{-1})^T$

$B_{12} = -A_{11}^{-1} A_{12}$ Prove $B_{12} = -R_1^{-1} R_1^{-T} A_{12}$

trivial b/c $A_{11}^{-1} = R_1^{-1} (R_1^{-T})$

prove $B_{21} = A_{21} R_1^{-1} R_1^{-T}$ given $B_{21} = A_{21} A_{11}^{-1}$

also trivial from

prove $B_{22} = -A_{21} R_1^{-1} R_1^{-T} A_{12} + R_2^T R_2$ given $B_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$

$=$

trivial b/c $R_1^{-1} R_1^{-T} = A_{11}^{-1}$ ✓

$$B = \begin{bmatrix} R_1^{-1} & 0 \\ A_{21} R_1^{-1} & R_2^T \end{bmatrix} \begin{bmatrix} R_1^{-T} & -R_1^{-T} A_{12} \\ 0 & R_2 \end{bmatrix} \; R$$

Complexity of computing R_1^{-1}, R_2^T, $A_{21}R_1^{-1}$.

Cholesky factor A_{11}, A_{22} takes $\frac{1}{3}n^3$ im going to ignore $2n^2$

R_2^T takes $\frac{1}{3}n^3$

R_1^{-1} takes $\frac{1}{3}n^3$ to find R_1, R_1 is UT

R_1^{-1} is solving $R_1 X = I$
column by column $= \frac{1}{3}n^3$

$A_{21}R_1^{-1} = 2n^3$ b/c $(n \times n)(n \times n)$

Total $= 2n^3 + n^3 = \boxed{3n^3}$

#4)

1. minimize $\lambda (a_1^T x - b_1)^2 + \sum_{i=2}^{m} (a_i^T x - b_i)^2 = (\lambda-1)(a_1^T x - b_1)^2 + \sum_{i=1}^{m} (a_i^T x - b_i)^2$

equivalent to minimize

$$\boxed{\left\| \begin{bmatrix} A \\ \sqrt{\lambda-1}\; a_1^T \end{bmatrix} x - \begin{bmatrix} b \\ \sqrt{\lambda-1}\; b_1 \end{bmatrix} \right\|^2 \quad \text{so that } e_1}$$

$$\hat{x}(\lambda) = \left(\begin{bmatrix} A^T & \sqrt{\lambda-1}\, a_1 \end{bmatrix} \begin{bmatrix} A \\ \sqrt{\lambda-1}\; a_1^T \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} A^T & \sqrt{\lambda-1}\, a_1 \end{bmatrix} \begin{bmatrix} b \\ \sqrt{\lambda-1}\, b_1 \end{bmatrix}$$

$$= \left(A^T A + (\lambda-1)\, a_1 a_1^T \right)^{-1} \cdot \left[A^T b + (\lambda-1)\, a_1 b_1 \right]$$

$A^T A \; \hat{x} = A^T b$ $\qquad \cancel{a_1^T a_1 \hat{x} = 2a_1^T b_1}$

$\left(A^T A + (\lambda-1)\, a_1 a_1^T \right) \hat{x}(\lambda) = A^T b + (\lambda-1) a_1 b_1 = \cancel{A^T A \hat{x} + (\lambda-1)}\, a_1 b_1$

$$A^T A \hat{x} + \frac{(\lambda-1)(b_1 - a_1^T \hat{x})}{1 + (\lambda-1)\, a_1^T (A^T A)^{-1} a_1} \cdot a_i +$$

2) $A = QR$

$$x = (A^T A)^{-1} A^T b = R^{-1} Q^T b$$

$$(A^T A)^{-1} a_1 = (R^T R)^{-1} a_1$$

$$= R^{-1} R^{-T} a_1$$

$R^{-T} a_1 = x \qquad R^T x = a_1$ can be computed in $\boxed{n^2}$ flops by back substitution.

$R^{-1} x = y \qquad Ry = x \quad \rightarrow n^2$ flops by back sub.

$R^{-1} R^{-T} a_1 = 2n^2$ flops.

$$a_1^T R^{-1} R^{-T} a_1 = \|(R^{-T} a_1)\|^2 = \|x\|^2 \approx \boxed{2n} \text{ flops.}$$

3) minimize $\|Ax - b\|^2 - (a_1^T x - b_1)^2 = \left\| \begin{bmatrix} A \\ -a_1^T \end{bmatrix} x - \begin{bmatrix} b \\ -b_1 \end{bmatrix} \right\|^2$

subject to $a_1^T x = b_1$

solution:

$$\begin{bmatrix} \begin{bmatrix} A^T & -a_1 \end{bmatrix} \begin{bmatrix} A \\ -a_1^T \end{bmatrix} & a_1 \\ a_1^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A^T & -a_1 \end{bmatrix} \begin{bmatrix} b \\ -b_1 \end{bmatrix} \\ b_1 \end{bmatrix}$$

$$\begin{bmatrix} A^TA + a_1a_1^T & a_1 \\ a_1^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A^Tb + a_1b_1 \\ b_1 \end{bmatrix}$$

$$(A^TA + a_1a_1^T)x + a_1 z = A^Tb + a_1b_1$$

$$a_1^T x = b_1$$

$z =$

$$\cancel{x = (A^TA + a_1a_1^T)^{-1}(A^Tb + a_1b_1 - a_1 z)}$$

$$A^TAx + \cancel{a_1b_1} + a_1 z = A^Tb + \cancel{a_1b_1}$$

$$a_1 z = A^Tb - A^TAx = (A^TA)(\hat{x} - x)$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$A^TAx = A^Tb - a_1 z$$

$$x = (A^TA)^{-1}(A^Tb - a_1 z)$$

$$x = \hat{x} - (A^TA)^{-1} a_1 z$$

$$= \boxed{\hat{x} - z(A^TA)^{-1}a_1}$$

= almost!! :(

PROBLEM 5

#1. $b=$ norm of columns are preserved so ||b|| is equal to ||last row of R||

#2 $A^T A \hat{x} = A^T b$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & \end{bmatrix} A\hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & & & & \end{bmatrix} b$$

m terms

first row $=$ $\underbrace{\text{sum}(A\hat{x})}_{m \text{ terms}}$ first row $=$ sum(b)

$$\text{avg} = \frac{\text{sum}(A\hat{x})}{m} = \frac{\text{sum}(b)}{m}$$

b/c first rows are equal

#3 $$\text{std}(b) = \sqrt{\frac{\|b - \text{avg}(b)\|^2}{m}}$$

I don't know how to do #1, but assuming that method exists,

1) calculate b from last colum of R

2) calculate $A\hat{x}$ from last colum R $\rightarrow$ $\text{avg}(A\hat{x}) = \text{avg}(b)$

std$(A\hat{x})$

1) calc $A\hat{x}$ from last row
2) calc $\text{avg}(A\hat{x})$ from $A\hat{x}$
3) calculate $\sqrt{\dfrac{\|A\hat{x} - \text{avg}(A\hat{x})\|^2}{m}}$

std$(b-A\hat{x})$

1) calc $b-A\hat{x}$ through part (a)

calc $\text{avg}(b-A\hat{x}) = \text{avg}(b) - \text{avg}(A\hat{x}) = 0$

calc $\sqrt{\dfrac{\|b-A\hat{x} - 0\|^2}{m}}$