Homework 2

Reading assignment: Chapters 6, 7, 8, 9, and sections 10.1, 10.2, 10.3 of the textbook.

Homework problems

1. Exercise A1.7.
2. Exercise A1.8.

   (Julia users can import the data using the command `include("orthregdata.m")`.)
3. Exercise A2.4.
4. Exercise A2.8.
5. Consider a product of m matrices

$$A_1 A_2 \cdots A_m, \tag{1}$$

where A_i has size $n_{i-1} \times n_i$. The total number of flops required to compute the result depends on the order in which we evaluate the matrix-matrix products. For example, for $m = 4$, we have the five possibilities

$$A_1(A_2(A_3A_4)) = A_1((A_2A_3)A_4) = (A_1A_2)(A_3A_4) = (A_1(A_2A_3))A_4 = ((A_1A_2)A_3)A_4.$$

In this problem we develop an efficient method for determining the optimal order for the matrix product in (1) and the number of flops if we use the optimal order.

We denote by c_{ij} the cost (optimal number of flops) of computing $A_iA_{i+1}\cdots A_j$, where $1 \leq i \leq j \leq m$. Clearly $c_{11} = c_{22} = \cdots = c_{mm} = 0$, and

$$c_{12} = 2n_0n_1n_2, \qquad c_{23} = 2n_1n_2n_3, \qquad \ldots.$$

(We use the simplification $(2q-1)pr \approx 2pqr$ for a product of a $p \times q$ and a $q \times r$ matrix.) We are interested in computing c_{1m}, the optimal number of flops for the entire product $A_1A_2\cdots A_m$.

(a) Explain why

$$c_{ij} = \min_{k=i,i+1,\ldots,j-1} (c_{ik} + c_{k+1,j} + 2n_{i-1}n_kn_j).$$

The minimum is over all values of k that satisfy $i \leq k < j$. For example, if $i = 1$ and $j = 4$,

$$c_{14} = \min\{c_{11} + c_{24} + 2n_0n_1n_4,\ c_{12} + c_{34} + 2n_0n_2n_4,\ c_{13} + c_{44} + 2n_0n_3n_4\}.$$

(b) The formula in part (a) suggests a recursive method for computing all the values of c_{ij}. We write the coefficients in a triangular table

$$\begin{array}{cccccc} c_{11} & c_{12} & c_{13} & \cdots & c_{1,m-1} & c_{1m} \\ & c_{22} & c_{23} & \cdots & c_{2,m-1} & c_{2m} \\ & & c_{33} & \cdots & c_{3,m-1} & c_{3m} \\ & & & \ddots & \vdots & \vdots \\ & & & & c_{m-1,m-1} & c_{m-1,m} \\ & & & & & c_{mm} \end{array}$$

and compute the entries diagonal by diagonal, as follows:

Define $c_{11} = \cdots = c_{mm} = 0$.
for $l = 1, \ldots, m-1$ **do**
  for $i = 1, \ldots, m-l$ **do**
    Compute

$$c_{i,i+l} = \min_{k=i,i+1,\ldots,i+l-1} (c_{ik} + c_{k+1,i+l} + 2n_{i-1}n_k n_{i+l}) \tag{2}$$

  end for
end for

An optimal order can be found by recording for each entry in the table a value of k that attains the minimum in (2).

Apply this algorithm to find the optimal order and number of flops for the product of four matrices $A_1A_2A_3A_4$ with dimensions

$$100 \times 5000 \times 10000 \times 1000 \times 10.$$

Also compare with the optimal order and the number flops required for the product $A_1A_2A_3$ of the first three matrices.