L. Vandenberghe
ECE133A

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Homework 3 solutions

1. Exercise T8.8. The interpolation conditions

$$\frac{c_1 + c_2 t_i + c_3 t_i^2}{1 + d_1 t_i + d_2 t_i^2} = y_i, \quad i = 1, \ldots, K,$$

can be written as linear equations in c_1, c_2, c_3, d_1, d_2:

$$c_1 + c_2 t_i + c_3 t_i^2 - y_i d_i t_i - y_i d_2 t_i^2 = y_i, \quad i = 1, \ldots, K.$$

If we define $x = (c_1, c_2, c_3, d_1, d_2)$, this can be written as $Ax = b$ with

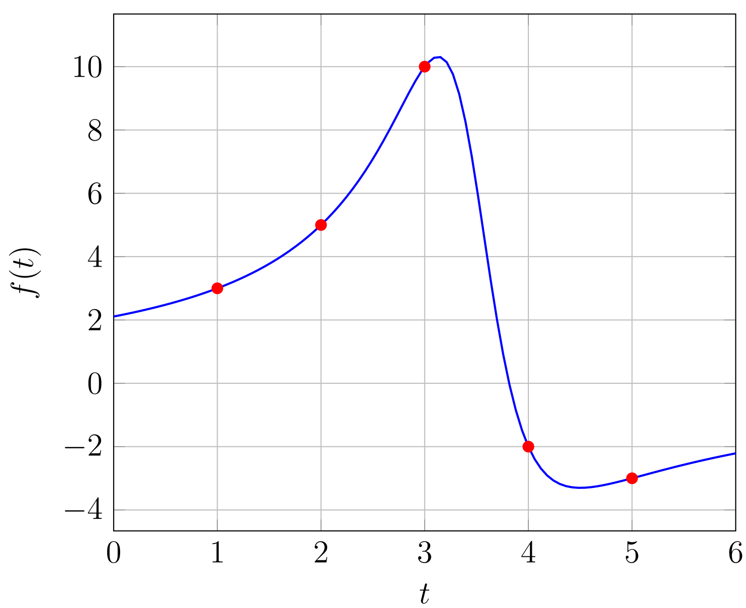
$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & -y_1 t_1 & -y_1 t_1^2 \\ 1 & t_2 & t_2^2 & -y_2 t_2 & -y_2 t_2^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_K & t_K^2 & -y_K t_K & -y_K t_K^2 \end{bmatrix}, \qquad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{bmatrix}.$$

For the problem data in the assignment the equation $Ax = b$ is

$$\begin{bmatrix} 1 & 1 & 1 & -3 & -3 \\ 1 & 2 & 4 & -10 & -20 \\ 1 & 3 & 9 & -30 & -90 \\ 1 & 4 & 16 & 8 & 32 \\ 1 & 5 & 25 & 15 & 75 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 10 \\ -2 \\ -3 \end{bmatrix}.$$

The solution is

$$c_1 = 2.10920, \quad c_2 = -0.54992, \quad c_3 = -0.00078, \quad d_1 = -0.56162, \quad d_2 = 0.08112.$$



2. Exercise A4.16.

(a)

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-2} & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-2} & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & t_m & t_m^2 & \cdots & t_m^{n-2} & t_m^{n-1} \\ 0 & 1 & 2t_1 & \cdots & (n-2)t_1^{n-3} & (n-1)t_1^{n-2} \\ 0 & 1 & 2t_2 & \cdots & (n-2)t_2^{n-3} & (n-1)t_2^{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 2t_m & \cdots & (n-2)t_m^{n-3} & (n-1)t_m^{n-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix}.$$

(b) The matrix is nonsingular. To show this, we verify that $Ax = 0$ only if $x = 0$. If $Ax = 0$, then the polynomial

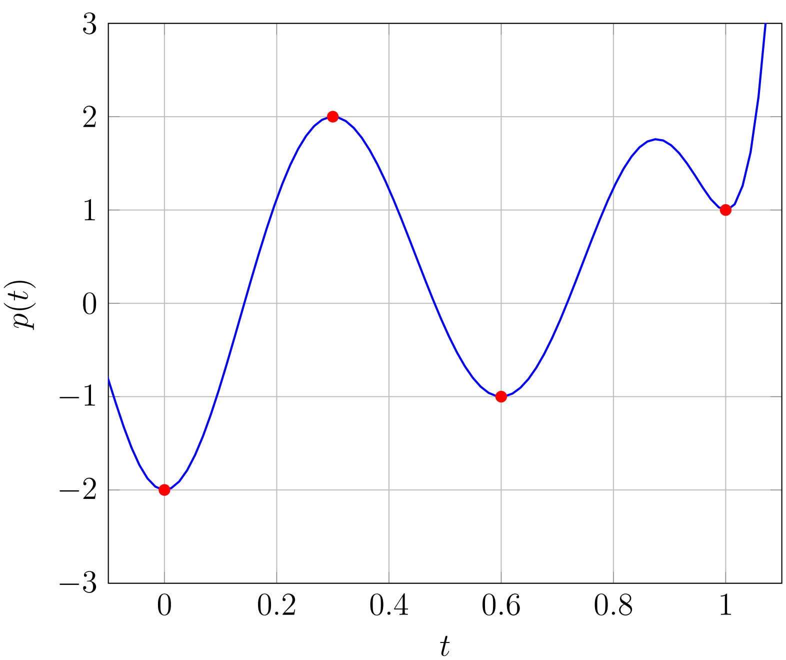
$$p(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$$

is zero and has zero derivative at the m points $t_1, \ldots, t_m$. Therefore each point t_i is a zero with multiplicity at least two. Counting roots with their multiplicity, we would have at least $n = 2m$ real roots. This is impossible for a polynomial of degree at most $n-1$, unless the polynomial is a constant zero.

(c)

$$x_1 = -2, \quad x_2 = 0, \quad x_3 = 134.74, \quad x_4 = -75.31,$$

$$x_5 = -1859.65, \quad x_6 = 4962.04, \quad x_7 = -4696.59, \quad x_8 = 1537.77.$$



3. Exercise T8.11. We square the two sides of each equation, and expand the norms:

$$x^T x - 2a_i^T x + a_i^T a_i = \rho_i^2, \quad i = 1, 2, 3, 4.$$

The quadratic term $x^T x$ can be eliminated by subtracting one equation from the three others. For example,

$$\begin{bmatrix} (a_1 - a_2)^T \\ (a_1 - a_3)^T \\ (a_1 - a_4)^T \end{bmatrix} x = \frac{1}{2} \begin{bmatrix} \rho_2^2 - \rho_1^2 + \|a_1\|^2 - \|a_2\|^2 \\ \rho_3^2 - \rho_1^2 + \|a_1\|^2 - \|a_3\|^2 \\ \rho_4^2 - \rho_1^2 + \|a_1\|^2 - \|a_4\|^2 \end{bmatrix}.$$

The solution for the data in the assignment is

$$x = (0.60471,\ 0.40475,\ -0.50349).$$

4. Exercise A4.13.

   (a) $a = 0$ and $a = n$.

   If $a = 0$, then $x = (1, -1, 0, \ldots, 0)$ is a nonzero vector that satisfies $Ax = 0$. If $a = n$, then $x = \mathbf{1}$ is a nonzero vector that satisfies $Ax = 0$. This shows that A is singular if $a = 0$ or $a = n$.

   To show that A is nonsingular for other values of a, we show that $Ax = 0$ implies $x = 0$. Suppose $Ax = ax - (\mathbf{1}^T x)\mathbf{1} = 0$. Taking the inner product with $\mathbf{1}$ gives $a(\mathbf{1}^T x) = n(\mathbf{1}^T x)$. If $a \neq 0$ and $n \neq a$, this is only possible if $\mathbf{1}^T x = 0$. But then $ax = (\mathbf{1}^T x)\mathbf{1}$ implies $x = 0$. This shows that A is nonsingular if $a \neq 0$ and $a \neq n$.

   (b) Suppose $a \neq n$ and $a \neq 0$. Following the hint we look for an inverse of the form $\beta I + \gamma \mathbf{1}\mathbf{1}^T$:

$$I = (aI - \mathbf{1}\mathbf{1}^T)(\beta I + \gamma \mathbf{1}\mathbf{1}^T) = a\beta I - (\beta + a\gamma - n\gamma)\mathbf{1}\mathbf{1}^T.$$

   This shows that $\beta = 1/a$ and $\gamma = \beta/(n - a)$, so

$$A^{-1} = \frac{1}{a}(I - \frac{1}{n - a}\mathbf{1}\mathbf{1}^T).$$

5. Exercise A4.15.

   (a)

$$\begin{aligned} C &= A(A+B)^{-1}B \\ &= A(A+B)^{-1}(A+B-A) \\ &= A(A+B)^{-1}(A+B) - A(A+B)^{-1}A \\ &= A - A(A+B)^{-1}A. \end{aligned}$$

(b) We start from the expression in part a:

$$\begin{aligned} C &= A - A(A+B)^{-1}A \\ &= (A+B)(A+B)^{-1}A - A(A+B)^{-1}A \\ &= (A+B-A)(A+B)^{-1}A \\ &= B(A+B)^{-1}A. \end{aligned}$$

(c) If A and B are invertible, then

$$\begin{aligned} C^{-1} &= (A(A+B)^{-1}B)^{-1} \\ &= B^{-1}(A+B)A^{-1} \\ &= B^{-1}AA^{-1} + B^{-1}BA^{-1} \\ &= B^{-1} + A^{-1}. \end{aligned}$$

6. Exercise A4.9. We use the definitions and properties

$$A^\dagger = (A^TA)^{-1}A^T, \qquad B^\dagger = B^T(BB^T)^{-1}, \qquad A^\dagger A = I, \qquad BB^\dagger = I.$$

(a) Follows from

$$YX = (B^\dagger A^\dagger)(AB) = B^\dagger(A^\dagger A)B = B^\dagger B = B^T(BB^T)^{-1}B.$$

(b) Follows from

$$XY = (AB)(B^\dagger A^\dagger) = A(BB^\dagger)A^\dagger = AA^\dagger = A(A^TA)^{-1}A^T.$$

(c) In part b we have shown that $XY = AA^\dagger$. The result follows from

$$Y(XY) = (B^\dagger A^\dagger)(AA^\dagger) = B^\dagger(A^\dagger A)A^\dagger = B^\dagger A^\dagger = Y.$$

(d) Similarly,

$$(XY)X = (AA^\dagger)(AB) = A(A^\dagger A)B) = AB = X.$$