HW 7 : A 10.1 abc, 10.14, 11.5, 11.8 bd, 11.26 ✓ NEVIN LIANG

10.1)

(a) Let $\begin{bmatrix} 1 & 1 \\ 0 & 0.95 \end{bmatrix} = A$ and $\begin{bmatrix} 0 \\ 0.1 \end{bmatrix} = b$

$s(t+1) = A\,s(t) + b\,u(t)$

$s(0) = \vec{0}$ $s(1) = b\,u(0)$ $s(2) = Ab\,u(0) + b\,u(1)$

$s(3) = A^2 b\,u(0) + Ab\,u(1) + b\,u(2)$

$s(4) = A^3 b u(0) + A^2 b u(1) + Ab u(2) + b u(3) \ldots$

$s(N) = A^{N-1} b u(0) + A^{N-2} b u(1) + \ldots + A^1 b u(N-2) + b u(N-1)$

$= \begin{bmatrix} A^{N-1} b & A^{N-2} b & A^{N-3} b & \ldots & A^1 b & b \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

Let $[u(0) ; u(1) ; u(2) \ldots ; u(N-1)] = \mathcal{U}$.

$$\begin{bmatrix} A^{N-1} b & A^{N-2} b & \ldots & A^1 b & b \end{bmatrix} \cdot \mathcal{U} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

and we want to minimize

$$E = \sum_{t=0}^{N-1} u(t)^2 = \|u\|^2$$

IF $C = \begin{bmatrix} A^{N-1} b & A^{N-2} b & A^{N-3} b & \ldots & A^1 b & b \end{bmatrix}$ and $d = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$,

we are trying to minimize $\|x\|^2$ subject to $Cx = d$. □

(b) & (c) on matlab next page. $\longrightarrow$

```
% 10.1b

N = 30;

A = [1 1; 0 0.95];
b = [0; 0.1];

C = zeros(2, N);

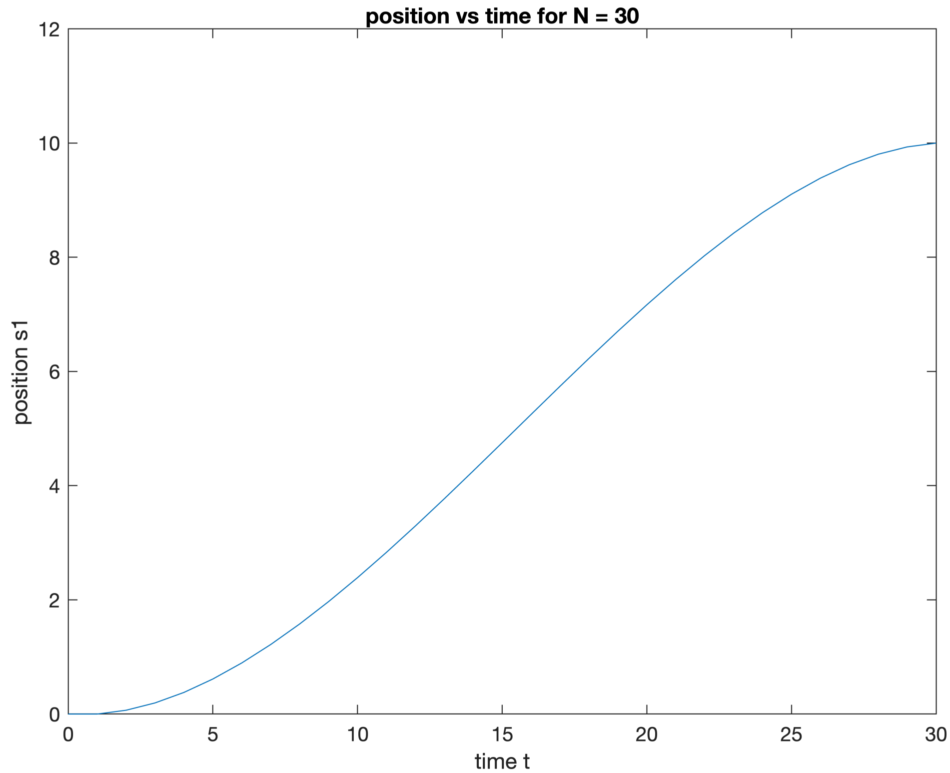
bi = b;
for i = N:-1:1
 C(:,i) = bi;
 bi = A * bi;
end

d = [10; 0];

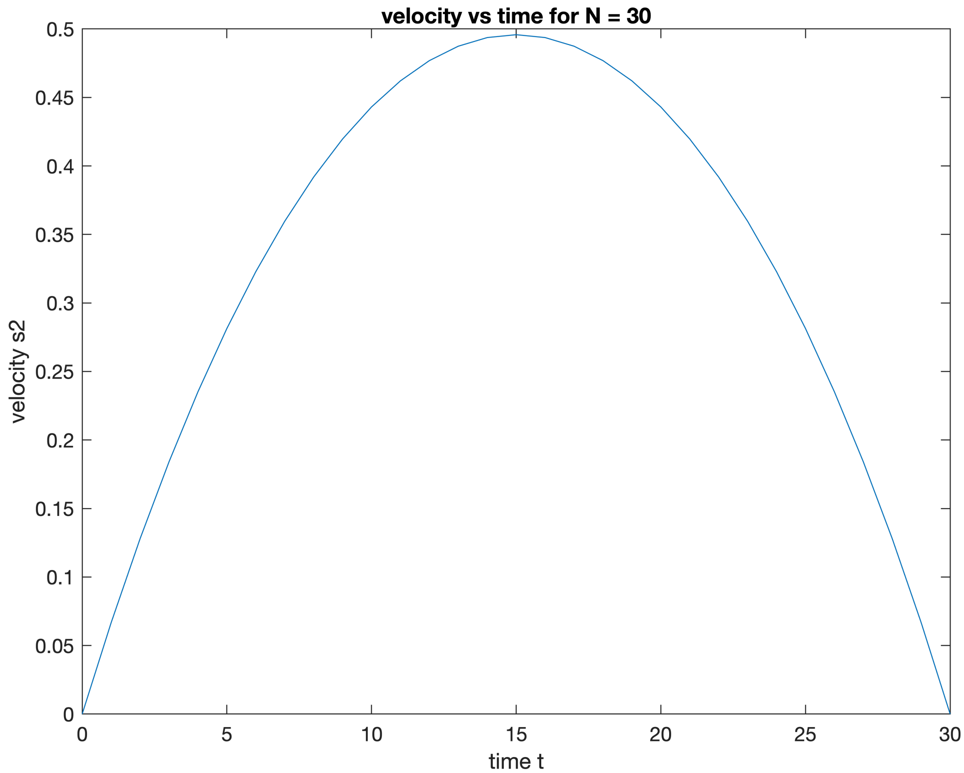
[Q, R] = qr(C', 0);
u = Q * (R' \ d);

s = zeros(2, N);
for i = 1:N
    s(:,i + 1) = A * s(:,i) + b * u(i);
end

plot(0:N, s(1,:));
xlabel('time t');
ylabel('position s1');
title('position vs time for N = 30');
```




```
plot(0:N, s(2,:));
xlabel('time t');
ylabel('velocity s2');
title('velocity vs time for N = 30');
```




```
% 10.1c

E = zeros(28, 1);
for N = 2:29
    A = [1 1; 0 0.95];
    b = [0; 0.1];

    C = zeros(2, N);

    bi = b;
    for i = N:-1:1
     C(:,i) = bi;
     bi = A * bi;
    end

    d = [10; 0];

    [Q, R] = qr(C', 0);
    u = Q * (R' \ d);
```

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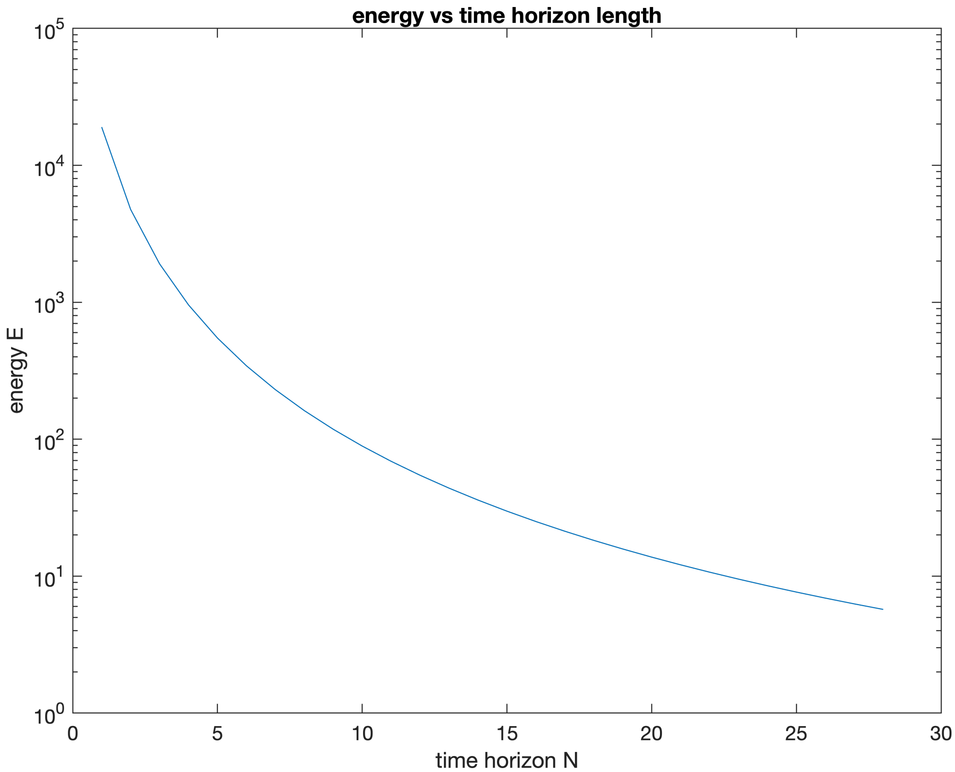
    E(N - 1) = norm(u)^2;
end

semilogy(1:28, E);
xlabel('time horizon N');
ylabel('energy E');
title('energy vs time horizon length');
```



10.14) A: $m \times n$ matrix with linearly ⫫ columns.

$$\hat{x}^{(i)} = \text{solution to } \begin{bmatrix} \text{minimize } \|Ax\|^2 \\ \text{s.t. } e_i^T x = -1 \end{bmatrix}$$

a) Prove $\hat{x}^{(i)} = \dfrac{-1}{e_i^T (A^T A)^{-1} e_i} (A^T A)^{-1} e_i$

Constrained Least squares formulation:

$$\text{minimize } \|Ax - b\|^2 \text{ s.t. } Cx = d$$

Solution is $\begin{bmatrix} A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A^T b \\ d \end{bmatrix}$

if a z exists

In our case $b = \vec{0}$, $C = e_i^T$ and $A = A$. $d = -1$.

$A^T A x + e_i z = A^T \cdot \vec{0} = \vec{0}$

$e_i^T x = -1$ → subject to condition.

$x = (A^T A)^{-1} \cdot (-e_i z)$ → A has lin ⫫ cols.

$e_i^T (A^T A)^{-1} \cdot (-e_i z) = -1$

$\underbrace{e_i^T (A^T A)^{-1} e_i}_{\text{constant}} \cdot z = 1$ → $z = \dfrac{1}{e_i^T (A^T A)^{-1} e_i}$

plug back into eq. 1:

$$x = \frac{-1}{e_i^T (A^T A)^{-1} e_i} (A^T A)^{-1} e_i$$

10.14)

b) $A^T A = (QR)^T QR$

$= R^T Q^T Q R$

1) $A = QR$: $2mn^2$ flops.

2) $(A^T A)^{-1} = R^{-1} R^{-T}$

$R^{-T} \Rightarrow R^T M = I$ col. by col. n^3

$N = R^{-1} M \Rightarrow RN = M$ col by col. n^3

$N = (A^T A)^{-1}$

$2n^3$ total

4) $\hat{x}^{(i)} = \frac{-1}{e_i^T N e_i} N e_i$

0 flops 0 flops

5) Once we find N in $O(2n^3 + mn^2)$

each $\hat{x}^{(i)}$ takes $O(1)$ to calculate.

$\hat{x}^{(i)}$ for $i \in$ 1–N

takes

O(2n^3 + 2mn^2)

1.5)

(a)
(b) Is $A = \frac{1}{\|a\|^2} aa^T$ positive semi-definite?

$$y^T A y = \frac{1}{\|a\|^2} y^T a a^T y = \frac{1}{\|a\|^2} \|a^T y\|^2$$

$$y^T a a^T y = (a^T y)^T (a^T y)$$

for any vector y, and nonzero vector a,

$$\|a^T y\| \geq 0 \quad \text{and} \quad \frac{1}{\|a\|^2} > 0$$

equality occurs when $y = \vec{0}$. IF $y \neq \vec{0}$, then $\|a^T y\| > 0$

Thus, A is positive semi-definite.

and ~~also positive definite~~ not positive definite b/c $a^T y$ is $=0$ ~~not~~ if a, y orthogonal.

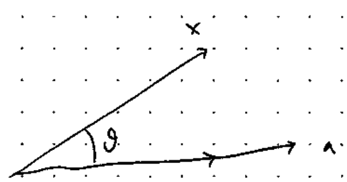
Is $B = I - A$ positive semi-definite or positive definite?

$$y^T (B) y = y^T (I - A) y = y^T y - y^T A y$$

Now, for all y, ~~$y^T A y \geq 0$ from ↑↑ prev section~~

$$y^T y - y^T \cdot \frac{1}{\|a\|^2} \cdot a a^T \cdot y = y^T y - \frac{1}{\|a\|^2} y^T a a^T y$$

$$= y^T y - \frac{1}{\|a\|^2} (a^T y)^2 = \|y\|^2 - \frac{1}{\|a\|^2} (a^T y)^2 \ ? \ 0$$



$$= \|y\|^2 \left(1 - \frac{1}{\|a\|^2 \|y\|^2} (a^T y)^2\right)$$

$$= \|y\|^2 \left(1 - \cos^2\theta\right)$$

$$= \|y\|^2 \sin^2\theta \geq 0$$

Thus, B is positive semi-definite

not positive definite $(\theta = 0)$

11.8)

b) $A = \begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & & \\ R_{12} & R_{22} & \\ R_{13} & R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & & R_{33} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ R_{12} & R_{22} & 0 \\ R_{13} & R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix}$

$R_{12} + 0 \cdot R_{22} = a \rightarrow R_{12} = a$

$R_{13} + 0 + 0 = 0 \rightarrow R_{13} = 0$

~~A is posi definite if $|a| \leq 1$~~

$= \begin{bmatrix} 1 & 0 & 0 \\ a & R_{22} & 0 \\ 0 & R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix}$

~~2x2 cholesky: $\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & \sqrt{1-a^2} \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & \sqrt{1-a^2} \end{bmatrix}$~~

(d) ~~$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ a & & \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & & \\ 0 & 0 & \end{bmatrix}$~~

~~factorizable for all a~~

~~$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$~~

$a \in \mathbb{R}$

$\boxed{|a| \leq \frac{\sqrt{2}}{2}}$

~~11.26)~~

$a^2 + R_{22}^2 = 1 \rightarrow R_{22} = \pm\sqrt{1-a^2}$

$0 + R_{22} \cdot R_{23} + 0 = a \rightarrow R_{23} = \frac{a}{\pm\sqrt{1-a^2}}$

$R_{23}^2 + R_{33}^2 = 1 \rightarrow R_{33} = \pm\sqrt{1 - \frac{a^2}{1-a^2}} = \pm\sqrt{\frac{1-2a^2}{1-a^2}}$

(d) $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ a & 0 & \pm\sqrt{1-a^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm\sqrt{1-a^2} \end{bmatrix}$

$\boxed{|a| \leq 1}$

$\sqrt{1} \Rightarrow \pm 1$

11.26)

(a) $A = \begin{bmatrix} B & -C^T \\ C & D \end{bmatrix} \overset{?}{=} \begin{bmatrix} R_{11}^T R_{11} & R_{11}^T R_{12} \\ -R_{12}^T R_{11} & -R_{12}^T R_{12} + R_{22}^T R_{22} \end{bmatrix}$

B is PD so $B = R_{11}^T R_{11}$ is possible.

$C = -R_{12}^T R_{11}$

since R_{11} has positive diagonal elements,

thus, since R_{11} is U.T. $\rightarrow$ invertible

$-R_{12}^T = C \cdot R_{11}^{-1}$

and $R_{12} = \left(-C \cdot R_{11}^{-1}\right)^T$

$D = -R_{12}^T R_{12} + R_{22}^T R_{22}$ ... we want to prove that R_{22} is upper triangular

$\underbrace{R^T R}_{\text{positive definite}} + \underbrace{R_{12}^T R_{12}}_{\text{positive semidefinite from:}} = R_{22}^T R_{22}$

$a^T R_{12}^T R_{12} \cdot a = \|(R_{12} \cdot a)\|^2 \geq 0$

for all a.

PD + PSD = PD so R_{22} exists and is U.T.

(b) Step 1: $B = R_{11}^T R_{11} \rightarrow R_{11}$ takes $\boxed{\frac{1}{3}n^3}$ by cholesky.

Step 2: $C = -R_{12}^T R_{11}$ $R_{11} \cdot \vec{X} = \vec{y}$ takes $O(N^2)$ to solve

$\hookrightarrow$ N colums $\Rightarrow$ $\boxed{\cancel{n^3}}$ mn^2 since $\vec{X}$ has m columns.

Step 3: $\underbrace{D +}_{m^2} \underbrace{R_{12}^T R_{12}}_{(n \times m)^T (n \times m) \Rightarrow 2m^2 n} = R_{22}^T R_{22} \Rightarrow$ cholesky's $= \frac{1}{3}m^3$

$\boxed{2mn^2 + \frac{1}{3}n^3 + \cancel{m^2} + \frac{1}{3}m^3}$ $\leftarrow$ only cubic terms.