ECE 141 – Midterm Solutions
Spring 2021

05/06/21

Duration: 1 hour and 40 minutes

Problem 1

In this problem we consider a pump subcutaneously injecting insulin to offset the glucose resulting from a meal.

1. Write down the differential equation that describes the dynamics of insulin knowing that:

    (a) its dynamics is governed by a compartmental model with two compartments described by x_1 and x_2;

    (b) the time derivative of x_1 is given by the difference between the insulin injection rate u and a term proportional to x_1 with constant of proportionality $k > 0$;

    (c) the time derivative of x_2 is given by the difference between a term proportional to x_1 and a term proportional to x_2, both with proportionality constant k.

    Solution:
    The first statement tells us that there are two states: x_1 and x_2. The second statement says that $\dot{x}_1 = u - kx_1$, and the third statement says $\dot{x}_2 = kx_1 - kx_2$. Putting these together, the differential equation in vector form is:

    $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -kx_1 + u \\ kx_1 - kx_2 \end{bmatrix}. \tag{1}$$

2. **(Version A and B)** If a step input is applied to the derived model (where u represents the input), will x_2 exhibit overshoot? Would your answer change if we use a different constant k for each compartment, i.e., x_1 terms are multiplied by k_1 and x_2 terms are multiplied by k_2?

(Version C and D) If a step input is applied to the derived model (where u represents the input), will x_1 or x_2 exhibit oscillatory behavior? Would your answer change if we use a different constant k for each compartment, i.e., x_1 terms are multiplied by k_1 and x_2 terms are multiplied by k_2?

Solution:
Applying the Laplace transform to the dynamical equations, the following transfer functions are obtained:

$$\frac{X_1}{U}(s) = \frac{1}{s+k}, \tag{2}$$
$$\frac{X_2}{X_1}(s) = \frac{k}{s+k}. \tag{3}$$

Then, the transfer function from u to x_2 is:

$$\frac{X_2}{U}(s) = \frac{k}{(s+k)^2}. \tag{4}$$

When u is a unit step, x_2 in the Laplace domain is:

$$X_2(s) = \frac{k}{s(s+k)^2}. \tag{5}$$

Since the poles are purely real, x_2 has neither oscillations nor overshoot.

If the proportionality constants are k_1 and k_2 instead, the transfer function from u to x_2 becomes:

$$\frac{X_2}{U}(s) = \frac{k_1}{(s+k_1)(s+k_2)}. \tag{6}$$

When u is a unit step, x_2 becomes:

$$X_2(s) = \frac{k_1}{s(s+k_1)(s+k_2)}. \tag{7}$$

Again, the poles are purely real, so x_2 has neither oscillations nor overshoot.

3. **(Version A)** Considering x_2 as the output, for which values of k will the rise time be smaller than 0.9 and the settling time be smaller than 13.8 when the input is a step?

   Solution:
   The transfer function from u to x_2 is:

$$\frac{X_2}{U}(s) = \frac{k}{(s+k)^2} = \frac{k}{s^2+2ks+k^2} = \frac{k}{s^2+2\zeta\omega_n s+\omega_n^2}. \tag{8}$$

By matching the terms, $\omega_n = k$ and $\zeta = 1$. Using the time domain specification equations for second-order systems, the rise time requirement is a requirement on ω_n. Specifically,

$$\begin{aligned} t_r &< 0.9, && (9)\\ \frac{1.8}{\omega_n} &< 0.9, && (10)\\ \omega_n &> \frac{1.8}{0.9}, && (11)\\ k &> 2. && (12) \end{aligned}$$

The settling time requirement specifies a condition on σ, which is the absolute value of the real part of the poles. In this case, since $\zeta = 1$, the system is critically damped, and we have that $\sigma = \omega_n = k$. Thus,

$$\begin{aligned} t_s &< 13.8, && (13)\\ \frac{4.6}{\sigma} &< 13.8, && (14)\\ \sigma &> \frac{4.6}{13.8}, && (15)\\ k &> \frac{1}{3}. && (16) \end{aligned}$$

Putting together both conditions, we get that $k > 2$.

(Version B) Considering x_2 as the output, for which values of k will the rise time be smaller than 0.45 and the settling time be smaller than 2.3 when the input is a step?

Solution:
The transfer function from u to x_2 is:

$$\frac{X_2}{U}(s) = \frac{k}{(s+k)^2} = \frac{k}{s^2 + 2ks + k^2} = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{17}$$

By matching the terms, $\omega_n = k$ and $\zeta = 1$. Using the time domain specification equations for second-order systems, the rise time requirement is a requirement on ω_n. Specifically,

$$\begin{aligned} t_r &< 0.45, && (18)\\ \frac{1.8}{\omega_n} &< 0.45, && (19)\\ \omega_n &> \frac{1.8}{0.45}, && (20)\\ k &> 4. && (21) \end{aligned}$$

The settling time requirement specifies a condition on σ, which is the absolute value of the real part of the poles. In this case, since $\zeta = 1$, the system is critically damped, and we have that $\sigma = \omega_n = k$. Thus,

$$\begin{aligned} t_s &< 2.3, && (22)\\ \frac{4.6}{\sigma} &< 2.3, && (23)\\ \sigma &> \frac{4.6}{2.3}, && (24)\\ k &> 2. && (25) \end{aligned}$$

Putting together both conditions, we get that $k > 4$.

(Version C) Considering x_2 as the output, for which values of k will the rise time be smaller than 7.2 and the settling time be smaller than 4.6 when the input is a step?

Solution:
The transfer function from u to x_2 is:

$$\frac{X_2}{U}(s) = \frac{k}{(s+k)^2} = \frac{k}{s^2 + 2ks + k^2} = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{26}$$

By matching the terms, $\omega_n = k$ and $\zeta = 1$. Using the time domain specification equations for second-order systems, the rise time requirement is a requirement on ω_n. Specifically,

$$\begin{aligned} t_r &< 7.2, && (27)\\ \frac{1.8}{\omega_n} &< 7.2, && (28)\\ \omega_n &> \frac{1.8}{7.2}, && (29)\\ k &> \frac{1}{4}. && (30) \end{aligned}$$

The settling time requirement specifies a condition on σ, which is the absolute value of the real part of the poles. In this case, since $\zeta = 1$, the system is critically damped, and we have that $\sigma = \omega_n = k$. Thus,

$$\begin{aligned} t_s &< 4.6, && (31)\\ \frac{4.6}{\sigma} &< 4.6, && (32)\\ \sigma &> \frac{4.6}{4.6}, && (33)\\ k &> 1. && (34) \end{aligned}$$

Putting together both conditions, we get that $k > 1$.

(Version D) Considering x_2 as the output, for which values of k will the rise time be smaller than 1.8 and the settling time be smaller than 9.2 when the input is a step?

Solution:
The transfer function from u to x_2 is:

$$\frac{X_2}{U}(s) = \frac{k}{(s+k)^2} = \frac{k}{s^2 + 2ks + k^2} = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{35}$$

By matching the terms, $\omega_n = k$ and $\zeta = 1$. Using the time domain specification equations for second-order systems, the rise time requirement is a requirement on ω_n. Specifically,

$$t_r < 1.8, \tag{36}$$
$$\frac{1.8}{\omega_n} < 1.8, \tag{37}$$
$$\omega_n > \frac{1.8}{1.8}, \tag{38}$$
$$k > 1. \tag{39}$$

The settling time requirement specifies a condition on σ, which is the absolute value of the real part of the poles. In this case, since $\zeta = 1$, the system is critically damped, and we have that $\sigma = \omega_n = k$. Thus,

$$t_s < 9.2, \tag{40}$$
$$\frac{4.6}{\sigma} < 9.2, \tag{41}$$
$$\sigma > \frac{4.6}{9.2}, \tag{42}$$
$$k > \frac{1}{2}. \tag{43}$$

Putting together both conditions, we get that $k > 1$.

4. **(Version A)** Assume now that we are interested in regulating the insulin concentration y related to x_2 by $\dot{y} = -\lambda y + x_2$. If we use an insulin injection rate of k, will y converge to a constant value? If so, which value?

Solution:
The transfer function from x_2 to y can be calculated by taking the Laplace transform of the given equation:

$$\frac{Y}{X_2}(s) = \frac{1}{s+\lambda}. \tag{44}$$

Combining this with the transfer function $\frac{X_2}{U}(s) = \frac{k}{(s+k)^2}$, we can calculate the overall transfer function:

$$\frac{Y}{U}(s) = \frac{k}{(s+\lambda)(s+k)^2}. \tag{45}$$

With the step input of $U(s) = \frac{k}{s}$, the output is then:

$$Y(s) = \frac{k^2}{s(s+\lambda)(s+k)^2}. \tag{46}$$

The poles of this function are all in the open left half plane with at most one at the origin, so the Final Value Theorem can be applied to find the steady state value of y.

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s), &(47)\\
&= \lim_{s\to 0} \frac{k^2}{(s+\lambda)(s+k)^2}, &(48)\\
&= \frac{1}{\lambda}. &(49)
\end{aligned}$$

Thus, y converges to the constant value of $\frac{1}{\lambda}$.

(Version B) Assume now that we are interested in regulating the insulin concentration y related to x_2 by $\dot{y} = -y + \lambda x_2$. If we use an insulin injection rate of $\frac{1}{\lambda}$, will y converge to a constant value? If so, which value?

Solution:
The transfer function from x_2 to y can be calculated by taking the Laplace transform of the given equation:

$$\frac{Y}{X_2}(s) = \frac{\lambda}{s+1}. \tag{50}$$

Combining this with the transfer function $\frac{X_2}{U}(s) = \frac{k}{(s+k)^2}$, we can calculate the overall transfer function:

$$\frac{Y}{U}(s) = \frac{\lambda k}{(s+1)(s+k)^2}. \tag{51}$$

With the step input of $U(s) = \frac{\lambda^{-1}}{s}$, the output is then:

$$Y(s) = \frac{k}{s(s+1)(s+k)^2}. \tag{52}$$

The poles of this function are all in the open left half plane with at most one at the origin, so the Final Value Theorem can be applied to find the steady state value of y.

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s), &(53)\\
&= \lim_{s\to 0} \frac{k}{(s+1)(s+k)^2}, &(54)\\
&= \frac{1}{k}. &(55)
\end{aligned}$$

Thus, y converges to the constant value of $\frac{1}{k}$.

(Version C) Assume now that we are interested in regulating the insulin concentration y related to x_2 by $\dot{y} = -\frac{1}{\lambda}y + x_2$. If we use an insulin injection rate of k, will y converge to a constant value? If so, which value?

Solution:
The transfer function from x_2 to y can be calculated by taking the Laplace transform of the given equation:

$$\frac{Y}{X_2}(s) = \frac{1}{s + \lambda^{-1}}. \tag{56}$$

Combining this with the transfer function $\frac{X_2}{U}(s) = \frac{k}{(s+k)^2}$, we can calculate the overall transfer function:

$$\frac{Y}{U}(s) = \frac{k}{(s + \lambda^{-1})(s + k)^2}. \tag{57}$$

With the step input of $U(s) = \frac{k}{s}$, the output is then:

$$Y(s) = \frac{k^2}{s(s + \lambda^{-1})(s + k)^2}. \tag{58}$$

The poles of this function are all in the open left half plane with at most one at the origin, so the Final Value Theorem can be applied to find the steady state value of y.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s), \tag{59}$$

$$= \lim_{s\to 0} \frac{k^2}{(s + \lambda^{-1})(s + k)^2}, \tag{60}$$

$$= \lambda. \tag{61}$$

Thus, y converges to the constant value of λ.

(Version D) Assume now that we are interested in regulating the insulin concentration y related to x_2 by $\dot{y} = -\lambda^2 y + x_2$. If we use an insulin injection rate of λk, will y converge to a constant value? If so, which value?

Solution:
The transfer function from x_2 to y can be calculated by taking the Laplace transform of the given equation:

$$\frac{Y}{X_2}(s) = \frac{1}{s + \lambda^2}. \tag{62}$$

Combining this with the transfer function $\frac{X_2}{U}(s) = \frac{k}{(s+k)^2}$, we can calculate the overall transfer function:

$$\frac{Y}{U}(s) = \frac{k}{(s + \lambda^2)(s + k)^2}. \tag{63}$$

With the step input of $U(s) = \frac{\lambda k}{s}$, the output is then:

$$Y(s) = \frac{\lambda k^2}{s(s + \lambda^2)(s + k)^2}. \tag{64}$$

The poles of this function are all in the open left half plane with at most one at the origin, so the Final Value Theorem can be applied to find the steady state value of y.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s), \tag{65}$$

$$= \lim_{s\to 0} \frac{\lambda k^2}{(s+\lambda^2)(s+k)^2}, \tag{66}$$

$$= \frac{1}{\lambda}. \tag{67}$$

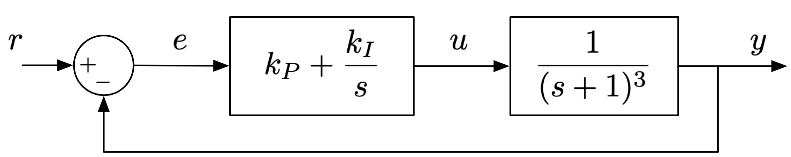
Thus, y converges to the constant value of $\frac{1}{\lambda}$.

5. Design a controller resulting in zero insulin concentration steady-state error to step inputs. Use $\lambda = k = 1$ for this question only.

Solution:
To ensure that steady-state error converges to zero, a proportional-integral controller can be used. Let the controller be of the form:

$$C(s) = k_P + \frac{k_I}{s}. \tag{68}$$

The closed-loop system now looks as shown:



The objective is to show that the closed-loop system is stable and that the error signal, $e = r - y$ converges to zero in the steady state when r is a unit step. The transfer function from the reference to the error is computed as:

$$\frac{E}{R}(s) = \frac{1}{1 + \left(k_P + \frac{k_I}{s}\right)\left(\frac{1}{(s+1)^3}\right)}, \tag{69}$$

$$= \frac{s(s+1)^3}{s^4 + 3s^3 + 3s^2 + (k_P+1)s + k_I}. \tag{70}$$

To ensure closed-loop stability, apply the Routh-Hurwitz table to the characteristic polynominal.

$$\left|\begin{array}{ccc} 1 & 3 & k_I \\ 3 & k_P+1 & \\ 8-k_P & 3k_I & \\ -k_P^2+7k_P+8-9k_I & & \\ k_I & & \end{array}\right.$$

Note that rows 3 and 4 have been multiplied by positive scalars to simplify the algebra, which does not change the stability requirements. From the last two rows, we require that:

$$0 < k_I < \frac{-k_P^2 + 7k_P + 8}{9}. \tag{71}$$

For k_I to exist, we require that $-k_P^2 + 7k_P + 8 > 0$, which is achieved when $k_P \in (-1, 8)$. The third row of the table requires that $k_P < 8$. Hence, the set of controller gains for which the closed-loop system is stable is:

$$\begin{cases} -1 < k_P < 8, \\ 0 < k_I < \frac{-k_P^2+7k_P+8}{9}. \end{cases} \tag{72}$$

For instance, picking $(k_P, k_I) = (2, 1)$ would stabilize the system. Now, we can apply the Final Value Theorem to show that the steady-state error is zero when the reference is a unit step.

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s), \tag{73}$$

$$= \lim_{s\to 0} \frac{s(s+1)^3}{s^4 + 3s^3 + 3s^2 + (k_P + 1)s + k_I}, \tag{74}$$

$$= 0. \tag{75}$$

Therefore, our controller stabilizes the closed-loop system while resulting in zero steady-state error to step inputs.

Problem 2

Solution Version A

Solution

1) By applying the block diagram algebra rules we obtain the transfer function:

$$\begin{aligned}
T(s) &= \frac{Y(s)}{R(s)} = \frac{10}{3} G_2 \frac{\frac{G_1}{1+G_1G_2}}{1+\frac{K_2}{1+K_1K_2}\frac{G_1}{1+G_1G_2}} \\
&= \frac{10}{3}\frac{G_2G_1(1+K_1K_2)}{(1+K_1K_2)(1+G_1G_2)+K_2G_1} \\
&= 30\frac{\frac{1}{s+2}\frac{1}{s+7}(1+K_1K_2)}{(1+K_1K_2)(1+\frac{1}{s+2}\frac{9}{s+7})+K_2\frac{1}{s+2}} \\
&= 30\frac{1+K_1K_2}{(1+K_1K_2)((s+2)(s+7)+9)+K_2(s+7)}.
\end{aligned} \tag{76}$$

2) The denominator polynomial of $T(s)$ is given by:

$$s^2(1+K_1K_2)+s(9+9K_1K_2+K_2)+23+23K_1K_2+7K_2,$$

and stability can be verified using the Routh–Hurwitz criterion. We first compute the corresponding table:

Routh-Hurwitz		
s^2	1+K$_1K_2$	23+23K$_1K_2+7K_2$
s^1	9+9K$_1K_2+K_2$	0
s	23+23K$_1K_2+7K_2$	0

Since all the entries in the first column need to be positive to ensure stability, we obtain the following 3 inequalities constraining the values of K_1 and K_2:

$$1+K_1K_2 > 0 \tag{77}$$
$$9+9K_1K_2+K_2 > 0 \tag{78}$$
$$23+23K_1K_2+7K_2 > 0. \tag{79}$$

The system is stable for any values of K_1 and K_2 satisfying the preceding inequalities.

3) We first address the steady state error requirement. When the input is a unit step we have $R(s) = 1/s$ and the error that results from applying this input is:

$$E(s) = [1 - T(s)]R(s).$$

In order to apply the Final Value Theorem, and compute $\lim_{t\to\infty} e(t)$, we need to ensure the poles of $sE(s)$ are stable. Since $sE(s) = 1 - T(s)$ and the denominator polynomial of $1 - T(s)$ is the same as the denominator polynomial of $T(s)$, the poles will be stable as long as inequalities (77)-(79) are satisfied. We now use the Final Value Theorem to obtain:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} 1 - T(s) = 1 - \lim_{s\to 0} T(s) = -\frac{7(1 + (K_1 - 1)K_2)}{23 + (7 + 23K_1)K_2}.$$

Hence, zero steady state error to step inputs requires the following equality to be satisfied:

$$1 + (K_1 - 1)K_2 = 0. \tag{80}$$

We now consider the overshoot and settling time requirements noticing that T has 2 poles and no zeros. By bringing the denominator polynomial:

$$s^2(1 + K_1K_2) + s(9 + 9K_1K_2 + K_2) + 23 + 23K_1K_2 + 7K_2,$$

into the form $s^2 + 2\zeta\omega_n s + \omega_n^2$ we conclude that:

$$\omega_n^2 = \frac{23 + 23K_1K_2 + 7K_2}{1 + K_1K_2}, \tag{81}$$

and:

$$2\zeta\omega_n = \frac{9 + 9K_1K_2 + K_2}{1 + K_1K_2}, \tag{82}$$

The settling time requirement provides the inequality:

$$\frac{4.6}{\zeta\omega_n} < 1 \Leftrightarrow \frac{9 + 9K_1K_2 + K_2}{1 + K_1K_2} > 2.3, \tag{83}$$

and the overshoot requirement provides:

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.03, \tag{84}$$

where ζ is obtained from (81) and (82).

In order for all the requirements to be met, the gains K_1 and K_2 must satisfy inequalities (77)-(79), (83), (84), and equality (80). If we choose $K_1 = -1$ we obtain $K_2 = 1/2$ by solving equality (80) for K_2. Therefore, this choice of gains satisfies (80). It is simple to verify that this choice also satisfies inequalities (77), (79), (83), and (84). To illustrate this point we consider (84). Substituting $K_1 = -1$ and $K_2 = 1/2$ in (81) we obtain $\omega_n = \sqrt{30}$ and using (82) leads to $\zeta = \sqrt{5/6}$. Hence $e^{-\frac{\pi\sqrt{5/6}}{\sqrt{1-5/6}}} = e^{-\sqrt{5}\pi} \approx 0.0009 < 0.03$.

Solution Version B

Solution

1) By applying the block diagram algebra rules we obtain the transfer function:

$$
\begin{aligned}
T(s) &= \frac{Y(s)}{R(s)} = 2.2G_2\frac{\frac{G_1}{1+G_1G_2}}{1+\frac{K_2}{1+K_1K_2}\frac{G_1}{1+G_1G_2}} \\
&= 2.2\frac{G_2G_1(1+K_1K_2)}{(1+K_1K_2)(1+G_1G_2)+K_2G_1} \\
&= 22\frac{\frac{1}{s+3}\frac{1}{s+3}(1+K_1K_2)}{(1+K_1K_2)(1+\frac{1}{s+3}\frac{10}{s+3})+K_2\frac{1}{s+2}} \\
&= 22\frac{1+K_1K_2}{(1+K_1K_2)((s+3)(s+3)+10)+K_2(s+3)}.
\end{aligned} \tag{85}
$$

2) The denominator polynomial of $T(s)$ is given by:

$$s^2(1+K_1K_2)+s(6+6K_1K_2+K_2)+19+19K_1K_2+3K_2,$$

and stability can be verified using the Routh–Hurwitz criterion. We first compute the corresponding table:

Routh-Hurwitz		
s^2	$1+\mathrm{K}_1K_2$	$19+19\mathrm{K}_1K_2+3K_2$
s^1	$6+6\mathrm{K}_1K_2+K_2$	0
s	$19+19\mathrm{K}_1K_2+3K_2$	0

Since all the entries in the first column need to be positive to ensure stability, we obtain the following 3 inequalities constraining the values of K_1 and K_2:

$$1+K_1K_2 > 0 \tag{86}$$
$$6+6K_1K_2+K_2 > 0 \tag{87}$$
$$19+19K_1K_2+3K_2 > 0. \tag{88}$$

The system is stable for any values of K_1 and K_2 satisfying the preceding inequalities.

3) We first address the steady state error requirement. When the input is a unit step we have $R(s) = 1/s$ and the error that results from applying this input is:

$$E(s) = [1 - T(s)]R(s).$$

In order to apply the Final Value Theorem, and compute $\lim_{t\to\infty} e(t)$, we need to ensure the poles of $sE(s)$ are stable. Since $sE(s) = 1 - T(s)$ and the denominator polynomial of $1 - T(s)$ is the same as the denominator polynomial of $T(s)$, the poles will be stable as long as inequalities (86)-(88) are satisfied. We now use the Final Value Theorem to obtain:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} 1 - T(s) = 1 - \lim_{s\to 0} T(s) = -\frac{3(1 + (K_1 - 1)K_2)}{19 + (3 + 19K_1)K_2}.$$

Hence, zero steady state error to step inputs requires the following equality to be satisfied:

$$1 + (K_1 - 1)K_2 = 0. \tag{89}$$

We now consider the overshoot and settling time requirements noticing that T has 2 poles and no zeros. By bringing the denominator polynomial:

$$s^2(1 + K_1K_2) + s(6 + 6K_1K_2 + K_2) + 19 + 19K_1K_2 + 3K_2,$$

into the form $s^2 + 2\zeta\omega_n s + \omega_n^2$ we conclude that:

$$\omega_n^2 = \frac{19 + 19K_1K_2 + 3K_2}{1 + K_1K_2}, \tag{90}$$

and:

$$2\zeta\omega_n = \frac{6 + 6K_1K_2 + K_2}{1 + K_1K_2}, \tag{91}$$

The settling time requirement provides the inequality:

$$\frac{4.6}{\zeta\omega_n} < 1.5 \Leftrightarrow \frac{6 + 6K_1K_2 + K_2}{1 + K_1K_2} > 9.2/1.5, \tag{92}$$

and the overshoot requirement provides:

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.05, \tag{93}$$

where ζ is obtained from (81) and (91).

In order for all the requirements to be met, the gains K_1 and K_2 must satisfy inequalities (86)-(88), (92), (84), and equality (80). If we choose $K_1 = -1$ we obtain $K_2 = 1/2$ by solving equality (89) for K_2. Therefore, this choice of gains satisfies (89). It is simple to verify that this choice also satisfies inequalities (86), (88), (92), and (93). To illustrate this point we consider (93). Substituting $K_1 = -1$ and $K_2 = 1/2$ in (90) we obtain $\omega_n = \sqrt{22}$ and using (91) leads to $\zeta = \frac{7}{2\sqrt{22}}$. Hence $e^{-\frac{\pi 7/2\sqrt{22}}{\sqrt{1-49/88}}} = e^{-\sqrt{5}\pi} \approx 0.029 < 0.05$.

Solution Version C

Solution

1) By applying the block diagram algebra rules we obtain the transfer function:

$$\begin{aligned}
T(s) &= \frac{Y(s)}{R(s)} = 2.5G_2\frac{\frac{G_1}{1+G_1G_2}}{1+\frac{K_2}{1+K_1K_2}\frac{G_1}{1+G_1G_2}} \\
&= 2.5\frac{G_2G_1(1+K_1K_2)}{(1+K_1K_2)(1+G_1G_2)+K_2G_1} \\
&= 20\frac{\frac{1}{s+3}\frac{1}{s+2}(1+K_1K_2)}{(1+K_1K_2)(1+\frac{1}{s+2}\frac{8}{s+3})+K_2\frac{1}{s+2}} \\
&= 20\frac{1+K_1K_2}{(1+K_1K_2)((s+2)(s+3)+8)+K_2(s+3)}. \qquad (94)
\end{aligned}$$

2) The denominator polynomial of $T(s)$ is given by:

$$s^2(1+K_1K_2)+s(5+5K_1K_2+K_2)+14+14K_1K_2+3K_2,$$

and stability can be verified using the Routh–Hurwitz criterion. We first compute the corresponding table:

Routh-Hurwitz		
s^2	$1+K_1K_2$	$14+14K_1K_2+2K_2$
s^1	$5+5K_1K_2+K_2$	0
s	$14+14K_1K_2+2K_2$	0

Since all the entries in the first column need to be positive to ensure stability, we obtain the following 3 inequalities constraining the values of K_1 and K_2:

$$\begin{aligned}
1+K_1K_2 &> 0 \qquad (95)\\
5+5K_1K_2+K_2 &> 0 \qquad (96)\\
14+14K_1K_2+3K_2 &> 0. \qquad (97)
\end{aligned}$$

The system is stable for any values of K_1 and K_2 satisfying the preceding inequalities.

3) We first address the steady state error requirement. When the input is a unit step we have $R(s) = 1/s$ and the error that results from applying this input is:

$$E(s) = [1 - T(s)]R(s).$$

In order to apply the Final Value Theorem, and compute $\lim_{t\to\infty} e(t)$, we need to ensure the poles of $sE(s)$ are stable. Since $sE(s) = 1 - T(s)$ and the denominator polynomial of $1 - T(s)$ is the same as the denominator polynomial of $T(s)$, the poles will be stable as long as inequalities (95)-(97) are satisfied. We now use the Final Value Theorem to obtain:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} 1 - T(s) = 1 - \lim_{s\to 0} T(s) = -\frac{6(1 + (K_1 - \frac{1}{2})K_2)}{14 + (3 + 14K_1)K_2}.$$

Hence, zero steady state error to step inputs requires the following equality to be satisfied:

$$1 + (K_1 - \frac{1}{2})K_2 = 0. \tag{98}$$

We now consider the overshoot and settling time requirements noticing that T has 2 poles and no zeros. By bringing the denominator polynomial:

$$s^2(1 + K_1K_2) + s(5 + 5K_1K_2 + K_2) + 14 + 14K_1K_2 + 3K_2,$$

into the form $s^2 + 2\zeta\omega_n s + \omega_n^2$ we conclude that:

$$\omega_n^2 = \frac{14 + 14K_1K_2 + 3K_2}{1 + K_1K_2}, \tag{99}$$

and:

$$2\zeta\omega_n = \frac{5 + 5K_1K_2 + K_2}{1 + K_1K_2}, \tag{100}$$

The settling time requirement provides the inequality:

$$\frac{4.6}{\zeta\omega_n} < 1.5 \Leftrightarrow \frac{5 + 5K_1K_2 + K_2}{1 + K_1K_2} > 9.2/1.5, \tag{101}$$

and the overshoot requirement provides:

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.03, \tag{102}$$

where ζ is obtained from (99) and (100).

In order for all the requirements to be met, the gains K_1 and K_2 must satisfy inequalities (95)-(97), (101), (102), and equality (98). If we choose $K_1 = -1$ we obtain $K_2 = 2/3$ by solving equality (98) for K_2. Therefore, this choice of gains satisfies (98). It is simple to verify that this choice also satisfies inequalities (95), (97), (101), and (102). To illustrate this point we consider (102). Substituting $K_1 = -1$ and $K_2 = 2/3$ in (99) we obtain $\omega_n = \sqrt{20}$ and using (100) leads to $\zeta = \frac{7}{2\sqrt{20}}$. Hence $e^{-\frac{\pi 7/2\sqrt{20}}{\sqrt{1-49/80}}} = e^{-\sqrt{5}\pi} \approx 0.019 < 0.03$.

Solution Version D

Solution

1) By applying the block diagram algebra rules we obtain the transfer function:

$$
\begin{aligned}
T(s) &= \frac{Y(s)}{R(s)} = \frac{27}{7} G_2 \frac{\frac{G_1}{1+G_1G_2}}{1 + \frac{K_2}{1+K_1K_2}\frac{G_1}{1+G_1G_2}} \\
&= \frac{27}{7}\frac{G_2G_1(1+K_1K_2)}{(1+K_1K_2)(1+G_1G_2)+K_2G_1} \\
&= 27\frac{\frac{1}{s+1}\frac{1}{s+5}(1+K_1K_2)}{(1+K_1K_2)(1+\frac{1}{s+1}\frac{7}{s+5})+K_2\frac{1}{s+2}} \\
&= 27\frac{1+K_1K_2}{(1+K_1K_2)((s+1)(s+5)+7)+K_2(s+5)}. \qquad (103)
\end{aligned}
$$

2) The denominator polynomial of $T(s)$ is given by:

$$s^2(1+K_1K_2) + s(6+6K_1K_2+K_2) + 12 + 12K_1K_2 + 5K_2,$$

and stability can be verified using the Routh–Hurwitz criterion. We first compute the corresponding table:

Routh-Hurwitz		
s^2	1+K$_1K_2$	12+12K$_1K_2+5K_2$
s^1	6+6K$_1K_2+K_2$	0
s	12+12K$_1K_2+5K_2$	0

Since all the entries in the first column need to be positive to ensure stability, we obtain the following 3 inequalities constraining the values of K_1 and K_2:

$$
\begin{aligned}
1+K_1K_2 &> 0 \qquad (104)\\
6+6K_1K_2+K_2 &> 0 \qquad (105)\\
12+12K_1K_2+5K_2 &> 0. \qquad (106)
\end{aligned}
$$

The system is stable for any values of K_1 and K_2 satisfying the preceding inequalities.

3) We first address the steady state error requirement. When the input is a unit step we have $R(s) = 1/s$ and the error that results from applying this input is:

$$E(s) = [1 - T(s)]R(s).$$

In order to apply the Final Value Theorem, and compute $\lim_{t\to\infty} e(t)$, we need to ensure the poles of $sE(s)$ are stable. Since $sE(s) = 1 - T(s)$ and the denominator polynomial of $1 - T(s)$ is the same as the denominator polynomial of $T(s)$, the poles will be stable as long as inequalities (104)-(106) are satisfied. We now use the Final Value Theorem to obtain:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} 1 - T(s) = 1 - \lim_{s\to 0} T(s) = -\frac{15(1 + (K_1 - \frac{1}{3})K_2)}{12 + (5 + 12K_1)K_2}.$$

Hence, zero steady state error to step inputs requires the following equality to be satisfied:

$$1 + (K_1 - \frac{1}{3})K_2 = 0. \tag{107}$$

We now consider the overshoot and settling time requirements noticing that T has 2 poles and no zeros. By bringing the denominator polynomial:

$$s^2(1 + K_1K_2) + s(6 + 6K_1K_2 + K_2) + 12 + 12K_1K_2 + 5K_2,$$

into the form $s^2 + 2\zeta\omega_n s + \omega_n^2$ we conclude that:

$$\omega_n^2 = \frac{12 + 12K_1K_2 + 5K_2}{1 + K_1K_2}, \tag{108}$$

and:

$$2\zeta\omega_n = \frac{6 + 6K_1K_2 + K_2}{1 + K_1K_2}, \tag{109}$$

The settling time requirement provides the inequality:

$$\frac{4.6}{\zeta\omega_n} < 1.3 \Leftrightarrow \frac{6 + 6K_1K_2 + K_2}{1 + K_1K_2} > 9.2/1.3, \tag{110}$$

and the overshoot requirement provides:

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.03, \tag{111}$$

where ζ is obtained from (108) and (109).

In order for all the requirements to be met, the gains K_1 and K_2 must satisfy inequalities (104)-(106), (110), (111), and equality (107). If we choose $K_1 = -1$ we obtain $K_2 = 3/4$ by solving equality (107) for K_2. Therefore, this choice of gains satisfies (107). It is simple to verify that this choice also satisfies inequalities (104), (106), (110), and (111). To illustrate this point we consider (111). Substituting $K_1 = -1$ and $K_2 = 3/4$ in (108) we obtain $\omega_n = \sqrt{27}$ and using (109) leads to $\zeta = \frac{9}{2\sqrt{27}}$. Hence $e^{-\frac{\pi 9/2\sqrt{27}}{\sqrt{1-81/108}}} = e^{-\sqrt{5}\pi} \approx 0.004 < 0.03$.