Homework 5

Solutions

May 11, 2021

Problems: 5.5c, 5.5e, 5.7c, 5.7e, 5.8e

5.5c: By completing the square in both the numerator and denominator of the loop transfer function, we see that

$$L(s) = \frac{s^2+2s+8}{s(s^2+2s+19)} = \frac{(s+1)^2+7}{s[(s+1)^2+9]}. \tag{1}$$

By inspection, the transfer function has three poles: 0, $-1 \pm 3j$, and two zeros: $-1 \pm \sqrt{7}j$. Since there is only one pole on the real axis, the portion to the left of the pole at 0 must belong to the root locus. The departure angle of this pole confirms this conclusion:

$$\phi_1 = \text{atan2}(-\sqrt{7}, 1) + \text{atan2}(\sqrt{7}, 1) - \text{atan2}(-3, 1) - \text{atan2}(3, 1) + 180^\circ = 180^\circ. \tag{2}$$

To determine the remaining branches, we only need to consider the pole and zero in the upper half plane as the root locus is symmetric about the real axis. The departure and arrival angles of the pole and zero, respectively, are:

$$\phi_2 = 90^\circ + 90^\circ - 90^\circ - \text{atan2}(3, -1) + 180^\circ \approx 162^\circ, \tag{3}$$

$$\psi_1 = -90^\circ + 90^\circ + \text{atan2}(\sqrt{7}, -1) - 90^\circ + 180^\circ \approx 201^\circ. \tag{4}$$

Using these angles, the root locus for $L(s)$ can be drawn.

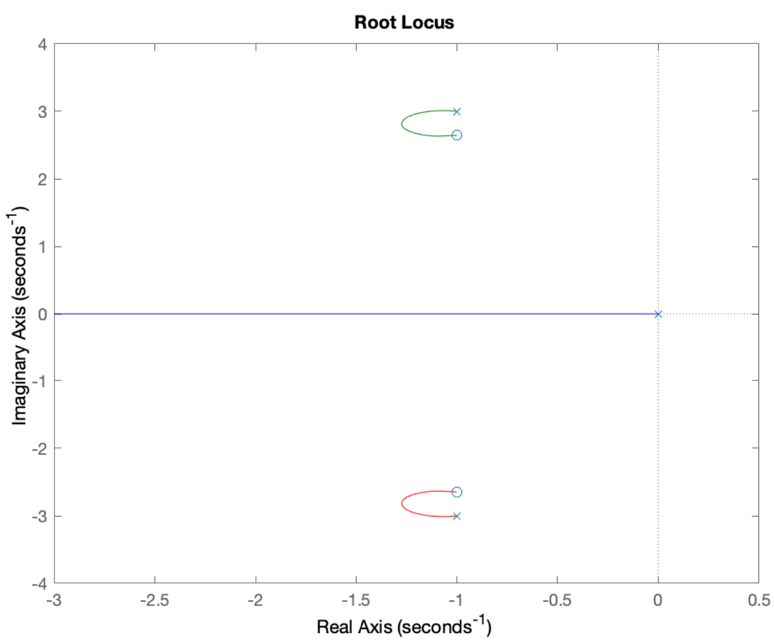

Figure 1: Root locus for $L(s) = \frac{s^2+2s+8}{s(s^2+2s+19)}$.

5.5e:

$$L(s) = \frac{s^2+1}{s(s^2+4)}. \tag{5}$$

By inspection, the loop transfer function has three poles: 0, $\pm 2j$, and two zeros: $\pm j$. Since there is only one pole on the real axis, the part of the real axis to the left of the pole must be a part of the root locus. The departure angle of the pole at the origin is:

$$\phi_1 = 90^\circ - 90^\circ + 90^\circ - 90^\circ + 180^\circ = 180^\circ, \tag{6}$$

which confirms this conclusion. The departure angle for the pole at $2j$ is:

$$\phi_2 = 90^\circ + 90^\circ - 90^\circ - 90^\circ + 180^\circ = 180^\circ. \tag{7}$$

The arrival angle for the zero at j is:

$$\psi_1 = -90^\circ + 90^\circ + 90^\circ - 90^\circ + 180^\circ = 180^\circ. \tag{8}$$

Using these angles, the root locus for $L(s)$ can be drawn as shown.

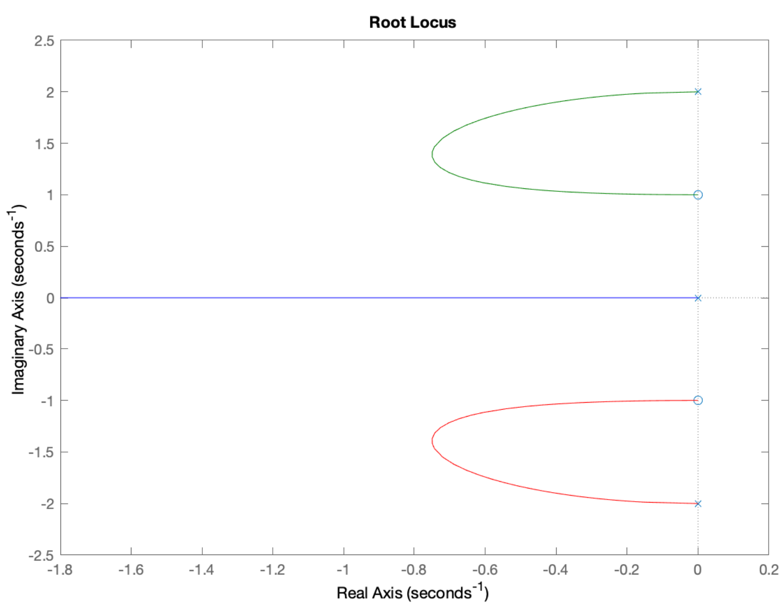

Figure 2: Root locus for $L(s) = \frac{s^2+1}{s(s^2+4)}$.

5.7c:

$$L(s) = \frac{(s+3)^2}{s^2(s+10)(s^2+6s+25)}. \tag{9}$$

The loop transfer function has five poles: 0, 0, -10, $-3 \pm 4j$, and two zeros: -3, -3. The center point $s = \alpha$ can be calculated as:

$$\alpha = \frac{[0+0+(-10)+(3+4j)+(3-4j)] - [(-3)+(-3)]}{5-2} = -\frac{10}{3}. \tag{10}$$

There are 3 asymptotic branches that are 120° apart. The portion of the real axis to the left of the pole at -10 has an odd number of poles and zeros to its right on the real axis, so that portion belongs on the real axis. To determine the other branches of the root locus, first consider the pole at $-3+4j$. Its departure angle can be calculated as:

$$\begin{aligned} \phi_1 &= 90^\circ + 90^\circ - \text{atan2}(4,-3) - \text{atan2}(4,-3) - 90^\circ - \text{atan2}(4,7) + 180^\circ, && (11) \\ &\approx -13^\circ. && (12) \end{aligned}$$

For the double pole at the origin, their departure angles can be calculated as:

$$\begin{aligned} 2\phi_2 &= 0^\circ + 0^\circ - \text{atan2}(3,-4) - 0^\circ - \text{atan2}(3,4) + 180^\circ, && (13) \\ \phi_2 &= 90^\circ, -90^\circ. && (14) \end{aligned}$$

Finally, the arrival angles for the double zero at -3 are calculated:

$$\begin{aligned} 2\psi_1 &= 180^\circ + 180^\circ - 90^\circ + 90^\circ + 0^\circ - 180^\circ, && (15) \\ \psi_1 &= 90^\circ, -90^\circ. && (16) \end{aligned}$$

Putting everything together, the root locus for $L(s)$ is drawn as shown.

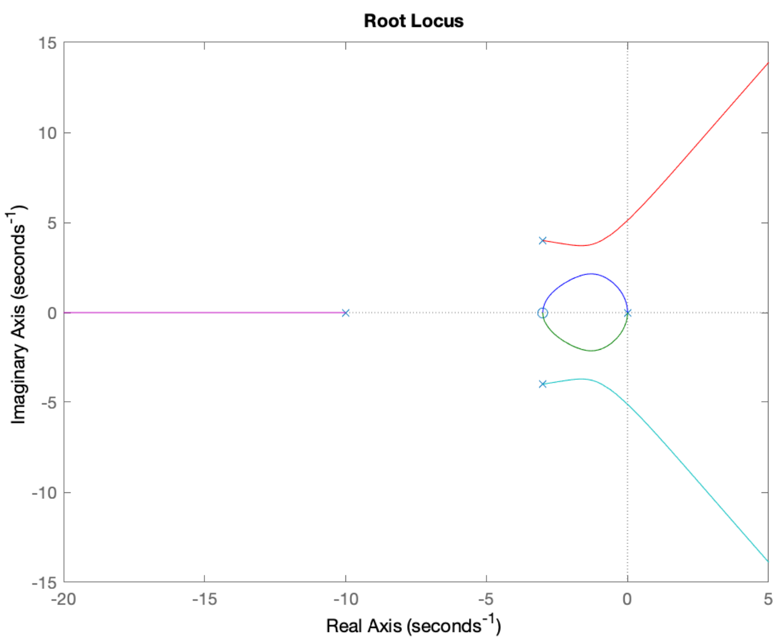

Figure 3: Root locus for $L(s) = \frac{(s+3)^2}{s^2(s+10)(s^2+6s+25)}$.

5.7e:

$$L(s) = \frac{(s+1)^2+1}{s^2(s+2)(s+3)}. \tag{17}$$

By inspection, there are four poles: 0, 0, -2, -3, and two zeros: $-1 \pm j$. The center point is calculated as:

$$\alpha = \frac{[0+0+(-2)+(-3)]-[(-1+j)+(-1-j)]}{4-2} = -\frac{3}{2}. \tag{18}$$

As there are only two asymptotic branches, they are 180° apart and run perpendicular to the real axis. The region between the poles at -3 and -2 belong to the root locus due to the odd number of poles to the right. The departure angles for the poles at -3 and -2, respectively, are:

$$\phi_2 = \text{atan2}(-1,-1) + \text{atan2}(1,-1) - 180^\circ - 180^\circ - 0^\circ + 3(180^\circ) = -180^\circ, \tag{19}$$

$$\phi_3 = \text{atan2}(-1,-2) + \text{atan2}(1,-2) - 180^\circ - 180^\circ - 180^\circ + 3(180^\circ) = 0^\circ. \tag{20}$$

Furthermore, the between these poles is a branching point from which the poles approach the two vertical asymptotes. The departure angle for the double pole at the origin can be calculated as:

$$2\phi_1 = \text{atan2}(-1,1) + \text{atan2}(1,1) - 0^\circ - 0^\circ + 180^\circ, \tag{21}$$

$$\phi_1 = 90^\circ, -90^\circ. \tag{22}$$

The arrival angles for the zero in the upper half plane is:

$$\psi_1 = \text{atan2}(1,-1) + \text{atan2}(1,-1) + \text{atan2}(1,1) + \text{atan2}(1,2) - 90^\circ - 180^\circ, \tag{23}$$

$$\approx 72^\circ. \tag{24}$$

Using these conclusions, the root locus can be drawn.

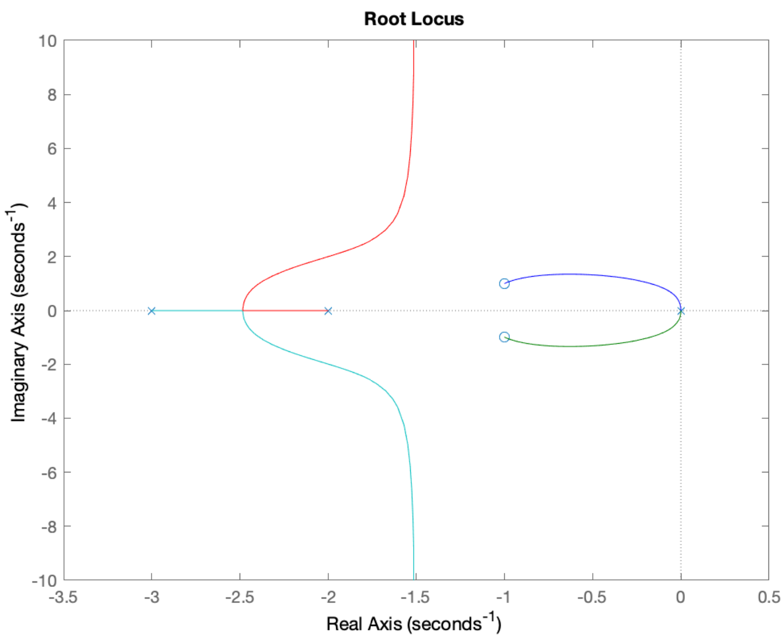

Figure 4: Root locus for $L(s) = \frac{(s+1)^2+1}{s^2(s+2)(s+3)}$.

5.8e:

$$L(s) = \frac{s+2}{s(s-1)(s+6)^2}. \tag{25}$$

The loop transfer function has four poles: -6, -6, 0, 1, and one zero at -2. The center point is then calculated to be:

$$\alpha = \frac{[0 + 1 + (-6) + (-6)] - (-2)}{4 - 1} = -3. \tag{26}$$

Since there are three more poles than zeros, there are three asymptotes $120°$ apart from each other. The region located between the poles at 0 and 1 belong to the root locus because there is an odd number of poles and zeros to the right of the region. The departure angles for the two poles confirm the result (departure angle for pole at 1 shown first):

$$\phi_1 = 0° - 0° - 0° - 0° + 180° = 180°, \tag{27}$$

$$\phi_2 = 0° - 180° - 0° - 0° + 180° = 0°. \tag{28}$$

In this region there is also a branching point as these these two poles approach two of the asymptotes. Next, the departure angles for the double pole at -6 can be calculated as:

$$2\phi_3 = 180° - 180° - 180° + 180°, \tag{29}$$

$$\phi_3 = 0°, 180°, \tag{30}$$

where the branch to the left is along one of the asymptotes. As the region between the poles at -6 and the zero at -2 has an odd number of poles and zeros on the real axis to its right, it belongs to the root locus. This can also be seen by calculating the arrival angle of the zero:

$$\psi_1 = 180° + 180° + 0° + 0° - 180° = 180°. \tag{31}$$

Putting all of these conclusions together, the root locus for $L(s)$ can be drawn.

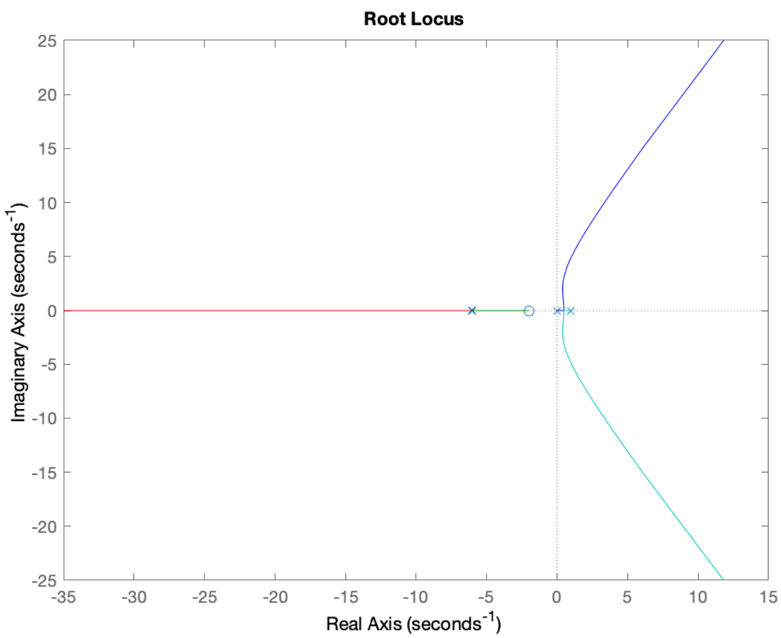

Figure 5: Root locus for $L(s) = \frac{s+2}{s(s-1)(s+6)^2}$.