Homework 3

Solutions

April 27, 2021

Problems: 3.53a, 3.53b, 3.54a, 3.54c, 3.57

3.53a: The closed-loop transfer function is:

$$\frac{KG(s)}{1+KG(s)}, \tag{1}$$

so the denominator of the transfer function is simplified to:

$$s^4 + 2s^3 + 3s^2 + 8s + 8. \tag{2}$$

Applying the Routh-Hurwitz criterion, we build the table:

$$\begin{array}{|lll} 1 & 3 & 8 \\ 2 & 8 & \\ a & b & \\ c & & \\ d & & \\ 0 & & \end{array}$$

The terms are calculated as:

$$a = -\frac{1}{2}\det\begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} = -1, \tag{3}$$

$$b = -\frac{1}{2}\det\begin{bmatrix} 1 & 8 \\ 2 & 0 \end{bmatrix} = 8, \tag{4}$$

$$c = -\frac{1}{a}\det\begin{bmatrix} 2 & 8 \\ a & b \end{bmatrix} = 24, \tag{5}$$

$$d = -\frac{1}{c}\det\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = 8. \tag{6}$$

The Routh-Hurwitz table now becomes:

$$\begin{array}{|ccc} 1 & 3 & 8 \\ 2 & 8 & \\ -1 & 8 & \\ 24 & & \\ 8 & & \\ 0 & & \end{array}$$

Since there are two sign changes in the first column, two of the poles are in the RHP. Hence, the closed-loop system is unstable.

3.53b: Applying similar calculations as the previous question, the characteristic polynomial of the closed-loop transfer function is:

$$s^3 + s^2 + 2s + 8. \tag{7}$$

The Routh-Hurwitz table for this polynominal becomes:

$$\begin{array}{|cc} 1 & 2 \\ 1 & 8 \\ a & \\ b & \\ 0 & \end{array}$$

The two terms are computed as:

$$a = -\frac{1}{1}\det\begin{bmatrix}1 & 2\\ 1 & 8\end{bmatrix} = -6, \tag{8}$$

$$b = -\frac{1}{a}\det\begin{bmatrix}1 & 8\\ a & 0\end{bmatrix} = 8. \tag{9}$$

The Routh-Hurwitz table is then:

$$\begin{array}{|cc} 1 & 2 \\ 1 & 8 \\ -6 & \\ 8 & \\ 0 & \end{array}$$

Since there are two sign changes in the first column, two of the poles are in the RHP. Hence, the closed-loop system is unstable.

3.54a: Using the polynominal $s^4 + 8s^3 + 32s^2 + 80s + 100$, the Routh-Hurwitz table is constructed as:

$$\begin{array}{|ccc} 1 & 32 & 100 \\ 8 & 80 & \\ a & b & \\ c & & \\ d & & \\ 0 & & \end{array}$$

The entries are calculated as:

$$a = -\frac{1}{8}\det\begin{bmatrix}1 & 32\\ 8 & 80\end{bmatrix} = 22, \tag{10}$$

$$b = -\frac{1}{8}\det\begin{bmatrix}1 & 100\\ 8 & 0\end{bmatrix} = 100, \tag{11}$$

$$c = -\frac{1}{a}\det\begin{bmatrix}8 & 80\\ a & b\end{bmatrix} = \frac{960}{22}, \tag{12}$$

$$d = -\frac{1}{c}\det\begin{bmatrix}a & b\\ c & 0\end{bmatrix} = 100. \tag{13}$$

This gives the table as:

$$\begin{array}{|ccc} 1 & 32 & 100 \\ 8 & 80 & \\ 22 & 100 & \\ \frac{960}{22} & & \\ 100 & & \\ 0 & & \end{array}$$

Since there are no sign changes in the first column, the polynominal has no roots with positive real part.

3.54c: Using the given polynominal $s^4 + 2s^3 + 7s^2 - 2s + 8$, the Routh-Hurwitz table is:

$$\begin{array}{|ccc} 1 & 7 & 8 \\ 2 & -2 & \\ a & b & \\ c & & \\ d & & \\ 0 & & \end{array}$$

The entries are calculated as:

$$a = -\frac{1}{2} \det \begin{bmatrix} 1 & 7 \\ 2 & -2 \end{bmatrix} = 8, \tag{14}$$

$$b = -\frac{1}{2} \det \begin{bmatrix} 1 & 8 \\ 2 & 0 \end{bmatrix} = 8, \tag{15}$$

$$c = -\frac{1}{a} \det \begin{bmatrix} 2 & -2 \\ a & b \end{bmatrix} = -4, \tag{16}$$

$$d = -\frac{1}{c} \det \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = 8. \tag{17}$$

This gives the table as:

$$\begin{array}{|ccc} 1 & 7 & 8 \\ 2 & -2 & \\ 8 & 8 & \\ -4 & & \\ 8 & & \\ 0 & & \end{array}$$

Since there are two sign changes in the first column, there are two poles in the RHP.

3.57: The closed-loop transfer function can be calculated as:

$$\frac{KK_o(s+z)}{s^3 + ps^2 + (KK_o - a^2)s + (KK_oz - a^2p)}. \tag{18}$$

Using the denominator, the Routh-Hurwitz table is constructed as:

$$\begin{array}{|cc} 1 & KK_o - a^2 \\ p & KK_oz - a^2p \\ b & \\ c & \\ 0 & \end{array}$$

The entries are calculated as:

$$b = \frac{p(KK_o - a^2) - (KK_oz - a^2p)}{p} = \frac{KK_o(p-z)}{p}, \tag{19}$$

$$c = KK_oz - a^2p. \tag{20}$$

For the closed-loop system to be stable, the first column of the Routh-Hurwitz table needs to be all positive. Thus, the conditions for stability are:

$$\begin{cases} p > 0, \\ KK_o(p-z) > 0, \\ KK_oz - a^2p > 0. \end{cases} \tag{21}$$