Homework 1

Solutions

April 13, 2021

Problems: 2.2, 2.12, 3.3a, 3.3b, 3.7f, 3.7j, 3.8b, 3.9b, 3.9c

2.2: For each of the two masses, we sum together all of the force contributions.

$$m_1\ddot{x}_1 = -K_1x_1 + K_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1), \tag{1}$$
$$m_1\ddot{x}_2 = -K_1x_2 - K_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1). \tag{2}$$

The dampener is the only element in the system that dissipates energy. Since the dampener is located between the two masses, any relative motion between the masses will decrease to zero. However, the two masses can still oscillate indefinitely as one entity.

2.12: Op-amp is ideal:
In this case, the equation governing the op-amp is $V_{out} = A(v_+ - v_-)$, where $A \to \infty$. Substituting in $v_+ = V_{in}$ and $v_- = V_{out}$, we get that:

$$V_{out} = A(V_{in} - V_{out}), \tag{3}$$
$$(A+1)V_{out} = AV_{in}, \tag{4}$$
$$V_{out} = \frac{A}{A+1}V_{in}. \tag{5}$$

Since $\lim_{A\to\infty} \frac{A}{A+1} = 1$, this results in $V_{out} = V_{in}$.

Op-amp is not ideal:
In this case, the equation of the op-amp is $V_{out} = \frac{10^7}{s+1}(v_+ - v_-)$. Carrying out the analysis:

$$V_{out} = \frac{10^7}{s+1}(V_{in} - V_{out}), \tag{6}$$
$$V_{out} = \frac{\frac{10^7}{s+1}}{\frac{10^7}{s+1} + 1}V_{in}, \tag{7}$$
$$V_{out} = \frac{10^7}{s + 10^7 + 1}V_{in}. \tag{8}$$

3.3a: Using the result $\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$ and linearity, we get:

$$F(s) = \frac{4s}{s^2 + 36}. \tag{9}$$

3.3b: Using the result $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$, frequency shifting: $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$, and linearity, we have:

$$F(s) = \frac{3}{s^2+9} + \frac{2s}{s^2+9} + \frac{3}{(s+1)^2+9}. \tag{10}$$

3.7f: We can write the given Laplace transform as:

$$F(s) = \frac{2(s+3)}{(s+1)(s^2+16)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+16}. \tag{11}$$

The goal is to calculate the constants A, B, C. By combining the two fractions, the numerator must satisfy:

$$A(s^2+16) + (Bs+C)(s+1) = 2s+6. \tag{12}$$

To solve for the three unknowns, we can use the three equations from the coefficients of the powers of s:

$$s^2: \quad A + B = 0, \tag{13}$$
$$s: \quad B + C = 2, \tag{14}$$
$$1: \quad 16A + C = 6. \tag{15}$$

Solving this linear system of equations, we get that $A = \frac{4}{17}, B = -\frac{4}{17}, C = \frac{38}{17}$. Hence,

$$F(s) = \frac{\frac{4}{17}}{s+1} + \frac{-\frac{4}{17}s + \frac{38}{17}}{s^2+16}, \tag{16}$$
$$= \frac{4}{17}\frac{1}{s+1} - \frac{4}{17}\frac{s}{s^2+16} + \left(\frac{38}{17}\right)\left(\frac{1}{4}\right)\frac{4}{s^2+16}, \tag{17}$$
$$\mathcal{L}^{-1}\{F\}(t) = \left(\frac{4}{17}e^{-t} - \frac{4}{17}\cos(4t) + \frac{19}{34}\sin(4t)\right)u(t). \tag{18}$$

3.7j: Note that $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = tu(t)$. Using the frequency shift property, we have that:

$$\mathcal{L}^{-1}\{F\}(t) = (t-1)u(t-1). \tag{19}$$

3.8b: Note that we can factor the denominator as $s^3 - 1 = (s-1)(s^2+s+1)$. Then,

$$F(s) = \frac{s^2+s+1}{s^3-1}, \tag{20}$$
$$= \frac{s^2+s+1}{(s-1)(s^2+s+1)}, \tag{21}$$
$$= \frac{1}{s-1}, \tag{22}$$
$$\mathcal{L}^{-1}\{F\}(t) = e^t u(t). \tag{23}$$

3.9b: Apply the Laplace transform to both sides of the equation, and isolate for $Y(s) := \mathcal{L}\{Y\}(t)$:

$$s^2Y(s) - sy(0) - \dot{y}(0) - 2sY(s) + 2y(0) + 4Y(s) = 0, \tag{24}$$
$$(s^2 - 2s + 4)Y(s) = sy(0) + \dot{y}(0) - 2y(0). \tag{25}$$

Substituting in the initial conditions $y(0) = 1$ and $\dot{y}(0) = 2$:

$$
\begin{aligned}
Y(s) &= \frac{s}{(s-1)^2+3}, && (26)\\
&= \frac{s-1+1}{(s-1)^2+3}, && (27)\\
&= \frac{s-1}{(s-1)^2+3} + \frac{\sqrt{3}}{3}\frac{\sqrt{3}}{(s-1)^2+3}, && (28)\\
y(t) &= e^t\cos\left(\sqrt{3}t\right) + \frac{\sqrt{3}}{3}e^t\sin\left(\sqrt{3}t\right). && (29)
\end{aligned}
$$

3.9c: Similar to before, apply the Laplace transform to both sides of the equation, substitute in the initial conditions, and isolate for $Y(s)$:

$$
\begin{aligned}
s^2Y(s) - sy(0) - \dot{y}(0) + sY(s) - y(0) &= \frac{1}{s^2+1}, && (30)\\
(s^2+s)Y(s) &= \frac{1}{s^2+1} + s + 2 + 1, && (31)\\
s(s+1)Y(s) &= \frac{s^3+3s^2+s+4}{s^2+1}, && (32)\\
Y(s) &= \frac{s^3+3s^2+s+4}{s(s+1)(s^2+1)}. && (33)
\end{aligned}
$$

Now, use partial fractions to split the denominator of $Y(s)$:

$$
Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}. \tag{34}
$$

The numerator satisfies the equation:

$$
A(s^3+s^2+s+1) + B(s^3+s) + (Cs+D)(s^2+s) = s^3+3s^2+s+4. \tag{35}
$$

The four equations that can be used to solve for the four unknowns are:

$$
\begin{aligned}
s^3:&\quad A+B+C = 1, && (36)\\
s^2:&\quad A+C+D = 3, && (37)\\
s:&\quad A+B+D = 1, && (38)\\
1:&\quad A = 4. && (39)
\end{aligned}
$$

Solving this linear system of equations yields: $A = 4, B = -\frac{5}{2}, C = -\frac{1}{2}, D = -\frac{1}{2}$. Then,

$$
\begin{aligned}
Y(s) &= \frac{4}{s} - \frac{\frac{5}{2}}{s+1} - \frac{\frac{1}{2}s+\frac{1}{2}}{s^2+1}, && (40)\\
&= \frac{4}{s} - \frac{5}{2}\frac{1}{s+1} - \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1}, && (41)\\
y(t) &= 4 - \frac{5}{2}e^{-t} - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t). && (42)
\end{aligned}
$$