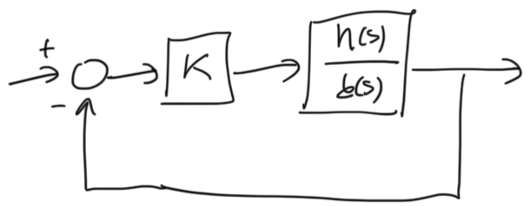
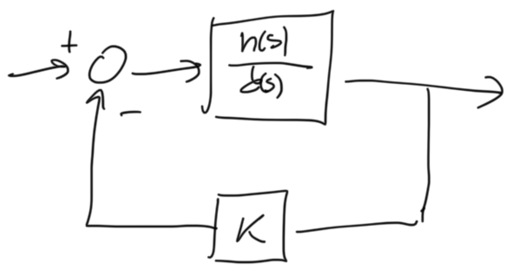
ECE 141

Lecture 9

Root Locus



$$\frac{K\frac{n}{d}}{1+K\frac{n}{d}} = \frac{Kn}{d+Kn}$$



$$\frac{\frac{n}{d}}{1+K\frac{n}{d}} = \frac{n}{d+Kn}$$

$$1+KL(s)=0 = 1+K\frac{n(s)}{d(s)}$$

Def: The root locus is the set of all values of s for which $1+KL(s)=0$ is satisfied as the parameter K varies from 0 to $+\infty$.

$$1+KL(s)=0 \iff KL(s)=-1$$

Assuming $K>0$ $\quad \angle L(s) = 180^\circ$

$$1+KL(s)=0 \implies \angle L(s) = 180^\circ$$

$$\angle L(s) = 180^\circ \implies 1+KL(s)=0 \;? \quad \text{for some } K>0$$

$$\angle L(s) = 180 \implies L(s) = -\alpha + 0j$$

$\alpha > 0$

$$1 + K(-\alpha + 0j) = 0$$

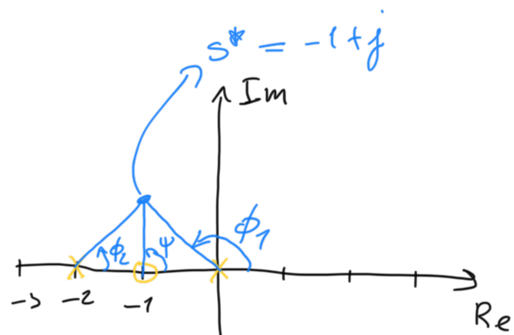
$$-\alpha K = -1$$

$$K = \frac{1}{\alpha}$$

$$1 + K L(s) = 0 \iff \angle L(s) = 180^\circ$$

$$L(s) = \frac{s+1}{s(s+2)}$$

$$s^* = -1 + j$$



$$\phi_1 = 90^\circ + 45^\circ$$

$$\phi_2 = 45^\circ$$

$$\psi = 90^\circ$$

$$\sum_i \psi_i - \sum_j \phi_j = 90^\circ - (90^\circ + 45^\circ + 45^\circ)$$
$$= -90^\circ$$

$$L(s^*) = L(-1+j) = \frac{-1+j+1}{(-1+j)(-1+j+2)} = \frac{j}{(-1+j)(1+j)}$$

$$= \frac{j}{j^2 - 1^2}$$

$$= \frac{j}{-1-1} = -\frac{j}{2}$$

$$\angle -j/2 = -90^\circ$$

$$1 + K \frac{s+1}{s(s-1)} \quad \text{(example)}$$

$a(s) + K\,b(s)$ denominator polynomial

Rule I: The n branches of the root locus start at the poles of $L(s)$ and m of the branches end at the zeros of $L(s)$.

n: # of poles

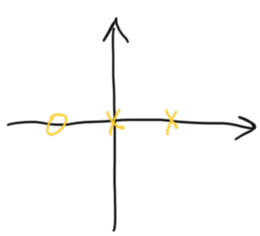
m: # of zeros

$a(s) + K\,b(s) = 0$

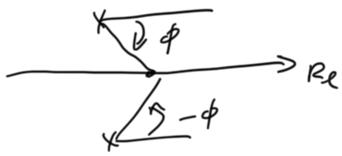
$K = 0 \Rightarrow \quad a(s) = 0$ solutions are the open loop poles. $L(s) = \frac{b(s)}{a(s)}$

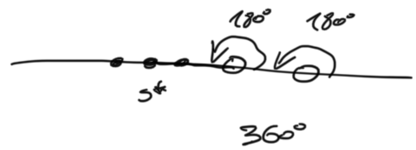
$K \to \infty \Rightarrow$ if $K\,b(s) \to \pm\infty$ then $a(s) \to \mp\infty \Rightarrow s \to \pm\infty$

$K\,b(s) \to \text{const.} \Rightarrow b(s) \to 0 \quad b(s) = 0$ (zeros)

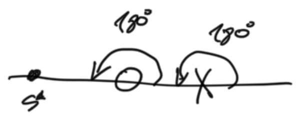


Rule II: The loci are on the real axis to the left of an odd number of poles and zeros.

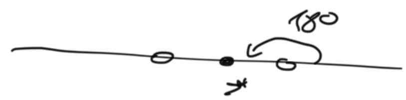




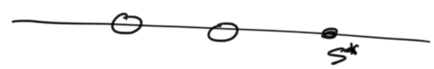
$\angle L(s^*) = 180°$? no!



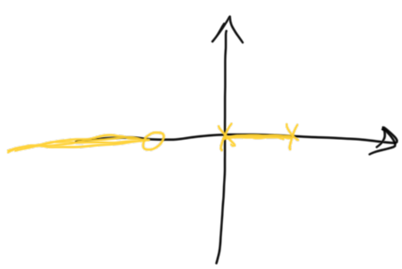
$$\angle L(s^*) = 180° - 180° = 0 \neq 180°$$



$$\angle L(s^*) = 180° + 0° = 180°$$



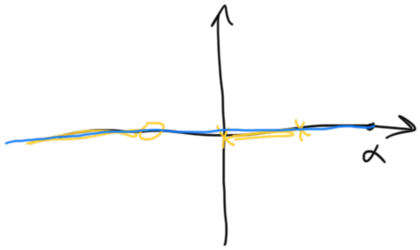
$$\angle L(s^*) = 0° + 0° = 0° \neq 180°$$



Rule III For large s and K, $n-m$ of the loci are asymptotic to lines at angles ϕ_l radiating out from the point $s = \alpha$ on the real axis where:

$$\phi_l = \frac{180° + 360°(l-1)}{n-m} \qquad l = 1, 2, \ldots, n-m$$

$$\alpha = \frac{\sum_i p_i - \sum_j z_j}{n-m}$$



$$\alpha = \frac{0+1-(-1)}{2-1} = \frac{2}{1} = 2$$

$$\phi_1 = \frac{180^\circ + 360^\circ \cdot 0}{2-1} = 180^\circ$$

Rule IV: The angles of departure of a branch of the locus from a pole of multiplicity q are given by:

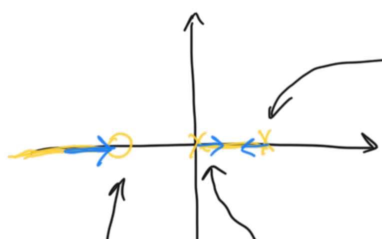
$$q\,\phi_{l,\mathrm{dep}} = \sum_i \psi_i - \sum_{j \neq l} \phi_j - 180^\circ - 360^\circ(l-1) \qquad l = 1, 2, \dots, q$$

and the angles of arrival of a branch at a zero of multiplicity are given by:

$$q\,\psi_{l,\mathrm{arr}} = \sum_j \phi_j - \sum_{i \neq l} \psi_i + 180^\circ + 360^\circ(l-1), \qquad l = 1, 2, \dots, q$$

$q = 1$

$$\phi_1 = 0^\circ - 0^\circ - 180^\circ - 360^\circ(0) = -180^\circ$$



$$\phi_1 = 0^\circ - 180^\circ - 180^\circ - 360^\circ \cdot 0 = 0^\circ$$

$$\psi_1 = 180^\circ + 180^\circ + 180^\circ + 360^\circ \cdot 0 = 180^\circ$$

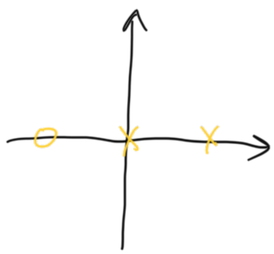
Rule V: The locus crosses the $j\omega$ axis at points where the Routh criterion shows a transition from roots in the left half-plane to the right half-plane.

$$1 + K\frac{s+1}{s(s-1)} \qquad s(s-1) + K(s+1) = s^2 + (K-1)s + K$$

2	1	K
1	$K-1$	
0	K	

$K=1$

$K=0$



$$s = j\omega_0$$

$$1 + K\frac{s+1}{s(s-1)} = 1 + \frac{j\omega_0+1}{j\omega_0(j\omega_0-1)} =$$