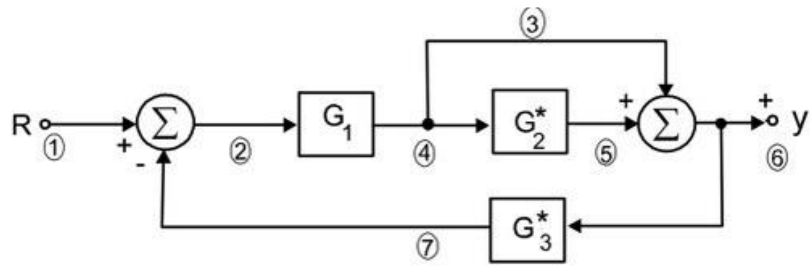
HOMEWORK 2 Solutions

3.21 a)

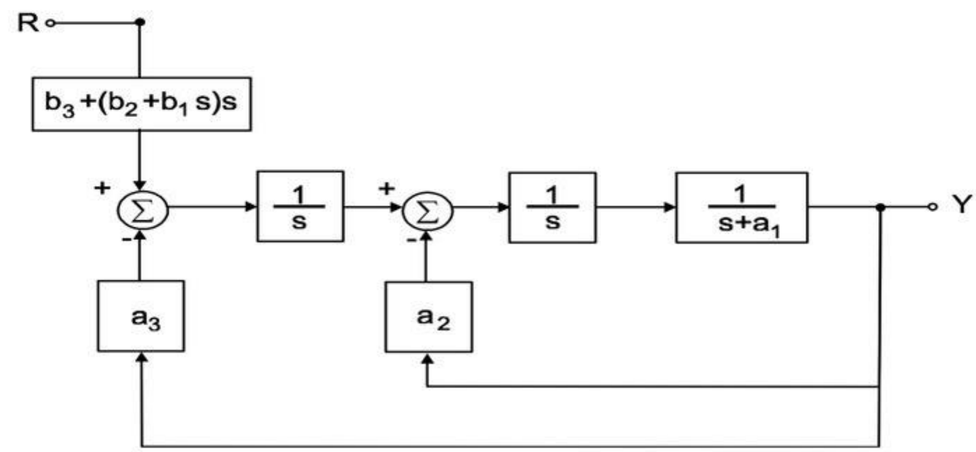


$$G_2^* = \frac{G_2}{1 - G_2H_2}$$
$$G_3^* = \frac{G_3}{1 - G_3H_3}$$

$$\frac{Y}{R} = \frac{G_1(1+G_2^*)}{1+G_1(1+G_2^*)G_3^*} = \frac{G_1(1-G_2H_2)(1-G_3H_3)+G_1G_2(1-G_3H_3)}{1+(1-G_2H_2)(1-G_3H_3)+G_1G_3(1-G_2H_2)+G_1G_2G_3}.$$

b)

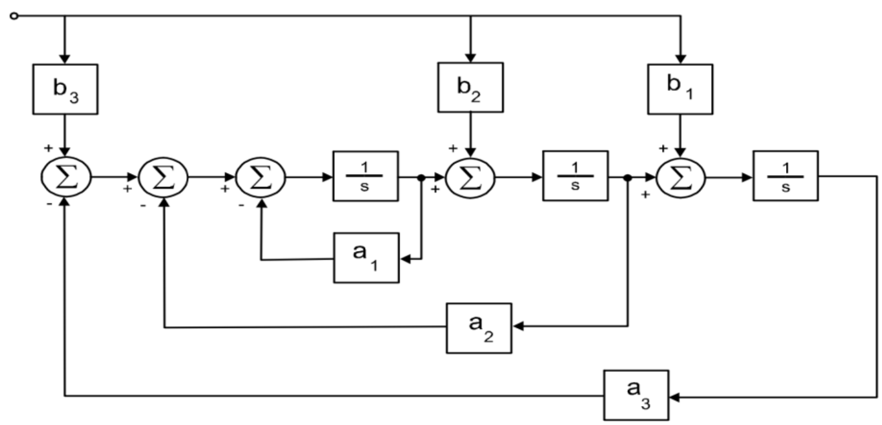
We move the summer on the right past the integrator to get $b_1 s$ and repeat to get $(b_2 + b_1 s)s$. Meanwhile we apply the feedback rule to the first inner loop to get $\frac{1}{s+a_1}$ as shown in the figure and repeat for the second and third loops to get:

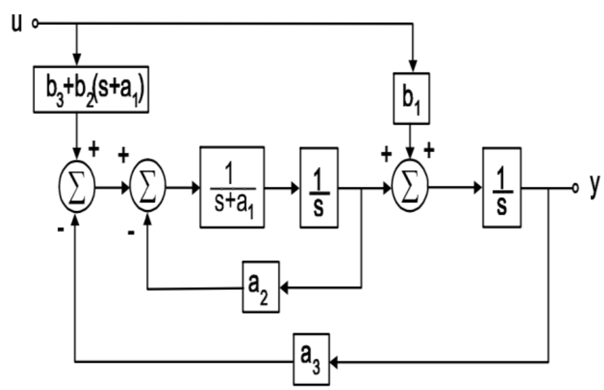


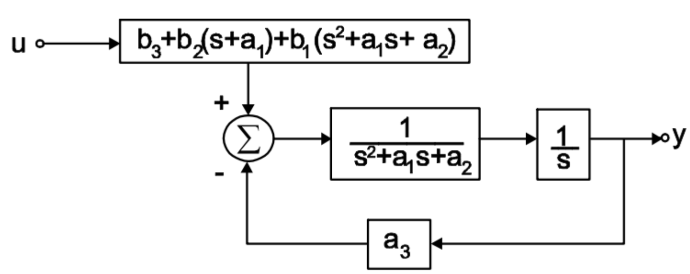
$$\frac{Y}{R} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

c)

Applying block diagram reduction: reduce innermost loop, shift b_2 to the b_3 node, reduce next innermost loop and continue systematically to obtain:

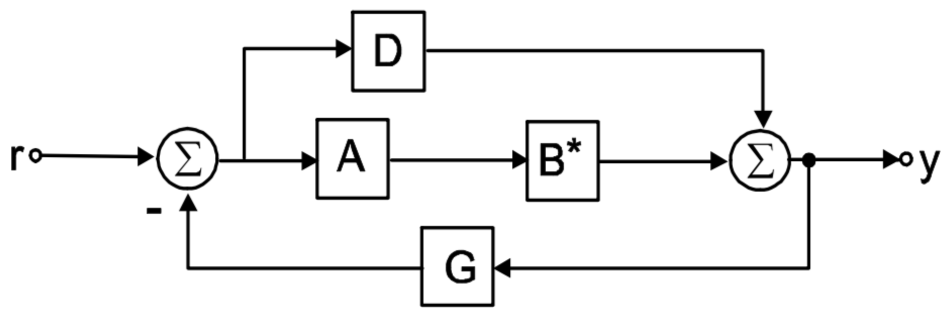






$$\frac{Y}{R} = \frac{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + a_2 b_1 + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

d)



$$B^* = \frac{B}{1+BH}$$

$$\frac{Y}{R} = \frac{D + AB^*}{1 + G(D + AB^*)} = \frac{D + DBH + AB}{1 + BH + GD + GBDH + GAB}.$$

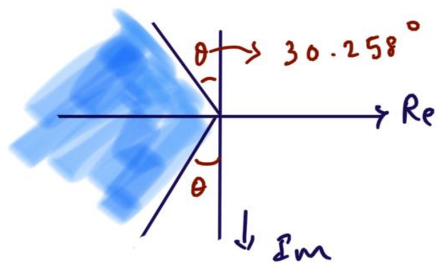
3.30) a)

(a) Overshoot $M_p = e^{-\pi \xi / \sqrt{1-\xi^2}}$

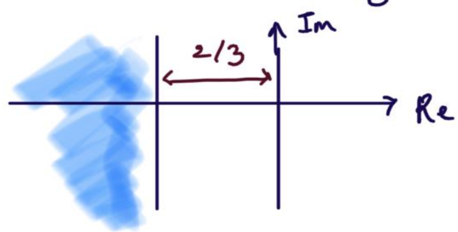
$$\Rightarrow \xi \leqslant \sqrt{\frac{\ln(M_p)^2}{\pi^2 + \ln(M_p)^2}} \quad \text{where } M_p = 0.16$$

$$\leqslant \sqrt{\frac{\ln(0.16)^2}{\pi^2 + \ln(0.16)^2}} \simeq 0.5039$$

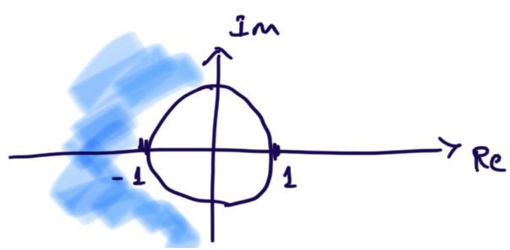
$$\therefore \theta = \sin^{-1}(0.5039) = 30.258^\circ$$



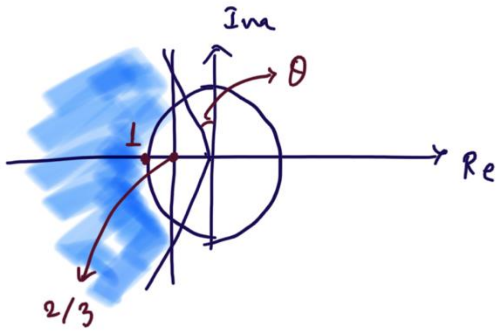
Settling time $t_s = \frac{4.6}{\delta} \leqslant 6.9\,s \Rightarrow \delta \geqslant 2/3$



Rising time $t_r = \frac{1.8}{\omega_n} \leqslant 1.8\,s \rightarrow \omega_n \geqslant 1$



∴ Region of acceptable closed-loop poles in the s-plane



b)

(b) $M_p = ?$

Given: $t_r = 1.8 s$

$t_s = 6.9 s$

$\Rightarrow \delta = 2/3$ and $\omega_n = 1$

$t_s = 4.6 / \xi \omega_n$

$6.9 = 4.6 / \xi (1) \Rightarrow \xi = 2/3$

$M_p = e^{-\pi \xi / \sqrt{1-\xi^2}}$

$= e^{-\pi (2/3) / \sqrt{1-(2/3)^2}}$

$M_p = 0.0602$

3.32)

(a)

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Equate the coefficients:

$$\begin{matrix} 2 = 2\zeta\omega_n \\ K = \omega_n^2 \end{matrix} \quad (*)$$

$$\Longrightarrow \quad \omega_n = \sqrt{K} \qquad \zeta = \frac{1}{\sqrt{K}}$$

We would need:

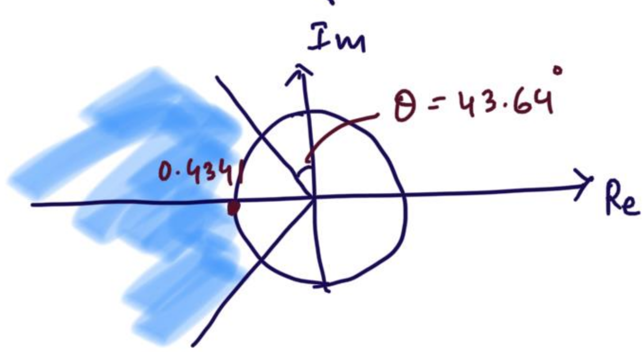
$$\frac{M_p\%}{100} = 0.05 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad \Longrightarrow \quad \zeta = 0.69$$

$$t_p = 1\,\text{sec} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad \Longrightarrow \quad \omega_n = 4.34$$

But the combination ($\zeta = 0.69$, $\omega_n = 4.34$) that we need is <u>not</u> possible by varying K alone. Observe that from equations (*) $\zeta\omega_n = 1 \neq 0.69 \times 4.34$

b)

Associated region in s-plane:



b) Let $M_p = 0.05\,r$ where r is the relaxation factor

$t_p = (1\,\text{sec})\,r$

$$M_p = e^{-\pi \xi/\sqrt{1-\xi^2}} = 0.05\,r \quad \text{---- (i)}$$

$$t_p = \pi/\omega_d = \pi/\omega_n\sqrt{1-\xi^2} = r \quad \text{---- (ii)}$$

$$\Rightarrow \frac{\pi}{\sqrt{1-\xi^2}} = r\,\omega_n$$

$$e^{-r\omega_n \xi} = 0.05\,r$$

where $\omega_n = 4.3410$

$\xi = 0.6901$

$$\Rightarrow r = 2.205$$

Let $d(s) = s^2 + 2s + \omega_n^2$

$$\begin{cases} 2 = 2\xi\omega_n \rightarrow \xi\omega_n = 1 \Rightarrow \omega_n = 1/\xi \\ \omega_n^2 = K \Rightarrow K = 1/\xi^2 \end{cases} \rightarrow \xi = 1/\sqrt{K}$$

-- (iii)

(i) and (iii) $\Rightarrow$

$$e^{-\pi/K/\sqrt{1-1/K}} = 0.05\,r$$

$$e^{-\pi/\sqrt{K-1}} = 0.05\,r$$

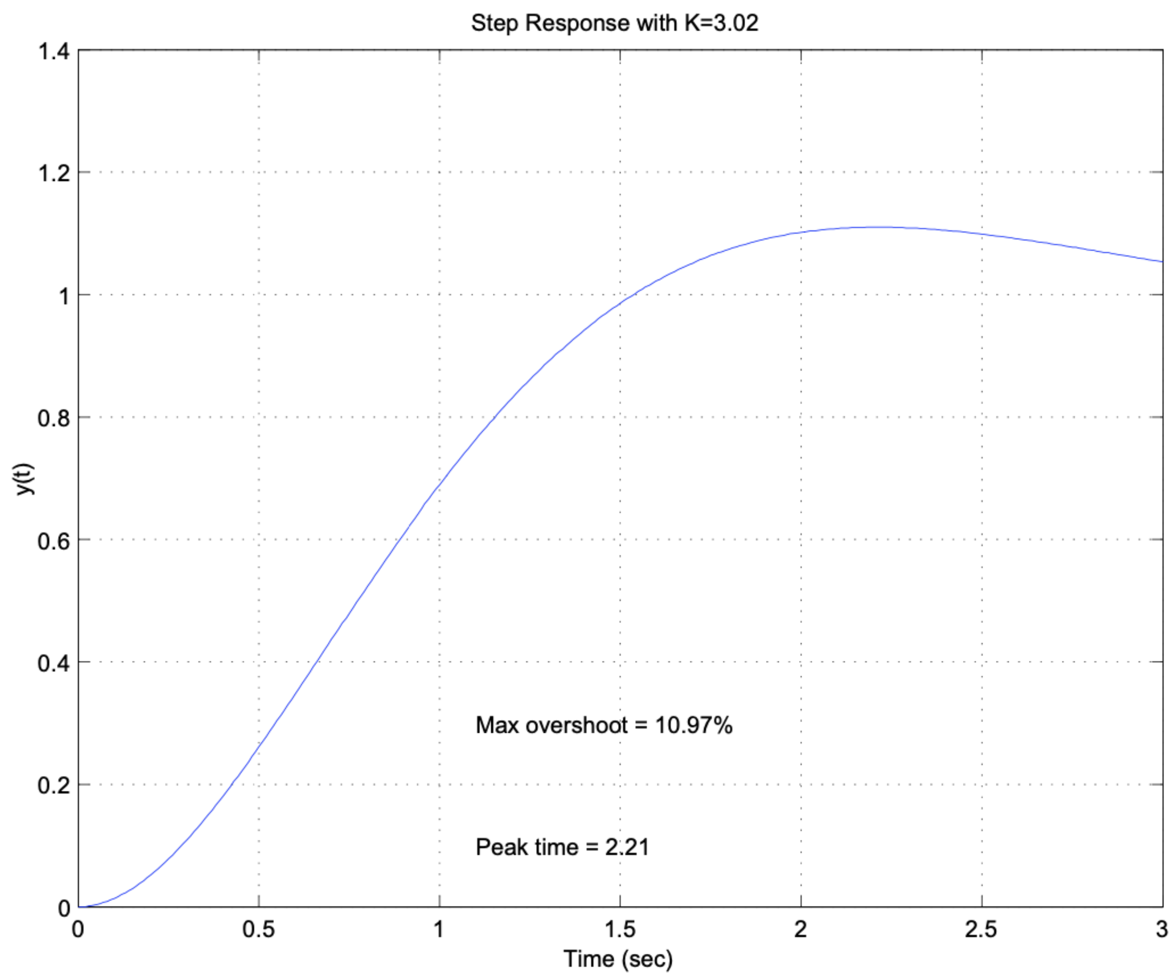
$$-\pi/\sqrt{K-1} = \ln(0.05\,r)$$

$$\boxed{K = 3.303}$$

(i) $\Rightarrow$ $M_p = 0.05\,r = 0.05(2.205) = 0.11025$

(ii) $\Rightarrow$ $t_p = r = 2.205$ sec c)

```
K=3.02;
num=[K];
den=[1, 2, K];
sys=tf(num,den);
t=0:.01:3;
y=step(sys,t);
plot(t,y);
yss = dcgain(sys);
Mp = (max(y) - yss)*100;
% Finding maximum overshoot
msg_overshoot = sprintf('Max overshoot = %3.2f%%', Mp);
% Finding peak time
idx = max(find(y==(max(y))));
tp = t(idx);
msg_peaktime = sprintf('Peak time = %3.2f', tp);
xlabel('Time (sec)');
ylabel('y(t)');
msg_title = sprintf('Step Response with K=%3.2f',K);
title(msg_title);
text(1.1, 0.3, msg_overshoot);
text(1.1, 0.1, msg_peaktime);
grid on;
```



Closed-loop step response

3.36)

$$J\ddot{\theta} + B\dot{\theta} = T_c$$

(a)

$$\begin{aligned}
J\Theta s^2 + B\Theta s &= T_c(s) \\
\frac{\Theta(s)}{T_c(s)} &= \frac{1}{Js+B} \\
J &= 600,000 kg \cdot m^2 \\
B &= 20,000 N \cdot m \cdot \sec \\
\frac{\Theta(s)}{T_c(s)} &= \frac{1.667 \times 10^{-6}}{s(s+\frac{1}{30})}
\end{aligned}$$

(b)

$$\begin{aligned}
\Theta(s) &= \frac{1.667 \times 10^{-6} K(\Theta_r - \Theta)}{s(s+\frac{1}{30})} \\
\frac{\Theta(s)}{\Theta_r(s)} &= \frac{1.667K \times 10^{-6}}{s^2 + \frac{1}{30}s + 1.667K \times 10^{-6}}
\end{aligned}$$

(c)

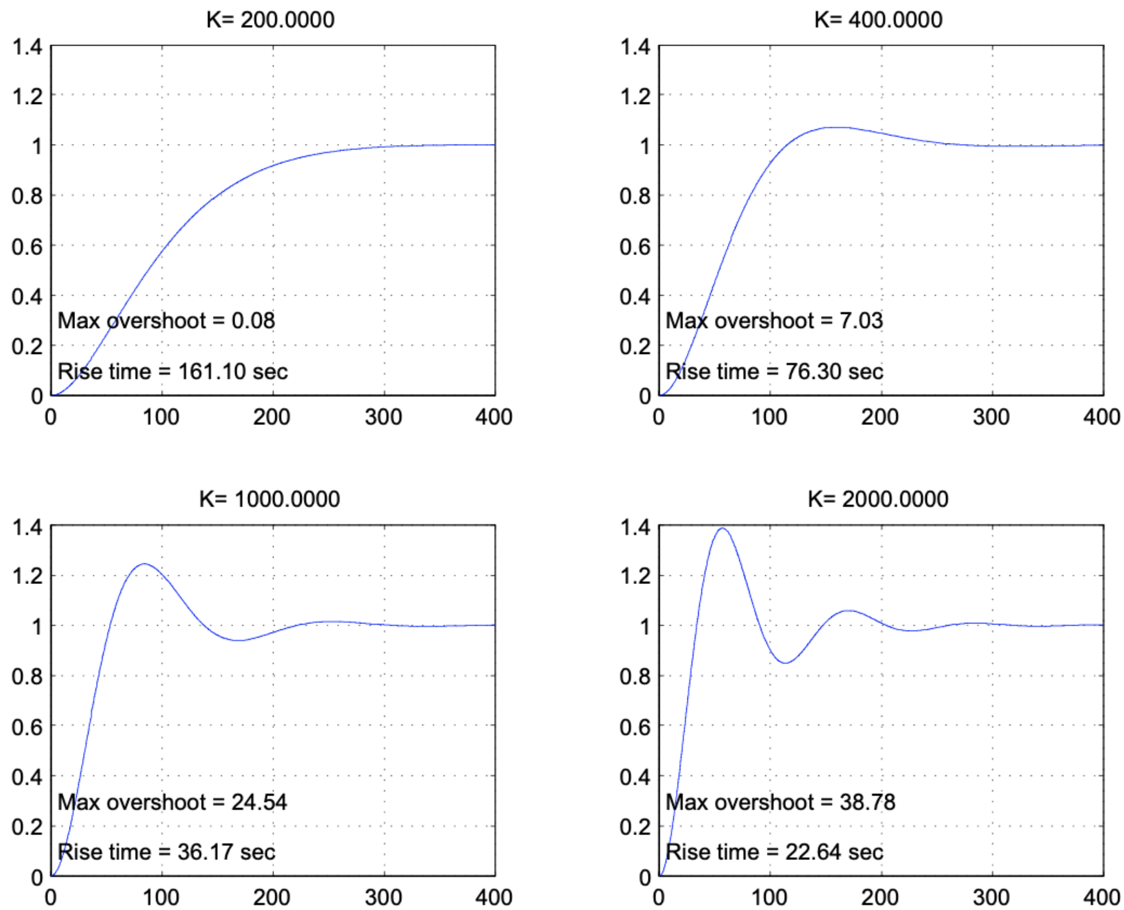
$$\begin{aligned}
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.1 \qquad (10\%) \\
\zeta &= 0.591 \\
Y(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
2\zeta\omega_n &= \frac{1}{30} \\
\omega_n &= \frac{\frac{1}{30}}{2(0.591)} = 0.0282 \ rads/\sec \\
\omega_n^2 &= 1.667K \times 10^{-6} \\
K &< 477
\end{aligned}$$

(d)

$$\begin{aligned}\omega_n &\geq \frac{1.8}{t_r}\\ \omega_n^2 &= 1.667K \times 10^{-6}\\ K &\geq 304\end{aligned}$$

(e)

Step Responses:



The results compare favorably with the predictions made in parts c and d. For K<477, the overshoot was less than 10 and the rise time was less than 80 seconds.