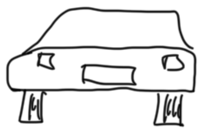
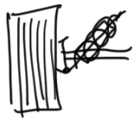
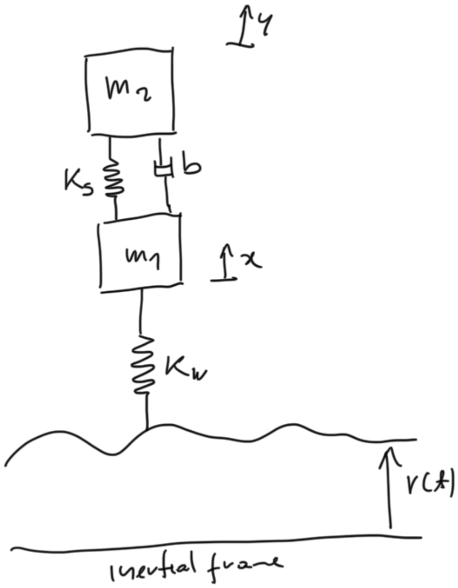
ECE 141
Lecture 1







$$m_2 \ddot{y} = \overbrace{-b(\dot{y} - \dot{x})}^{\text{damper}} - \overbrace{k_s(y - x)}^{\text{spring}}$$

$$m_1 \ddot{x} = -b(\dot{x} - \dot{y}) - k_s(x - y) - k_w(x - r)$$

$$\mathcal{L}\{m_2 \ddot{y}\} = \mathcal{L}\{-b(\dot{y} - \dot{x}) - k_s(y - x)\}$$

Review of Laplace transform

$f : \mathbb{R} \to \mathbb{R}$

$$\mathcal{L}\{f\} = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt \qquad s = \sigma + j\omega$$

One sided Laplace transform

<u>Step function</u>

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad 1(t) \text{ book's notation}$$

$$\mathcal{L}\{u\} = \int_{0^-}^{\infty} u(t)\, e^{-st}\, dt = \int_{0^-}^{\infty} e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty}$$

$$= \underbrace{\lim_{t \to +\infty} \frac{e^{-st}}{-s}}_{0} - \underbrace{\lim_{t \to 0} \frac{e^{-st}}{-s}}_{-\frac{1}{s}} = - - \frac{1}{s} = \frac{1}{s}$$

when $\sigma > 0$,

Impulse function

$\delta(t)$ defined by the sifting property: $\displaystyle\int_{-\infty}^{+\infty} f(t)\, \delta(t-\tau)\, dt = f(\tau)$

$$\mathcal{L}\{\delta\} = \int_{0^-}^{+\infty} \delta(t)\, e^{-st}\, dt = e^{-s0} = 1$$

Properties of Laplace function

Superposition

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$
$$= \alpha F(s) + \beta G(s), \quad \forall_{\alpha, \beta \in \mathbb{R}}$$

Time-delay

$$\mathcal{L}\{f(t-\tau)\} = e^{-s\tau} \mathcal{L}\{f(t)\} = e^{-s\tau} F(s)$$

Time-scaling

$$\mathcal{L}\{f(at)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

Shifts in the s-domain

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{?\} = F(s+a)$$

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Differentiation

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = ? \qquad g \overset{\text{def.}}{=} \frac{df}{dt} \Rightarrow \frac{d^2f}{dt^2} = \frac{dg}{dt}$$

$$\mathcal{L}\left\{\frac{dg}{dt}\right\} = sG(s) - g(0^-)$$

$$= sG(s) - \frac{df}{dt}(0^-)$$

$$\downarrow$$

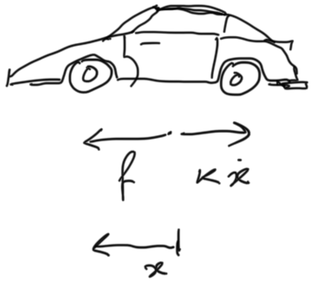
$$\mathcal{L}\{g(t)\} = \mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{dg}{dt}\right\} = s^2F(s) - sf(0^-) - \frac{df}{dt}(0^-)$$

$$\mathcal{L}\left\{\frac{d^mf}{dt^m}\right\} = s^mF(s) - s^{m-1}f(0^-) - s^{m-2}\frac{df}{dt}(0^-) - \dots - \frac{d^{m-1}f}{dt^{m-1}}(0^-)$$

Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$



$$m\ddot{x} = f - \kappa\dot{x}$$

$$\ddot{x} = \frac{1}{m}f - \frac{\kappa}{m}\dot{x}$$

$$\mathcal{L}\{\ddot{x}\} = \mathcal{L}\left\{\frac{1}{m}f - \frac{\kappa}{m}\dot{x}\right\}$$

$$s^2 X(s) = \frac{1}{m}F(s) - \frac{\kappa}{m}X(s)s$$ (assuming x(0

$$X(s)\left[s^2 + \frac{\kappa}{m}s\right] = \frac{1}{m}F(s)$$

$$X(s) = \frac{1/m}{s^2 + s\frac{\kappa}{m}}F(s)$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1/m}{s^2 + s\frac{\kappa}{m}}F(s)\right.$$

Inverse Laplace transform

zeros
poles
roots

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \dots + b_{m+1}}{s^n + a_1 s^{n-1} + \dots + a_n}$$

monic polynomial

$m \le n$ proper transfer function
$m < n$ strictly proper

always the case
for physical systems

$$H(s) = K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

$$H(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \dots + \frac{c_n}{s - p_n}$$ partial fraction expansion

$$\mathcal{L}^{-1}\{H(s)\} = c_1 \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-p_1}\right\}} + c_2 \mathcal{L}^{-1}\left\{\frac{1}{s-p_2}\right\} + \dots + c_n \mathcal{L}^{-1}\left\{\frac{1}{s-\right.$$

$$e^{p_1 t} u(t)$$

$$= c_1 e^{p_1 t} u(t) + c_2 e^{p_2 t} u(t) + \dots + c_n e^{p_n t} u(t)$$

$$Y(s) = \boxed{\frac{s+1}{s(s+2)} = \frac{c_1}{s} + \frac{c_2}{s+2}}$$

$$\frac{s+1}{s+2} = c_1 + \frac{c_2 s}{s+2}$$

for $s=0$ $\quad \frac{1}{2} = c_1$

$$\frac{s+1}{s} = \frac{s+2}{s} c_1 + c_2$$

for $s=-2$ $\quad \frac{-1}{-2} = c_2$

$$c_2 = (s+2)\, Y(s) \Big|_{s=-2}$$

$$c_1 = s\, Y(s) \Big|_{s=0}$$

$$Y(s) = \frac{1/2}{s} + \frac{1/2}{s+2} \Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2} u(t) + \frac{1}{2} e^{-2t} u(t)$$

Back to example

$$x(t) = \mathcal{L}^{-1}\left\{ \frac{1/m}{s^2 + s\frac{k}{m}} F(s) \right\}$$

Assume $f(t) = c\,\delta(t)$

$$F(s) = c$$

$$\mathcal{L}^{-1}\left\{\frac{c/m}{\underbrace{s^2+s\frac{k}{m}}_{s\left(s+\frac{k}{m}\right)}}\right\} = \mathcal{L}^{-1}\left\{\frac{c_1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{c_2}{s+\frac{k}{m}}\right\}$$

$$c_1 = s\,X(s)\Big|_{s=0} = \frac{c/m}{k/m} = \frac{c}{k}$$

$$c_2 = \left(s+\frac{k}{m}\right)X(s)\Big|_{s=-\frac{k}{m}} = \frac{c/m}{-k/m} = -\frac{c}{k}$$

$$x(t) = \underbrace{\frac{c}{k}u(t)}_{\frac{c}{k}e^{0t}u(t)} - \frac{c}{k}e^{-\frac{k}{m}t}u(t)$$

$$\lim_{t\to\infty} x(t) = \frac{c}{k} \quad //$$

How to compute this limit without computing $\mathcal{L}^{-1}$?

Final Value Theorem

If all the poles of $sY(s)$ are on the left half of the s-plane, then:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s).$$

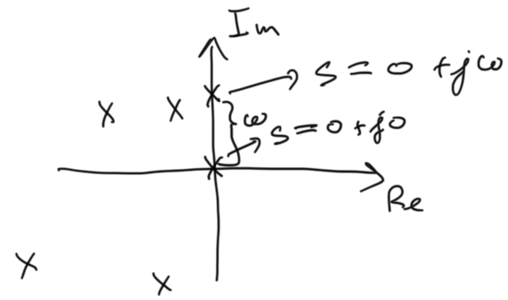
$$Y(s) = X(s) = \frac{c/m}{s(s+k/m)} \qquad sX(s) = \frac{c/m}{s+\frac{k}{m}}$$

$$p = -\frac{k}{m}$$

$$\frac{c/m}{} \qquad \frac{c/}{} \qquad e^{p_i t}$$

$$\lim_{s \to 0} s X(s) = \lim_{s \to 0} \frac{\quad}{s + \frac{k}{m}} = \frac{c/m}{k/m}$$

$$= \frac{c}{k}$$



$s = j\omega$

$= -j\omega$

$\sin(\omega t)$