Homework 7

SOLUTIONS

MAY 25, 2021

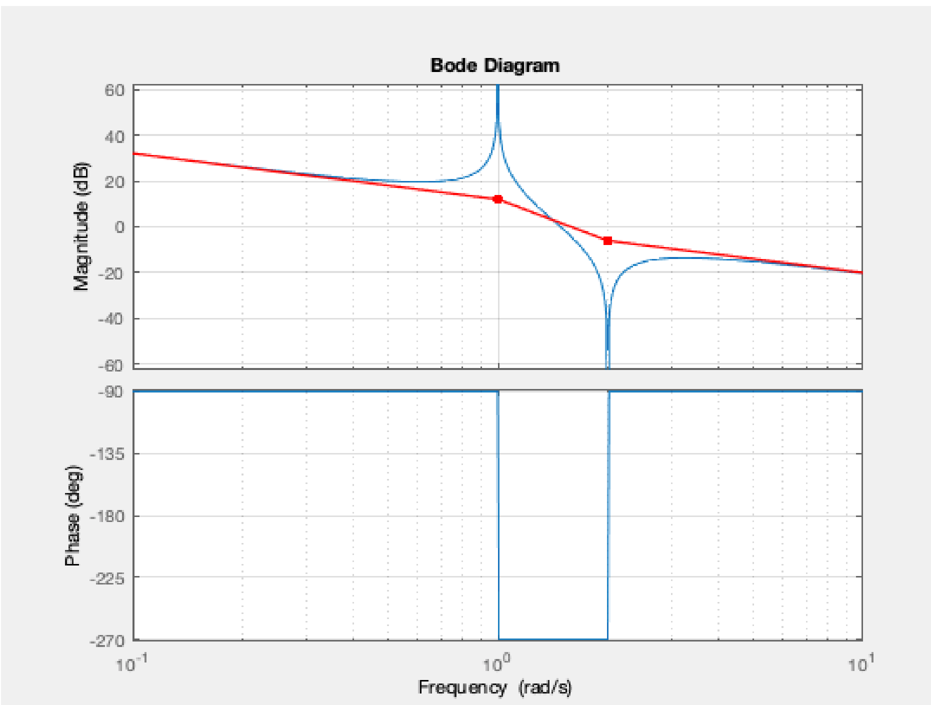
Problems: 6.5e, 6.7b, 6.16b, 6.17c

6.5e: Begin by evaluating $L(s)$ along $s = j\omega$ and factor out any constants to put the transfer function in a simpler form to draw the Bode plot:

$$L(s) = \frac{s^2+4}{s(s^2+1)}, \tag{1}$$

$$L(j\omega) = \frac{4\left(\frac{(j\omega)^2}{4}+1\right)}{j\omega[(j\omega)^2+1]}. \tag{2}$$

Observe that for the two quadratic expressions in the numerator and denominator, the damping ratio is 0, so we expect peaks at $-\infty$ for the expression in the numerator, and ∞ for the expression in the denominator. The factor of 4 means that the asymptotic Bode plot intersects the $\omega = 1$ axis at $20\log(4) \approx 12\,\text{dB}$. The bandwidths for the quadratic expressions in the denominator and numerator are $\omega = 1$ and $\omega = 2$, respectively. Putting everything together, the Bode plot is shown below.

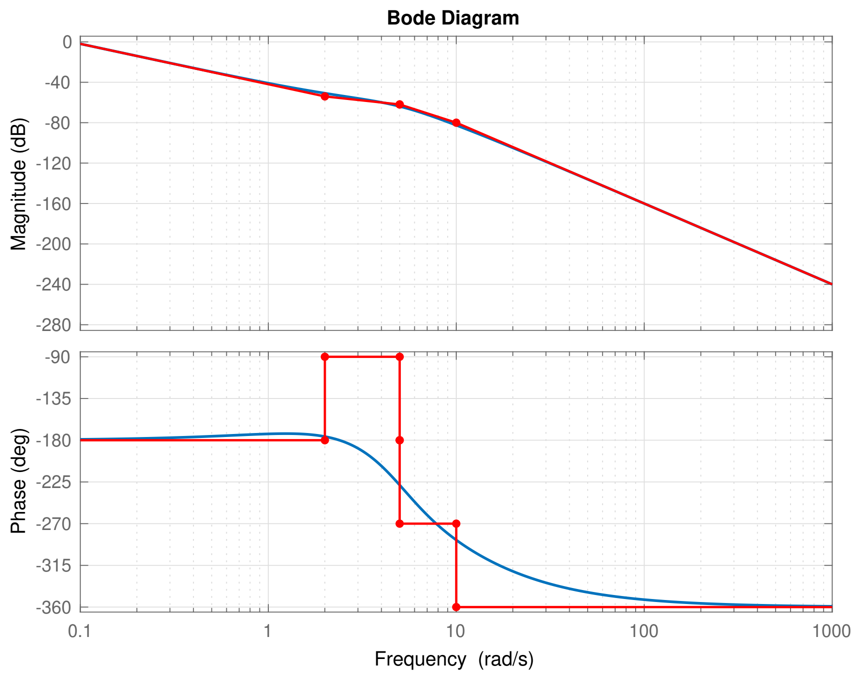


6.7b: Similar to the previous question, consider $L(s)$ along $s = j\omega$ and factor out any coefficients:

$$L(s) = \frac{s+2}{s^2(s+10)(s^2+6s+25)}, \tag{3}$$

$$L(j\omega) = \frac{1}{125} \frac{\frac{j\omega}{2}+1}{(j\omega)^2\left(\frac{j\omega}{10}+1\right)\left(\frac{(j\omega)^2}{25}+\frac{6(j\omega)}{25}+1\right)}. \tag{4}$$

The quadratic expression in the numerator has $\omega_n = 5$, so $\zeta = 0.6$. The magnitude of $L(j\omega)$ at $\omega = 1$ is $20\log\left(\frac{1}{125}\right) \approx -42\,\text{dB}$. The Bode plot is shown below.

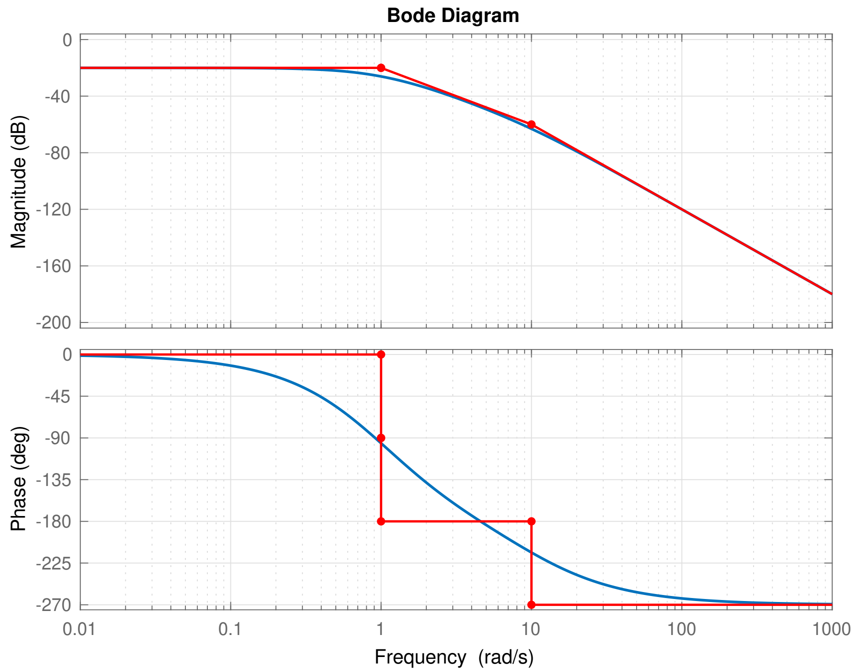


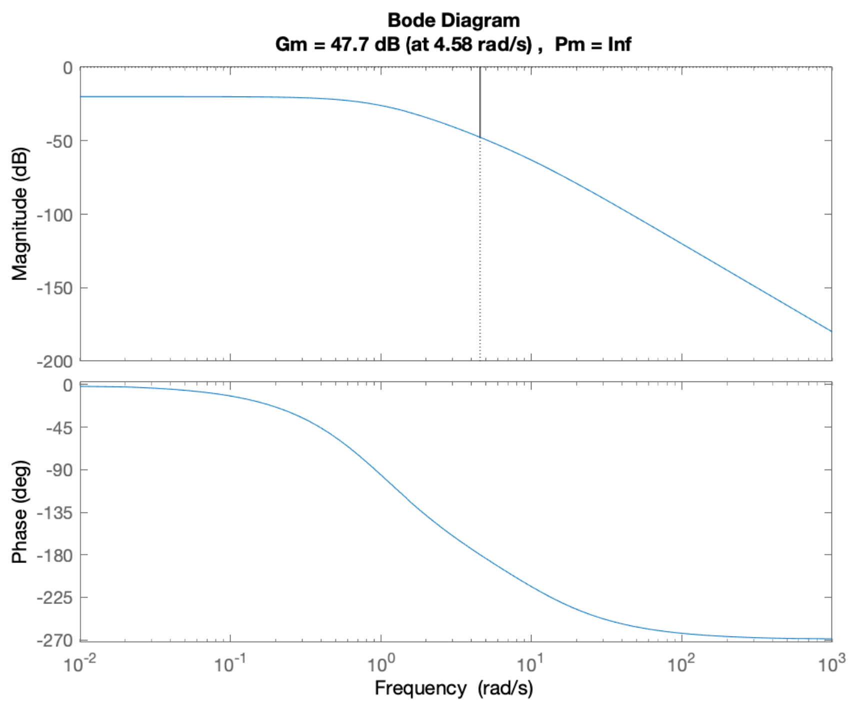
6.16b: The $(s+1)^2$ term in the denominator of $KG(s)$ corresponds to a second-order system with $\omega_n = 1$ and $\zeta = 1$. At $s = j\omega$, $KG(s)$ becomes:

$$KG(s) = \frac{K}{(s+10)(s+1)^2}, \tag{5}$$

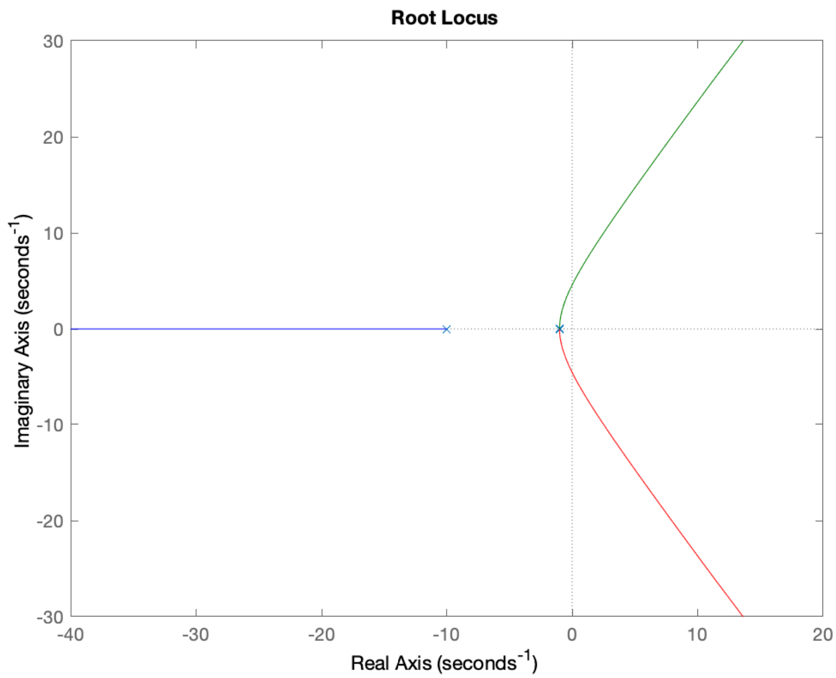
$$KG(j\omega) = \frac{K}{10\left(\frac{j\omega}{10}+1\right)\left[(j\omega)^2+2(j\omega)+1\right]}. \tag{6}$$

The Bode plot of $KG(s)$ with $K = 1$ is shown below. The gain margin is approximately $47.7\,\text{dB}$, which means that setting $K \approx 242$ would result in instability of the closed-loop system.





To verify this, consider the root locus for $KG(s)$. The transfer function has three poles and no zeros, so the center point occurs at $\alpha = \frac{-10+(-1)+(-1)}{3} = -4$ and there are three asymptotes. The departure angle for the pole at -10 is $180°$. The departure angles for the double pole at -1 are $\pm 90°$. From drawing the root locus, we see that there is an upper limit of K that ensures closed-loop stability. To determine this value, we can apply the Routh criterion to the characteristic polynominal: $(s+10)(s+1)^2 + K$.



Once expanded, the characteristic polynominal is $s^3 + 12s^2 + 21s + K + 10$. From the Routh criterion, the three conditions for stability become:

$$\begin{cases} 12 > 0, \\ K + 10 > 0, \\ (12)(21) > K + 10. \end{cases} \tag{7}$$

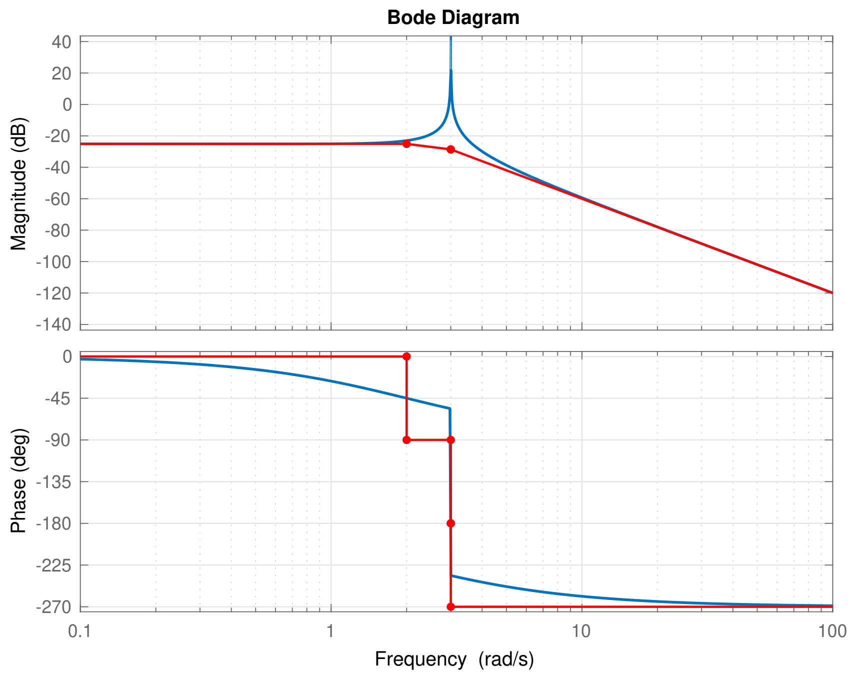
Hence, the range of K for closed-loop stability is $K \in (-10, 242)$, where the upper limit of 242 agrees with our computation from the Bode plot.

6.17c: Considering $KG(s)$ along $s = j\omega$:

$$KG(s) = \frac{K}{(s+2)(s^2+9)}, \tag{8}$$

$$KG(j\omega) = \frac{K}{18\left(\frac{j\omega}{2} + 1\right)\left(\frac{(j\omega)^2}{9} + 1\right)}. \tag{9}$$

At low frequencies, the Bode plot is $20 \log\left(\frac{1}{18}\right) \approx -25.1\,\text{dB}$. The second-order expression in the denominator has a bandwidth of 3 rads/s and a damping ratio of 0, which results in a peak to ∞ at $\omega = 3$. Using this information, the Bode plot for $KG(s)$ with $K = 1$ is shown below.

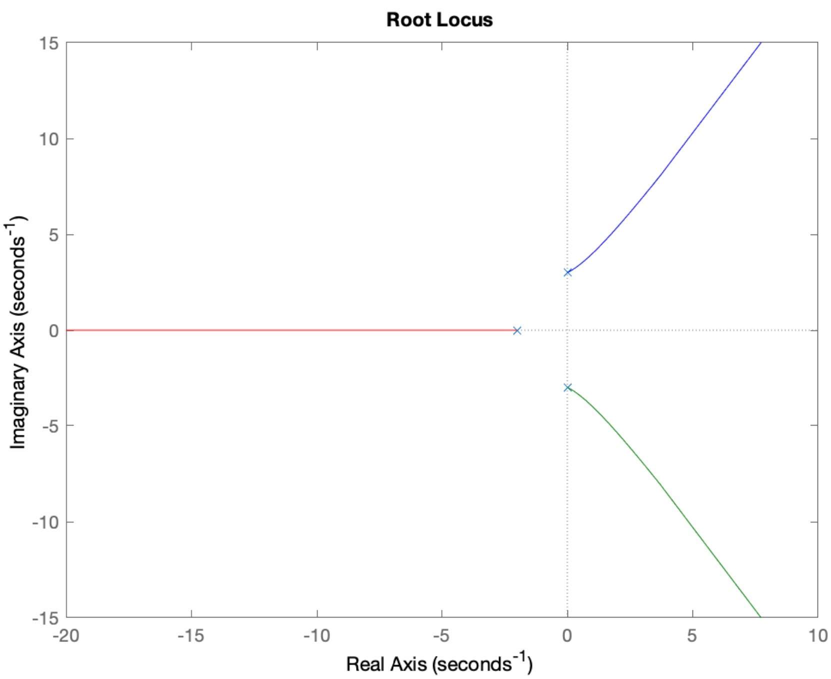


When the phase of $KG(j\omega)$ is -180°, the magnitude is ∞. Thus, there is no K that would result in a stable closed-loop system. To verify this, we can look at the root locus of $KG(s)$. The transfer function has three poles and no zeros, so there are three asymptotes that intersect at a center point at $\alpha = \frac{-2+(3j)+(-3j)}{3} = -\frac{2}{3}$. The departure angle for the pole at -2 is 180°. The departure angle for the pole at $3j$ is calculated as:

$$\phi_1 = -(90^\circ + \operatorname{atan2}(3,2)) + 180^\circ, \tag{10}$$

$$\approx 33.7^\circ. \tag{11}$$

Since this departure angle is in the interval $(-90^\circ, 90^\circ)$, the branch of the root locus from this pole goes towards the right half plane and off towards the asymptote. Thus, no value of $K > 0$ can stabilize the closed-loop system. The root locus is shown below.


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
15
10
5
0
-5
-10
-15
-20
-15
-10
-5
0
5
10