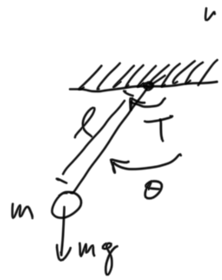
ECE 141

Lecture 8



$$\underline{\underline{m l^2 \ddot{\theta} = T - mg \sin(\theta) l}}$$

normal form $\dot{x} = f(x, u)$ $\quad x \in \mathbb{R}^n$ (example $n=2$) → state

$u \in \mathbb{R}^m$ (example $m=1$) → input

$$\ddot{\theta} = \frac{1}{m l^2} T - \frac{g}{l} \sin \theta$$

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{x}_1 &= \dot{\theta} = x_2 \\ \dot{x}_2 &= \ddot{\theta} = \frac{1}{m l^2} u - \frac{g}{l} \sin x_1 \end{aligned}$$

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ \frac{1}{m l^2} u - \frac{g}{l} \sin x_1 \end{bmatrix}}_{f(x, u)}$$

$$\boxed{\frac{dx}{dt} = \dot{x} = f(x, u)}$$

linear differential equation when f is a linear function

nonlinear differential equation when f is a nonlinear function linear : $f(x,u) = Ax + Bu = [A \mid B] \begin{bmatrix} x \\ \hline u \end{bmatrix}$

Linearization around an equilibrium

The pair (x_e, u_e) is an equilibrium for $\dot{x} = f(x,u)$ if $f(x_e, u_e) = 0$.

$x(t) \equiv x_e$ solution of $\dot{x} = f(x,u)$?

$$\frac{d\,x(t)}{dt} = \frac{d\,x_e}{dt} = 0 \qquad f(x_e, u_e) = 0$$

$g : \mathbb{R} \to \mathbb{R}$

$$g(y) = g(y_0) + \left.\frac{\partial g}{\partial y}\right|_{y=y_0} (y - y_0) + \frac{1}{2} \left.\frac{\partial^2 g}{\partial y^2}\right|_{y=y_0} (y-y_0)^2 + \dots$$

$x(t) = x_e + \delta x(t) \rightarrow$ small deviation from x_e

$u(t) = u_e + \underbrace{\delta u(t)}$ $\rightarrow$ small deviation from u_e

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left(x_e + \delta x(t)\right) = \frac{d}{dt} x_e + \frac{d}{dt} \delta x(t) = \frac{d}{dt} \delta x(t)$$

$$f(x(t), u(t)) = \underbrace{f(x_e, u_e)}_{0} + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_e \\ u=u_e}} \underbrace{(x - x_e)}_{\delta x} + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_e \\ u=u_e}} \underbrace{(u - u_e)}_{\delta u} + \dots$$

$$\cong \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_e \\ u=u_e}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_e \\ u=u_e}} \delta u$$

$$= A\,\delta x + B\,\delta u \quad ②$$

$$\boxed{\frac{d}{dt}\,\delta x(t) = A\,\delta x(t) + B\,\delta u(t)}$$

Note: A and B are functions of x_e and u_e

Example: $f(x,u) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}u \end{bmatrix}$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x = x_e \\ u = u_e}}$$

Equilibrium pair: $f(x_e, u_e) = 0$

$$\begin{cases} \boxed{x_{e2} = 0} \\ -\frac{g}{l}\sin x_{e1} + \frac{1}{ml^2}u_e = 0 \end{cases}$$

$x_{e1} = 0 \Rightarrow u_e = 0$ $\qquad$ $x_{e1} = \frac{\pi}{4} \Rightarrow \frac{1}{ml^2}u_e = \frac{g}{l}\sin\left(\frac{\pi}{4}\right)$

$\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0\right)$ $\qquad$ $\left(\begin{bmatrix} \pi/4 \\ 0 \end{bmatrix}, gml\frac{\sqrt{2}}{2}\right)$ $\qquad$ $u_e = gml\frac{\sqrt{2}}{2}$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0\right)} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & 0 \end{bmatrix}\right|_{\substack{x_1 = 0 \;\; \frac{\pi}{4} \\ x_2 = 0 \\ u = 0 \;\; gml\frac{\sqrt{2}}{2}}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0\right)} = \left.\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right|_{x_1 = 0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\frac{}{ml^2}\right] \Bigg| \begin{array}{l} x_2=0 \\ u=0 \end{array} \qquad \left[\frac{}{ml^2}\right]$$

$$\frac{d}{dt}\delta x = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u \qquad \begin{cases} \frac{d}{dt}\delta x_1 = \delta x_2 \\ \frac{d}{dt}\delta x_2 = -\frac{g}{l}\delta x_1 + \frac{u}{ml^2} \end{cases}$$

$$\delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \qquad A\,\delta x = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \begin{bmatrix} \delta x_2 \\ -\frac{g}{l}\delta x_1 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}\Bigg|_{\begin{array}{l} x = x_e \\ u = u_e \end{array}} \qquad \Bigg|_{\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ u = 0 \end{array}}$$

Lotka-Volterra predator-prey

$\dot{x}_1 = -x_1 + x_1 x_2 - x_1 u$ predators

$\dot{x}_2 = x_2 - x_1 x_2$ preys

$(x_e, u_e) = ?$

$$f(x_e, u_e) = 0 \iff \begin{cases} 0 = -x_1 + x_1 x_2 - x_1 u \\ 0 = x_2 - x_1 x_2 \end{cases} \iff \begin{cases} \overline{} \\ 0 = x_2(1 - x_1) \end{cases}$$

$$\iff \begin{cases} \overline{} \\ x_2 = 0 \lor x_1 = 1 \end{cases} \qquad \begin{cases} 0 = -1 + x_2 - u \\ x_1 = 1 \end{cases}$$

$\Rightarrow u_1 = 1$

if $x_2 = 2 \Rightarrow u = 1$

$$x_e = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad u_e = 1$$

$$f(x,u) = \begin{bmatrix} -x_1 + x_1 x_2 - x_1 u \\ x_2 - x_1 x_2 \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x = x_e \\ u = u_e}} = \left.\begin{bmatrix} -1 + x_2 - u & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}\right|_{\substack{x_1 = 1 \\ x_2 = 2 \\ u = 1}} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x = x_e \\ u = u_e}} = \left.\begin{bmatrix} -x_1 \\ 0 \end{bmatrix}\right|_{x_1 = 1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{cases} \dfrac{d}{dt}\delta x_1 = \delta x_2 - \delta u \\[2ex] \dfrac{d}{dt}\delta x_2 = -2\,\delta x_1 \end{cases}$$

$$\frac{dx}{dt} = f(x,u) \iff \boxed{\int_0^t \frac{dx}{dt}\,d\tau} = \int_0^t f(x(\tau), u(\tau))\,d\tau$$

$$x(t) - x(0) = \int_0^t f(x(\tau), u(\tau))\,d\tau$$

$$\boxed{x(t) = x(0) + \int_0^t f(x(\tau), u(\tau))\,d\tau}$$

initial
condition

$$\boxed{\begin{aligned} x(t) &= x_e + \delta x(t) \\ u(t) &= u_e + \delta u(t) \end{aligned}}$$

$$u \longrightarrow \boxed{\dot{x} = f(x, u)} \longrightarrow x$$

$$\delta u \longrightarrow \boxed{\frac{d\,\delta x}{dt} = A\,\delta x + B\,\delta u} \longrightarrow \delta x$$