(2)

$$S = \frac{1}{N} \sum_{n=1}^{N} (x_n - \bar{x})(x_n - \bar{x})^T$$

$$\bar{x} = \begin{bmatrix} 2+0+-2+0 \\ 2-2+0+0 \\ 0+2+0+-2 \end{bmatrix} /4 = \vec{0}$$

$$= \frac{1}{N} \sum_{n=1}^{N} x_n x_n^T = \frac{1}{4}\left(\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\left\| \begin{matrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{matrix} \right\| = 0 \Rightarrow (2-\lambda)(4-4\lambda+\lambda^2-1) - 1\cdot(2-\lambda) + 0 = 0$$

$$(2-\lambda)(2-4\lambda+\lambda^2) = 0$$

$$\lambda = 2, \quad \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$$

Largest $\lambda = 2+\sqrt{2}$ $\longrightarrow$ $\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$-\sqrt{2}\,a + b = 0, \quad a - b\sqrt{2} - c = 0, \quad -b - c\sqrt{2} = 0$

Solve, to get $a = b/\sqrt{2}$, $c = -b/\sqrt{2}$, $b = b$. Nomalize: $u_1 = \begin{bmatrix} 1/2 \\ \sqrt{2}/2 \\ -1/2 \end{bmatrix}$

$\text{proj}_u x_i = \frac{x_i \cdot u}{\|u\|^2} \cdot \vec{u}$

Next larger $\lambda = 2 \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$u_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\quad \text{proj}_u x_i = \frac{x_i \cdot u}{1^2} = x_i^T u$ wrt $\vec{u}$ $\quad b = 1 \quad a = 1 \quad c = -1$

$$\boxed{\begin{aligned} x_1 &\to (1+\sqrt{2}, \sqrt{2}) \\ x_2 &\to (-\sqrt{2}-1, -\sqrt{2}) \\ x_3 &\to (-1, -\sqrt{2}) \\ x_4 &\to (1, -\sqrt{2}) \end{aligned}}$$