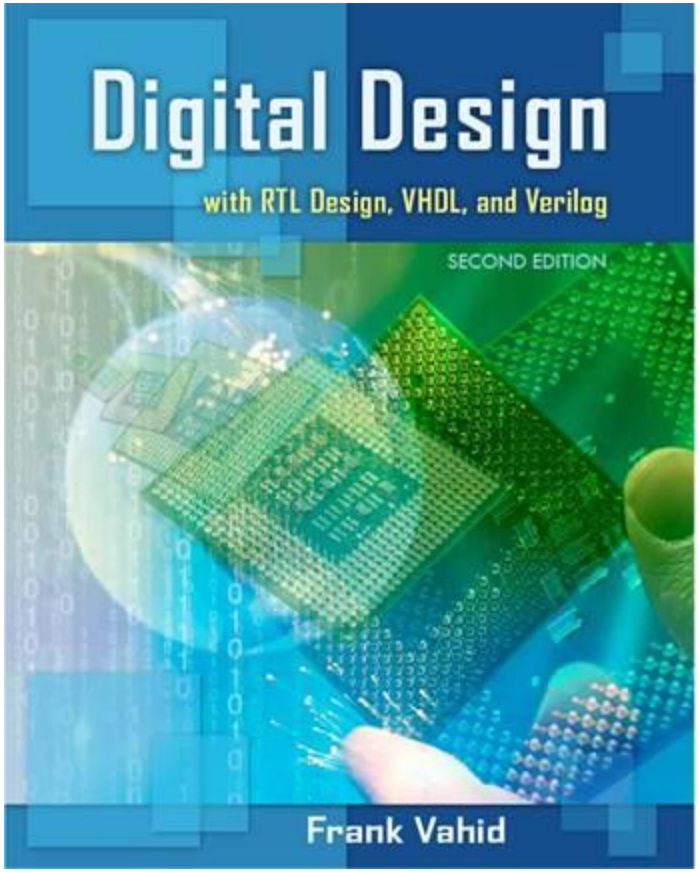


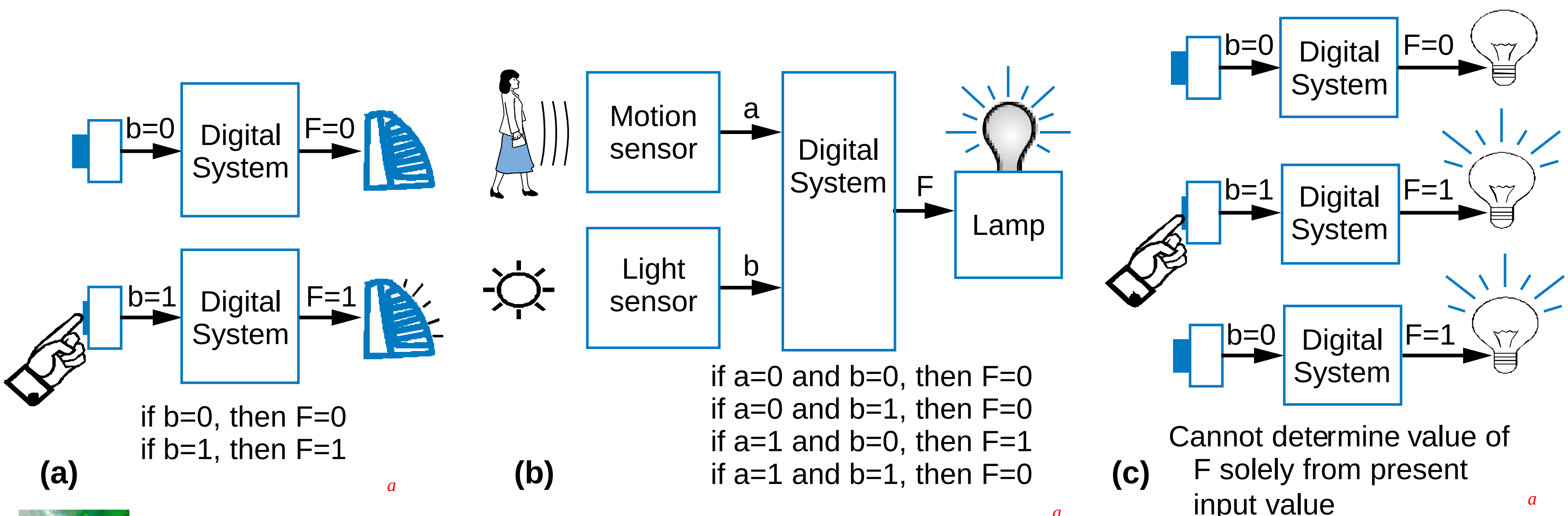
Digital Design

Chapter 2:
Combinational Logic Design

Slides to accompany the textbook *Digital Design, with RTL Design, VHDL, and Verilog*, 2nd Edition,
by Frank Vahid, John Wiley and Sons Publishers, 2010.
http://www.ddvahid.com

Introduction

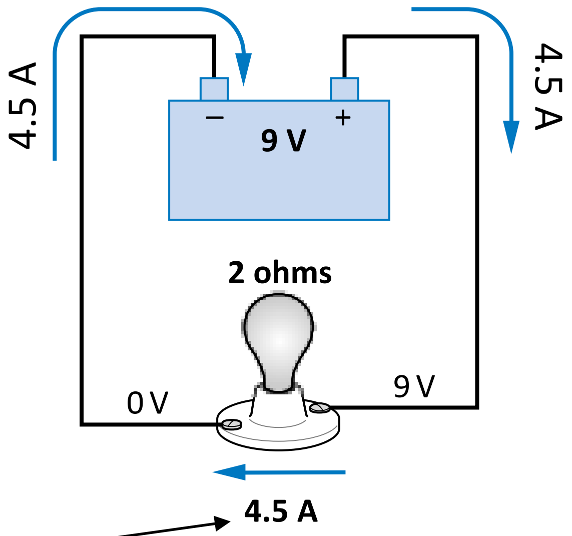
- Let’s learn to design digital circuits, starting with a simple form of circuit:
  - *Combinational circuit*
    - Outputs depend solely on the *present **combination** of the circuit inputs’ values*
    - Vs. sequential circuit: Has “memory” that impacts outputs too



Note: Slides with animation are denoted with a small red "a" near the animated items

Switches

- Electronic switches are the basis of binary digital circuits
  - Electrical terminology
    - ***Voltage***: Difference in electric potential between two points (volts, V)
      - Analogous to water pressure
    - ***Resistance***: Tendency of wire to resist current flow (ohms, Ω)
      - Analogous to water pipe diameter
    - ***Current***: Flow of charged particles (amps, A)
      - Analogous to water flow
    - V = I * R (Ohm's Law)
      - 9 V = I * 2 ohms
      - I = 4.5 A

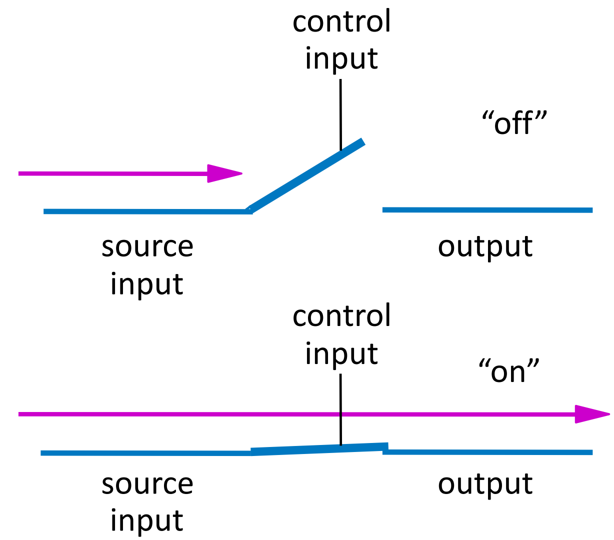


a

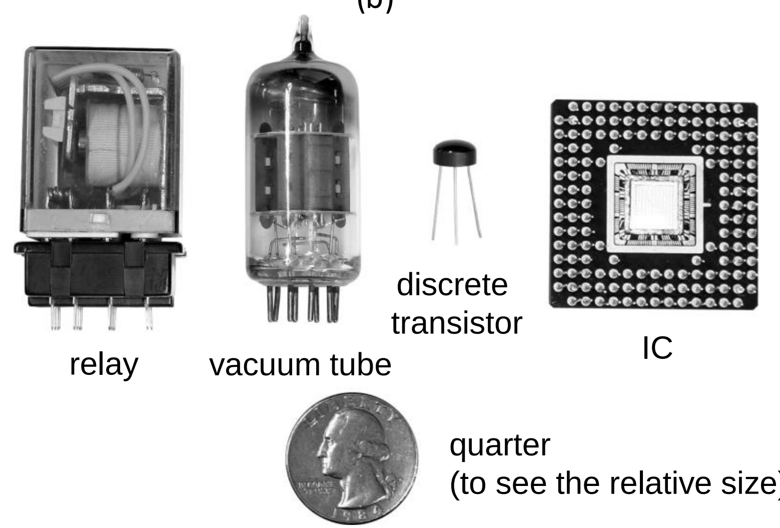
If a 9V potential difference is applied across a 2 ohm resistor, then 4.5 A of current will flow.

Switches

- A switch has three parts
  - Source input, and output
    - Current tries to flow from source input to output
  - Control input
    - Voltage that controls whether that current can flow
- The amazing shrinking switch
  - 1930s: Relays
  - 1940s: Vacuum tubes
  - 1950s: Discrete transistor
  - 1960s: Integrated circuits (ICs)
    - Initially just a few transistors on IC
    - Then tens, hundreds, thousands...

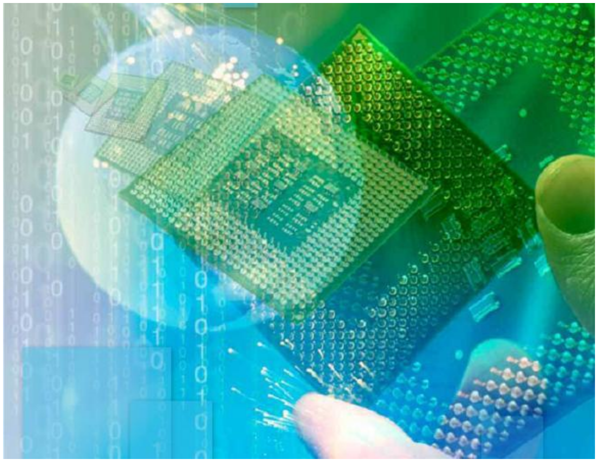


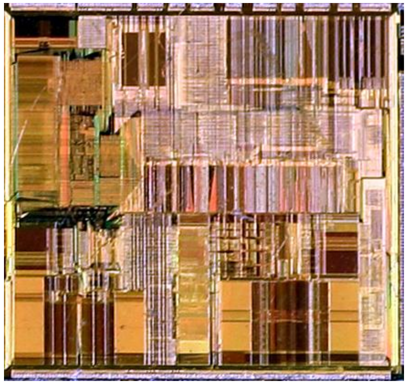
(b)



Moore's Law

- IC capacity doubling about every 18 months for several decades
  - Known as "Moore's Law" after Gordon Moore, co-founder of Intel
    - Predicted in 1965 predicted that components per IC would double roughly every year or so
  - Book cover depicts related phenomena
    - For a particular number of transistors, the IC area shrinks by half every 18 months
      - Consider how much shrinking occurs in just 10 years (try drawing it)
      - Enables incredibly powerful computation in incredibly tiny devices
  - Today's ICs hold *billions* of transistors
    - The first Pentium processor (early 1990s) needed only 3 million

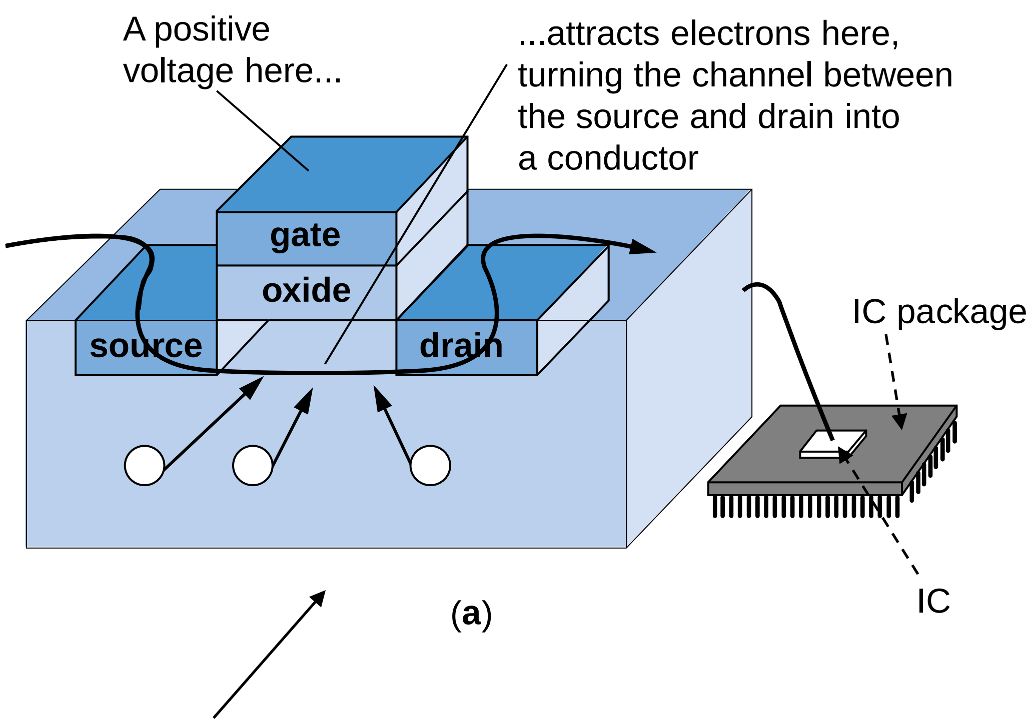




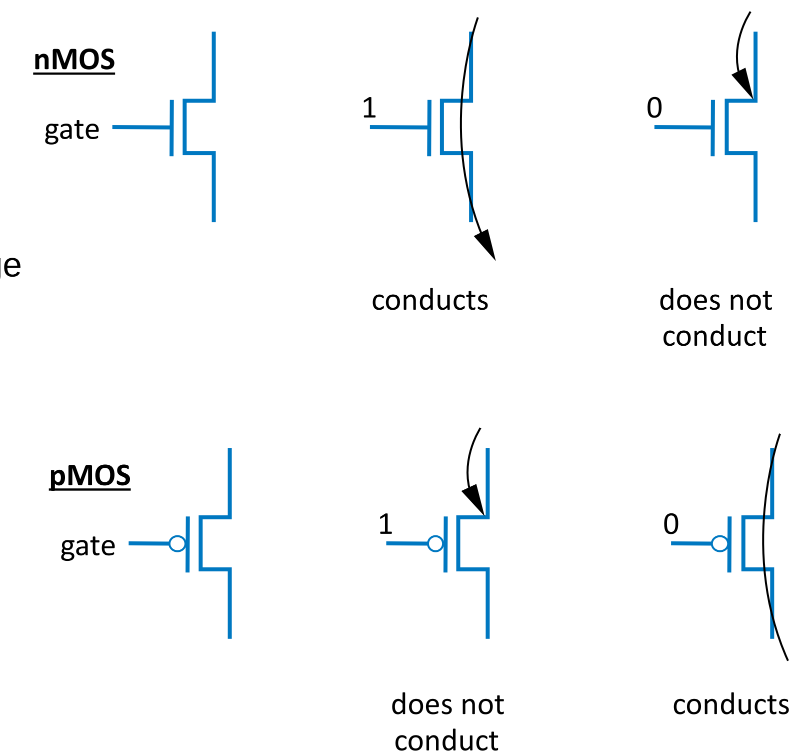
An Intel Pentium processor IC having millions of transistors

The CMOS Transistor

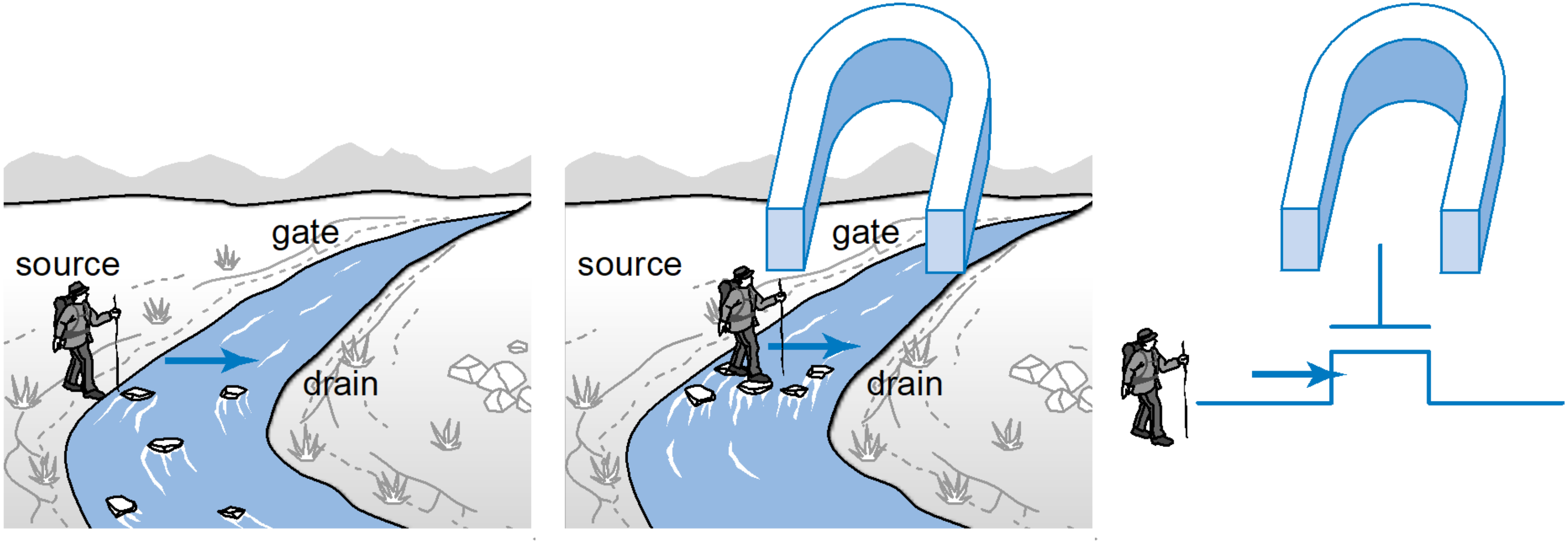
- CMOS transistor
  - Basic switch in modern ICs



Silicon -- not quite a conductor or insulator:
Semiconductor



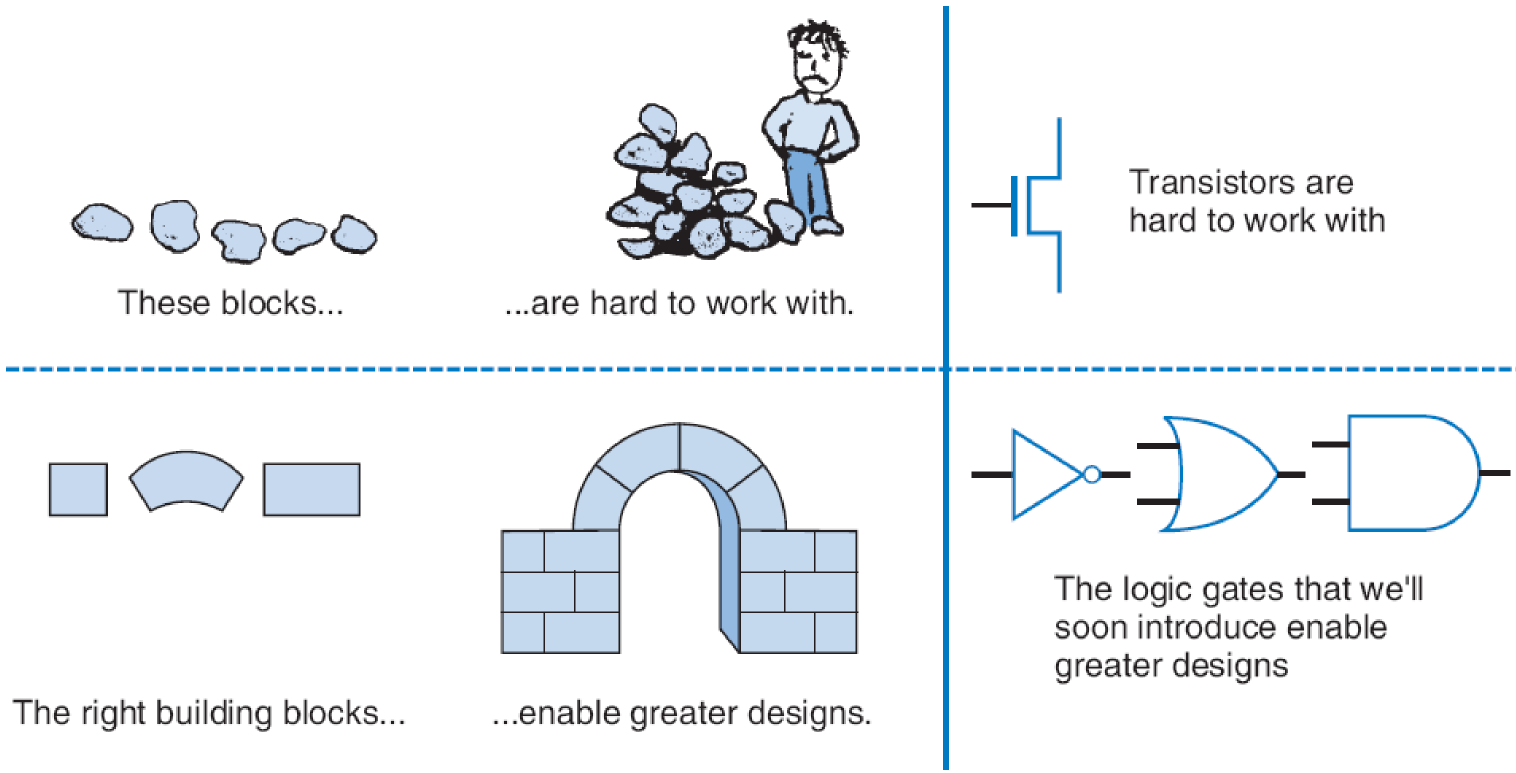
CMOS Transistor Analogy



Boolean Logic Gates

Building Blocks for Digital Circuits

(Because Switches are Hard to Work With)



- "Logic gates" are better digital circuit building blocks than switches (transistors)
  - Why?...

Boolean Algebra and its Relation to Digital Circuits

- To understand the benefits of “logic gates” vs. switches, we should first understand Boolean algebra
- “Traditional” algebra
  – Variables represent real numbers (x, y)
  – Operators operate on variables, return real numbers (2.5*x + y - 3)
- ***Boolean Algebra***
  – Variables represent 0 or 1 only
  – Operators return 0 or 1 only
  – Basic operators
    - AND: *a AND b* returns 1 only when both a=1 and b=1
    - OR: *a OR b* returns 1 if either (or both) a=1 or b=1
    - NOT: *NOT a* returns the opposite of a (1 if a=0, 0 if a=1)

a	b	AND
0	0	0
0	1	0
1	0	0
1	1	1

a

a	b	OR
0	0	0
0	1	1
1	0	1
1	1	1

a	NOT
0	1
1	0

Boolean Algebra and its Relation to Digital Circuits

- Developed mid-1800's by George Boole to formalize human thought
  - Ex: "I'll go to lunch if Mary goes OR John goes, AND Sally does not go."
    - Let F represent my going to lunch (1 means I go, 0 I don't go)
    - Likewise, m for Mary going, j for John, and s for Sally
    - Then **F = (m OR j) AND NOT(s)**
  - Nice features
    - Formally evaluate
      - m=1, j=0, s=1 --> F = (1 OR 0) AND NOT(1) = 1 AND 0 = **<u>0</u>**
    - Formally transform
      - F = (m and NOT(s)) OR (j and NOT(s))
        - » Looks different, but same function
        - » We'll show transformation techniques soon
    - Formally prove
      - Prove that if Sally goes to lunch (s=1), then I don't go (F=0)
      - F = (m OR j) AND NOT(1) = (m OR j) AND 0 = 0

a	b	AND
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR
0	0	0
0	1	1
1	0	1
1	1	1

a	NOT
0	1
1	0

Evaluating Boolean Equations

- Evaluate the Boolean equation **F = (a AND b) OR (c AND d)** for the given values of variables a, b, c, and d:
  - Q1: a=1, b=1, c=1, d=0.
    - Answer: F = (1 AND 1) OR (1 AND 0) = 1 OR 0 = 1.
  - Q2: a=0, b=1, c=0, d=1.
    - Answer: F = (0 AND 1) OR (0 AND 1) = 0 OR 0 = 0.
  - Q3: a=1, b=1, c=1, d=1.
    - Answer: F = (1 AND 1) OR (1 AND 1) = 1 OR 1 = 1.

a	b	AND
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR
0	0	0
0	1	1
1	0	1
1	1	1

a	NOT
0	1
1	0

Converting to Boolean Equations

- a • Convert the following English statements to a Boolean equation
  - Q1. a is 1 and b is 1.
    - Answer: F = a AND b
  - Q2. either of a or b is 1.
    - Answer: F = a OR b
  - Q3. a is 1 and b is 0.
    - Answer: F = a AND NOT(b)
  - Q4. a is not 0.
    - Answer:
      - (a) Option 1: F = NOT(NOT(a))
      - (b) Option 2: F = a

Converting to Boolean Equations

- Q1. A fire sprinkler system should spray water if high heat is sensed and the system is set to enabled.
  - Answer: Let Boolean variable h represent "high heat is sensed," e represent "enabled," and F represent "spraying water." Then an equation is: F = h AND e.
- Q2. A car alarm should sound if the alarm is enabled, and either the car is shaken or the door is opened.
  - Answer: Let a represent "alarm is enabled," s represent "car is shaken," d represent "door is opened," and F represent "alarm sounds." Then an equation is: F = a AND (s OR d).
  - (a) Alternatively, assuming that our door sensor d represents "door is closed" instead of open (meaning d=1 when the door is closed, 0 when open), we obtain the following equation: F = a AND (s OR NOT(d)).

Relating Boolean Algebra to Digital Design

Boolean algebra (mid-1800s)

Boole's intent: formalize human thought

Switches (1930s)

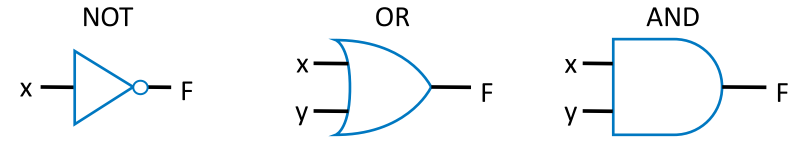
For telephone switching and other electronic uses

Shannon (1938)

Showed application of Boolean algebra to design of switch-based circuits

Digital design

Symbol



Truth table

NOT

x	F
0	1
1	0

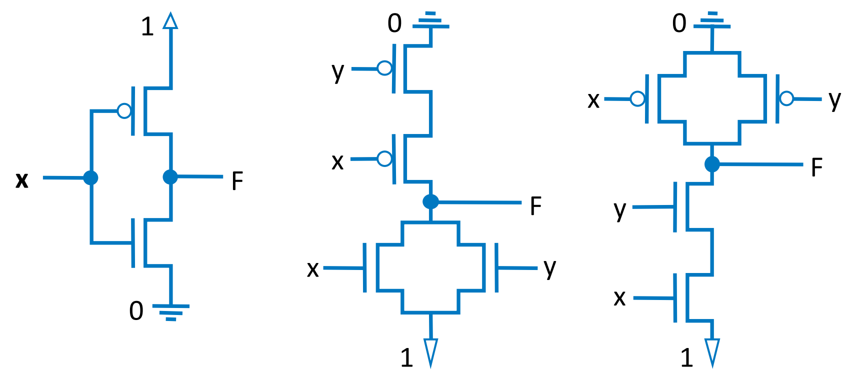
OR

x	y	F
0	0	0
0	1	1
1	0	1
1	1	1

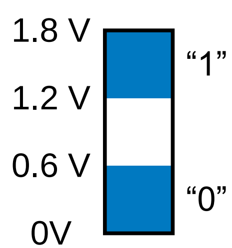
AND

x	y	F
0	0	0
0	1	0
1	0	0
1	1	1

Transistor circuit



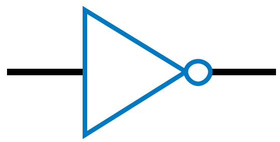
- Implement Boolean operators using transistors
  - Call those implementations ***logic gates***.
  - ***Lets us build circuits by doing math*** -- powerful concept



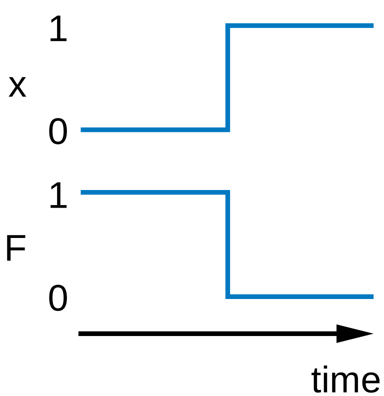
1 and 0 each actually corresponds to a voltage range

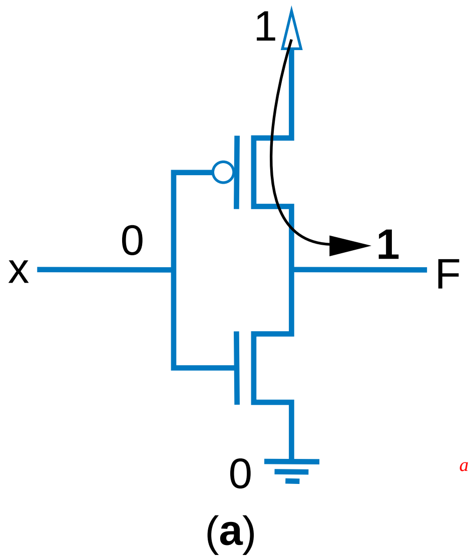
Next slides show how these circuits work. Note: The above OR/AND implementations are inefficient; we'll show why, and show better ones, later.

NOT gate

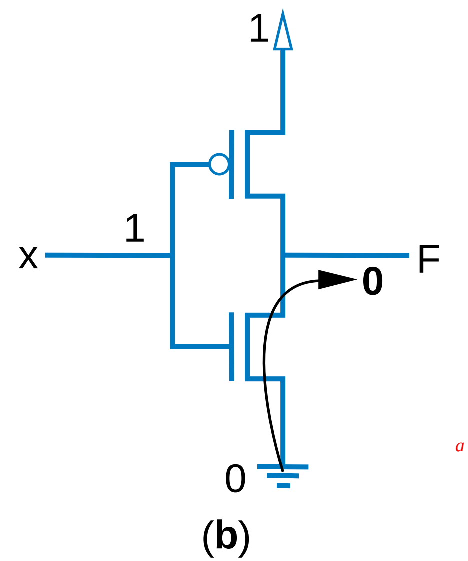


x	F
0	1
1	0



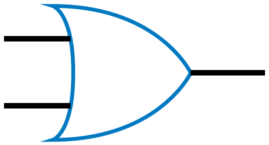


When the input is 0

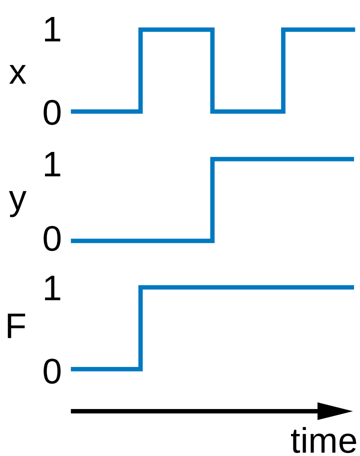


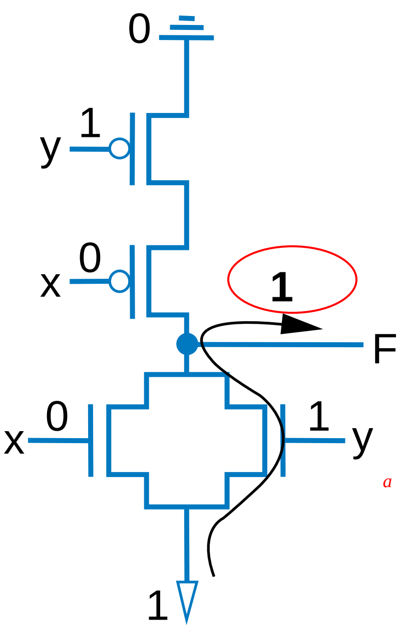
When the input is 1

OR gate



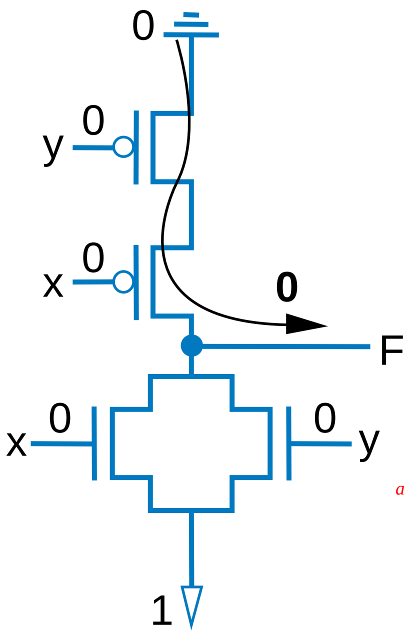
x	y	F
0	0	0
0	1	1
1	0	1
1	1	1





(a)

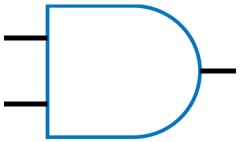
When an input is 1



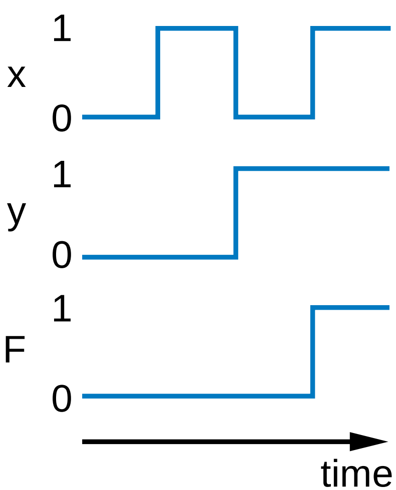
(b)

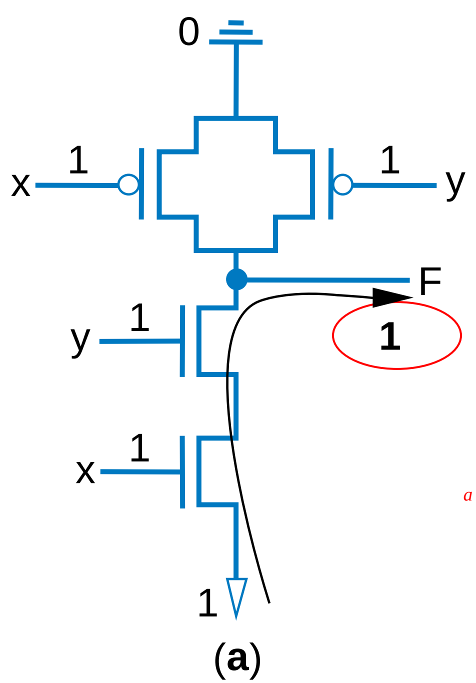
When both inputs are 0

AND gate

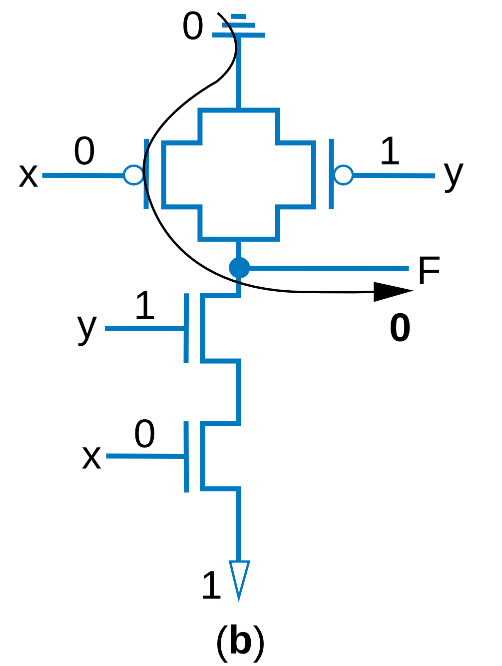


x	y	F
0	0	0
0	1	0
1	0	0
1	1	1



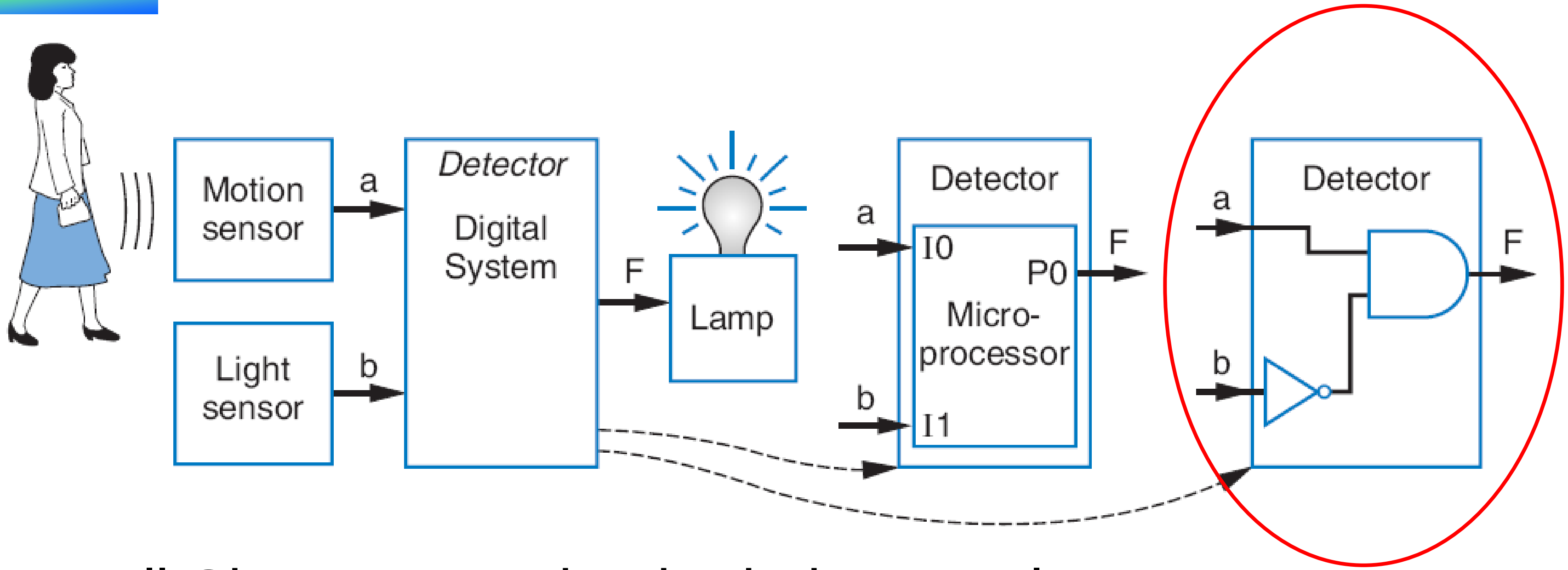


When both inputs are 1



When an input is 0

Building Circuits Using Gates



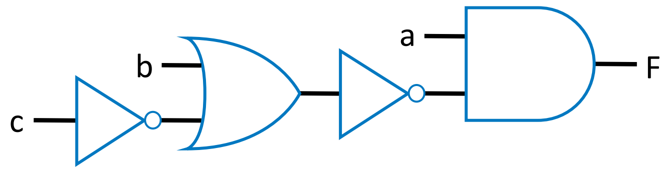
- Recall Chapter 1 motion-in-dark example
  - Turn on lamp (F=1) when motion sensed (a=1) and no light (b=0)
  - F = a AND NOT(b)
  - Build using logic gates, AND and NOT, as shown
  - We just built our first digital circuit!

Example: Converting a Boolean Equation to a Circuit of Logic Gates

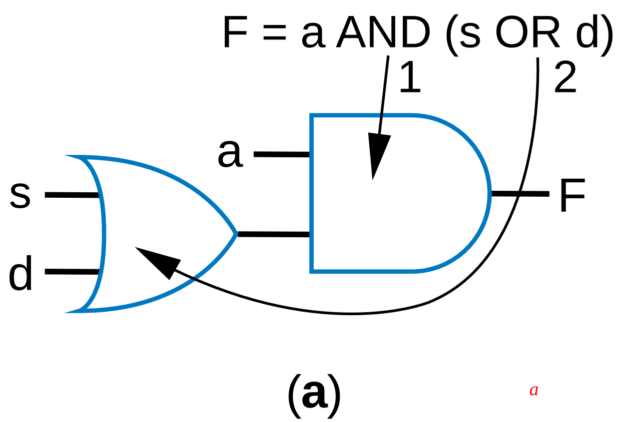
Start from the output, work back towards the inputs

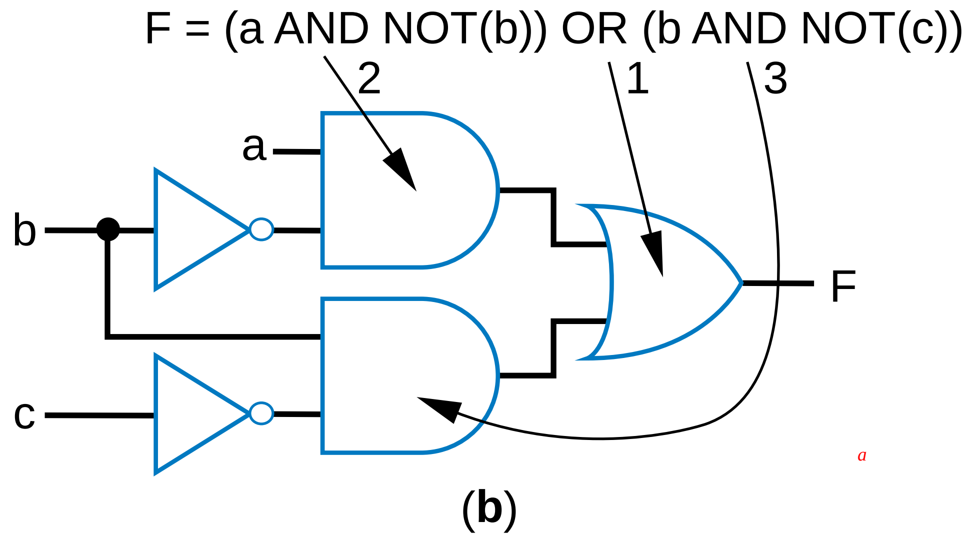
- Q: Convert the following equation to logic gates:
  F = a AND NOT(b OR NOT(c))

a



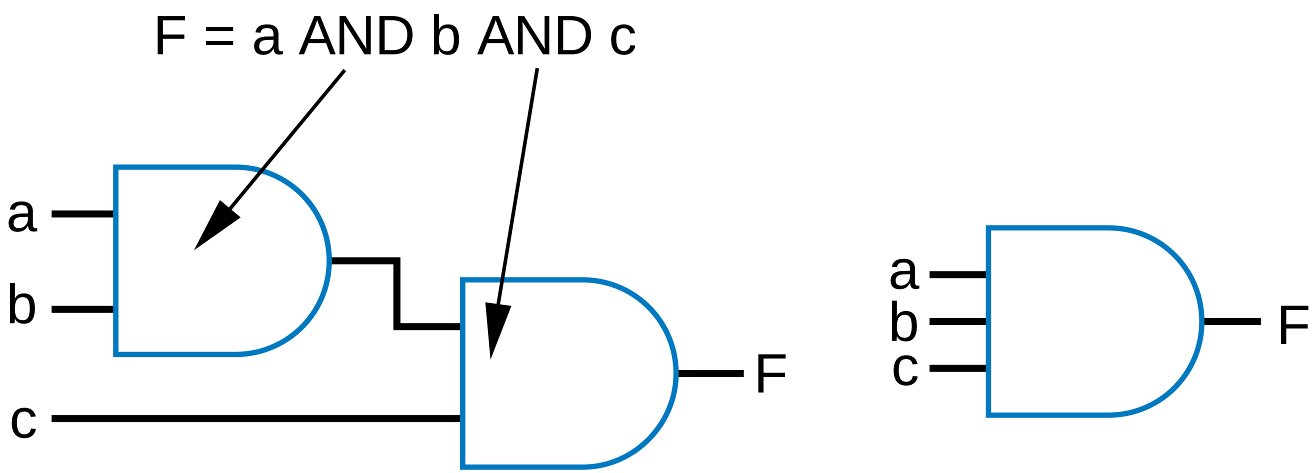
More examples





Start from the output, work back towards the inputs

Using gates with more than 2 inputs



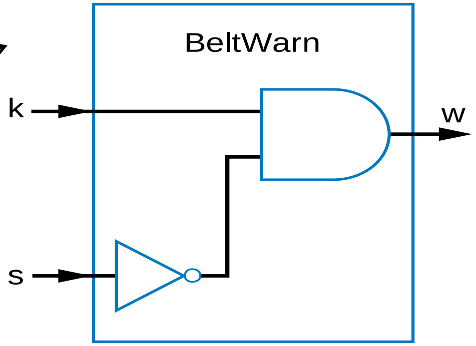
(a) **(b)**

Can think of as AND(a,b,c)

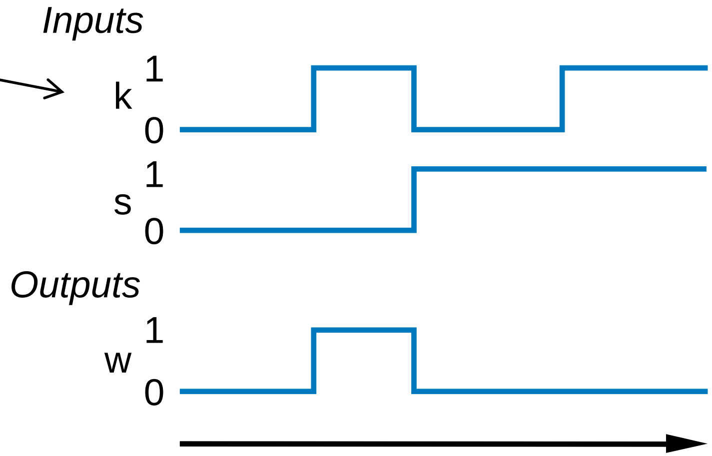
Example: Seat Belt Warning Light System

- Design circuit for warning light
- Sensors
  - s=1: seat belt fastened
  - k=1: key inserted
- Capture Boolean equation
  - seat belt not fastened, and key inserted
- Convert equation to circuit

w = NOT(s) AND k



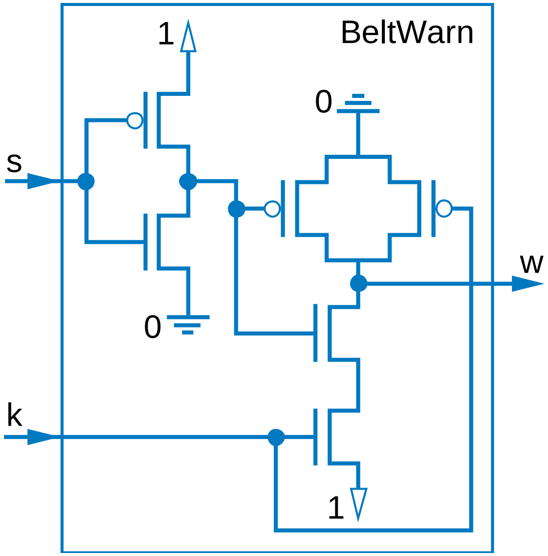
- *Timing diagram* illustrates circuit behavior
  - We set inputs to any values
  - Output set according to circuit



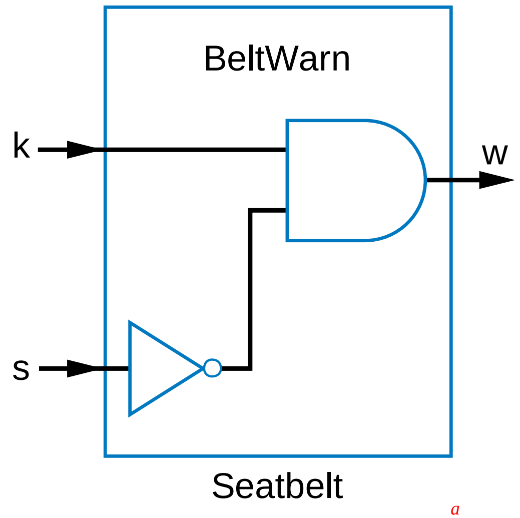
Gates vs. switches

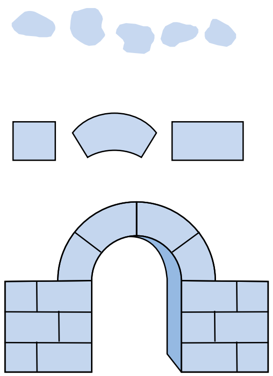
Notice

- Boolean algebra enables easy capture as equation and conversion to circuit
  - How design with switches?
  - Of course, logic gates are built from switches, but we think at level of logic gates, not switches



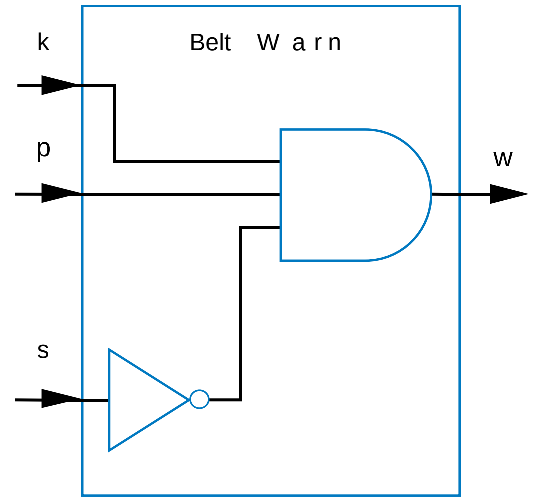
w = NOT(s) AND k



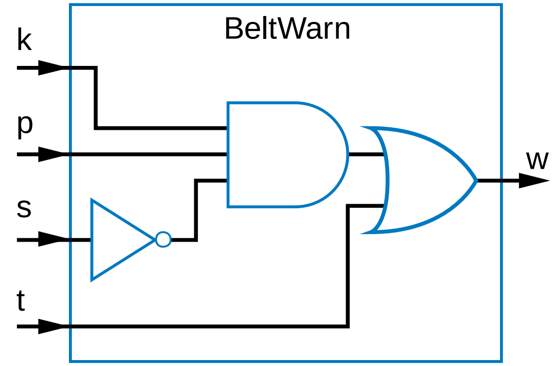


More examples: Seat belt warning light extensions

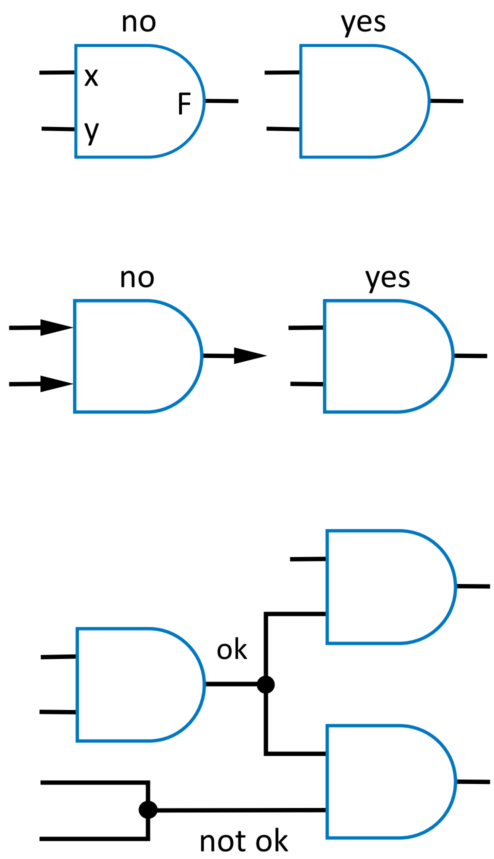
- Only illuminate warning light if person is in the seat (p=1), and seat belt not fastened and key inserted
- w = p AND NOT(s) AND k



- Given t=1 for 5 seconds after key inserted. Turn on warning light when t=1 (to check that warning lights are working)
- w = (p AND NOT(s) AND k) OR t



Some Gate-Based Circuit Drawing Conventions



a

a

Boolean Algebra

- By defining logic gates based on Boolean algebra, we can *use algebraic methods to manipulate circuits*
- Notation: Writing a AND b, a OR b, NOT(a) is cumbersome
  - Use symbols: a * b (or just ab), a + b, and a’
    - Original: w = (p AND NOT(s) AND k) OR t
    - New: w = ps’k + t
      - Spoken as “w equals p and s prime and k, or t”
      - Or just “w equals p s prime k, or t”
      - s’ known as “complement of s”
    - While symbols come from regular algebra, ***don’t*** say “times” or “plus”
      - "product" and "sum" are OK and commonly used

Boolean algebra precedence, highest precedence first.

Symbol	Name	Description
( )	Parentheses	Evaluate expressions nested in parentheses first
’	NOT	Evaluate from left to right
*	AND	Evaluate from left to right
+	OR	Evaluate from left to right

Boolean Algebra Operator Precedence

- Evaluate the following Boolean equations, assuming a=1, b=1, c=0, d=1.
  - Q1. F = a * b + c.
    - Answer: * has precedence over +, so we evaluate the equation as F = (1 *1) + 0 = (1) + 0 = 1 + 0 = 1.
  - Q2. F = ab + c.
    - Answer: the problem is identical to the previous problem, using the shorthand notation for *.
  - Q3. F = ab'.
    - Answer: we first evaluate b' because NOT has precedence over AND, resulting in F = 1 * (1') = 1 * (0) = 1 * 0 = 0.
  - Q4. F = (ac)'.
    - Answer: we first evaluate what is inside the parentheses, then we NOT the result, yielding (1*0)' = (0)' = 0' = 1.
  - Q5. F = (a + b') * c + d'.
    - Answer: Inside left parentheses: (1 + (1')) = (1 + (0)) = (1 + 0) = 1. Next, * has precedence over +, yielding (1 * 0) + 1' = (0) + 1'. The NOT has precedence over the OR, giving (0) + (1') = (0) + (0) = 0 + 0 = 0.

Boolean algebra precedence, highest precedence first.

Symbol	Name	Description
( )	Parentheses	Evaluate expressions nested in parentheses first
'	NOT	Evaluate from left to right
*	AND	Evaluate from left to right
+	OR	Evaluate from left to right

Boolean Algebra Terminology

- Example equation: **F(a,b,c) = a'bc + abc' + ab + c**
- ***Variable***
  - Represents a value (0 or 1)
  - Three variables: a, b, and c
- ***Literal***
  - Appearance of a variable, in true or complemented form
  - Nine literals: a', b, c, a, b, c', a, b, and c
- ***Product term***
  - Product of literals
  - Four product terms: a'bc, abc', ab, c
- ***Sum-of-products***
  - Equation written as OR of product terms only
  - Above equation is in sum-of-products form. "F = (a+b)c + d" is not.

Boolean Algebra Properties

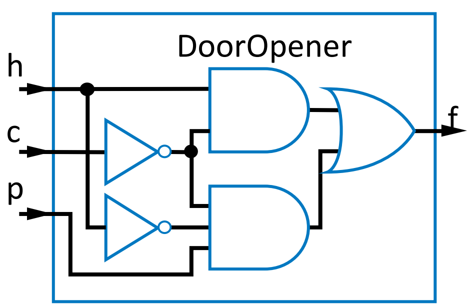
- Commutative
  - $a + b = b + a$
  - $a * b = b * a$
- Distributive
  - $a * (b + c) = a * b + a * c$
    - Can write as: $a(b+c) = ab + ac$
  - $a + (b * c) = (a + b) * (a + c)$
    - (This second one is tricky!)
    - Can write as: $a+(bc) = (a+b)(a+c)$
- Associative
  - $(a + b) + c = a + (b + c)$
  - $(a * b) * c = a * (b * c)$
- Identity
  - $0 + a = a + 0 = a$
  - $1 * a = a * 1 = a$
- Complement
  - $a + a' = 1$
  - $a * a' = 0$
- To prove, just evaluate all possibilities

Example uses of the properties

- Show abc' equivalent to $c'ba$.
  - Use commutative property:
    - $a*b*c' = a*c'*b = c'*a*b = c'*b*a$
- Show $abc + abc' = ab$.
  - Use first distributive property
    - $abc + abc' = ab(c+c')$.
  - Complement property
    - Replace $c+c'$ by 1: $ab(c+c') = ab(1)$.
  - Identity property
    - $ab(1) = ab*1 = ab$.
- Show $x + x'z$ equivalent to $x + z$.
  - Second distributive property
    - Replace $x+x'z$ by $(x+x')*(x+z)$.
  - Complement property
    - Replace $(x+x')$ by 1,
  - Identity property
    - replace $1*(x+z)$ by $x+z$.

Example that Applies Boolean Algebra Properties

- Want automatic door opener circuit (e.g., for grocery store)
  - Output: $f=1$ opens door
  - Inputs:
    - $p=1$: person detected
    - $h=1$: switch forcing hold open
    - $c=1$: key forcing closed
  - Want open door when
    - $h=1$ and $c=0$, or
    - $h=0$ and $p=1$ and $c=0$
  - Equation: $f = hc' + h'pc'$



- Can the circuit be simplified?

$f = hc' + h'pc'$

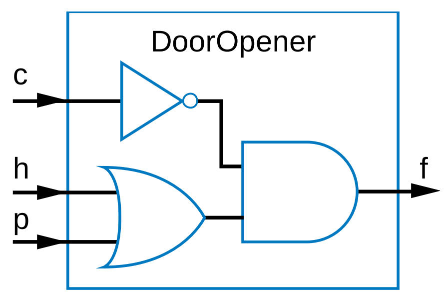
$f = c'h + c'h'p$ (by the commutative property)

$f = c'(h + h'p)$ (by the first distrib. property)

$f = c'((h+h')*(h+p))$ (2nd distrib. prop.; tricky one)

$f = c'((1)*(h + p))$ (by the complement property)

$f = c'(h+p)$ (by the identity property)

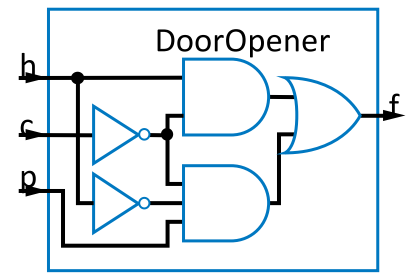


Simplified circuit

Simplification of circuits is covered in Sec. 2.11 / Sec 6.2.

Example that Applies Boolean Algebra Properties

- Found inexpensive chip that computes:
  - f = c'hp + c'hp' + c'h'p
  - Can we use it for the door opener?
    - Is it the same as f = hc' + h'pc'?
- Apply Boolean algebra:



- Commutative
  - a + b = b + a
  - a * b = b * a
- Distributive
  - a * (b + c) = a * b + a * c
  - a + (b * c) = (a + b) * (a + c)
- Associative
  - (a + b) + c = a + (b + c)
  - (a * b) * c = a * (b * c)
- Identity
  - 0 + a = a + 0 = a
  - 1 * a = a * 1 = a
- Complement
  - a + a' = 1
  - a * a' = 0

f = c'hp + c'hp' + c'h'p

f = c'h(p + p') + c'h'p (by the distributive property)

f = c'h(1) + c'h'p (by the complement property)

f = c'h + c'h'p (by the identity property)

f = hc' + h'pc' (by the commutative property)

a

Same! Yes, we can use it.

Boolean Algebra: Additional Properties

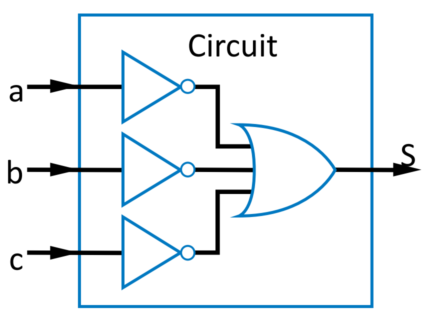
- Null elements
  - $a + 1 = 1$
  - $a * 0 = 0$
- Idempotent Law
  - $a + a = a$
  - $a * a = a$
- Involution Law
  - $(a')' = a$
- DeMorgan's Law
  - $(a + b)' = a'b'$
  - $(ab)' = a' + b'$
  - *Very useful!*
- To prove, just evaluate all possibilities

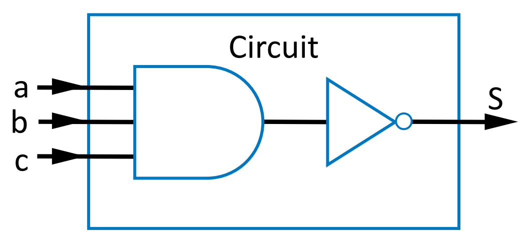
Example Applying DeMorgan's Law

$(a + b)' = a'b'$
$(ab)' = a' + b'$

Aircraft lavatory sign example

- Behavior
  - Three lavatories, each with sensor (a, b, c), equals 1 if door locked
  - Light "Available" sign (S) if any lavatory available
- Equation and circuit
  - $S = a' + b' + c'$
- Transform
  - $(abc)' = a'+b'+c'$ (by DeMorgan's Law)
  - $S = (abc)'$
- New circuit





- Alternative: Instead of lighting "Available," light "Occupied"
  - Opposite of "Available" function $S = a' + b' + c'$
  - So $S' = (a' + b' + c')'$
    - $S' = (a')' * (b')' * (c')'$ (by DeMorgan's Law)
    - $S' = a * b * c$ (by Involution Law)
  - Makes intuitive sense
    - Occupied if all doors are locked

Example Applying Properties

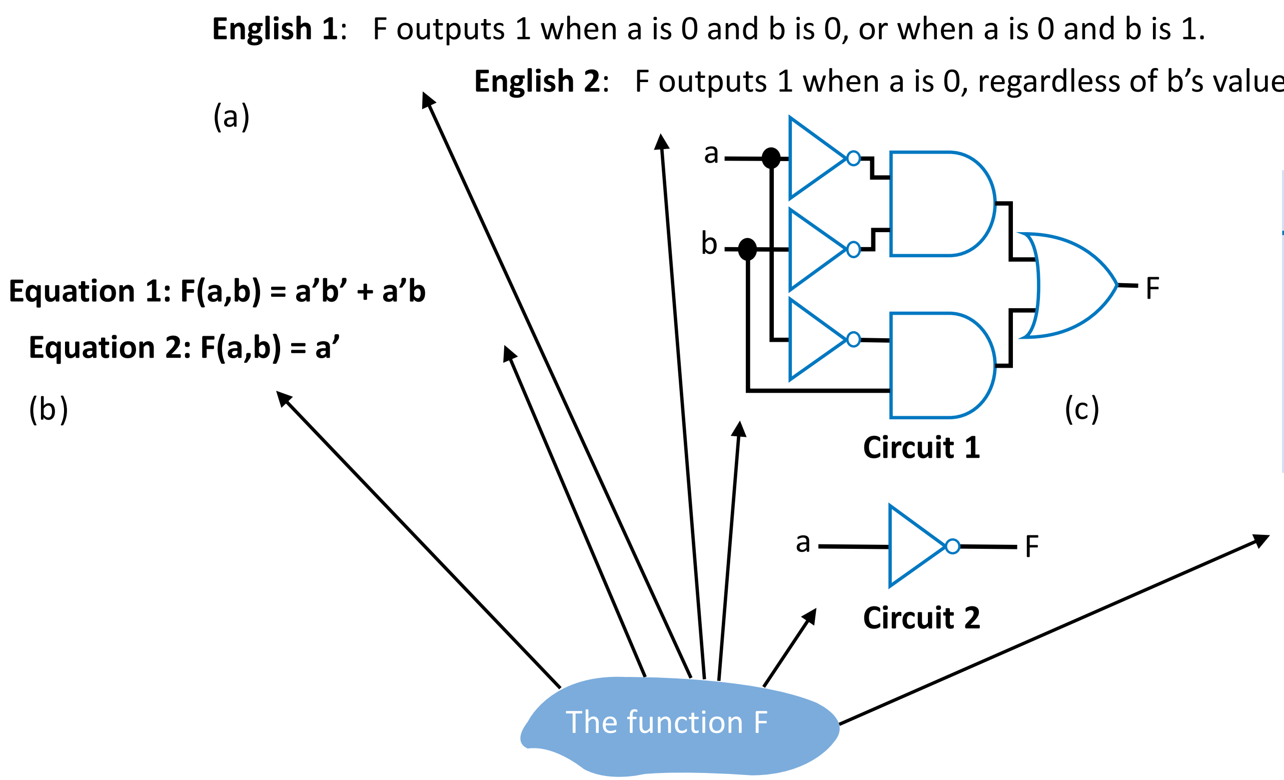
- Commutative
  - a + b = b + a
  - a * b = b * a
- Distributive
  - a * (b + c) = a * b + a * c
  - a + (b * c) = (a + b) * (a + c)
- Associative
  - (a + b) + c = a + (b + c)
  - (a * b) * c = a * (b * c)
- Identity
  - 0 + a = a + 0 = a
  - 1 * a = a * 1 = a
- Complement
  - a + a’ = 1
  - a * a’ = 0
- Null elements
  - a + 1 = 1
  - a * 0 = 0
- Idempotent Law
  - a + a = a
  - a * a = a
- Involution Law
  - (a’)’ = a
- DeMorgan’s Law
  - (a + b)’ = a’b’
  - (ab)’ = a’ + b’

- For door opener f = c'(h+p) , *prove* door stays closed (f=0) when c=1
  - f = c'(h+p)
  - *Let c = 1* (door forced closed)
  - f = 1'(h+p)
  - f = 0(h+p)
  - f = 0h + 0p (by the distributive property)
  - f = 0 + 0 (by the null elements property)
  - f = 0

Complement of a Function

- Commonly want to find complement (inverse) of function F
  - 0 when F is 1;   1 when F is 0
- Use DeMorgan’s Law repeatedly
  - Note: DeMorgan’s Law defined for more than two variables, e.g.:
    - (a + b + c)' = a’b’c’
    - (abc)' = (a' + b' + c’)
- Find complement of f where f = w'xy + wx'y'z'
  - f ' = (w'xy + wx'y'z')'
  - f ' = (w'xy)'(wx'y'z')'   (by DeMorgan’s Law)
  - f ' = (w+x'+y')(w'+x+y+z)   (by DeMorgan’s Law)
- Can then expand into sum-of-products form

Representations of Boolean Functions



a	b	F
0	0	1
0	1	1
1	0	0
1	1	0

Truth table

(d)

- A function can be represented in different ways
  - Above shows seven representations of the same functions F(a,b), using four different methods: English, Equation, Circuit, and Truth Table

Truth Table Representation of Boolean Functions

- Define value of F for each possible combination of input values
  - 2-input function: 4 rows
  - 3-input function: 8 rows
  - 4-input function: 16 rows
- Q: Use truth table to define function F(a,b,c) that is 1 when abc is 5 or greater in binary

a	b	F
0	0	
0	1	
1	0	
1	1	

(a)

a	b	c	F
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

(b)

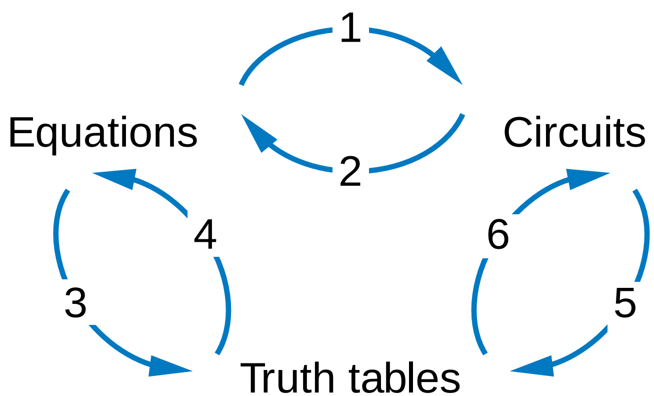
a	b	c	d	F
0	0	0	0	
0	0	0	1	
0	0	1	0	
0	0	1	1	
0	1	0	0	
0	1	0	1	
0	1	1	0	
0	1	1	1	
1	0	0	0	
1	0	0	1	
1	0	1	0	
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	

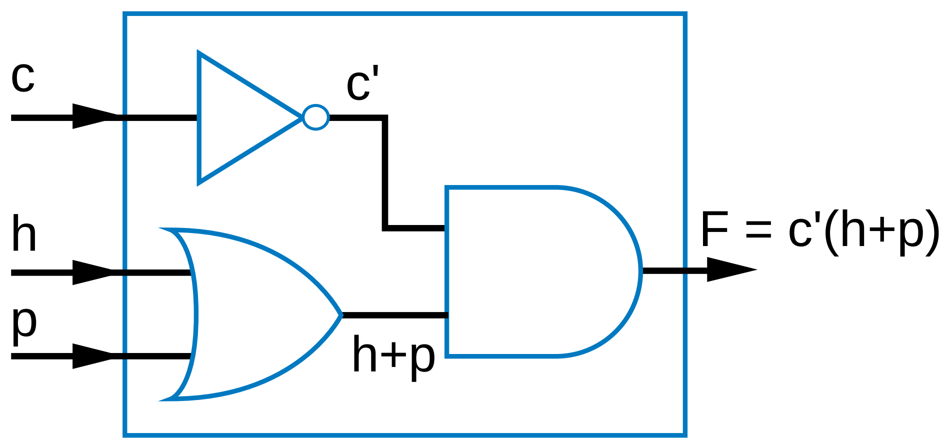
(c)

a	b	c	F
0	0	0	**0**
0	0	1	**0**
0	1	0	**0**
0	1	1	**0**
1	0	0	**0**
1	0	1	**1**
1	1	0	**1**
1	1	1	**1**

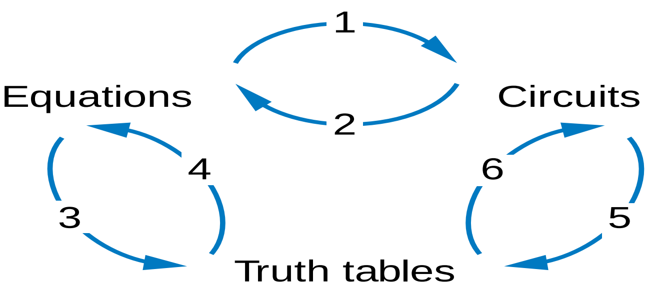
Converting among Representations

- Can convert from any representation to another
- Common conversions
  - Equation to circuit (we did this earlier)
  - Circuit to equation
    - Start at inputs, write expression of each gate output





Converting among Representations



- More common conversions
  - Truth table to equation (which we can then convert to circuit)
    - Easy–just OR each input term that should output 1
  - Equation to truth table
    - Easy—just evaluate equation for each input combination (row)
    - Creating intermediate columns helps

Inputs		Outputs	Term
a	b	F	F = sum of
0	0	1	a'b'
0	1	1	a'b
1	0	0	
1	1	0	

F = a'b' + a'b

Q: Convert to equation

a	b	c	F	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	1	ab'c
1	1	0	1	abc'
1	1	1	1	abc

F = ab'c + abc' + abc

Q: Convert to truth table: F = a'b' + a'b

Inputs				Output
a	b	a' b'	a' b	F
0	0	1	0	**1**
0	1	0	1	**1**
1	0	0	0	**0**
1	1	0	0	**0**

Example: Converting from Truth Table to Equation

- Parity bit: Extra bit added to data, intended to enable detection of error (a bit changed unintentionally)
  - e.g., errors can occur on wires due to electrical interference
- Even parity: Set parity bit so total number of 1s (data + parity) is even
  - e.g., if data is 001, parity bit is 1 → 0011 has even number of 1s
- Want equation, but easiest to start from truth table for this example

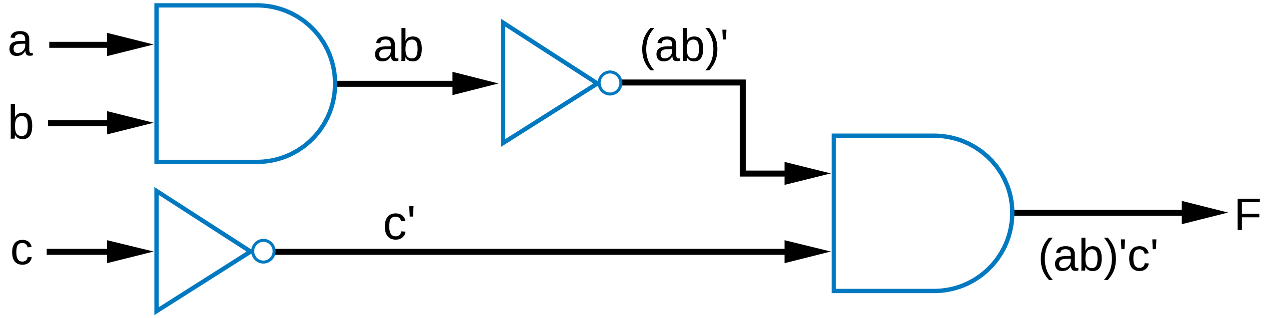
a	b	c	P
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Convert to eqn.

$$P = a'b'c + a'bc' + ab'c' + abc$$

Example: Converting from Circuit to Truth Table

- First convert to circuit to equation, then equation to table



Inputs						Outputs
a	b	c	ab	(ab)'	c'	F
0	0	0	0	1	1	1
0	0	1	0	1	0	0
0	1	0	0	1	1	1
0	1	1	0	1	0	0
1	0	0	0	1	1	1
1	0	1	0	1	0	0
1	1	0	1	0	1	0
1	1	1	1	0	0	0

Standard Representation: Truth Table

- How can we determine if two functions are the same?
  - Recall automatic door example
    - Same as f = hc' + h'pc'?
    - Used algebraic methods
    - But if we failed, does that prove *not* equal? No.
- Solution: Convert to truth tables
  - Only ONE truth table representation of a given function
    - ***Standard*** representation—for given function, only one version in standard form exists

f = c'hp + c'hp' + c'h'

f = c'h(p + p') + c'h'p

f = c'h(1) + c'h'p

f = c'h + c'h'p

(what if we stopped here?)

f = hc' + h'pc'

Q: Determine if F=ab+a' is same function as F=a'b'+a'b+ab, by converting each to truth table first

F = ab + a'

a	b	F
0	0	1
0	1	1
1	0	0
1	1	1

Same

F = a'b' + a'b + ab

a	b	F
0	0	1
0	1	1
1	0	0
1	1	1

a

Truth Table Canonical Form

- Q: Determine via truth tables whether ab+a' and (a+b)' are equivalent

F = ab + a′

a	b	F
0	0	1
0	1	1
1	0	0
1	1	1

F = (a+b)′

a	b	F
0	0	1
0	1	0
1	0	0
1	1	0

Not equivalent

a

Canonical Form – Sum of Minterms

- Truth tables too big for numerous inputs
- Use standard form of equation instead
  - Known as ***canonical form***
  - Regular algebra: group terms of polynomial by power
    - $ax^2 + bx + c$ $(3x^2 + 4x + 2x^2 + 3 + 1 \dashrightarrow 5x^2 + 4x + 4)$
  - Boolean algebra: create sum of minterms
    - ***Minterm***: product term with every function literal appearing exactly once, in true or complemented form
    - Just multiply-out equation until sum of product terms
    - Then expand each term until all terms are minterms

Q: Determine if F(a,b)=ab+a’ is equivalent to F(a,b)=a’b’+a’b+ab, by converting first equation to canonical form (second already is)

F = ab+a’ (already sum of products)

a F = ab + a’(b+b’) (expanding term)

F = ab + a’b + a’b’ (Equivalent – same three terms as other equation)

Canonical Form – Sum of Minterms

- Q: Determine whether the functions G(a,b,c,d,e) = abcd + a'bcde and H(a,b,c,d,e) = abcde + abcde' + a'bcde + a'bcde(a' + c) are equivalent.

G = abcd + a'bcde

G = abcd(e+e') + a'bcde

G = abcde + abcde' + a'bcde

G = a'bcde + abcde' + abcde *(sum of minterms form)*

Equivalent

a

H = abcde + abcde' + a'bcde + a'bcde(a' + c)

H = abcde + abcde' + a'bcde + a'bcdea' + a'bcdec

H = abcde + abcde' + a'bcde + a'bcde + a'bcde

H = abcde + abcde' + a'bcde

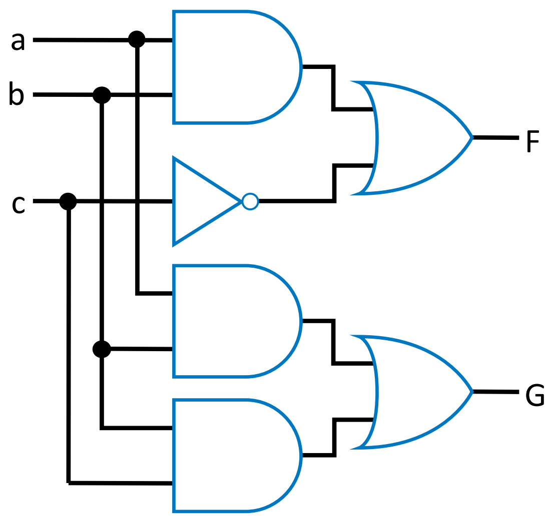
H = a'bcde + abcde' + abcde

Compact Sum of Minterms Representation

- List each minterm as a number
- Number determined from the binary representation of its variables' values
  - a'bcde corresponds to 01111, or 15
  - abcde' corresponds to 11110, or 30
  - abcde corresponds to 11111, or 31
- Thus, H = a'bcde + abcde' + abcde can be written as:
  - $H = \sum m(15,30,31)$
  - "H is the sum of minterms 15, 30, and 31"

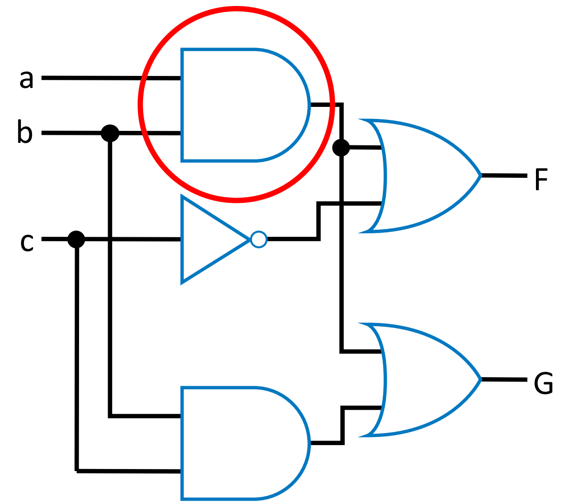
Multiple-Output Circuits

- Many circuits have more than one output
- Can give each a separate circuit, or can share gates
- Ex: $F = ab + c'$, $G = ab + bc$



(a)

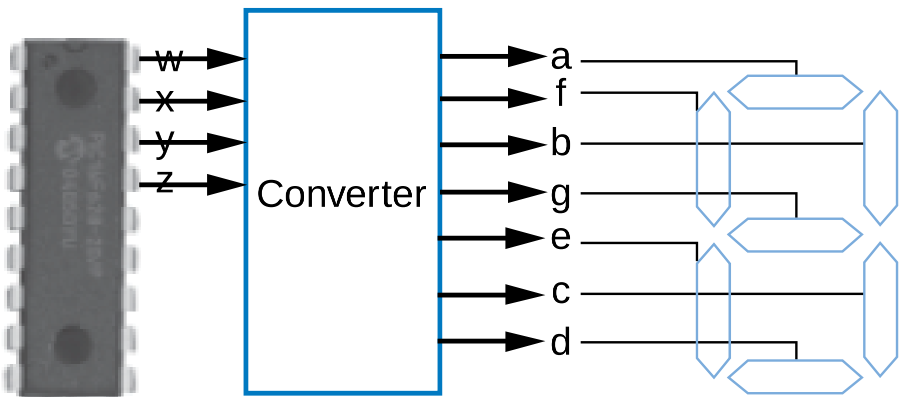
Option 1: Separate circuits

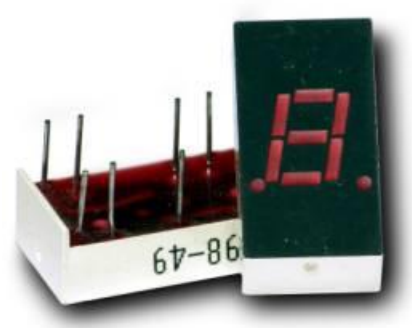


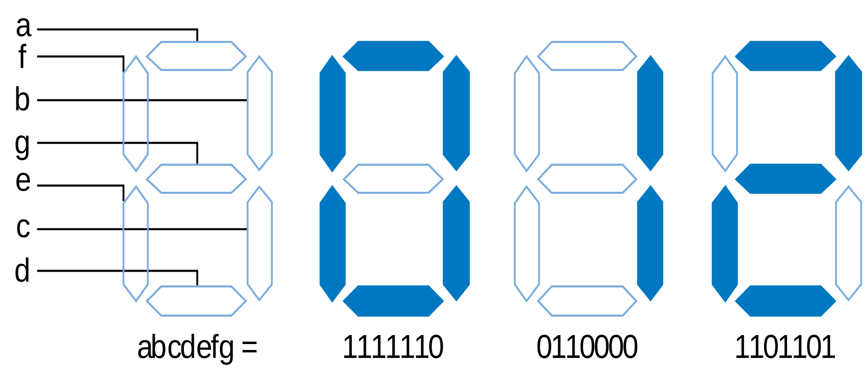
(b)

Option 2: Shared gates

Multiple-Output Example:
BCD to 7-Segment Converter





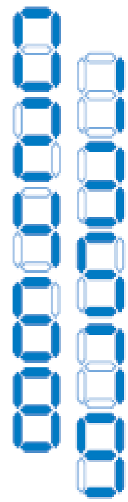


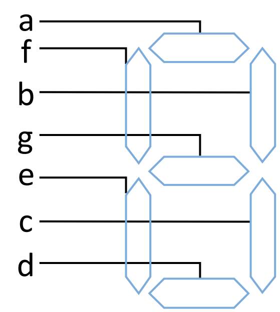
(a) (b)

Multiple-Output Example:
BCD to 7-Segment Converter

TABLE 2-4 4-bit binary number to seven-segment display truth table

w	x	y	z	a	b	c	d	e	f	g
0	0	0	0	1	1	1	1	1	1	0
0	0	0	1	0	1	1	0	0	0	0
0	0	1	0	1	1	0	1	1	0	1
0	0	1	1	1	1	1	1	0	0	1
0	1	0	0	0	1	1	0	0	1	1
0	1	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	0	0	0	0
1	0	0	0	1	1	1	1	1	1	1
1	0	0	1	1	1	1	1	0	1	1
1	0	1	0	0	0	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0





$a = w'x'y'z' + w'x'yz' + w'x'yz + w'xy'z + w'xyz' + w'xyz + wx'y'z' + wx'y'z$

$b = w'x'y'z' + w'x'y'z + w'x'yz' + w'x'yz + w'xy'z' + w'xyz + wx'y'z' + wx'y'z$

…

Combinational Logic Design Process

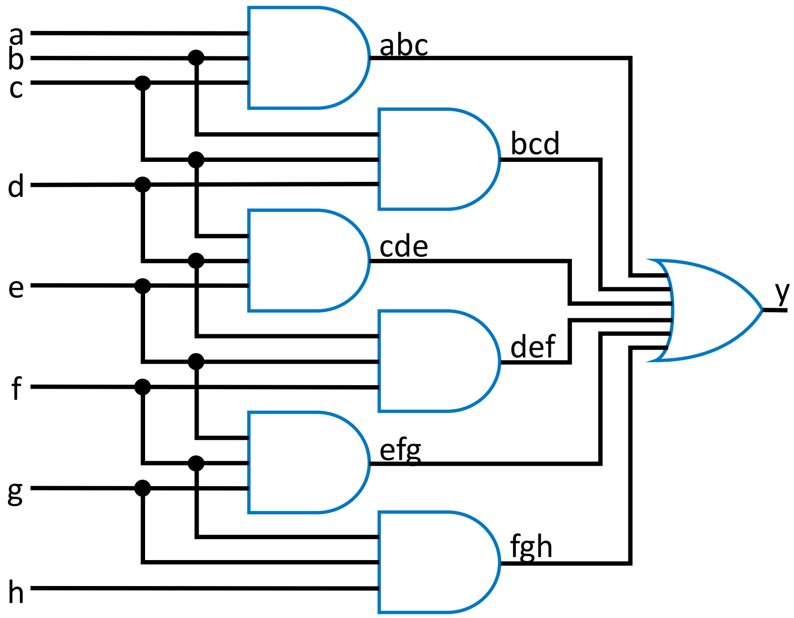
Step		Description
Step 1: Capture behavior	**Capture** the function	Create a truth table or equations, ***whichever is most natural for the given problem***, to describe the desired behavior of each output of the combinational logic.
Step 2: Convert to circuit	2A: **Create** equations	This substep is only necessary if you captured the function using a truth table instead of equations. Create an equation for each output by ORing all the minterms for that output. Simplify the equations if desired.
	2B: **Implement** as a gate-based circuit	For each output, create a circuit corresponding to the output's equation. (Sharing gates among multiple outputs is OK optionally.)

Example: Three 1s Pattern Detector

- Problem: Detect three consecutive 1s in 8-bit input: abcdefgh
    - 00**111**01 → 1
    - 10101011 → 0
    - **111**10000 → 1
  - **Step 1: Capture** the function
    - Truth table or equation?
      - Truth table too big: 2^8=256 rows
      - Equation: create terms for each possible case of three consecutive 1s
    - **y = abc + bcd + cde + def + efg + fgh**
  - **Step 2a: Create** equation -- already done
  - **Step 2b: Implement** as a gate-based circuit

a

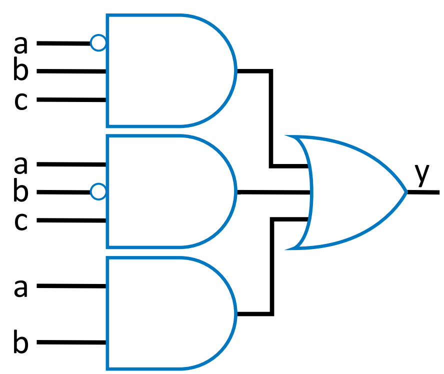
a

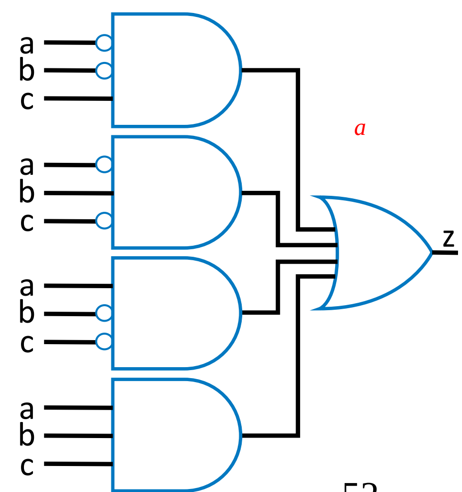


Example: Number of 1s Counter

- Problem: Output in binary on two outputs yz the # of 1s on three inputs
    - 010 → 01
    - 101 → 10
    - 000 → 00
- – **Step 1: Capture** the function
    - Truth table or equation?
        - – Truth table is straightforward
- – **Step 2a: Create** equations
    - y = a’bc + ab’c + abc’ + abc
    - z = a’b’c + a’bc’ + ab’c’ + abc
    - Optional: Let's simplify y:
        - – y = a'bc + ab'c + ab(c' + c) = a'bc + ab'c + ab
- – **Step 2b: Implement** as a gate-based circuit

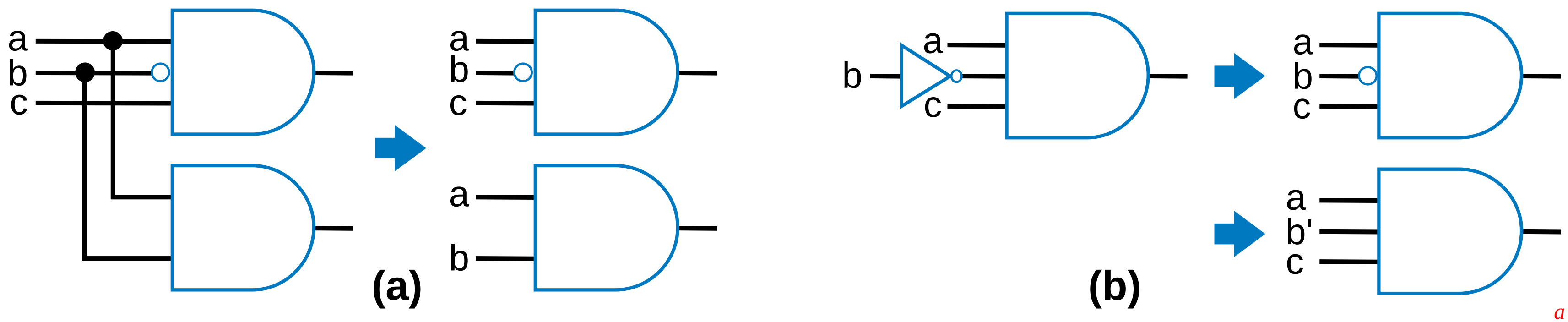
Inputs			(# of 1s)	Outputs	
a	b	c		y	z
0	0	0	(0)	0	0
0	0	1	(1)	0	1
0	1	0	(1)	0	1
0	1	1	(2)	1	0
1	0	0	(1)	0	1
1	0	1	(2)	1	0
1	1	0	(2)	1	0
1	1	1	(3)	1	1





Simplifying Notations

- Used in previous circuit

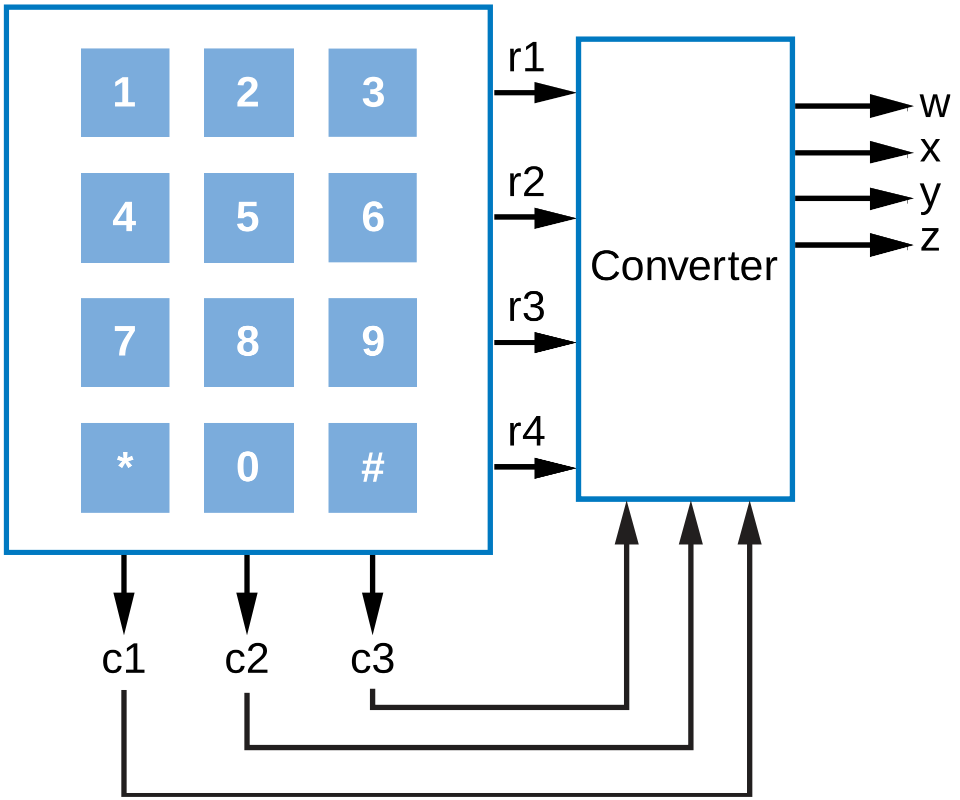


List inputs multiple times
→ Less wiring in drawing

Draw inversion bubble rather than inverter. Or list input as complemented.

Example: Keypad Converter

- Keypad has 7 outputs
  - One per row
  - One per column
- Key press sets one row and one column output to 1
  - Press "5" → r2=1, c2=1
- Goal: Convert keypad outputs into 4-bit binary number
  - 0-9 → 0000 to 1001
  - * → 1010, # → 1011
  - nothing pressed: 1111



Example: Keypad Converter

- Step 1: Capture behavior
  - Truth table too big (2^7 rows); equations not clear either
  - Informal table can help

TABLE 2.7 Informal table for the 12-button keypad to 4-bit code converter.

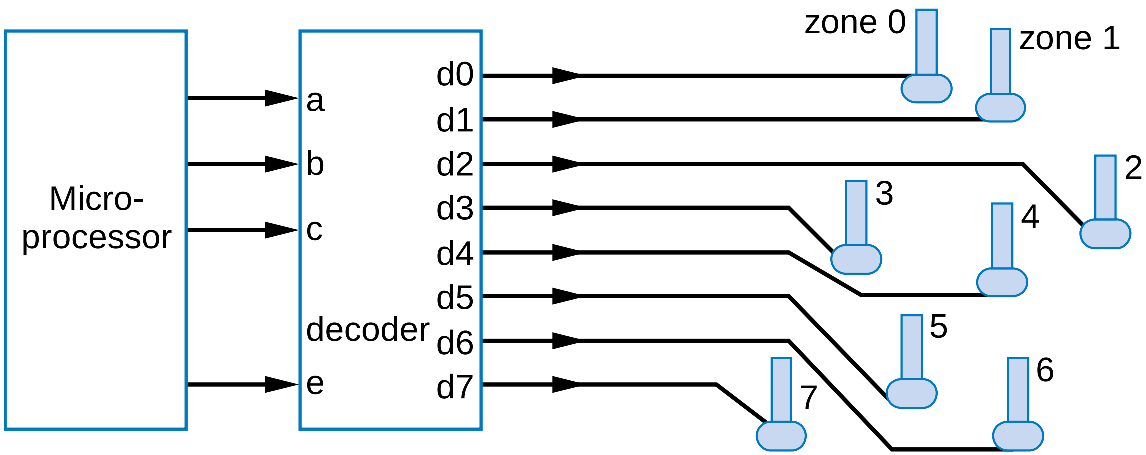
Button	Signals		4-bit code outputs			
			w	x	y	z
1	r1	c1	0	0	0	1
2	r1	c2	0	0	1	0
3	r1	c3	0	0	1	1
4	r2	c1	0	1	0	0
5	r2	c2	0	1	0	1
6	r2	c3	0	1	1	0
7	r3	c1	0	1	1	1
8	r3	c2	1	0	0	0
9	r3	c3	1	0	0	1
*	r4	c1	1	0	1	0
0	r4	c2	0	0	0	0
#	r4	c3	1	0	1	1
(none)			1	1	1	1

Step 2b: Implement as circuit *(note sharable gates)* ...

$$w = r3c2 + r3c3 + r4c1 + r4c3 + r1'r2'r3'r4'c1'c2'c3'$$
$$x = r2c1 + r2c2 + r2c3 + r3c1 + r1'r2'r3'r4'c1'c2'c3'$$
$$y = r1c2 + r1c3 + r2c3 + r3c1 + r4c1 + r4c3 + r1'r2'r3'r4'c1'c2'c3'$$
$$z = r1c1 + r1c3 + r2c2 + r3c1 + r3c3 + r4c3 + r1'r2'r3'r4'c1'c2'c3'$$

Example: Sprinkler Controller

- Microprocessor outputs which zone to water (e.g., cba=110 means zone 6) and enables watering (e=1)
- Decoder should set appropriate valve to 1



Step 1: Capture behavior

$d0 = a'b'c'e$

$d1 = a'b'ce$

$d2 = a'bc'e$

$d3 = a'bce$

$d4 = ab'c'e$

$d5 = ab'ce$

$d6 = abc'e$

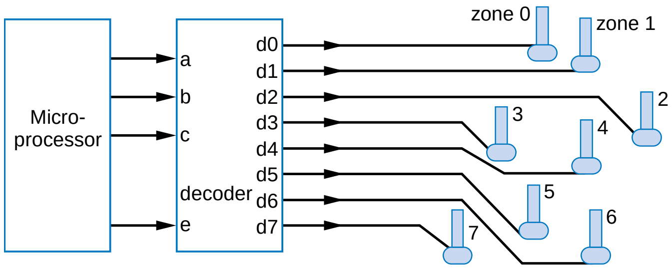
$d7 = abce$

a

Equations seem like a natural fit

Example: Sprinkler Controller

- Step 2b: Implement as circuit



$d0 = a'b'c'e$

$d1 = a'b'ce$

$d2 = a'bc'e$

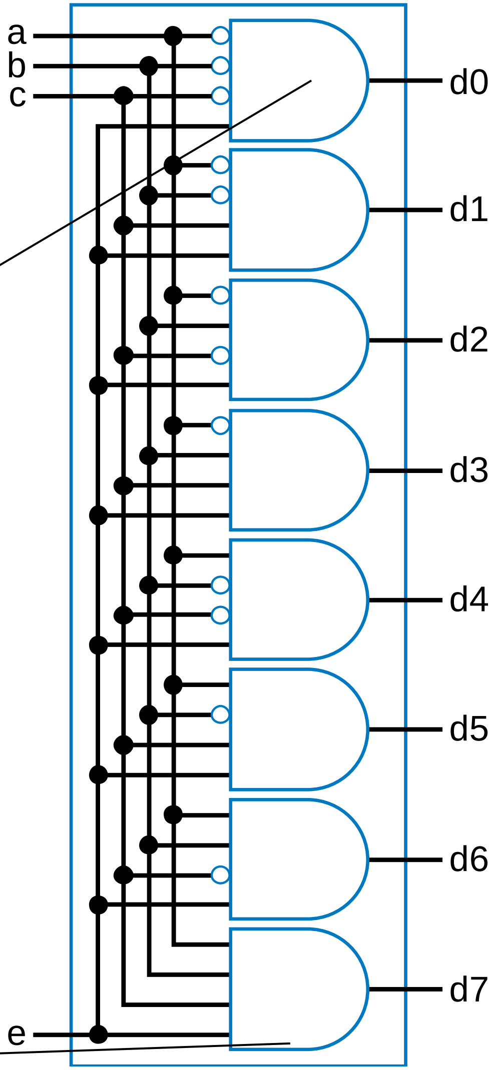
$d3 = a'bce$

$d4 = ab'c'e$

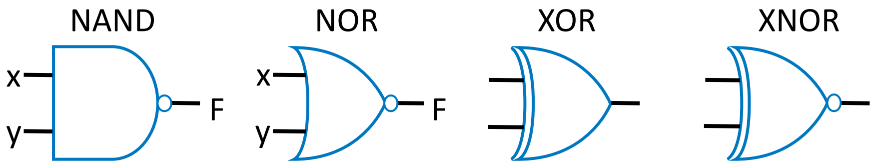
$d5 = ab'ce$

$d6 = abc'e$

$d7 = abce$



More Gates

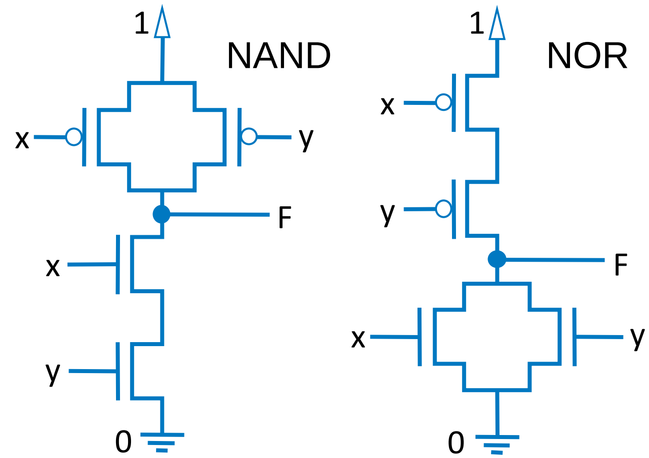


x	y	F (NAND)
0	0	1
0	1	1
1	0	1
1	1	0

x	y	F (NOR)
0	0	1
0	1	0
1	0	0
1	1	0

x	y	F (XOR)
0	0	0
0	1	1
1	0	1
1	1	0

x	y	F (XNOR)
0	0	1
0	1	0
1	0	0
1	1	1

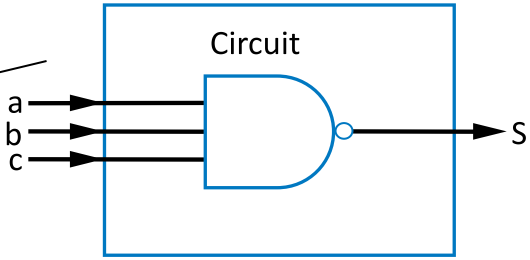


- NAND: Opposite of AND ("NOT AND")
- NOR: Opposite of OR ("NOT OR")
- XOR: Exactly 1 input is 1, for 2-input XOR. (For more inputs -- odd number of 1s)
- XNOR: Opposite of XOR ("NOT XOR")

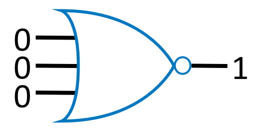
- NAND same as AND with power & ground switched
  - nMOS conducts 0s well, but not 1s (reasons beyond our scope) – so NAND is more efficient
- Likewise, NOR same as OR with power/ground switched
- NAND/NOR more common
- AND in CMOS: NAND with NOT
- OR in CMOS: NOR with NOT

More Gates: Example Uses

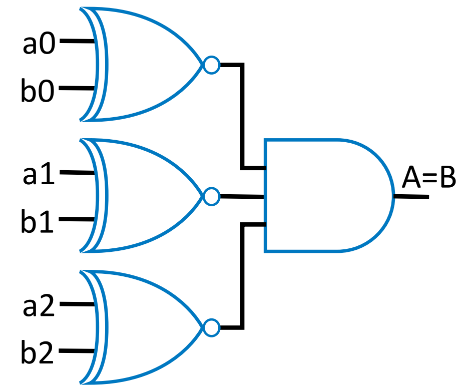
- Aircraft lavatory sign example
  - S = (abc)’



- Detecting all 0s
  - Use NOR



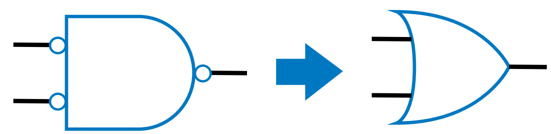
- Detecting equality
  - Use XNOR



- Detecting odd # of 1s
  - Use XOR
  - Useful for generating “parity” bit common for detecting errors

Completeness of NAND

- Any Boolean function can be implemented *using just NAND gates.* Why?
  - Need AND, OR, and NOT
  - NOT: 1-input NAND (or 2-input NAND with inputs tied together)
  - AND: NAND followed by NOT
  - OR: NAND preceded by NOTs



  - Thus, NAND is a universal gate
    - Can implement any circuit using just NAND gates
- Likewise for NOR

Number of Possible Boolean Functions

- How many possible functions of 2 variables?
  - 2^2 rows in truth table, 2 choices for each
  - $2^{(2^2)} = 2^4 = 16$ possible functions
- N variables
  - 2^N rows
  - $2^{(2^N)}$ possible functions

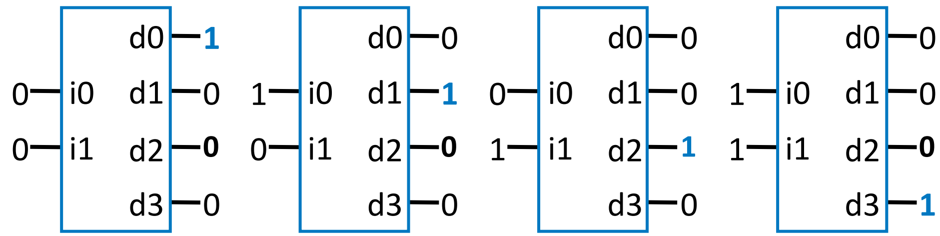
a	b	F	
0	0	0 or 1	2 choices
0	1	0 or 1	2 choices
1	0	0 or 1	2 choices
1	1	0 or 1	2 choices

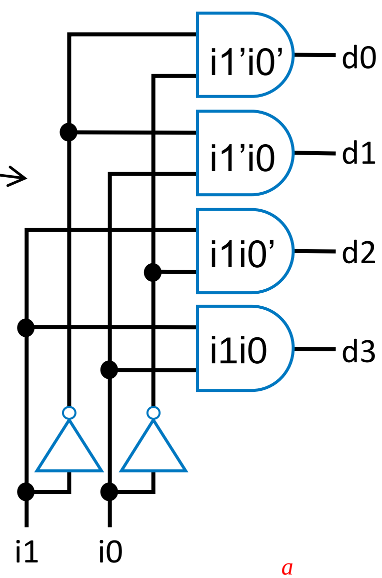
$2^4 = 16$
possible functions

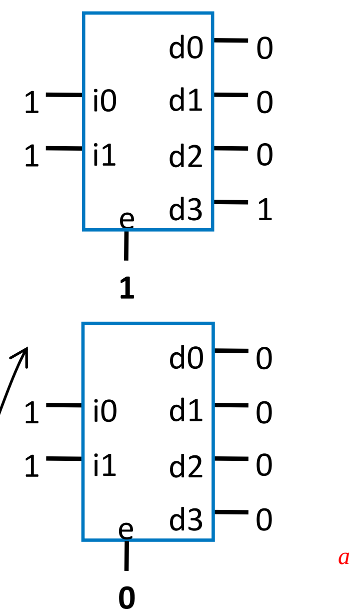
a	b	f0	f1	f2	f3	f4	f5	f6	f7	f8	f9	f10	f11	f12	f13	f14	f15
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
0	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**
1	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**
1	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
		0	a AND b		a		b	a XOR b	a OR b	a NOR b	a XNOR b	b'		a'		a NAND b	1

Decoders and Muxes

- **Decoder**: Popular combinational logic building block, in addition to logic gates
  - Converts input binary number to one high output
- 2-input decoder: four possible input binary numbers
  - So has four outputs, one for each possible input binary number
- Internal design
  - AND gate for each output to detect input combination
- Decoder with enable e
  - Outputs all 0 if e=0
  - Regular behavior if e=1
- n-input decoder: 2^n outputs

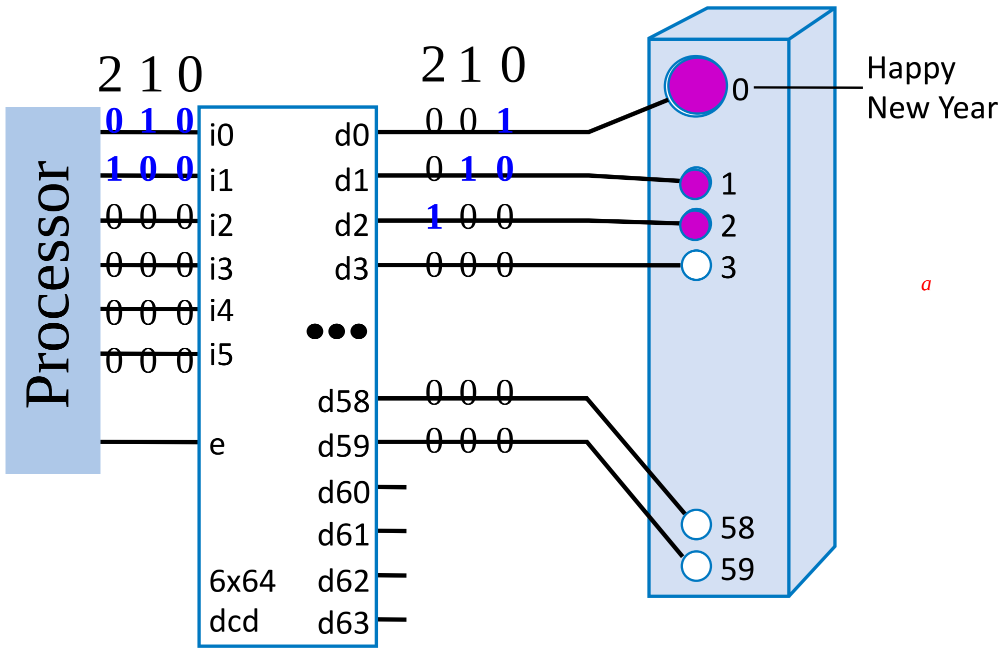






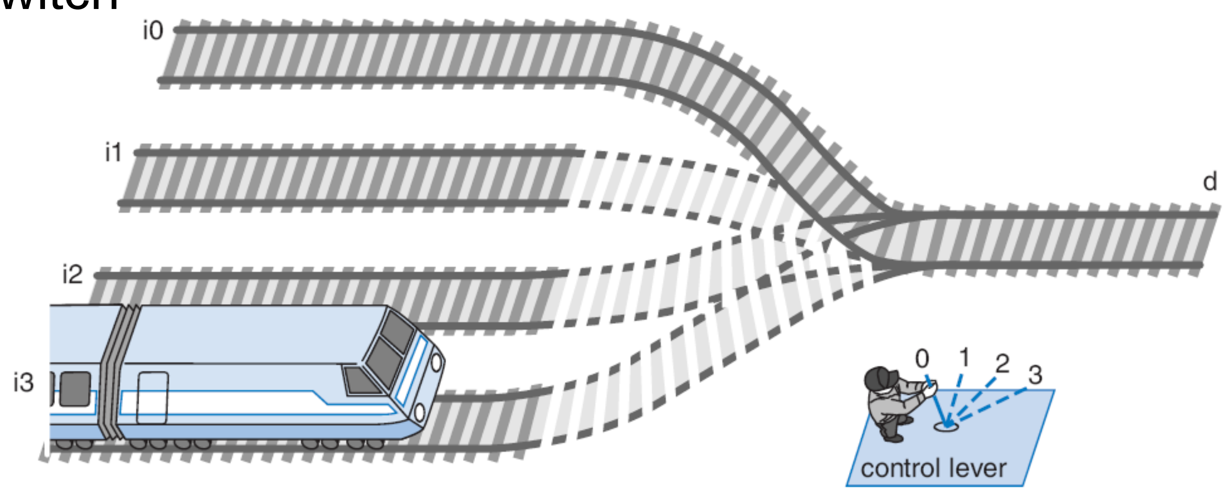
Decoder Example

- New Year's Eve Countdown Display
  - Microprocessor counts from 59 down to 0 in binary on 6-bit output
  - Want illuminate one of 60 lights for each binary number
  - Use 6x64 decoder
    - 4 outputs unused

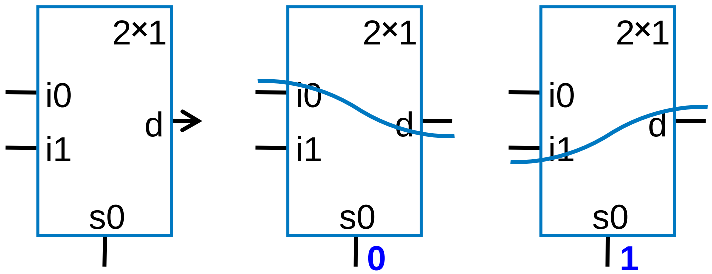


Multiplexor (Mux)

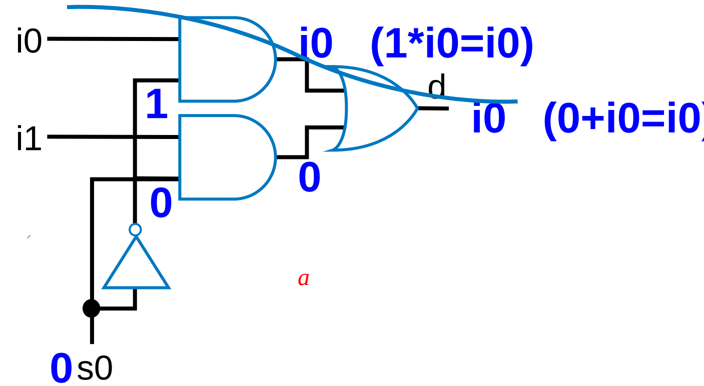
- Mux: Another popular combinational building block
  - Routes one of its N data inputs to its one output, based on binary value of select inputs
    - 4 input mux → needs 2 select inputs to indicate which input to route through
    - 8 input mux → 3 select inputs
    - N inputs → $\log_2(N)$ selects
  - Like a rail yard switch

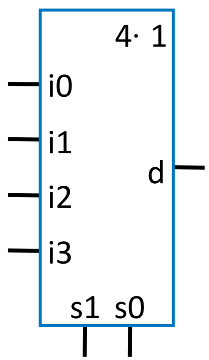


Mux Internal Design

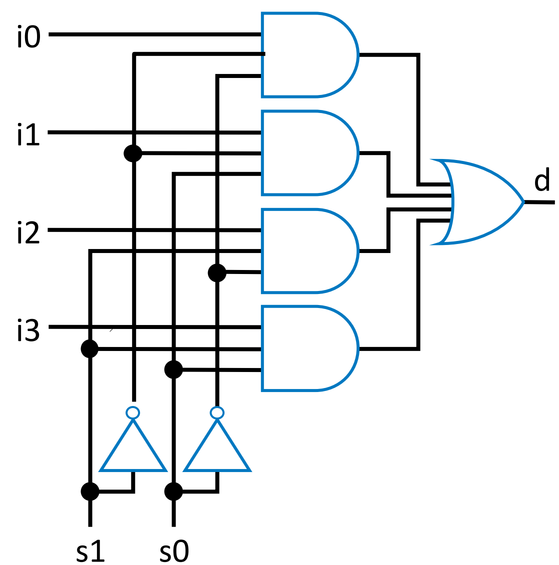


2x1 mux



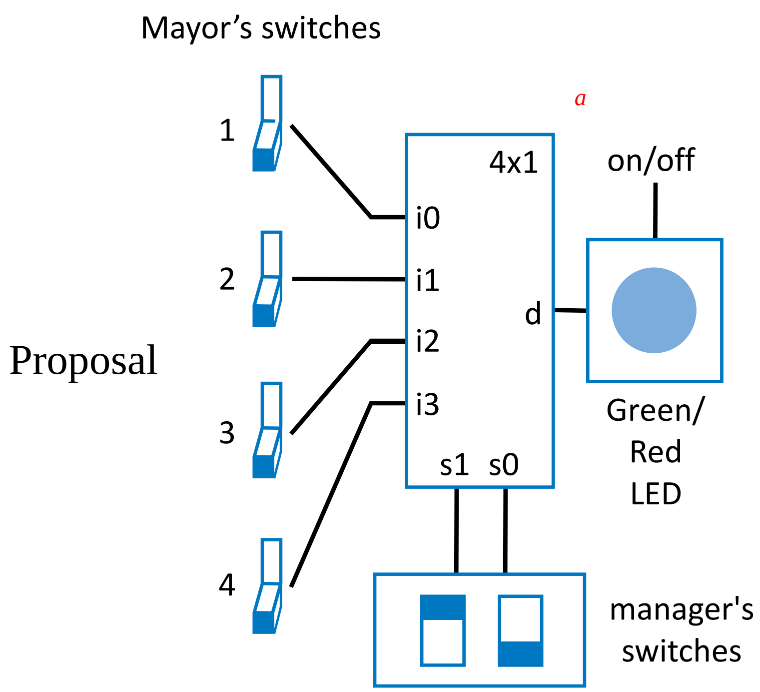


4x1 mux

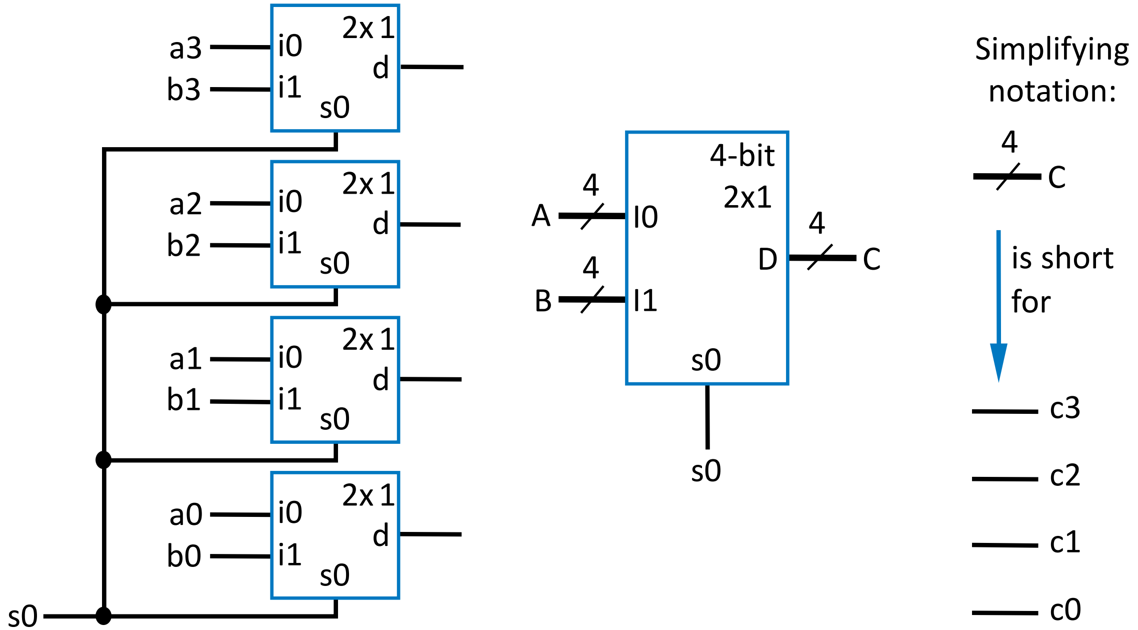


Mux Example

- City mayor can set four switches up or down, representing his/her vote on each of four proposals, numbered 0, 1, 2, 3
- City manager can display any such vote on large green/red LED (light) by setting two switches to represent binary 0, 1, 2, or 3
- Use 4x1 mux

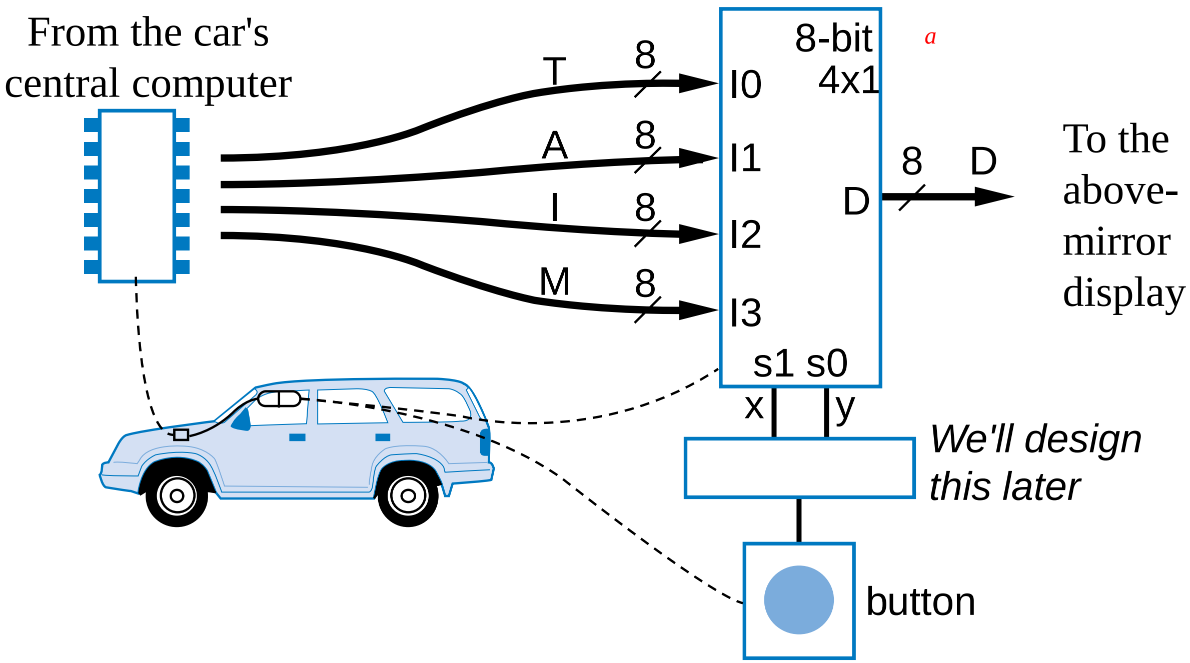


Muxes Commonly Together – N-bit Mux



- Ex: Two 4-bit inputs, A (a3 a2 a1 a0), and B (b3 b2 b1 b0)
  - 4-bit 2x1 mux (just four 2x1 muxes sharing a select line) can select between A or B

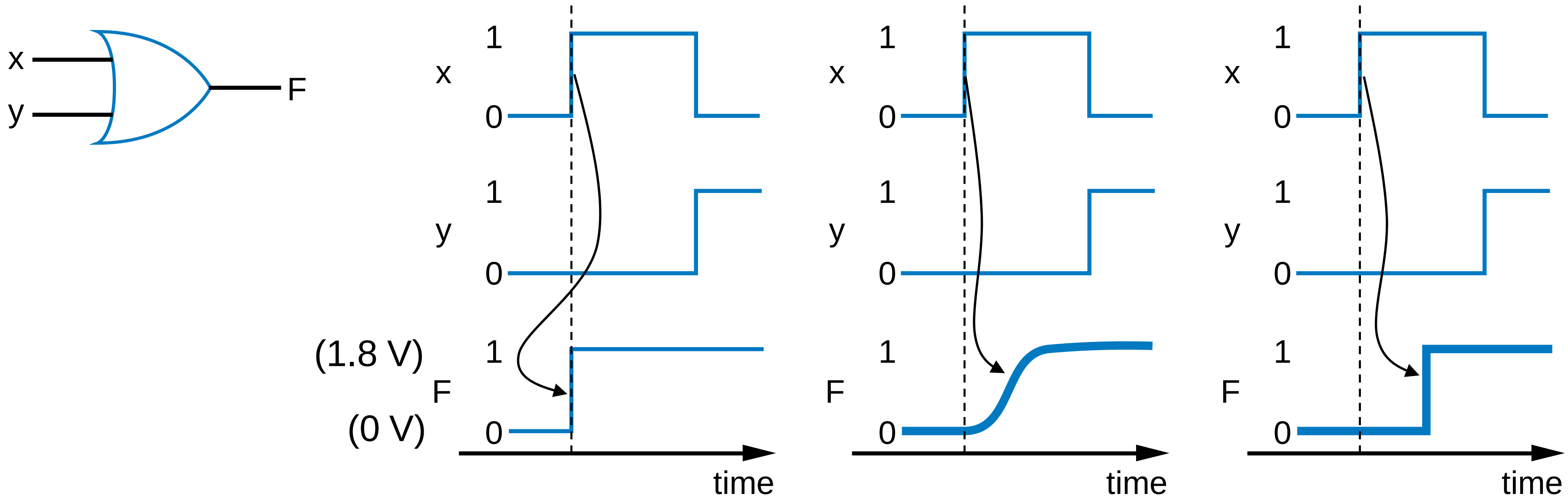
N-bit Mux Example



- Four possible display items
  - Temperature (T), Average miles-per-gallon (A), Instantaneous mpg (I), and Miles remaining (M) – each is 8-bits wide
  - Choose which to display on D using two inputs x and y
    - Pushing button sequences to the next item
  - Use 8-bit 4x1 mux

Additional Considerations

Non-Ideal Gate Behavior -- Delay



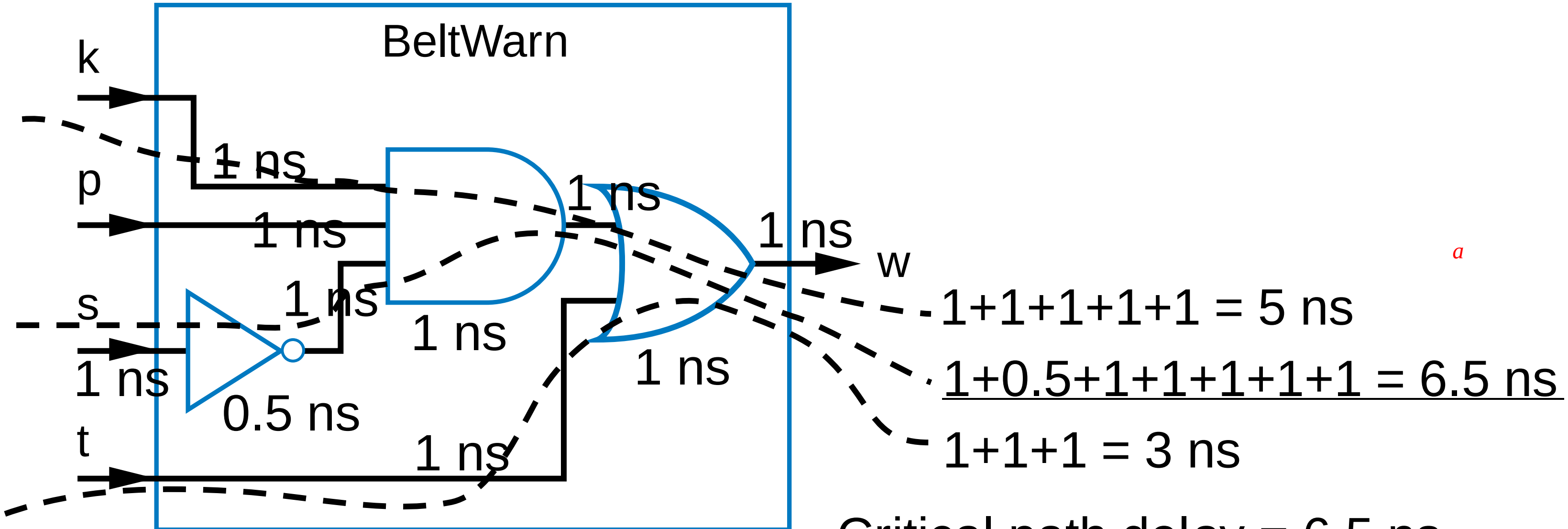
(**a**) *ideal*

(**b**) *more realistic*

(**c**) *with delay but otherwise ideal*

- Real gates have some delay
  - Outputs don't change immediately after inputs change

Circuit Delay and Critical Path

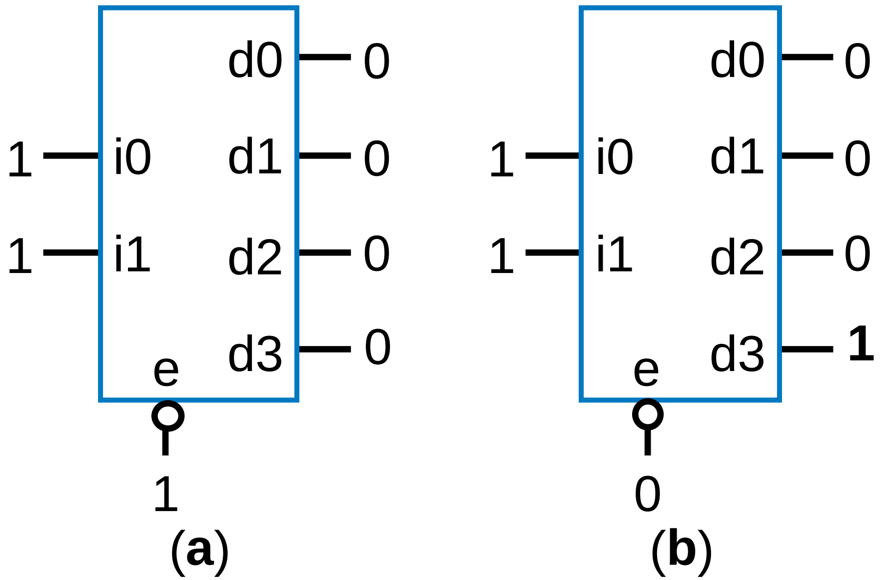


Critical path delay = 6.5 ns
Hence, circuit's delay is 6.5 ns

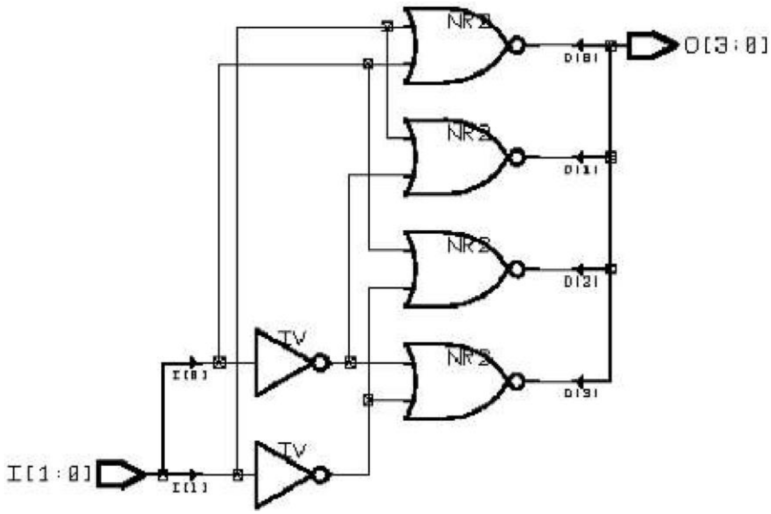
- Wires also have delay
- Assume gates and wires have delays as shown
- Path delay – time for input to affect output
- Critical path – path with longest path delay
- Circuit delay – delay of critical path

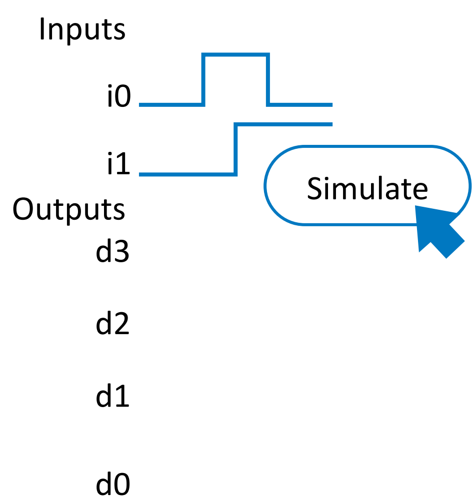
Active Low Inputs

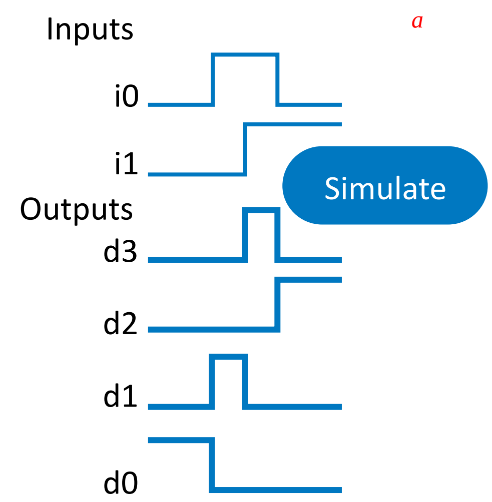
- Data inputs: flow through component (e.g., mux data input)
- Control input: influence component behavior
  - Normally active high – 1 causes input to carry out its purpose
  - Active low – Instead, 0 causes input to carry out its purpose
  - Example: 2x4 decoder with active low enable
    - 1 disables decoder, 0 enables
  - Drawn using inversion bubble



Schematic Capture and Simulation







- **Schematic capture**
  - Computer tool for user to capture logic circuit graphically
- **Simulator**
  - Computer tool to show what circuit outputs would be for given inputs
    - Outputs commonly displayed as **waveform**

Chapter Summary

- Combinational circuits
  - Circuit whose outputs are function of present inputs
    - No “state”
- Switches: Basic component in digital circuits
- Boolean logic gates: AND, OR, NOT – Better building block than switches
  - Enables use of Boolean algebra to design circuits
- Boolean algebra: Uses true/false variables/operators
- Representations of Boolean functions: Can translate among
- Combinational design process: Translate from equation (or table) to circuit through well-defined steps
- More gates: NAND, NOR, XOR, XNOR also useful
- Muxes and decoders: Additional useful combinational building blocks