Gebze Technical University

CSE 321 - HW3

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a) $T(n) = 27T(n/3) + n^2$

$T(n/3) = 27T(n/3/3) + (n/3)^2$

$T(n) = 27T(n/9) + \frac{n^2}{9}$

$T(n) = 27(27T(n/9) + \frac{n^2}{9}) + n^2$

$= 729T(n/9) + 3n^2 + n^2$

$T(n/9) = 27T(n/3 \cdot \frac{1}{9}) + (\frac{n}{9})^2$

$T(n) = 729(27T(n/27) + \frac{n^2}{81}) + 3n^2 + n^2$

$19683T(n/27) + 9n^2 + 3n^2 + n^2$

Therefore, general form

$T(n) = 27^k T(\frac{n}{3^k}) + n^2(3^{k-1} + 3^{k-2} \dots 3^0)$

$= 27^k T(\frac{n}{3^k}) + n^2 \sum_{i=0}^{k-1} 3^i$

$\sum_{i=0}^{n-1} k^i = \frac{1-k^n}{1-k}$

$= 27^k T(\frac{n}{3^k}) + n^2 (\frac{1-3^k}{1-3})$

Lets say: constant

$T(1)=1$

$T(\frac{n}{3^k}=1) = 1$

$n = 3^k$

$\boxed{\log_3 n = k}$

$T(n) = 27^{\log_3 n} \cdot T(\frac{n}{3^{\log_3 n}}) + (\frac{1-3^{\log_3 n}}{1-3})$

$= n^3 \cdot T(\frac{n}{n}) + (\frac{1-n}{1-3})$

$= n^3 \cdot T(1) - \frac{1-n}{2} \longrightarrow \Theta(n^3)$

b) $T(n) = 9T(\frac{n}{4}) + n$ $\quad T(\frac{n}{4}) = 9 \cdot T(\frac{n}{4} \cdot \frac{1}{4}) + \frac{n}{4}$

$T(n) = 9T(\frac{n}{4}) + n$

$T(n) = 81 \cdot T(\frac{n}{16}) + (\frac{n}{4})$

$T(\frac{n}{16}) = 81T(\frac{n}{16} \cdot \frac{1}{16}) + \frac{n}{16}$

Therefore general form.

$T(n) = 9^k T(\frac{n}{4^k}) + n((\frac{9}{4})^{k-1} + (\frac{9}{4})^{k-2} \quad \frac{9}{4} + 1)$

$= 9^k T(\frac{n}{4^k}) + n(\frac{1-(\frac{9}{4})^k}{1-\frac{9}{4}})$

I suppose that $T(1)=1$

$T(\frac{n}{4^k}=1) = 1$

$n = 4^k$

$\boxed{k = \log_4 n}$

$\Rightarrow T(n) = 9^{\log_4 n} \cdot T(\frac{n}{4^{\log_4 n}}) + n(\frac{1-(\frac{9}{4})^{\log_4 n}}{1-\frac{9}{4}})$

$= 9^{\log_4 n} T(1) + n \cdot (\frac{1 - \frac{9^{\log_4 n}}{n}}{\frac{-5}{4}})$

$= 9^{\log_4 n} \cdot T(1) + \frac{n - 9^{\log_4 n}}{\frac{-5}{4}}$

$= 9^{\log_4 n} \cdot 1 + \frac{4n - 4 \cdot 9^{\log_4 n}}{-5}$

$\boxed{\Theta(n^{\log_4 9})}$

c) $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}$

* master teorem can be applied;

$$\left.\begin{matrix} 2 > 0 \\ 4 > 1 \\ \frac{1}{2} \geq 0 \end{matrix}\right\} \to \log_b^a \to \log_4^2 = \frac{1}{2} \quad d = \frac{1}{2}$$

$d = \log_b^a$

$\Theta(\sqrt{n} \log n)$

master teorem

$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^d)$

$a > 0$, $b > 1$, $d \geq 0$

$$T(n) \begin{cases} \Theta(n^d) & \text{if } d > \log_b^a \\ \Theta(n^d \log n) & \text{if } d = \log_b^a \\ \Theta(n^{\log_b^a}) & \text{if } d < \log_b^a \end{cases}$$

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d) $T(n) = 2T(\sqrt{n}) + 1$

$T(n) = 2T(\sqrt{n}) + 1$

$\to T(\sqrt{n}) = 2T(n^{\frac{1}{4}}) + 1$

$T(n) = 2 \cdot 2 \cdot T(n^{\frac{1}{4}}) + 2 \cdot 1 + 1$

$= 4 \cdot T(n^{\frac{1}{4}}) + 2 + 1$

$\Rightarrow T(n) = 2^k T(n^{\frac{1}{2^k}}) + (2^{k-1} + 2^{k-2} + \dots + 2 + 1)$

$= 2^k \cdot T(n^{\frac{1}{2^k}}) + \left(\frac{1-2^k}{1-2}\right)$

$= \log_2 n \cdot T(1) + \frac{(1 - \log_2 n)}{-1}$ $\quad \Theta(\log n)$

$T(2) = 1$

$T\left(n^{\frac{1}{2^k}} = 2\right) = 1 \quad \log_n 2 = \frac{1}{2^k}$

$2^k = \log_2 n \quad k = \log_2 \log_2 n \quad = \log_2 n + \log_2 n - 1 = 2\log_2 n - 1$

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e) $T(n) = 2T(n-2) \quad T(0) = 1, \ T(1) = 1$

$$\left.\begin{matrix} T(2) = 2 \cdot T(0) = 2 \\ T(3) = 2 \cdot T(1) = 2 \\ T(4) = 2 \cdot T(2) = 4 \\ T(5) = 2 \cdot T(4) = 4 \end{matrix}\right\}$$

$T \to 1, 1, 2, 2, 4, 4 \dots$

$$T(n) \left.\begin{cases} n \text{ is odd} & 2^{\frac{n-1}{2}} \\ n \text{ is even} & 2^{\frac{n}{2}} \end{cases}\right\} \Theta(2^n)$$

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f)

$T(n) = 4T\left(\frac{n}{2}\right) + n, \quad T(1) = 1$

master teorem can be applied

$a > 0 \quad 4 > 0$

$b > 1 \quad 2 > 1$

$d \geq 0 \quad 1 \geq 0$

$\log_b^a = \log_2 4 = 2 > 1 \quad$ if $\log_b^a > d$

$\Theta(n^{\log_2 4}) = \Theta(n^2)$

3)

$T(n) = 2T(\sqrt[3]{n}) + 1, \quad T(1)=1$

$T(\sqrt[3]{n}) = 2T(n^{\frac{1}{9}}) + 1$

$T(n) = 2(2T(n^{\frac{1}{9}}) + 1) + 1$

$= 4T(n^{\frac{1}{9}}) + 2 + 1$

$T(n) = 4(2T(n^{\frac{1}{27}}) +) + 2 + 1$

$= 8T(n^{\frac{1}{27}}) + 4 + 2 + 1$

$2^k T(n(\frac{1}{3})^k = 1$

$n(\frac{1}{3})^k = 1$

$\log_n 2 = (\frac{1}{3})^k$

$\log_3 n = 3^k \quad \boxed{\log_3 \log_3 n = k}$

therefore general form

$T(n) = 2^k T((n^{\frac{1}{3}})^k) + (2^{k-1} + 2^{k-2} + \dots 2^1 + 1)$

$= 2^{\log_3 \log_3 n} T(n^{\frac{\log_3 \log_3 n}{3}}) + 2.2^{\log \log n} - 1 \quad \Rightarrow (\log_3 n)^{\log_3 2}$

$\Theta(2^{\log_3 \log_3 n})$

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2)

$T(n) = n.T(\frac{n}{2}) + 1$

$T(\frac{n}{2}) = \frac{n}{2} T(\frac{n}{4}) + 1$

$T(n) = \frac{n^2}{2} . T(\frac{n}{4}) + \frac{n}{2}$

$T(\frac{n}{4}) = \frac{n}{4} . T(\frac{n}{8}) + 1$

$T(n) = (\frac{n}{2})^k . T(\frac{n}{2^k}) + (\frac{n}{2})^{k-1} + (\frac{n}{2})^{k-2} + \dots \frac{n}{2} + 1$

supposed that

$T(1) = 1 \qquad n = 2^k \quad k = \log_2 n$

$T(\frac{n}{2^k} = 1) = 1$

$$= (\frac{n}{2})^{\log_2 n} . T(1) + \frac{(\frac{n}{2})^{\log_2 n} - 1}{(\frac{n}{2}) - 1}$$

$$= \frac{n^{\log_2 n}}{n} + \frac{\frac{n^{\log_2 n}}{n} - 1}{\frac{n-2}{2}}$$

$$= n^{\log_2 n - 1} + \frac{(n^{\log_2 n} - n) . 2}{n-2} \Rightarrow \Theta(n^{\log n})$$

3) $T(n) = 3T\left(\frac{2n}{3}\right) + 1$ → coming from swap part

$T\left(\frac{2n}{3}\right) = 3 \cdot T\left(\left(\frac{2}{3}\right)^2 n\right) + 1$

$T(n) = 3\left(3 \cdot T\left(\left(\frac{2}{3}\right)^2 n\right) + 1\right) + 1$

$= 3^2 \cdot T\left(\left(\frac{2}{3}\right)^2 n\right) + 3 + 1$

$T(n) = 3^k \cdot T\left(\left(\frac{2}{3}\right)^k n\right) + \underbrace{3^{k-1} + 3^{k-2} \dots + 1}_{3^{k-1}}$

$= 3^k \cdot T\left(\left(\frac{2}{3}\right)^k n\right) + \frac{1-3^k}{1-3}$

$T(1) = 0$ ← supposed

$T\left(\left(\frac{2}{3}\right)^k n = 1\right) = 1$

$\left(\frac{2}{3}\right)^k = \frac{1}{n}$

$n = \left(\frac{3}{2}\right)^k$

$\boxed{\log_{1,5} n = k}$

$= \underbrace{3^{\log_{1,5} n} \cdot T(\overset{0}{\overbrace{1}})}_{\text{zero}} + \frac{1-3^{\log_{1,5} n}}{1-3} \Rightarrow T(n) = \Theta\left(n^{\log_{1,5} 3}\right)$

$= \Theta\left(n^{2,709}\right)$

4)

* we know;

average case of the quick-sort $= \Theta(n \log n)$

the best case of the insertion sort $= \Theta(n)$ → it does not rely on the size of the input.

<u>In insertion sort:</u>

algorithm must swap each not ordered connected item pairs.

* For analyzing;

need to calculate number of swaps of all permutation inputs. It can be possible if the number of input is not constant. Therefore we need to determine expected number of swaps for calculating average case.

Expected value is equal to sum of probability of existence of each sub-part

<u>Sub-parts:</u>

$\left.\begin{array}{l} \text{element } k > \text{element } k-1 \\ \quad \text{"} \quad k < \quad \text{"} \quad k-1 \end{array}\right\}$ in normal distribution both situation exist with $\left(\frac{1}{2}\right)$ probabilities

so, in average case we need to know n.

K swap operations → $K = \sum \frac{1}{2}$ which is also → $\frac{n}{2}$ so average case becomes $\Theta(n * \frac{n}{2}) = \Theta(n^2)$

<u>Result</u>

I did 100 test for test part to calculate average case. Each array contains 10 elements

<u>Analyze result:</u>

$\left.\begin{array}{l} \text{average swap of quick sort} = 17.8 \\ \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{insort} \quad \text{"} = 22.29 \end{array}\right\}$ Insertion sort so much better performans than its theoretical expectation

That results $\Theta(n^2)$ is not a tight boundary, any better boundary can be found for the average-case

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a) $T(n) = 5T(\frac{n}{3}) + n^2$

master teorem can be applied

$\log_b^a = \log_3^5$

$\boxed{\log_3^5 < 2}$

$O(n^2)$

b) $T(n) = 2T(\frac{n}{2}) + n^2$

$\log_b^a = \log_2^2$ $\boxed{d=2}$

$\log_2^2 < 2$ $\Theta(n^2)$

c) $T(n) = T(n-1) + n$

$T(n-1) = T(n-2) + (n-1)$

$T(n) = T(n-2) + (n-1) + n$

$T(n-2) = T(n-3) + n-2$

$T(n) = T(n-3) + (n-2) + (n-1) + n$

$T(n) = T(n-k) + n.k - \sum_{i=1}^{k} i$

$= T(n-k) + n.k - \frac{k.(k+1)}{2}$

$T(1) = 1$ ← supposed

$T(n-k=1) = 1$

$n = k$

$T(n) = T(1) + n^2 - \frac{n^2-n}{2}$

$= 1 + \frac{n^2-n}{2}$

$\boxed{T(n) = \Theta(n^2)}$

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