CSE 321 - Introduction to Algorithm - Fall 2020

Homework 1 - Solution

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1. Question

a)

$f(n) = \log_2 n^2 + 1$

$g(n) = n$

$\rightarrow \lim_{n\to\infty} \frac{\log_2 n^2 + 1}{n} = \frac{\infty}{\infty}$ L'hospital's Rule can be applied

$\rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{2.\left(\frac{1}{n.\ln 2}\right)}{1}$

$= \lim_{n\to\infty} \frac{2.0}{1} = \lim_{n\to\infty} \frac{0}{1} = 0$ //

* According to growth rate $f(n) \in o(n)$ and:

$o(n) \subset O(n)$ True

b) $f(n) = \sqrt{n.(n+1)}$

$g(n) = n$

$\rightarrow \lim_{n\to\infty} \frac{\sqrt{n.(n+1)}}{n} = \lim_{n\to\infty} \frac{\sqrt{n.(n+1)}}{n} \cdot \frac{\sqrt{n.(n+1)}}{\sqrt{n.(n+1)}} = \lim_{n\to\infty} \frac{n^2+n}{\sqrt{n^4+n^3}}$

$\rightarrow \lim_{n\to\infty} \frac{(n^2+n) \cdot \frac{1}{n^2}}{\sqrt{n^4+n^3} \cdot \frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1+0}{1} = 1$ //

* According to growth rate $f(n) \in \sim(n)$ and all sub growth functions are subset of $\Omega(n)$.

$\sim(n) \longrightarrow \Omega(n)$ so the expression is;

True

c) $f(n) = n^{n-1}$

$g(n) = n^n$

$$\lim_{n\to\infty} \frac{n^{n-1}}{n^n} = \lim_{n\to\infty} \frac{n^n}{n^n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

* According to growth rate all sub item of little $o(n^n)$ not also item of $\Theta(n)$. Expression is <u>False</u>.

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d) $f(n) = 2^n + n^3$

$g(n) = 4^n$

$$\lim_{n\to\infty} \frac{2^n+n^3}{4^n} = \lim_{n\to\infty} \frac{2^n+n^3}{2^n \cdot 2^n} = \lim_{n\to\infty} \frac{2^n}{4^n} + \frac{n^3}{4^n}$$

$$= \lim_{n\to\infty} \frac{1}{2^n} + \lim_{n\to\infty} \frac{n^3}{4^n}$$

$$= 0 + \lim_{n\to\infty} \frac{n^3}{4^n}$$

$$= \lim_{n\to\infty} \frac{3n^2}{\ln 4 \cdot 4^n} = \frac{\infty}{\infty} \quad \text{(L'Hospital)}$$

$$= \lim_{n\to\infty} \frac{6n^2}{(\ln 4)^2 \cdot 4^n} \quad \text{still same}$$

$$= \lim_{n\to\infty} \frac{12n}{(\ln 4)^3 \cdot 4^n} \quad \text{" "}$$

$$= \lim_{n\to\infty} \frac{12}{(\ln 4)^4 \cdot 4^n} = \lim_{n\to\infty} \frac{12}{\infty} = 0$$

* According to growth rate all sub item of little $o(4^n)$ is also item of $O(4^n)$. Expression is <u>True</u>.

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e) $f(n) = 2\log_3^2 \sqrt{n}$

$g(n) = 3\log_2 n^2$

$\rightarrow \lim_{n\to\infty} \frac{2\log_3^2\sqrt{n}}{3\log_2 n^2} = \lim_{n\to\infty} \frac{\frac{2}{3}\cdot\log_3 n}{6\cdot\log_2 n} = \lim_{n\to\infty} \frac{1}{9}\cdot\frac{\log_3 n}{\log_2 n} = \lim_{n\to\infty} \frac{1}{9}\cdot\frac{\log n}{\log 3}\cdot\frac{\log 2}{\log n}$

$\rightarrow \lim_{n\to\infty} \frac{\log 2}{9\log 3} = \frac{1}{9}\cdot\log_3 2$ // (constant)

* According to growth rate, all sub item of $\Theta(3\log_2 n^2)$ is not only item of $O(3\log_2 n^2)$, it should also equal. So expression is False.

f)

$f(n) = \log_2\sqrt{n}$

$g(n) = (\log_2 n)^2$

$\lim_{n\to\infty} \frac{\log_2\sqrt{n}}{(\log_2 n)^2} = \lim_{n\to\infty} \frac{\frac{1}{2}\cdot\log_2 n}{\log_2 n\cdot\log_2 n} = \lim_{n\to\infty} \frac{\frac{1}{2}}{\log_2 n}$

$= \lim_{n\to\infty} \frac{1}{2}\cdot\log_n 2$

$= \lim_{n\to\infty} \frac{1}{2}\cdot 0 = 0$ //

* According to growth rate, all sub item of $o((\log_2 n)^2)$, therefore their growth rate is in different asymptotic order. Expression is False

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② - Question

$\log n < \sqrt{n} < n^2 < n^2 \log n < n^3 = 8^{\log n} < 2^n < 10^n$

1) $\log n < \sqrt{n}$

$$\lim_{n\to\infty} \frac{\log_2 n}{n} \Rightarrow \lim_{n\to\infty} \left(\frac{\frac{1}{n \ln 2}}{\frac{1}{2\sqrt{n}}} \right) \Rightarrow \lim_{n\to\infty} \left(\frac{2}{\ln 2 \sqrt{n}} \right) = \frac{2}{\ln 2} \cdot \underbrace{\lim_{n\to\infty} \frac{1}{\sqrt{n}}}_{0}$$

$\therefore f(n) \in o(g(n))$

2) $\sqrt{n} < n^2$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2} = \lim_{n\to\infty} \frac{\sqrt{n}}{(\sqrt{n})^4} = \lim_{n\to\infty} \frac{1}{(\sqrt{n})^3} = 0$$

$\therefore f(n) \in o(g(n))$

3) $n^2 < n^2 \log n$

$$\lim_{n\to\infty} \frac{n^2}{n^2 \log_2 n} \Rightarrow \lim_{n\to\infty} \frac{1}{\log_2 n} = 0$$

$\therefore f(n) \in o(g(n))$

4) $n^2 \log n < n^3$

$$\lim_{n\to\infty} \frac{n^2 \log_2 n}{n^3} = \lim_{n\to\infty} \frac{\log_2 n}{n} = \frac{\infty}{\infty} \text{ L'Hopital } \left(\frac{\frac{1}{n \cdot \ln 2}}{1} \right) = \frac{1}{\ln 2} \lim_{x\to\infty} \frac{1}{x} = 0$$

$f(n) \in o(g(n))$

5) $n^3 = 8^{\log n}$

$n^3 = n^{\log_2 8} \} 3$

$n^3 = n^3 \quad f(n) \in \Theta(g(n))$

6) $8^{\log n} < 2^n$

$$\lim_{n\to\infty} \frac{8^{\log n}}{2^n} = \lim_{n\to\infty} \frac{x^3}{2^x} \Rightarrow \text{L'Hospital} = \lim_{n\to\infty} \frac{3x^2}{2^x \ln 2} = \lim_{n\to\infty} \frac{6x}{2^n \cdot (\ln^2(2))} \Rightarrow$$

$$\rightarrow \frac{6}{\ln^3(2)} \cdot \underbrace{\lim_{n\to\infty} \left(\frac{1}{2^x}\right)}_{0} = 0 \quad f(n) \in o(g(n))$$

7) $2^n < 10^n$

$$\lim_{n\to\infty} \frac{2^n}{10^n} = \lim_{n\to\infty} \frac{1}{5^n} = 0 //$$

$f(n) \in o(g(n))$

* All little o's prove that the one of the phrase is absolutely lower complexity than other. It occurs at the through the same way. And also there is one equal complexity.

3. Question

a)

In this algorithm there is no changing on the borders of the algorithm which is denoted by 'i'.
And also, there is no escape statement such as return or another function call.

$\Rightarrow$ Loop will repeat as size of array times with no doubt.
therefore, complexity must be Θ(size of array)
if size of array is denoted by 'n';

complexity = $\Theta(n)$

b)

Firstly, we need to update the for loop to determine. According the updated 'i', count will be increasing from;

($i=2$; $n \geq i$; $i = i^2 + k$ can be ignored.)

* In loop, it takes several values;

$2, 2^k, (2^k)^k \dots = 2, 2^k, (2^k)^k \dots = 2^{k^2}, \dots, 2^{k^{\log_k \log n}}$

$\therefore \log_k(\log(n))$ iteration in total

Total complexity $\longrightarrow O(\log(\log n))$

4. Question a) $\sum_{i=1}^{n} i^2 \log i$

$\int_0^n i^2 \log i \, di \leq f(n) \leq \int_1^{n+1} i^2 \log i \, di$

$\frac{1}{3} i^3 \log_2(i) - \int \frac{i^2}{3\ln 2} dn \leq f(n) \leq \frac{1}{3} i^3 \log_2 i - \int \frac{i^2}{3\ln 2} di$

$\frac{1}{3} i^3 \log_2 i - \frac{i^3}{9\ln 2} \Big|_0^n \leq f(n) \leq \frac{1}{3} i^3 \log i - \frac{i^3}{9\ln 2} \Big|^{n+1}$

$\frac{i^3(3\ln i - 1)}{9} \Big|_0^n \leq f(n) \leq \frac{i^3(3\ln i - 1)}{9} \Big|^{n+1}$

$= \Theta(n^3 \log n)$

b) $\sum_{i=1}^{n} i^3$

$$\int_0^n i^3 di \le f(n) \le \int_1^{n+1} i^3 di \Rightarrow \frac{i^4}{4} + c \Big|_0^n \le f(n) \le \frac{i^4}{4} + c \Big|_1^{n+1}$$

$$\frac{n^4}{4} \le f(n) \le \frac{(n+1)^4}{4} \Rightarrow \Theta(n^4)$$

c) $\sum_{i=1}^{n} \frac{1}{(2\sqrt{i})}$ (non-incresing)

$$\int_1^{n+1} \frac{1}{2\sqrt{i}} di \le f(n) \le \int_0^n \frac{1}{2\sqrt{i}} di \Rightarrow \sqrt{i} \Big|_1^{n+1} \le f(n) \le \sqrt{i} \Big|_0^n$$

$$\sqrt{n+1} - \sqrt{1} \le f(n) \le \sqrt{n} \Rightarrow \Theta(\sqrt{n})$$

d) $\sum_{i=1}^{n} \frac{1}{i}$ (non-increasing) $\Rightarrow \int_1^{n+1} \frac{1}{di} \le f(n) \le \int_0^{n+1} \frac{1}{i}$

$$\ln i \Big|_1^{n+1} \le f(n) \le \ln i \Big|_0^{n+1}$$

$\ln(n+1) \le f(n) \le \infty$

upper bound cannot be examined

Therefore $\Rightarrow$ $H(n) = 1 + \sum_{i=2}^{n} \frac{1}{i} \Rightarrow H(n) \le 1 + \int_1^n f(x) dx$

$H(n) \le 1 + \ln(x) \Big|_1^n$

$H(n) \le \ln(n) + 1$

$\therefore \ln(n+1) \le H(n) \le \ln(n) + 1$

Both upper lower bounds are defined in terms of

$H(n) \in \Theta(\log n)$ //

5. Question

```
function LinearSearch( L[1:n], k )
    for i:=1 to n do
        if (L[i]=k) then
            return (i)
        end if
    end for
    return ∅
end
```

* Linear search goes through given list. And check the certain element in the list whether it matches. Lets analyze for Best, worst and average case

Best case : If searched item is located as first element $B(n) = 1 \in \underline{O(1)}$

Worst case : if k is not in the element, it will loop over entire list

$w(n) = n \in \underline{O(n)}$

Average Case : $\sum_{i=1}^{n} 1.p$ $\sum_{i=1}^{n} \left(\frac{p}{n}\right) = \frac{n(n+1)}{2} \cdot \frac{p}{n} = \frac{(n+1).p}{2}$

$\underline{\Theta(n)}$