Snell's Law at Core-Mantle Boundary

Difference in velocities of P-waves and why the difference is

P-waves, or primary waves, travel at different velocities in the mantle and core due to differences in the physical properties of these layers. The mantle is composed primarily of silicate materials, which are denser and more rigid than the molten or partially molten iron-nickel alloy in the outer core. This variation in composition leads to a higher P-wave velocity in the mantle compared to the core. In simpler terms, the same reason why sound travels faster in solids than liquids.

Why Snell's law is a valid thing here

Snell's Law applies here exactly as the analog as it is done in geometrical optics. It describes the relationship between the angles and velocities of waves as they pass through boundaries between two different media. At the core-mantle boundary, this law is valid because (rather obviously) the Earth is packed, as in it is a true interface without any weird gaps or breaks in the spaces. There is also the deeper thing about all natural transmissions following Fermat's principle of least action.

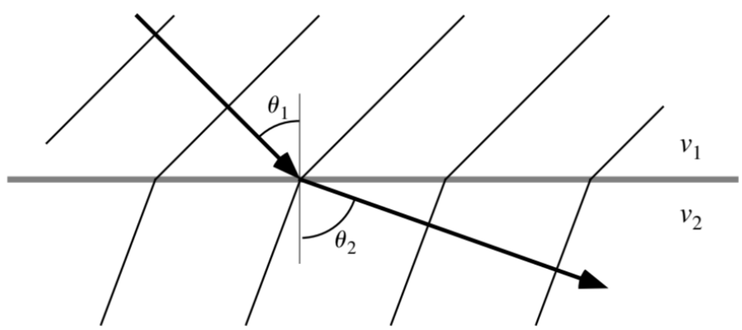

Figure 4.2 A plane wave crossing a horizontal interface between two homogeneous half-spaces. The higher velocity in the bottom layer causes the wavefronts to be spaced further apart.

Figure 1: Snell's Law

Ray geometry at core mantle boundary

Snell's law applies straightforward:

$$\frac{\sin i}{v_1} = \frac{\sin r}{v_2},$$

where i is the angle of incidence, r is the angle of refraction, v_1 is the P-wave velocity in the mantle, and v_2 is the P-wave velocity in the core.

In the seismic case, we often write $u_1 = 1/v_1$ ande $u_2 = 1/v_2$, which are the respective *slowness*, the analog of the refractive index, rephrasing Snell's law as:

$$u_1 \sin i = u_2 \sin r \equiv p,$$

where p is the apparent horizontal slowness of the wavefront.

Bonus: Setting up at the spherical boundary

The above is the straight line assumption: perfect for working out angles for a single wave path at a single interface, but a nightmare for working out derivatives or integrals. Because as you can see below, the refracted angle is not exactly the incidence angle for the next layer.

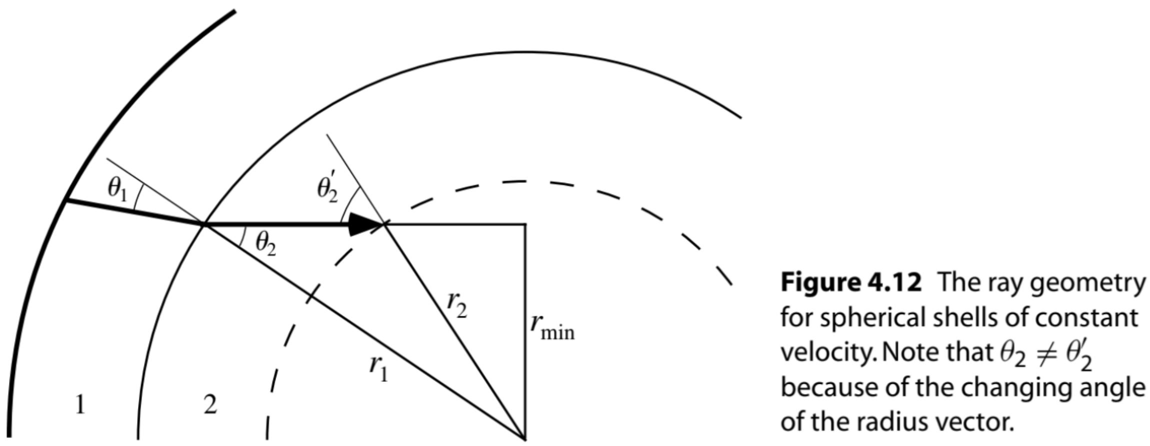

Figure 2: Your caption here.

To sort it, out, we have from Snell's law

$$u_1 \sin\theta_1(r_1) = u_2 \sin\theta_2(r_1) \tag{1}$$

As the ray travels through shell 2, note that the incidence angle changes ($\theta_2(r_1) \neq \theta_2(r_2)$). If we project the ray to its "turning point" as if layer 2 continued down, we see from the geometry of the triangles that we can express the incidence angle within layer 2 as a function of radius:

$$\sin\theta_2(r) = r_{\min}/r. \tag{4.37}$$

Thus $\theta_2(r_1)$ in (4.36) is related to $\theta_2(r_2)$ by the expression:

$$r_1 \sin\theta_2(r_1) = r_2 \sin\theta_2(r_2). \tag{2}$$

or

$$\sin\theta_2(r_1) = (r_2/r_1)\sin\theta_2(r_2). \tag{3}$$

We obtain the generalization of Snell's law for spherically symmetric media:

$$r_1 u_1 \sin\theta_1 = r_2 u_2 \sin\theta_2. \tag{4.39}$$

In this case, the ray parameter p becomes

$$p_{\text{sph}} = ru\sin\theta. \tag{4.40}$$

Sample Calculation

Given:

- Mantle P-wave velocity, $v_1 = 10$ km/s
- Core P-wave velocity, $v_2 = 8$ km/s
- Angle of incidence, $i = 30°$

Using Snell's Law:

$$\frac{\sin i}{v_1} = \frac{\sin r}{v_2}$$

Substitute the values:

$$\frac{\sin 30°}{10} = \frac{\sin r}{8}$$

Solve for $\sin r$:

$$\sin r = \frac{8 \times \sin 30°}{10} = \frac{8 \times 0.5}{10} = 0.4$$

Find r:

$$r = \arcsin(0.4) \approx 23.58°$$

Thus, the angle of refraction is approximately 23.6°.