# Module 1

Module 1 is primarily based on the wave seismic theory which is useful in polarization of P and S waves which helps many scientists in analyzing the damage (impact) , in earthquake prediction and in studying the internal structure of the planet.

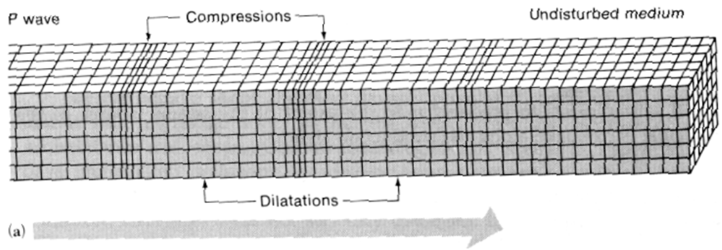
## P and S waves

### Characteristics of P waves

- **Nature:** P waves are compressional (longitudinal) waves where particles of the medium move parallel to the direction of wave propagation. They are purely the contribution of the scalar potential ($\phi$) due to the gradient:

$$\vec{u}_p = \nabla\phi$$

  Conceptually, the gradient represents the direction of maximum change. Since P waves have wave motion in the direction of propagation, the direction of the drop in the scalar potential's gradient determines the wave displacement. This motion is illustrated below:




- **Speed:** P waves are the fastest seismic waves and, hence, the first to be detected by seismographs.

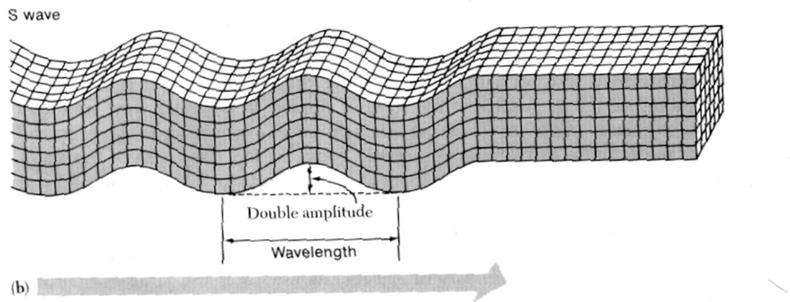
- **Material Propagation:** P waves travel through solids, liquids, and gases.

### Characteristics of S waves

- **Nature:** S waves are shear (transverse) waves where particles of the medium move perpendicular to the direction of wave propagation. Their displacement is governed purely by the vector potential ($\vec{\psi}$), satisfying:

$$\vec{u}_s = \nabla \times \vec{\psi}$$

Conceptually, curl represents net rotation: the net transverse motion at a point. Notice how at the slices of the material for a shear wave, there has to be a net difference in movement on the fore and aft sides for the wave to move on. And there is no motion in direction of propagation.




- **Speed:** S waves are slower than P waves but still propagate significantly faster than surface waves. They are the second to be detected on seismographs.
- **Material Propagation:** S waves travel only through solids, as their propagation relies on the rigidity (shear strength) of the material. They cannot travel through fluids or gases because these media lack shear resistance.

## Differences Between P Waves and S Waves

| **Aspect** | **P Waves (Primary Waves)** | **S Waves (Secondary Waves)** |
|---|---|---|
| Nature | Longitudinal (compressional) | Transverse (shear) |
| Direction of Particle Motion | Parallel to wave propagation | Perpendicular to wave propagation |
| Speed | Faster (first to arrive) | Slower (second to arrive) |
| Material Propagation | Travel through solids, liquids, and gases | Travel only through solids |
| Destructive Potential | Less Destructive | More destructive |

Table 1: Summary of Differences Between P Waves and S Waves

### Explanations of Key Differences

**Why are P-waves faster?** For the vast majority of homogeneous dense solids, the compressive stress is higher than the transverse stress for the same amount of strain. Mathematically, the corresponding moduli ($K > \mu$; bulk modulus is greater than shear modulus). Now, to get a feel for this, consider why a large marshmallow bobs up and down faster than if it is hit side to side.

**Why can't S-waves go through fluids?** Try shearing a fluid. Turn on a large cross-sectioned water faucet and place a plate to shift the water flow to the right. Notice how the entire liquid stream is **not** bending slightly right from the outlet. Fluids conform to their container (liquids don't change volume, but gases can freely expand in all directions), thus fluids give **0 shear stress**.

**Why are S-waves more destructive?** Well, an earthquake occurs vertically above the hypocenter. So, rotate the images of P and S waves by 90°. P wave motion is constrained by gravity, does not shift building foundations, and—as we have said—moduli of compressive motion for solids are more than shear, so the foundation takes this better. They are the early quake vibrations animals can feel.

S waves, on the other hand, cause side-to-side motion of the building foundations. This motion works against the shear moduli, and the buildings suffer. Also, an earthquake's high energy output can often liquefy parts of the crust. Since fluids have little to no shear strength, there is nothing to resist an S-wave wildly disturbing the layers against the liquefied layer, though compressive resistance still applies for fluids.

# Calculating P and S wave velocities

## P-wave velocity

The velocity of P waves ($\alpha$) in a medium is given by:

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

The Lamé parameters can also be expressed in terms of the bulk modulus ($K$) and the shear modulus ($\mu$):

$$\lambda = K - \frac{2}{3}\mu$$

Substituting $\lambda$ in the original equation, the P-wave velocity can also be written in terms of the bulk modulus:

$$\alpha = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$

- $\alpha$: P-wave velocity (speed of compressional waves in the medium)
- $K$: Bulk modulus, which measures a material's resistance to uniform compression
- $\mu$: Shear modulus, which quantifies the material's rigidity or resistance to pure shearing
- $\rho$: Density of the medium, representing its mass per unit volume
- $\lambda$: First Lamé parameter, which relates to the compressibility of the material but has no easy intuitive physical meaning

## S-wave Velocity

The velocity of S waves ($\beta$) in a medium is given by:

$$\beta = \sqrt{\frac{\mu}{\rho}}$$

The shear modulus ($\mu$) can be expressed in terms of the Lamé parameters and the bulk modulus ($K$) as:

$$\mu = \frac{3(K - \lambda)}{2}$$

However, the most commonly used form directly involves $\mu$ and $\rho$:

$$\beta = \sqrt{\frac{\mu}{\rho}}$$

- $\beta$: S-wave velocity (speed of shear waves in the medium)
- $\mu$: Shear modulus, which quantifies the material's rigidity or resistance to pure shearing
- $\rho$: Density of the medium, representing its mass per unit volume

## Calculation of sample P and S Wave Velocities

Given:

- $K = 2.5 \times 10^{10}\,\mathrm{Pa}$
- $\mu = 1 \times 10^{10}\,\mathrm{Pa}$
- $\rho = 3 \times 10^{3}\,\mathrm{kg/m^3}$

The formula for P-wave velocity in terms of bulk modulus ($K$) is:

$$\alpha = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$

Substitute the values:

$$\alpha = \sqrt{\frac{2.5 \times 10^{10} + \frac{4}{3}(1 \times 10^{10})}{3 \times 10^3}}$$

Calculate step-by-step:

$$\alpha = \sqrt{\frac{2.5 \times 10^{10} + 1.333 \times 10^{10}}{3 \times 10^3}}$$

$$\alpha = \sqrt{\frac{3.833 \times 10^{10}}{3 \times 10^3}}$$

$$\alpha = \sqrt{1.277 \times 10^7}$$

$$\alpha \approx 3574.4\,\mathrm{m/s}$$

The formula for S-wave velocity is:

$$\beta = \sqrt{\frac{\mu}{\rho}}$$

Substitute the values:

$$\beta = \sqrt{\frac{1 \times 10^{10}}{3 \times 10^3}}$$

Calculate step-by-step:

$$\beta = \sqrt{3.333 \times 10^6}$$

$$\beta \approx 1825.7\,\mathrm{m/s}$$

- **P-wave velocity ($\alpha$):** $3574\,\mathrm{m/s}$
- **S-wave velocity ($\beta$):** $1825\,\mathrm{m/s}$

# Deriving P-Wave and S-wave Equations for Isotropic Medium

The basic stress wave equation is:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

Where:

- $\rho$: Density of the material
- $u_i$: Displacement component in the $i$-direction
- $\sigma_{ij}$: Stress tensor
- $x_j$: Spatial coordinate in the $j$-direction

Hooke's Law: The constitutive relation given by Hooke's law in index notation is:

$$\sigma_{ij} = \lambda \delta_{ij} \nabla \cdot \vec{u} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Where:

- $\lambda$: First Lamé parameter
- $\mu$: Shear modulus (second Lamé parameter)
- $\delta_{ij}$: Kronecker delta (1 if $i = j$, 0 otherwise)
- $\nabla \cdot \vec{u}$: Divergence of the displacement vector ($\nabla \cdot \vec{u} = \frac{\partial u_k}{\partial x_k}$)
- $\frac{\partial u_i}{\partial x_j}$: Strain tensor component due to displacement gradients

## Understanding Index Notation

Index notation, also called Einstein notation, is a compact and efficient way to represent mathematical expressions, particularly in physics and engineering. It relies on the idea of summation over repeated indices and is extremely useful when dealing with vectors and tensors.

Let's break it down:

### Basics

In index notation:
- Different indices say $i$ and $j$ mean they are *independent* of each other. Same indices mean they are the same.
-So what about when $i \neq \jmath$ or other constraints? That's what Kronecker delta's and Levi-Civita stuff are for. - $u_i$ refers to the components of a vector $\mathbf{u}$. Here, $i$ can take values like 1, 2, 3, representing the x, y, and z components.
- $\partial_i$ denotes partial differentiation with respect to the $i$-th coordinate. For example, $\partial_1$ means differentiation with respect to x.
- $\delta_{ij}$ is the Kronecker delta, which is 1 if $i = j$ and 0 otherwise. It's often used to simplify expressions.
- Repeated indices imply summation. For example, $\partial_j u_j$ means $\sum_{j=1}^{3} \partial_j u_j$, summing over the index $j$.

### Applying It to Hooke's Law

Now, looking at the expression:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \sigma_{ij}$$

This tells us the acceleration (left-hand side) is related to the divergence of the stress tensor $\sigma_{ij}$ (right-hand side). It also tells you *implicitly* that $i \neq \jmath$.
-But you said they were independent? Where is my kronecker delta? -Stress tensor is explicitly through a 2D section. so $\sigma_{ii}$... does not exist (well you take it to be 0)

Expanding $\sigma_{ij}$:

$$\sigma_{ij} = \lambda \delta_{ij} \partial_k u_k + \mu(\partial_i u_j + \partial_j u_i)$$

Here: - $\lambda$ and $\mu$ are material constants (Lamé parameters). - $\delta_{ij}\partial_k u_k$ essentially picks out the $i$-th component due to the Kronecker delta. - $\partial_i u_j + \partial_j u_i$ represents the symmetric part of the strain tensor, related to how the material deforms.

Taking the derivative $\partial_j$:

$$\partial_j \left[\lambda \delta_{ij} \partial_k u_k + \mu(\partial_i u_j + \partial_j u_i)\right]$$

-Wait, I don't understand why the kronecker is here exactly? -Well, $\delta_k$ is not really a variable but an operator. Technically $\partial_k u_k$ is a single variable. We want to signify that when an operation will take place on here (ONLY) the coincident vector component is relevant. The kronecker delta is for the $\partial_j$ that will go in front. This involves applying the derivative to each term, keeping in mind that repeated indices mean summation!

## Momentum Equation Derivation

First we write Hooke's law in index notation and take the $\partial_j$ of it

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \sigma_{ij} =$$

$$\partial_j \left[ \lambda \delta_{ij} \partial_k u_k + \mu \left( \partial_i u_j + \partial_j u_i \right) \right]$$

Now, we assume that $\lambda$ and $\mu$, the Lame parameters of the material do not change across the medium; i.e., it is perfectly homogenous. This is not exactly true but close enough and acceptable to get the basic wave equations.

$$\implies \rho \frac{\partial^2 u_i}{\partial t^2} = \lambda \partial_i \partial_k u_k + \mu \partial_i \partial_j u_j + \mu \partial_j \partial_j u_i$$

Now the first and second term of the above are equivalent; since k and j are independent of i (as per index notation). And note the third term basically the laplacian summed across indices.

Defining $\ddot{\mathbf{u}} = \frac{\partial^2 \mathbf{u}}{\partial t^2}$, the equation becomes:

$$\rho \ddot{\mathbf{u}} = (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}.$$

Using the vector identity:

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u},$$

we simplify $\nabla^2 \mathbf{u}$ to:

$$\nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}).$$

Substituting this into the equation, we obtain:

$$\rho \ddot{\mathbf{u}} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}). \tag{1}$$

the momentum equation for wave motion in a packed homogenous isotropic medium.

## Deriving Wave Equations

### P waves

Taking divergence on both sides of equation (1), of the second term is $\mu \nabla \cdot (\nabla \times (\nabla \times \mathbf{u}))$, which is 0 since divergence of a curl is 0.

$$\frac{\partial^2 (\nabla \cdot \mathbf{u})}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \nabla^2 (\nabla \cdot \mathbf{u})$$

or

$$\nabla^2 (\nabla \cdot \mathbf{u}) - \frac{1}{\alpha^2} \frac{\partial^2 (\nabla \cdot \mathbf{u})}{\partial t^2} = 0,$$

where the $P$-wave velocity, $\alpha$, is given by

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho}.$$

Hence,

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

**S waves**

Taking curl on both sides of equation (1), of the first term is $\nabla \times (\nabla(\nabla \cdot \mathbf{u}))$, which is 0 since curl of a gradient is 0.

$$\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} = -\frac{\mu}{\rho}\nabla \times \nabla \times (\nabla \times \mathbf{u}).$$

Now the laplacian vector identity says: $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u},$. But note that $\nabla(\nabla \cdot (\nabla \times \mathbf{u}))$ is 0 for the divergence of the curl inside, and so:

$$\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} = \frac{\mu}{\rho}\nabla^2(\nabla \times \mathbf{u}),$$

or

$$\nabla^2(\nabla \times \mathbf{u}) - \frac{1}{\beta^2}\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} = 0,$$

where the $S$-wave velocity, $\beta$, is given by

$$\beta^2 = \frac{\mu}{\rho}.$$

Hence,

$$\beta = \sqrt{\frac{\mu}{\rho}}$$