CSCI 303, Homework 0 (Warm-up)

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1 Problem One

Prove by induction that

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$$

for all $n \geq 1$.

Proof:

Base Case: If $n = 1$, then $(1 \cdot 1!) = (1+1)! - 1$ which simplifies to $1 = 1$

Inductive hypothesis: For all $n \geq 1$, $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$.

Inductive step: Assume that the inductive hypothesis is true for n. We then need to prove that $\sum_{i=1}^{n+1} i \cdot i! = ((n+1)+1)! - 1$. Because $\sum_{i=1}^{n+1} i \cdot i!$ is $\sum_{i=1}^{n} i \cdot i! + ((n+1) \cdot (n+1)!)$, the inductive hypothesis allows us to make the following substitution and then solve algebraically.

$$\begin{aligned}
((n+1)+1)! - 1 &= \sum_{i=1}^{n} i \cdot i! + ((n+1) \cdot (n+1)!) && (1)\\
&= ((n+1)! - 1) + ((n+1) \cdot (n+1)!) && (2)\\
&= (n+1)! + (n+1)!(n+1) - 1 && (3)\\
&= (n+1)!(1 + (n+1)) - 1 && (4)\\
&= (n+1)!(n+2) - 1 && (5)\\
(n+2)! - 1 &= (n+2)! - 1 && (6)
\end{aligned}$$

Conclusion: By the principle of induction, the inductive hypothesis is true for all $n \geq 1$

2 Problem Two

Prove by induction that

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

for all $n \geq 1$.

Proof:

Base Case: If $n = 1$, then $\frac{1}{1^2} \leq 2 - \frac{1}{1}$ which simplifies to $1 \leq 1$

Inductive hypothesis: For all $n \geq 1$, $\sum\limits_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Inductive step: Assume that the inductive hypothesis is true for n. We then need to prove that $\sum\limits_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}$. Because $\sum\limits_{i=1}^{n+1} \frac{1}{i^2}$ is $\sum\limits_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$, and by the induction hypothesis, $\sum\limits_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$, then if we substitute $\sum\limits_{i=1}^{n} \frac{1}{i^2}$ with $2 - \frac{1}{n}$ and the resulting inequality is algebraically true, we know that $\sum\limits_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}$ is also true because the original term is less than or equal to that which is being substituted. And so,

$$(2 - \frac{1}{n}) + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1} \tag{7}$$

$$\frac{1}{(n+1)^2} - \frac{1}{n} \leq -\frac{1}{n+1} \tag{8}$$

$$\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1} \tag{9}$$

$$1 \leq \frac{(n+1)^2}{n} - (n+1) \tag{10}$$

$$1 \leq \frac{n^2 + 2n + 1}{n} - n - 1 \tag{11}$$

$$1 \leq n + 2 + \frac{1}{n} - n - 1 \tag{12}$$

$$1 \leq 1 + \frac{1}{n} \tag{13}$$

Conclusion: By the principle of induction, the inductive hypothesis is true for all $n \geq 1$