Algebraic Geometry 1

Lecture notes based on the lecture in algebraic geometry by Prof. Dr. Eike Lau in summer semester 2025

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0 Introduction

The area of algebraic geometry originates with the study of solutions to polynomial equations and systems of polynomial equations in multiple variables. This lecture begins with the more classical approach to algebraic geometry and later transitions to the more modern approach.

In the following lecture we denote with k an algebraically closed field unless stated otherwise. The classical view of algebraic geometry mainly deals with the following objects:

Definition 0.1: Algebraic sets

Let $k[T_1, \ldots, T_n]$ be the set set of polynomials in variables $T_1, \ldots, T_n$ with coefficients in k. For $M \subseteq k[T_1, \ldots, T_n]$ we denote the zero locus of M by

$$V(M) = \{x \in k^n \mid f(x) = 0, \ \forall f \in M\}$$

We call such subsets of k^n algebraic sets.

In the case of $n = 1$ the algebraic sets are easy to determine.

Example 0.2

In the case of $n = 1$ the only algebraic sets of k are finite sets and k itself. For that we can distinguish the cases $(M) = (0)$ and $(M) \neq (0)$. In the first case all it immediately follows that $M = \{0\}$. Hence we get

$$V(M) = \{x \in k \mid f(x) = 0, \ \forall f \in M\} = \{x \in k \mid 0 = 0\} = k$$

In the second case the zero locus can be written as the intersection of finite sets

$$V(M) = \{x \in k \mid f(x) = 0, \ \forall f \in M\} = \bigcap_{f \in M} \{x \in k \mid f(x) = 0\}$$

and is hence also finite.

In this setting affine varieties will be algebraic sets with some geometric structure and varieties in general will be glued from affine varieties. In the modern setting the objects will be schemes. Affine schemes will be some generalization of affine varieties with the idea of replacing the ring $k[T_1, \ldots, T_n]$ with arbitrary rings. Schemes in general will then glued from affine schemes.

The starting goal of the lecture will be establishing a correspondence between the sets

$$\{\text{subsets of } k[T_1, \ldots, T_n]\} \longleftrightarrow \{\text{subsets of } k^n\},$$

where the correspondence from left to right is given by the operator V defined previously. For the correspondence for right to left we define the following operator:

Definition 0.3

Let $X \subseteq k^n$ the we define the corresponding ideal to be

$$I(X) = \{f \in k[T_1, \dots, T_n] \mid f(x) = 0,\ \forall x \in X\}.$$

It is easy to check that for all $X \subseteq k^n$ this definition does indeed yield an ideal. Assume that $f, g \in I(X)$ and $x \in X$ then $(f+g)(x) = f(x) + g(x) = 0 + 0 = 0$. Dropping now the assumption that $g \in I(X)$ and assuming $g \in k[T_1, \dots, T_n]$ instead we get that $(g \cdot f)(x) = g(x)f(x) = g(x) \cdot 0 = 0$. Therefore $I(X)$ is an ideal in $k[T_1, \dots, T_n]$.

The goal at first will now be to understand properties of this correspondence and more specifically to characterize all $M \subseteq k[T_1, \dots, T_n]$ for which the algebraic set $V(M)$ is non-empty.

1 Commutative Rings and Ideals

In the following lecture we will always consider a ring to be commutative and to have a unit.

Definition 1.1

Let R be a ring. A subset $I \subseteq R$ is called an ideal if $I \subseteq R$ is a subgroup with respect to addition and $a \in R$ and $b \in I$ implies $ab \in I$.

Assume just the first property for $I \subseteq R$. So let $I \subseteq R$ be a subgroup with respect to addition. So we get a homomorphism of abelian groups

$$\begin{aligned} \pi : R &\longrightarrow {}^{R}\!/\!_{I} \\ a &\longmapsto a + I \end{aligned}$$

and the following lemma:

Lemma 1.2

The abelian group R/I carries a ring structure such that π is a ring homomorphism if and only if I is an ideal.

Proof. Assume R/I carries a well defined multiplication such that π is a ring homomorphism. Let now $a \in R$ and $b \in I$. So it follows that

$$\pi(ab) = \pi(a)\pi(b) = (a + I) \cdot (b + I) = (a + I) \cdot (0 + I) = 0 + I$$

and on the other hand $\pi(ab) = ab + I$. Hence $ab \in I$ and I is an ideal.
Conversely if I is an ideal we can define the multiplication structure on R/I by $(x+I)(y+I) = xy+I$. It is easy to check that this definition does not depend on the chosen representatives $x, y \in R$ and that π becomes a ring homomorphism.

$\square$

Example 1.3

For $n \in \mathbb{N}$ the ideal $n\mathbb{Z} \subset \mathbb{Z}$ results in the quotient ring $\mathbb{Z}/n\mathbb{Z}$.

A common quotient ring we will encounter is given by a subset $M \subset k[T_1, \dots, T_n]$ and the ideal (M). So the smallest ideal in $k[T_1, \dots, T_n]$, which contains M. The ring is then $k[T_1, \dots, T_n]/(M)$.

It turns out that this quotient construction satisfies the following universal property:

Lemma 1.4: Universal property of the quotient ring

Let $R \xrightarrow{\varphi} S$ be a ring homomorphism and $I \subseteq R$ be an ideal. If $\varphi(I) = \{0\}$ then there exists a unique ring homomorphism $\overline{\varphi} : R/I \to S$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & S \\ & R/I \arrow[ur, dashed, "\exists!\, \overline{\varphi}"'] & \end{tikzcd}$$

Proof. We define the morphism $\overline{\varphi}$ by $\overline{\varphi}(a+I) = \varphi(a)$. First it should be mentioned that if $a, b \in R$ with $a - b \in I$ then

$$\overline{\varphi}(b+I) = \varphi(b) + \varphi(a-b) = \varphi(a) = \overline{\varphi}(a+I)$$

and the definition $\overline{\varphi}$ does not depend on representatives. Second it should be checked that $\overline{\varphi}$ is indeed a ring homomorphism. So we compute for $a, b \in R$ that

$$\overline{\varphi}((a+I)+(b+I)) = \overline{\varphi}(a+b+I) = \varphi(a+b) = \varphi(a) + \varphi(b) = \overline{\varphi}(a+I) + \overline{\varphi}(b+I).$$

The computation for multiplication is analogous. Lastly it is easy to see that $\overline{\varphi}(1+I) = \varphi(1) = 1$ and so $\overline{\varphi}$ is a ring homomorphism. The fact that $\varphi = \overline{\varphi} \circ \pi$ immediately follows from the definition. It only remains to check that $\overline{\varphi}$ is unique with this property. Let $\tilde{\varphi} : R/I \to S$ be another ring homomorphism such that $\varphi = \tilde{\varphi} \circ \pi$. Then it follows that

$$\tilde{\varphi}(a+I) = \varphi(a) = \overline{\varphi}(a+I)$$

and hence that $\overline{\varphi}$ is unique.

□

1.1 Noetherian rings

Definition 1.5

A ring R is called noetherian if every ideal $I \subseteq R$ is finitely generated. That is for every ideal $I \subseteq R$ there exists a finite subset $M \subseteq R$ such that I is the smallest ideal which contains M. We denote this with $I = (M)$.

This definition is characterized by an eqivalent property stated in the following lemma.

Lemma 1.6: Ascending chain condition

A ring R is noetherian if and only if for every ascending chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \cdots$$

of ideals $I_j \subseteq R$ there exists an index $n \in \mathbb{N}$ such that for all $m \geq n$ we have $I_m = I_n$.

Proof. Assume that R is not noetherian and let $I \subseteq R$ be an ideal that is not finitely generated. Then for every finitely generated ideal $I' \subseteq I$ there exists an element $f \in I \setminus I'$. We construct an ascending chain of ideals by defining $I_0 := \{0\}$ and for every $n \in \mathbb{N}$ choosing an elemnt $f_n \in I \setminus I_n$ and setting $I_{n+1} := (I_n \cup \{f_n\})$. By definition this is an ascending chain of ideals that at no step has an equality.

Assume now that R is noetherian and consider an ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \cdots .$$

Then the set $I = \bigcup_{n\in\mathbb{N}} I_n$ is an ideal. For $a, b \in I$ there exists an $n \in \mathbb{N}$ such that $a, b \in I_n$ and hence $a + b \in I_n$ which implies $a + b \in I_n$. Similarly if $a \in R$ and $b \in I$ then there is an $n \in \mathbb{N}$ such that $b \in I_n$ and thus $ab \in I_n$ which implies $ab \in I$.

Due to R being noetherian we get that I is finitely generated. Let $\{f_1, \ldots, f_n\} \subset I$ be a finite generating set. We then define

$$.m := \max\{l \in \mathbb{N} \mid \exists i \in \{1, \ldots, n\} :\ f_i \notin I_l\}.$$

Since for every f_i there exists an index $l \in \mathbb{N}$ such that $f_i \in I_l$ this subset of $\mathbb{N}$ is bounded from above and hence contains a largest element. Now from $I_{m+1} \subseteq I$ and $\{f_1, \ldots, f_n\} \subseteq I_{m+1}$ follows $I_{m+1} = I$ and hence

$$I_{m+1} = I_{m+2} = I_{m+3} = I_{m+4} = \cdots .$$

□

There are some ways to easily construct noetherian rings out of known noetherian rings. The following lemma gives one such way.

Lemma 1.7

If $R \overset{\pi}{\twoheadrightarrow} R'$ is a surjective ring homomorphism and R is noetherian then R' is noetherian.

Proof. Let $I \subseteq R'$ be an ideal. Then $\pi^{-1}(I)$ is an ideal and finitely generated by assumption. Let $\{f_1, \ldots, f_n\} \subseteq \pi^{-1}(I)$ be a generating set of $\pi^{-1}(I)$. Then we claim that $I = (\pi(f_1), \ldots, \pi(f_n))$. Clearly we have $I \supseteq (\pi(f_1), \ldots, \pi(f_n))$, since $\pi(f_i) \in I$ is equivalent to $f_i \in \pi^{-1}(I)$. If $f' \in I$ then there exists $f \in R$ with $\pi(f) = f'$ and so $f \in \pi^{-1}(I)$. Therefore f can be written as $f = \sum_{i=1}^n g_i f_i$ with some $g_i \in R$. It follows that $f' = \pi(f) = \sum_{i=1} \pi(g_i)\pi(f_i) \in (\pi(f_1), \ldots, \pi(f_n))$.

With the equality shown it follows that I is finitely generated.

□

Another way of finding new noetherian rings is given by the following theorem.

Theorem 1.8: Hilbert's basis theorem

If R is noetherian then $R[T]$ is noetherian.

Before we prove the theorem we formulate a few conclusions from the theorem and the previous lemma. Additionally we need a preliminary lemma in the proof of the theorem.

Corollary 1.9

If R is noetherian and $n \in \mathbb{N}$ then $R[T_1, \dots, T_n]$ is noetherian. If $I \subset R[T_1, \dots, T_n]$ is an ideal then $R[T_1, \dots, T_n]/I$ is noetherian.

Proof. The first part of the statement follows from the theorem and induction by n. The second part of the statement follows from the lemma and the first part of the statement. □

Lemma 1.10

Let R be a ring and the ideal $I \subseteq R$ be finitely generated. Let furthermore $M \subseteq I$ be a generating set of I. Then there exists a finite subset $M' \subseteq M$ that generates I

Proof. Let $\{f_1, \dots, f_m\}$ be a finite generating set for I. Then for every $i \in \{1, \dots, n\}$ we can write

$$f_i = \sum_{g \in M} b_{ig} g$$

while almost all b_{ig} vanish. So the set

$$M' \coloneqq \{g \in M \mid \exists i \in \{1, \dots, n\} : \ b_{ig} \neq 0\}$$

is finite and we claim that M' generates I. The inclusion $(M') \subseteq I$ is clear. So assume that $h \in I$. Becuase the f_i generate I it follows that there exist $a_1, \dots, a_n \in R$ such that

$$h = \sum_{i=1}^{n} a_i f_i = \sum_{i=1}^{n} \sum_{g \in M'} a_i b_{ig} g = \sum_{g \in M'} \left(\sum_{i=1}^{n} a_i b_{ig} \right) g$$

and hence $h \in (M')$ □

Proof of Hilbert's basis theorem. Assume $R[T]$ is not noetherian and let $I \subseteq R[T]$ be an ideal which is not finitely generated. Choose a polynomial $f_1 \in I$ of minimal degree n_1. For every $i \in \mathbb{N}$ choose a polynomial $f_i \in I \setminus (f_1, \dots, f_{i-1})$ of minimal degree n_i. Note that for $i \leq j$ we get $n_i \leq n_j$. For all $i \in \mathbb{N}$ we denote the leading coefficient of f_i with a_i. Since R is noetherian it follows that $(a_i \mid i \in \mathbb{N}) \subseteq R$ is finitely generated. By Lemma 1.10 there exists an $m \in \mathbb{N}$ such that $(a_1, \dots, a_m) = (a_i \mid i \in \mathbb{N})$ and in particular there are $b_1, \dots, b_m \in R$ such that

$$a_{m+1} = \sum_{i=1}^{m} b_i a_i.$$

With that we set

$$g \coloneqq f_{m+1} - \sum_{i=1}^{m} b_i T^{n_{m+1}-n_i} f_i$$

Then we get that the n_{m+1}-th coefficient of g vanishes and hence that g is of lower degree than f_{m+1}. At the same time it must be the case that $g \in I \setminus (f_1, \ldots, f_m)$, because if $g \in (f_1, \ldots, f_m)$ then

$$f_{m+1} = g + \sum_{i=1}^{m} b_i T^{n_{m+1}-n_i} f_i \in (f_1, \ldots, f_m)$$

and that is not the case by the choice of f_{m+1}. But now g being of lower degree than f_{m+1} and $g \in I \setminus (f_1, \ldots, f_m)$ contradicts the minimality of n_{m+1}. So the assumption that I is not finitely generated must have been wrong.

□

Remark 1.11

We just showed that $k[T_1, T_2]$ and $\mathbb{Z}[T]$ are noetherian. But even though every ideal I in these rings is finitely generated, we can choose I to require arbitrarily large (finite) generating sets.

Proof. □

1.2 Units

Definition 1.12

Let R be a ring. We call the elements of the set

$$R^* = \{x \in R \mid \exists y \in R : \ xy = 1\}$$

the units in R.

Remark 1.13

Given $x \in R*$ the corresponding $y \in R$ with $xy = 1$ is unique. We denote this element with x^{-1}.

Proof. Let $y, y' \in R$ with $xy = xy' = 1$. Then we have $y = (xy')y = (xy)y' = y'$. □

Definition 1.14

Let R be a ring. A subset $S \subseteq R$ is called multiplicative if $1 \in S$ and for all $a, b \in S$ it follows that $ab \in S$.

Consider a ring R and a multiplicative subset $S \subseteq R$. The aim is to construct the ring of fractions which have only elements of R as enumerators and elements of S as denominatos.

Lemma 1.15

Define the relation $\sim$ on $R \times S$ by

$$(x, s) \sim (y, t) \quad :\Leftrightarrow \quad \exists u \in S: \; xtu = ysu.$$

Then $\sim$ is an equivalence relation whose equivalence classes we denote by $[(x, s)]_\sim = \frac{x}{s}$.

Proof. Form the definition reflexivity and symmetry are clear. Transitivity requires a short calculation. Let $(x, s) \sim (y, t)$ and $(y, t) \sim (z, u)$. so there exist $a, b \in S$ with $xta = ysa$ and $yub = ztb$. It follows that

$$xu(tab) = xta(ub) = ysa(ub) = yub(sa) = ztb(sa) = zs(tab)$$

and so with $tab \in S$ we get $(x, s) \sim (z, u)$. $\square$

Lemma 1.16

For a ring R and a multiplicaive subset $S \subseteq R$ we denote with

$$S^{-1}R = \left\{ \frac{x}{s} \mid x \in R, \; s \in S \right\}$$

the set of equivalence classes from the previous lemma. On this set we define an addition and a multiplication by

$$\frac{x}{s} + \frac{y}{t} = \frac{xt + ys}{st}$$

and

$$\frac{x}{s} \cdot \frac{y}{s} = \frac{xy}{st}$$

respectively. Then $S^{-1}R$ together with thes operations is a ring which we call the localization of R by S.

Proof. It is clear that the addition an multiplication are commutative and associative. Furthermore it is easy to see that $\frac{0}{1}$ is a neutral element with respect to addition and $\frac{1}{1}$ is neutral with respect to multiplication. It remains to check thhat both operations are well-defined and that multiplicaton distributes over addition. Let $(x, s) \sim (x', s')$ and $(y, t) \sim (y', t')$. So there are $u, v \in S$ such that $xs'u = x'su$ and $yt'v = y'tv$. Then we get

$$\frac{x}{s} + \frac{y}{t} = \frac{xt + ys}{st} = \frac{xts't'uv + yss't'uv}{sts't'uv} = \frac{x'sutt'v + y'tvss'u}{sts't'uv} = \frac{x't' + y's'}{s't'} = \frac{x'}{s'} + \frac{y'}{t'}$$

and similarly we compute

$$\frac{x}{s} \cdot \frac{y}{t} = \frac{xy}{st} = \frac{xs'uyt'v}{sts't'uv} = \frac{x'suy'tv}{sts't'uv} = \frac{x'y'}{s't'} = \frac{x'}{s'} \cdot \frac{y'}{t'}.$$

To check the distributivity let $x, y, z \in R$ and $s, t, u \in S$. Then we get

$$\frac{x}{s}\left(\frac{y}{t} + \frac{z}{u}\right) = \frac{x(yu+zt)}{stu} = \frac{xyu+xzt}{stu} = \frac{xyus+xzts}{s^2tu} = \frac{xy}{st} + \frac{xz}{su} = \frac{x}{s}\frac{y}{t} + \frac{x}{s}\frac{z}{u}.$$

□

We get that the localization satisfies a universal property.

Lemma 1.17: Universal property of the localization

For a ring R and a multiplicative subset $S \subseteq R$ consider the ring homomorphism

$$\begin{aligned} j : R &\longrightarrow S^{-1}R \\ x &\longmapsto \frac{x}{1} \end{aligned}$$

If $\varphi : R \to R'$ is a ring homomorphism such that $\varphi(S) \subseteq (R')^*$ then there exists a unique ring homomorphism $\overline{\varphi} : S^{-1}R \to R'$ such that the diagram

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "j"'] & & R' \\ & S^{-1}R \arrow[ur, dashed, "\exists!\,\overline{\varphi}"'] & \end{tikzcd}$$

commutes.

Proof. First we construct the homomorphism $\overline{\varphi}$. Since for every $s \in S$ the image $\varphi(s)$ is invertible in R' we define

$$\overline{\varphi}\left(\frac{x}{s}\right) = \varphi(x)\varphi(s)^{-1}.$$

Then clearly

$$\varphi(x) = \varphi(x)\varphi(1)^{-1} = \overline{\varphi}\left(\frac{x}{1}\right) = \overline{\varphi}(j(x)).$$

Futhermore it follows immediately form the fact that φ is a ring homomorphism that $\overline{\varphi}$ is also one. It remains to check that φ is well-defined. Let $(x, s) \sim (x', s')$ by the element $u \in S$. Then we get

$$\overline{\varphi}\left(\frac{x}{s}\right) = \varphi(x)\varphi(s)^{-1} = \varphi(xs'u)\varphi(ss'u)^{-1} = \varphi(x'su)\varphi(ss'u)^{-1} = \varphi(x')\varphi(s')^{-1} = \overline{\varphi}\left(\frac{x'}{s'}\right).$$

Let $\tilde{\varphi}$ be another homomorphism that makes the diagram commute. Then it follows that

$$\overline{\varphi}\left(\frac{x}{s}\right) = \varphi(x)\varphi(s)^{-1} = \tilde{\varphi}\left(\frac{x}{1}\right)\tilde{\varphi}\left(\frac{s}{1}\right)^{-1} = \tilde{\varphi}\left(\frac{x}{1}\right)\tilde{\varphi}\left(\frac{1}{s}\right) = \tilde{\varphi}\left(\frac{x}{s}\right).$$

□

Similarly to the quotient ring the localization also preserves the property of being noetherian.

Lemma 1.18

If R is a noetherian ring and $S \subseteq R$ is a multiplicative subset then $S^{-1}R$ is noetherian.

Proof. From Hilbert's basis theorem it follows that $R[T]$ is noetherian. Considering the surjective homomorphism $R[T] \to S^{-1}R$, which sends constants $x \in R[T]$ to x and sends T to $\frac{1}{s}$ we get that $S^{-1}R$ is noetherian. $\square$

1.3 Prime ideals and maximal ideals

Definition 1.19

Let R be a ring. An ideal $I \subseteq R$ is called a prime ideal if the quotient ring R/I is a domain.

We have the following equivalent characterization of prime ideals:

Lemma 1.20

Let R be a ring. An ideal $I \subseteq R$ is a prime ideal if and only if for all $a, b \in R$ with $ab \in I$ it follows $a \in I$ or $b \in I$.

Proof. Let first R/I be a domain and $a, b \in R$ be such that $ab \in I$. Then it follows that in the quotient R/I we have $(a + I)(b + I) = ab + I = 0 + I$ and because R/I is a domain it follows that $a + I = 0 + I$ or $b + I = 0 + I$. Hence $a \in I$ or $b \in I$.

Conversely assume that $I \subseteq R$ is a prime ideal and let $(a + I)(b + I) = 0 + I$. So it follows that $ab \in I$ and because I is prime we have $a \in I$ or $b \in I$. So $a + I = 0 + I$ or $b + I = 0 + I$ making R/I a domain.

$\square$

Definition 1.21

Let R be a ring. An ideal $I \subseteq R$ is called maximal if $I \neq R$ and for all ideals $J \subseteq R$ with $I \subseteq J$ it follows that $J = I$ or $J = R$.

Remark 1.22

For any ideal $I \subseteq R$ the projection homomorphism

$$\begin{aligned} \pi : R &\longrightarrow R/I \\ x &\longmapsto x + I \end{aligned}$$

induces a bijection between the set of ideals $J \subseteq R$ with $I \subseteq J$ and the set of ideals of the quotient ring R/I. This happens by mapping an ideal $\overline{J} \subseteq R/I$ to $\pi^{-1}(\overline{J})$. It is easy to check that this is a homomorphism and that the inverse map is given by mapping an ideal $I \subseteq J \subseteq R$ to $\pi(J)$, which is an ideal because the projection is surjective.

This remark becomes useful in formulating the following important characterization of maximal ideals.

Lemma 1.23

Let R be a ring and $I \subseteq R$ be an ideal. Then I is maximal if and only if the quotient ring R/I is a field.

Proof. Assume that I is maximal. This is the case if and only if there only exist two ideals $J \subseteq R$ with IJ. By the previous remark this is equivalent to R/I only having two ideals, which must be $(0), (1) \subseteq R/I$. This is the case if and only if R/I is a field. $\square$

Theorem 1.24

If R is a ring and $R \neq 0$ then R contains a maximal ideal.

Proof. Let $\mathcal{M}$ be the set of all ideals $I \subseteq R$, which are not R itself. Then we get that $\mathcal{M} \neq \varnothing$ since $(0) \in \mathcal{M}$. If $\mathcal{M}' \subseteq \mathcal{M}$ is a totally ordered subset then the union $I' :=_{I \in \mathcal{M}'} I$ is an ideal and hence an upper bound. By Zorns Lemma it follows that $\mathcal{M}$ contains a maximal ideal.

$\square$

2 Algebraic Sets

In the following chapter k always denotes an algebraically closed field.

Definition 2.1

Let $M \subseteq k[\underline{T}] := k[T_1, \ldots, T_n]$. We define $V(M)$ to be the zero locus of all elements of M. That means

$$V(M) := \{x \in k^n \mid f(x) = 0 \, \forall f \in M\}.$$

For a subset $X \subseteq k^n$ we define

$$I(X) := \{f \in k[\underline{T}] \mid f(x) = 0 \, \forall x \in X\}.$$

This gives us a correspondence between subsets of $k[\underline{T}]$ and subsets of k^n.

Right away there are some statements that can be observed about these definitions.

Remark 2.2

((*i*)) The operators V and I are inclusion reversing. That means if $M \subseteq M' \subseteq k[\underline{T}]$ then it follows that $V(M) \supseteq V(M')$ and similarly if $X \subseteq X' \subseteq k^n$ then $I(X) \supseteq I(X')$. Furthermore for all $X \subseteq k^n$ it holds that $X \subset V(I(X))$.

((*ii*)) For all $X \subseteq k^n$ the set $I(X) \subseteq k[\underline{T}]$ is an ideal.

Proof. (i): Let $M \subseteq M' \subseteq k[\underline{T}]$ and $X \subseteq X' \subseteq k^n$. Then for a point $x \in k^n$ then if $f(x) = 0$ for all $f \in M'$ then in particular $f(x) = 0$ for all $f \in M$. Similarly if some $f \in k[\underline{T}]$ vanishes everywhere on X' then in particular it vanishes everywhere on X.
Lastly if $x \in X$ and $f \in I(X)$ then $f(x) = 0$ by definition of $I(X)$ and hence $X \subset V(I(X))$.

(ii): Now let $f_1, f_2 \in I(X)$ and $x \in X$. Then it follows that $(f_1+f_2)(x) = f_1(x)+f_2(x) = 0+0 = 0$. If now $g \in k[\underline{T}]$ then $(gf_1)(x) = g(x)f_1(x) = g(x) \cdot 0 = 0$. Hence $I(X)$ is an ideal in $k[\underline{T}]$.

□

With a similar proof to the second statement of the remark one can also show that for a subset $M \subset k[\underline{T}]$ we have $V(M) = V((M))$.

Definition 2.3

A subset $X \subset k^n$ is called algebraic if $X = V(M)$ for some subset $M \subseteq k[\underline{T}]$. We call an ideal $I \subseteq k[\underline{T}]$ admissable if and only if $I = I(X)$ for some $X \subseteq k^n$.

It should be remarked that for $X \subseteq k^n$ there exists $M \subseteq k[\underline{T}]$ with $X = V(M)$ if and only if there exists an ideal $I \subseteq k[\underline{T}]$ such that $X = V(I)$.

Lemma 2.4

Let $\mathcal{A}$ be the set of algebraic subsets of k^n and $\mathcal{B}$ be the set of admissable ideals of $k[\underline{T}]$. Then the maps $I|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ and $V|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ are bijections and inverses of one another.

Proof. Let X be an algebraic subset. We always have $X \subseteq V(I(X))$. Since X is algebraic there exists an ideal $J \subseteq k[\underline{T}]$ such that $X = V(J)$. Then we claim that $J \subseteq I(V(J))$. If $f \in J$ and $x \in V(J)$ then $f(x) = 0$ by definition. It follows that $f \in I(V(J))$ and hence our claim. From this and the fact that V is inclusion reversing we conclude that $X = V(J) \supseteq V(I(V(J))) = V(I(X))$. □

Inn the following chapter we seek to understand what algebraic sets and admissable ideals are and we are looking for alternative ways to characterize them. The following statement gives a bit more structure to the algebraic subsets of k^n.

Lemma 2.5

The algebraic subsets of k^n are the closed subsets of a topology, which we call the Zariski topology on k^n.

Proof. It is easy to see that $V(\{1\}) = \varnothing$ and $V(\{0\}) = k^n$. Let $\{M_i\}_i$ be a collection of subsets of $k[\underline{T}]$ and $M_1, M_2 \in k[\underline{T}]$. Then we claim that $V(\bigcup_i M_i) = \bigcap_i V(M_i)$ and

$$V(M_1M_2) = V(\{f_1f_2 \mid f_1 \in M_1,\ f_2 \in M_2\}) = V(M_1) \cup V(M_2).$$

We first check that the first statement holds. If $x \in V(\bigcup_i M_i)$ then if $f \in M_i$ for some i it follows that $f(x) = 0$. Hence $x \in V(M_i)$, where i was chosen arbitrarily. So we get $V(\bigcup_i M_i) \subseteq \bigcap_i V(M_i)$. On the other hand if $x \in V(M_i)$ for all i then $f(x) = 0$ for all $f \in \bigcup_i M_i$. Therefore $V(\bigcup_i M_i) \supseteq \bigcap_i V(M_i)$.
Now we check that the second statement holds. If $x \in V(M_1)$ and we have $f \in M_1$ and $g \in M_2$ respectively it follows that $(fg)(x) = f(x)g(x) = g(x) \cdot 0 = 0$. Hence $x \in V(M_11M_2)$. On the other hand if $x \notin V(M_1) \cup V(M_2)$ there must exist $f \in M_1$ and $g \in M_2$ such that $f(x) \neq 0$ and $g(x) \neq 0$. So $(fg)(x) = f(x)g(x) \neq 0$, since $k[\underline{T}]$ is a domain. Therefore $x \notin V(M_1M_2)$. □

Example 2.6

In an example from the introduction we have already seen that the Zariski topology on k^1 is the cofinite topology. Recall that we have shown the closed sets to be all finite sets and k itself.

2.1 First properties of the Zariski topology

We notice that the Zariski topology is (T1). That means that singletons $\{x\}$, $x \in k^n$ are closed sets. Given $x = (x_1, \ldots, x_n) \in k^n$ we have

$$V(T_1 - x_1, \ldots, T_n - x_n) = \{x\}.$$

We give the ideal corresponding to $\{x\}$ a special name.

Definition 2.7

For a point $x \in k^n$ we write $\mathfrak{m}_x = I(\{x\})$

We can immediately observe that for a point $x = (x_1, \dots, x_n) \in k^n$ $\mathfrak{m}_x$ is the kernel of the evaluation homomorphism

$$\begin{aligned} \mathrm{ev}_x : k[T_1, \dots, T_n] &\longrightarrow k \\ f &\longmapsto f(x) = f(x_1, \dots, x_n). \end{aligned}$$

With the help of this view on $\mathfrak{m}_x$ we can colclude the following lemma.

Lemma 2.8

Let $x = (x_1, \dots, x_n) \in k^n$. Then $\mathfrak{m}_x = (T_1 - x_1, \dots, T_n - x_n)$.

Proof. We denote the ideal $(T_1 - x_1, \dots, T_n - x_n)$ with I. We have already shown that $J \subseteq I(V(J))$ for all ideals $J \subseteq k[\underline{T}]$. In particular $I \subseteq \mathfrak{m}_x$.

□