In the following chapter k always denotes an algebraically closed field.

Definition -1.1

Let $M \subseteq k[\underline{T}] \coloneqq k[T_1, \ldots, T_n]$. We define $V(M)$ to be the zero locus of all elements of M. That means

$$V(M) \coloneqq \{x \in k^n \mid f(x) = 0 \; \forall f \in M\}.$$

For a subset $X \subseteq k^n$ we define

$$I(X) \coloneqq \{f \in k[\underline{T}] \mid f(x) = 0 \; \forall x \in X\}.$$

This gives us a correspondence between subsets of $k[\underline{T}]$ and subsets of k^n.

Right away there are some statements that can be observed about these definitions.

Remark -1.2

((i)) The operators V and I are inclusion reversing. That means if $M \subseteq M' \subseteq k[\underline{T}]$ then it follows that $V(M) \supseteq V(M')$ and similarly if $X \subseteq X' \subseteq k^n$ then $I(X) \supseteq I(X')$. Furthermore for all $X \subseteq k^n$ it holds that $X \subset V(I(X))$.

((ii)) For all $X \subseteq k^n$ the set $I(X) \subseteq k[\underline{T}]$ is an ideal.

Proof. (i): Let $M \subseteq M' \subseteq k[\underline{T}]$ and $X \subseteq X' \subseteq k^n$. Then for a point $x \in k^n$ then if $f(x) = 0$ for all $f \in M'$ then in particular $f(x) = 0$ for all $f \in M$. Similarly if some $f \in k[\underline{T}]$ vanishes everywhere on X' then in particular it vanishes everywhere on X.
Lastly if $x \in X$ and $f \in I(X)$ then $f(x) = 0$ by definition of $I(X)$ and hence $X \subset V(I(X))$.

(ii): Now let $f_1, f_2 \in I(X)$ and $x \in X$. Then it follows that $(f_1+f_2)(x) = f_1(x)+f_2(x) = 0+0 = 0$. If now $g \in k[\underline{T}]$ then $(gf_1)(x) = g(x)f_1(x) = g(x) \cdot 0 = 0$. Hence $I(X)$ is an ideal in $k[\underline{T}]$.

□

With a similar proof to the second statement of the remark one can also show that for a subset $M \subset k[\underline{T}]$ we have $V(M) = V((M))$.

Definition -1.3

A subset $X \subset k^n$ is called algebraic if $X = V(M)$ for some subset $M \subseteq k[\underline{T}]$. We call an ideal $I \subseteq k[\underline{T}]$ admissable if and only if $I = I(X)$ for some $X \subseteq k^n$.

It should be remarked that for $X \subseteq k^n$ there exists $M \subseteq k[\underline{T}]$ with $X = V(M)$ if and only if there exists an ideal $I \subseteq k[\underline{T}]$ such that $X = V(I)$.

Lemma -1.4

Let $\mathcal{A}$ be the set of algebraic subsets of k^n and $\mathcal{B}$ be the set of admissable ideals of $k[\underline{T}]$. Then the maps $I|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ and $V|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ are bijections and inverses of one another.

Proof. Let X be an algebraic subset. We always have $X \subseteq V(I(X))$. Since X is algebraic there exists an ideal $J \subseteq k[\underline{T}]$ such that $X = V(J)$. Then we claim that $J \subseteq I(V(J))$. If $f \in J$ and $x \in V(J)$ then $f(x) = 0$ by definition. It follows that $f \in I(V(J))$ and hence our claim. From this and the fact that V is inclusion reversing we conclude that $X = V(J) \supseteq V(I(V(J))) = V(I(X))$. $\square$