The area of algebraic geometry originates with the study of solutions to polynomial equations and systems of polynomial equations in multiple variables. This lecture begins with the more classical approach to algebraic geometry and later transitions to the more modern approach.

In the following lecture we denote with k an algebraically closed field unless stated otherwise. The classical view of algebraic geometry mainly deals with the following objects:

Definition 0.1: Algebraic sets

Let $k[T_1, \dots, T_n]$ be the set set of polynomials in variables $T_1, \dots, T_n$ with coefficients in k. For $M \subseteq k[T_1, \dots, T_n]$ we denote the zero locus of M by

$$V(M) = \{x \in k^n \mid f(x) = 0, \ \forall f \in M\}$$

We call such subsets of k^n algebraic sets.

In the case of $n = 1$ the algebraic sets are easy to determine.

Example 0.2

In the case of $n = 1$ the only algebraic sets of k are finite sets and k itself. For that we can distinguish the cases $(M) = (0)$ and $(M) \neq (0)$. In the first case all it immediately follows that $M = \{0\}$. Hence we get

$$V(M) = \{x \in k \mid f(x) = 0, \ \forall f \in M\} = \{x \in k \mid 0 = 0\} = k$$

In the second case the zero locus can be written as the intersection of finite sets

$$V(M) = \{x \in k \mid f(x) = 0, \ \forall f \in M\} = \bigcap_{f \in M} \{x \in k \mid f(x) = 0\}$$

and is hence also finite.

In this setting affine varieties will be algebraic sets with some geometric structure and varieties in general will be glued from affine varieties. In the modern setting the objects will be schemes. Affine schemes will be some generalization of affine varieties with the idea of replacing the ring $k[T_1, \dots, T_n]$ with arbitrary rings. Schemes in general will then glued from affine schemes.

The starting goal of the lecture will be establishing a correspondence between the sets

$$\{\text{subsets of } k[T_1, \dots, T_n]\} \longleftrightarrow \{\text{subsets of } k^n\},$$

where the correspondence from left to right is given by the operator V defined previously. For the correspondence for right to left we define the following operator:

Definition 0.3

Let $X \subseteq k^n$ the we define the corresponding ideal to be

$$I(X) = \{f \in k[T_1, \ldots, T_n] \mid f(x) = 0,\ \forall x \in X\}.$$

It is easy to check that for all $X \subseteq k^n$ this definition does indeed yield an ideal. Assume that $f, g \in I(X)$ and $x \in X$ then $(f+g)(x) = f(x) + g(x) = 0 + 0 = 0$. Dropping now the assumption that $g \in I(X)$ and assuming $g \in k[T_1, \ldots, T_n]$ instead we get that $(g \cdot f)(x) = g(x)f(x) = g(x) \cdot 0 = 0$. Therefore $I(X)$ is an ideal in $k[T_1, \ldots, T_n]$.

The goal at first will now be to understand propertiesof this correspondence and more specifically to aswer the question for which $M \subseteq k[T_1, \ldots, T_n]$ the algebraic set $V(M)$ is non-empty.