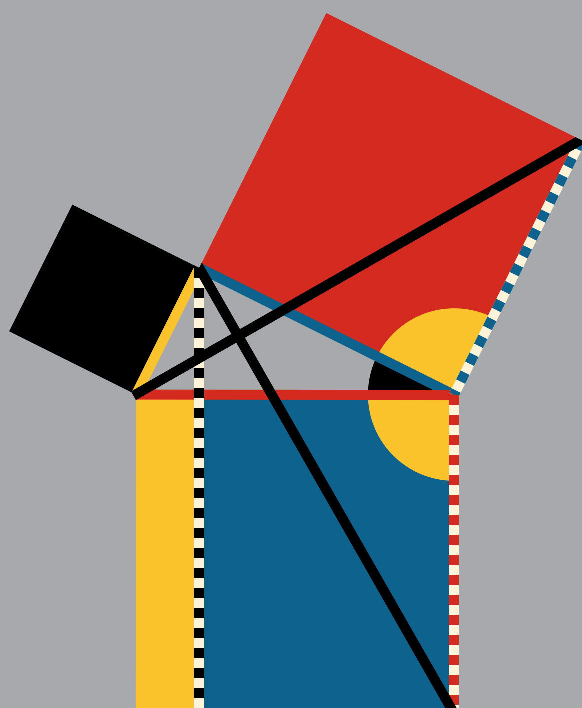
BYRNE'S EUCLID

The First Six Books of the *Elements* of Euclid



Newell Jensen

2023

Byrne's

Euclid

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Dedicated to the COLOURFUL memory of my father, WILLIAM HENRY JENSEN.

Preface

Let no one destitute of geometry enter my doors.

— Plato

Sire, there is no royal road to geometry.

— Euclid

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

— G. H. Hardy

Legend has it that the words, *"Let no one destitute of geometry enter my doors"* were inscribed in the doorway entering Plato's Academy in Athens. Euclid, a renowned ancient Greek mathematician of Alexandria and contemporary of Plato, is widely regarded as the '*Father of geometry*' for his contributions to the field as well as his monumental treatise of 13 books, the *Elements*. A mathematical and logical masterpiece, the *Elements* is a collection of definitions, postulates, propositions (consisting of theorems, problems and constructions), as well as the logical proofs of these propositions. The *Elements* has been referred to as the most successful and influential textbook ever written.

According to the ancient Greek historian Proclus, when the King of Egypt asked Euclid if there was an easier way to learn geometry, Euclid famously replied: *"There is no royal road to geometry."* A well-known quote, often used to emphasize the importance of hard work, discipline, and persistence in the pursuit of knowledge and understanding, it suggests that there are no shortcuts to true understanding, and that the only way to master a subject is through diligent study and practice.

Enter Oliver Byrne. An Irish-born civil engineer and surveyor, Byrne (1810-1880) is best known for his illustrated edition of the first six books of Euclid's *Elements*, published in 1847 under the title *Byrne's Euclid*. Byrne's edition is noted for its distinctive use of colour-coded diagrams and symbols, intended to make the complex concepts of Euclidean geometry more accessible and understandable to a wider audience. Each of the book's geometric figures is rendered in bold primary colors, with different colors being used to distinguish between various parts of the figure and to indicate different types of lines and angles. Byrne's edition of the *Elements* was a commercial success and went through several editions in the years following its initial publication. While the book's approach to illustrating geometry was unconventional for its time, it has since become a popular and influential work in the field of graphic design, as well as a fascinating example of the intersection between mathematics, art, and visual communication.

Overall, *Byrne's Euclid* represents a unique and innovative approach to the study of geometry – one that combines technical precision with a bold and imaginative visual style. Inspiring generations of students and scholars to approach the study of mathematics with a sense of creativity and wonder, it remains an important and enduring work in the history of mathematical literature. The book's use of colorful diagrams and illustrations, along with concise and straightforward explanations, can make it easier for students to understand abstract concepts and develop a deeper appreciation for the beauty and elegance of mathematics. Additionally, studying *Byrne's Euclid* can help students develop problemsolving skills and logical reasoning, which are valuable not just in mathematics but in many areas of life.

My first experience with *Byrne's Euclid* occurred while I was working through Euclid's *Elements* and searching on the Internet for more information about a particular proposition, when I came across Nicholas Rougeux's exquisite reproduction of Oliver Byrne's celebrated work — `https://c82.net/euclid`. Awed by the stunning beauty of the *Elements* and its logical precision, as well as Byrne's masterful and imaginative approach, I was filled with inspiration to create this book. Both G. H. Hardy's quote and *Byrne's Euclid*, underscore the creative and aesthetic dimensions of mathematics. Hardy's quote highlights how mathematicians, like painters or poets, create enduring patterns with ideas, while *Byrne's Euclid*, visually showcases the beauty of mathematical concepts through intricate illustrations. Together, they remind us that mathematics is not solely a logical pursuit, but also a richly imaginative and expressive one. It is my hope that this rendition of *Byrne's Euclid* continues in this spirit.

While Byrne's original work featured the elegant *Caslon* typeface, I have chosen to use the open source *EB Garamond* typeface for my edition. While the two typefaces share many similarities in ligatures and glyphs, those who aren't typography experts may not even notice the difference. For example, both *Caslon* and *EB Garamond* are serif typefaces, which means they have small decorative lines at the ends of each letter stroke. Classic and elegant, both have had a long history in printing and publishing.

In terms of their specific design features, these typefaces have similar letter shapes and proportions. For example, they both have a lowercase "a" with a curved tail, and a lowercase "g" with a descending loop. They also both have a tall and narrow uppercase "H", and a diagonal crossbar on the uppercase "A". Finally, both typefaces feature ligatures (two or more letters that are joined together into a single glyph), such as "fi" and "fl", which have a similar design and placement in the two typefaces.

First time readers of *Byrne's Euclid* should be made aware that the long s (ſ) is used throughout the book. The long s (ſ) is a letterform of the Latin alphabet that was commonly used in Europe from the Middle Ages until the 19^{th} century. It looks like a lowercase ”s”, but with a longer, more elongated shape, resembling an ”f” without its crossbar. In printed materials from the time, the long s was used in place of a normal ”s” at the beginning or in the middle of a word, but not at the end of a word or after certain letters like ”m”, ”n”, or ”u.”

A stylistic convention, the use of the long s was thought to make text easier to read and more aesthetically pleasing, as it allowed letters to be more closely spaced and made words look more uniform in appearance. However, as printing technology evolved and more uniform letter spacing became possible, the long s fell out of use and was gradually replaced by the modern short ”s” in the 19^{th} century. Despite its decline in usage, the long s can still be found in some historic texts and remains a fascinating example of the evolution of written language over time. Here is an illustration of the long s,

ſorts = sorts ≠ forts

caſe = case ≠ cafe

I wish to express my gratitude and appreciation to the creators and contributors of LaTeX, whose powerful and versatile document preparation system has been instrumental in typesetting this book. I am also deeply grateful to Nicholas Rougeux for his generous permission to utilize the Scalable Vector Graphics figures from his website. Additionally, I would like to express my gratitude to all the logicians and mathematicians whose shoulders we all stand on. And finally, to you, the reader. It's my sincere hope and desire that you enjoy this book and the logical truths found herein. I encourage you to study and practice the propositions so that you may walk your own road to geometry.

Newell Jensen, 2023

Reason is immortal, all else is mortal.

— Pythagoras

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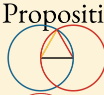
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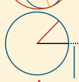
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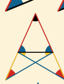
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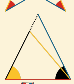
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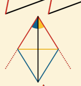
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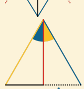
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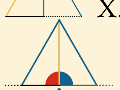
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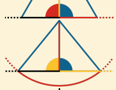
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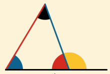
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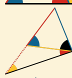
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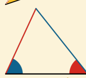
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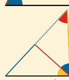
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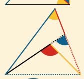
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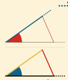
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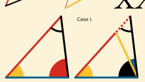

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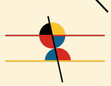
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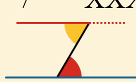
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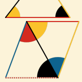
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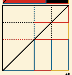
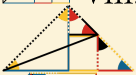
 VIII. Theorem 83

 IX. Theorem 84

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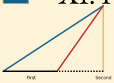
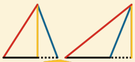
 XII. Problem 90

 XIII. Problem 91

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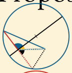
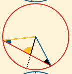
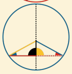
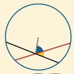
 I. Problem 97

 II. Theorem 98

 III. Theorem. 99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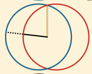
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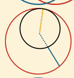
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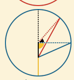
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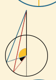
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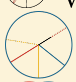
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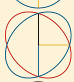
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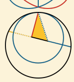
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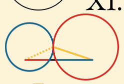
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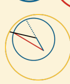
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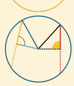
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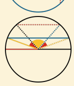
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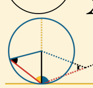
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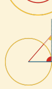
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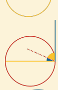
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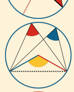
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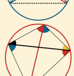
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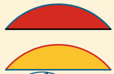
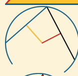
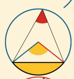
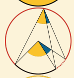
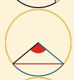
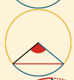
 XXIV. Theorem 129

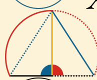
 XXV. Theorem 130

 XXVI. Theorem 131

 XXVII. Theorem 132

 XXVIII. Theorem. 133

 XXIX. Theorem 134

 XXX. Problem 135

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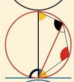
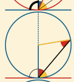
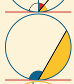
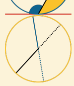
 XXXII. Theorem 138

 XXXIII. Problem 139

 XXXIV. Problem 140

 XXXV. Theorem 141

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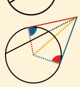 XXXVII. Problem 145

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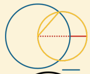
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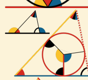
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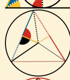
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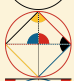
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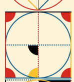
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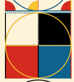
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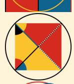
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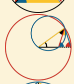
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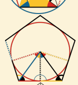
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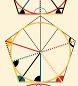
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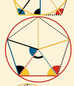
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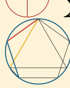
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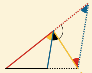
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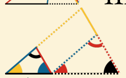
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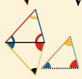
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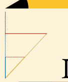
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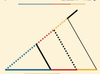
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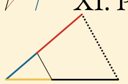
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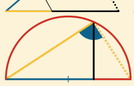
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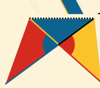
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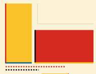
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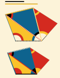
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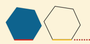
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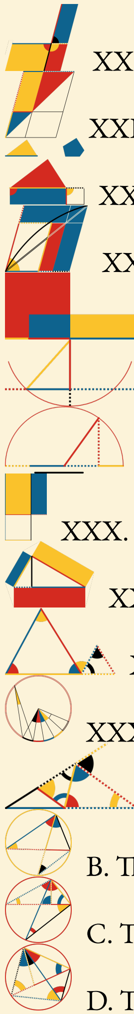

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INTRODUCTION.

THE arts and ſciences have become ſo extenſive, that to facilitate their acquirement is of as much importance as to extend their boundaries. Illuſtration, if it does not ſhorten the time of ſtudy, will at leaſt make it more agreeable. This work has a greater aim than mere illuſtration; we do not introduce colours for the purpoſe of entertainment, or to amuſe *by certain combinations of tint and form*, but to aſſiſt the mind in its reſearches after truth, to increaſe the facilities of inſtruction, and to diffuſe permanent knowledge. If we wanted authorities to prove the importance and uſefulneſs of geometry, we might quote every philoſopher ſince the days of Plato. Among the Greeks, in ancient, as in the ſchool of Peſtalozzi and others in recent times, geometry was adopted as the beſt gymnaſtic of the mind. In fact, Euclid's Elements have become, by common conſent, the baſis of mathematical ſcience all over the civilized globe. But this will not appear extraordinary, if we conſider that this ſublime ſcience is not only better calculated than any other to call forth the ſpirit of inquiry, to elevate the mind, and to ſtrengthen the reaſoning faculties, but alſo it forms the beſt introduction to moſt of the uſeful and important vocations of human life. Arithmetic, land-ſurveying, menſuration, engineering, navigation, mechanics, hydroſtatics, pneumatics, optics, phyſical aſtronomy, &c. are all dependent on the propoſitions of geometry.

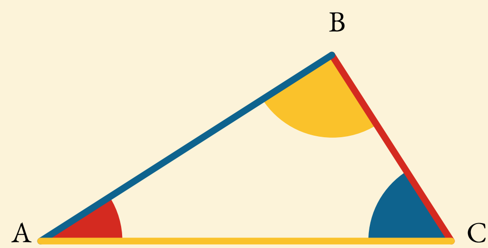
Much however depends on the firſt communication of any ſcience to a learner, though the beſt and moſt eaſy methods are ſeldom adopted. Propoſitions are placed before a ſtudent, who though having a ſufficient underſtanding, is told juſt as much about them on entering at the very threſhold of the ſcience, as gives him a prepoſſeſſion moſt unfavourable to his future ſtudy of this delightful ſubject; or "the formalities and paraphernalia of rigour are ſo oſtentatiouſly put forward, as almoſt to hide the reality. Endleſs and perplexing repetitions, which do not confer greater exactitude on the reaſoning, render the demonſtrations involved and obſcure, and conceal from the view of the ſtudent the conſecution of evidence."

Thus an averſion is created in the mind of the pupil, and a ſubject ſo calculated to improve the reaſoning powers, and give the habit of cloſe thinking, is degraded by a dry and rigid courſe of inſtruction into an unintereſting exerciſe of the memory. To raiſe the curioſity, and to awaken the liſtleſs and dormant powers of younger minds ſhould be the aim of every teacher; but where examples of excellence are wanting, the attempts to attain it are but few, while eminence excites attention and produces imitation. The object of this Work is to introduce a method of teaching geometry, which has been much approved of by many ſcientific men in this country, as well as in France and America. The plan here adopted forcibly appeals to the eye, the moſt ſenſitive and the moſt comprehenſive of our external organs, and its pre-eminence to imprint it ſubject on the mind is ſupported by the incontrovertible maxim expreſſed in the well known words of Horace:—

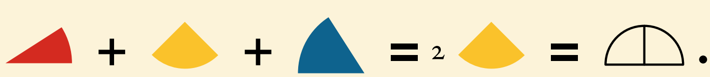
> *Segnius irritant animos demiſſa per aurem*
> *Quàm quæ ſunt oculis ſubjecta fidelibus.*
> A feebler impreſs through the ear is made,
> Than what is by the faithful eye conveyed.

All language conſiſts of repreſentative ſigns, and thoſe ſigns are the beſt which effect their purpoſes with the greateſt preciſion and diſpatch. Such for all common purpoſes are the audible ſigns called words, which are ſtill conſidered as audible, whether addreſſed immediately to the ear, or through the medium of letters to the eye. Geometrical diagrams are not ſigns, but the materials of geometrical ſcience, the object of which is to ſhow the relative quantities of their parts by a proceſs of reaſoning called Demonſtration. This reaſoning has been generally carried on by words, letters, and black or uncoloured diagrams but as the uſe of coloured ſymbols, ſigns, and diagrams in the linear arts and ſciences, renders the proceſs of reaſoning more preciſe, and the attainment more expeditious, they have been in this inſtance accordingly adopted.

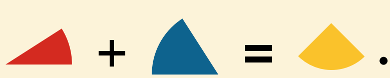
Such is the expedition of this enticing mode of communicating knowledge, that the Elements of Euclid can be acquired in leſs than one third the time uſually employed, and the retention by the memory is much more permanent; theſe facts have been aſcertained by numerous experiments made by the inventor, and ſeveral others who have adopted his plans. The particulars of which are few and obvious; the letters annexed to points, lines, or other parts of a diagram are in fact but arbitrary names, and repreſent them in the demonſtration; inſtead of theſe, the parts being differently coloured, are made to name themſelves, for their forms in correſponding colours repreſent them in the demonſtration. In order to give a better idea of this ſyſtem, and of the advantages gained by its adoption, let us take a right angled triangle, and expreſs ſome of its properties both by colours and the method generally employed.



Some of the properties of the right angled triangle ABC, expreſſed by the method generally employed.

1. 

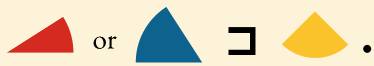
That is, the red angle added to the yellow angle added to the blue angle, equal twice the yellow angle, equal two right angles.

2. 

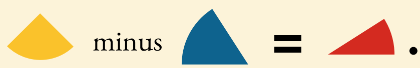
Or in words, the red angle added to the blue angle, equal the yellow angle.

3. 

The yellow angle is greater than either the red or blue angle.

4. 

Either the red or blue angle is leſs than the yellow angle.

5. 

In other terms, the yellow angle made leſs by the blue angle equal the red angle.

6. 

That is, the ſquare of the yellow line is equal to the ſum of the ſquares of the blue and red lines.

In oral demonſtrations we gain with colours this important advantage, the eye and the ear can be addreſſed at the ſame moment, ſo that for teaching geometry, and other linear arts and ſciences, in claſſes, the ſyſtem is the beſt ever propoſed, this is apparent from the examples juſt given.

Whence it is evident that a reference from the text to the diagram is more rapid and ſure, by giving the forms and colours of the parts, or by naming the parts and their colours, than naming the parts and letters on the diagram. Beſides the ſuperior ſimplicity, this ſyſtem is likewiſe conſpicuous for concentration, and wholly excludes the injurious through prevalent practice of allowing the ſtudent to commit the demonſtration to memory; until reaſon, and fact, and proof only make impreſſions on the underſtanding.

Again, when lecturing on the principles or properties of figures, if we mention the colour of the part or parts referred to, as in ſaying, the red angle, the blue line, or lines, &c. the part or parts thus named will be immediately ſeen by all in the claſs at the ſame inſtant; not ſo if we ſay the angle ABC, the triangle PFQ, the figure EGKt, and ſo on; for the letters muſt be traced one by one before the ſtudents arrange in their minds the particular magnitude referred to, which often occaſions confuſion and error, as well as loſs of time. Alſo if the parts which are given as equal, have the ſame colours in any diagram, the mind will not wander from the object before it; that is, ſuch an arrangement preſents an ocular demonſtration of the parts to be proved equal, and the learner retains the data throughout the whole of the reaſoning. But whatever may be the advantages of the preſent plan, if it be not ſubſtituded for, it can always be made a powerful auxiliary to the other methods, for the purpoſe of introduction, or of a more ſpeedy reminiſcence, or of more permanent retention by the memory.

The experience of all who have formed ſyſtems to impreſs facts on the underſtanding, agree in proving that coloured repreſentations, as pictures, cuts, diagrams, &c. are more eaſily fixed in the mind than mere ſentences unmarked by any peculiarity. Curious as it may appear, poets ſeem to be aware of this fact more than mathematicians; many modern poets allude to this viſible ſyſtem of communicating knowledge, one of them has thus expreſſed himſelf:

> Sounds which addreſs the ear are loſt and die
> In one ſhort hour, but theſe which ſtrike the eye,
> Live long upon the mind, the faithful ſight
> Engraves the knowledge with a beam of light.

This perhaps may be reckoned the only improvement which plane geometry has received ſince the days of Euclid, and if there were any geometers of note before that time, Euclid's ſucceſs has quite eclipſed their memory, and even occaſioned all good things of that kind to be aſſigned to him; like Æſop among the writers of Fables. It may alſo be worthy of remark, as tangible diagrams afford the only medium through which geometry and other linear arts and ſciences can be taught to the blind, this viſible ſyſtem is no leſs adapted to the exigencies of the deaf and dumb.

Care muſt be taken to ſhow that colour has nothing to do with the lines, angles, or magnitudes, except merely to name them. A mathematical line, which is length without breadth, cannot poſſeſs colour, yet the junction of the two colours on the ſame plane gives a good idea of what is meant by a mathematical line; recollect we are ſpeaking familiarly, ſuch a junction is to be underſtood and not the colour, when we ſay the black line, the red line or lines, &c.

Colours and coloured diagrams may at firſt appear a clumſy method to convey proper notations of the properties and parts of mathematical figures and magnitudes, however they will be found to afford a means more refined and extenſive than any that has been hitherto propoſed.

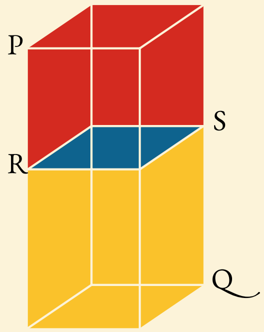
We ſhall here define a point, a line, and a surface, and demonſtrate a propoſition in order to ſhow the truth of this aſſertion.

A point is that which has poſition, but not magnitude; or a point is poſition only, abſtracted from the conſideration of length, breadth, and thickneſs. Perhaps the following deſcription is better calculated to explain the nature of a mathematical point to thoſe who have not acquired the idea, than the above ſpecious definition.

Let three colours meet and cover a portion of the paper, where they meet is not blue, nor is it yellow, nor is it red, as it occupies no portion of the plane, for if it did, it would belong to the blue, the red, or the yellow part; yet it exiſts, and has poſition without magnitude, ſo that with a little reflection, this junction of three colours on a plane gives a good idea of a mathematical point.

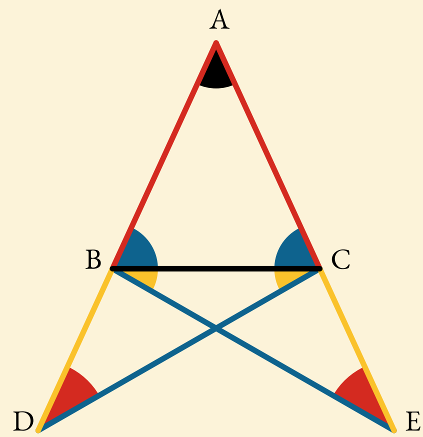
A line is length without breadth. With the aſſiſtance of colours, nearly in the ſame manner as before, an idea of a line may be thus given:—

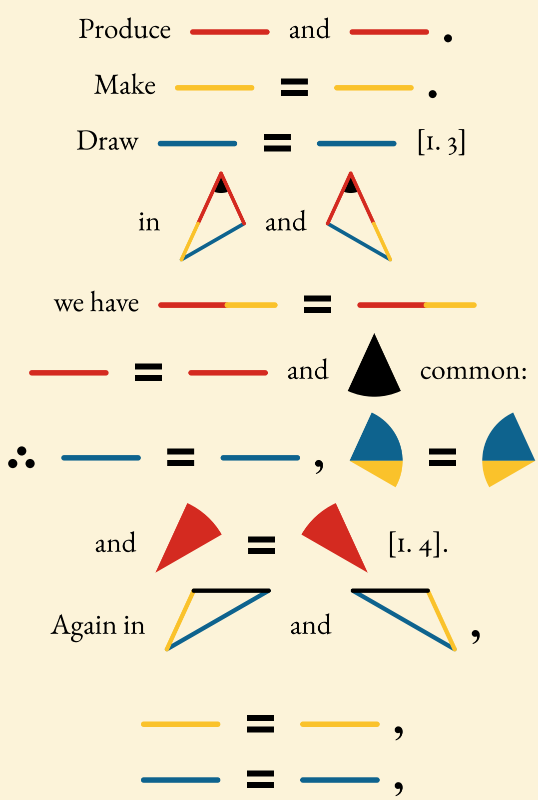
Let two colours meet and cover a portion of the paper; where they meet is not red, nor is it blue; therefore the junction occupies no portion of the plane, and therefore it cannot have breadth but only length: from which we can readily form an idea of what is meant by a mathematical line. For the purpoſe of illuſtration, one colour differing from the colour of the paper, or plane upon which it is drawn, would have been ſufficient; hence in future, if we ſay the red line, the blue line, or lines, &c. it is the junctions with the plane upon which they are drawn are to be underſtood.

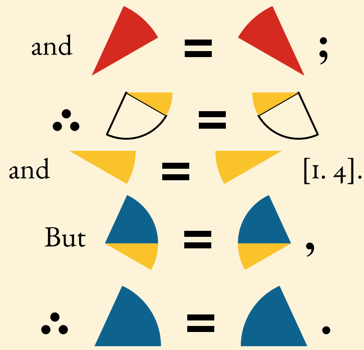


Surface is that which has length and breadth without thickneſs. When we conſider a ſolid body (PQ), we perceive at once that it has three dimensions, namely:—length, breadth, and thickneſs; ſuppoſe one part of this ſolid (PS) to be red, and the other part (QR) yellow, and that the colours be diſtinct without commingling, the blue surface (RS) which ſeparates theſe parts, or which is the ſame thing, that which divides the ſolid without loſs of material, muſt be without thickneſs, and only poſſeſſes length and breadth; this plainly appears from reaſoning, ſimilar to that juſt employed in defining, or rather deſcribing a point and a line.

The propoſition which we have ſelected to elucidate the manner in which the principles are applied is the fifth of the firſt Book. In an iſoſceles triangle ABC, the internal angles at the baſe ABC, ACB are equal, and when the ſides AB, AC are produced, the external angles at the baſe BCE, CBD are alſo equal.







Q.E.D.

By annexing Letters to the Diagram.

Let the equal ſides AB and AC be produced through the extremities BC, of the third ſide, and in the produced part BD of either, let any point D be aſſumed, and from the other let AE be cut off equal to AD [I. 3]. Let the points E and D, ſo taken in the produced ſides, be connected by ſtraight lines DC and BE with the alternate extremities of the third ſide of the triangle. In the triangles DAC and EAB the ſides DA and AC are reſpectively equal to EA and AB, and the included angle A is common to both triangles. Hence [I. 4] the line DC is equal to BE, the angle ADC to the angle AEB, and the angle ACD to the angle ABE; if from the equal lines AD and AE the equal ſides AB and AC be taken, the remainders BD and CE will be equal. Hence in the triangles BDC and CEB, the ſides BD and DC are reſpectively equal to CE and EB, and the angles D and E included by thoſe ſides are alſo equal. Hence [I. 4] the angles DBC and ECB, which are thoſe included by the third ſide BC and the productions of the equal ſides AB and AC are equal. Alſo the angles DCB and EBC are equal if thoſe equals be taken from the angles DCA and EBA before proved equal, the remainders, which are the angles ABC and ACB oppoſite to the equal ſides, will be equal.

Therefore in an iſoſceles triangle, &c. Q.E.D.

Our objeƈt in this place being to introduce the ſyſtem rather than to teach any particular ſet of propoſitions, we have therefore ſeleƈted the foregoing out of the regular courſe. For ſchools and other public places of inſtruƈtion, dyed chalks will anſwer to deſcribe diagrams, &c. for private uſe coloured pencils will be found very convenient.

We are happy to find that the Elements of Mathematics now forms a conſiderable part of every ſound female education, therefore we call the attention of thoſe intereſted or engaged in the education of ladies to this very attraƈtive mode of communicating knowledge, and to the ſucceeding work for its future development.

We ſhall for the preſent conclude by obſerving, as the ſenſes of ſight and hearing can be ſo forcibly and inſtantaneously addreſſed alike with one thouſand as with one, *the million* might be taught geometry and other branches of mathematics with great eaſe, this would advance the purpoſe of education more than any thing that *might* be named, for it would teach the people how to think, and not what to think; it is in this particular the great error of education originates.

THE ELEMENTS OF EUCLID
BOOK I.

DEFINITIONS.

I.

A *point* is that which has no part.

II.

A *line* is length without breadth.

III.

The extremities of a line are points.

IV.

A ſtraight or right line is that which lies evenly between its extremities.

V.

A ſurface is that which has length and breadth only.

VI.

The extremities of a ſurface are lines.

VII.

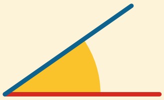
A plane ſurface is that which lies evenly between its extremities.

VIII.

A plane angle is the inclination of two lines to one another, in a plane, which meet together, but are not in the ſame direction.

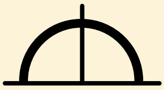
IX.

A plane rectilinear angle is the inclination of two ſtraight lines to one another, which meet together, but are not in the ſame ſtraight line.



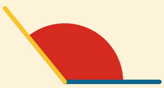
X.

When one ſtraight line ſtanding on another ſtraight line makes the adjacent angles equal, each of theſe angles is called a *right angle*, and each of theſe lines is ſaid to be *perpendicular* to the other.

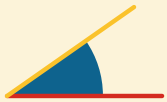


XI.

An obtuſe angle is an angle greater than a right angle.



XII.



An acute angle is leſs than a right angle.

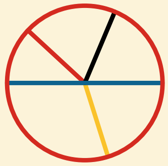
XIII.

A term or boundary is the extremity of any thing.

XIV.

A figure is a ſurface encloſed on all ſides by a line or lines.

XV.



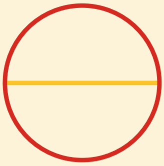
A circle is a plane figure, bounded by one continued line, called its circumference or periphery; and having a certain point within it, from which all ſtraight lines drawn to its circumference are equal.

XVI.

This point (from which the equal lines are drawn) is called the centre of the circle.

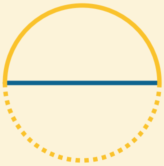
XVII.

A diameter of a circle is a ſtraight line drawn through the centre, terminated both ways in the circumference.



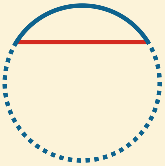
XVIII.

A ſemicircle is the figure contained by the diameter, and the part of the circle cut off by the diameter.



XIX.

A ſegment of a circle is a figure contained by a ſtraight line, and the part of the circumference which it cuts off.



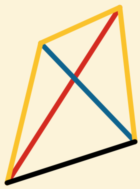
XX.

A figure contained by ſtraight lines only, is called a rectilinear figure.

XXI.

A triangle is a rectilinear figure included by three ſides.

XXII.

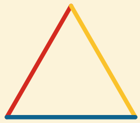


A quadrilateral figure is one which is bounded by four ſides. The ſtraight lines and connecting the vertices of the oppoſite angles of a quadrilateral figure, are called its diagonal.

XXIII.

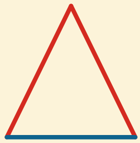
A polygon is a rectilinear figure bounded by more than four ſides.

XXIV.



A triangle whoſe three ſides are equal, is ſaid to be equilateral.

XXV.



A triangle which has only two ſides equal is called an iſoſceles triangle.

XXVI.

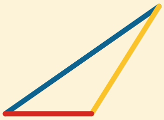
A ſcalene triangle is one which has no two ſides equal.

XXVII.

A right angled triangle is that which has a right angle.

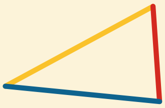
XXVIII.

An obtuſe angled triangle is that which has an obtuſe angle.



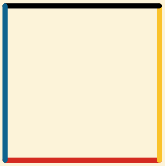
XXIX.

An acute angled triangle is that which has three acute angles.



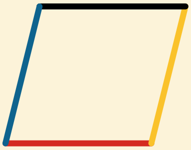
XXX.

Of four-ſided figures, a ſquare is that which has all its ſides equal, and all its angles right angles.

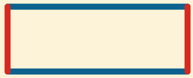


XXXI.

A rhombus is that which has all its ſides equal, but its angles are not right angles.

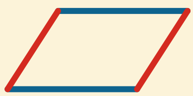


XXXII.



An oblong is that which has all its angles right angles, but has not all its ſides equal.

XXXIII.



A rhomboid is that which has its oppoſite ſides equal to one another, but all its ſides are not equal, nor its angles right angles.

XXXIV.

All other quadrilateral figures are called trapeziums.

XXXV.

Parallel ſtraight lines are ſuch as are in the ſame plane, and which being produced continually in both directions, would never meet.

POSTULATES.

I.

Let it be granted that a ſtraight line may be drawn from any one point to any other point.

II.

Let it be granted that a finite ſtraight line may be produced to any length in a ſtraight line.

III.

Let it be granted that a circle may be deſcribed with any centre at any diſtance from that centre.

AXIOMS.

I.

Magnitudes which are equal to the ſame are equal to each other.

II.

If equals be added to equals the ſums will be equal.

III.

If equals be taken away from equals the remainders will be equal.

IV.

If equals be added to unequals the ſums will be unequal.

V.

If equals be taken away from unequals the remainders will be unequal.

VI.

The doubles of the ſame or equal magnitudes are equal.

VII.

The halves of the ſame or equal magnitudes are equal.

VIII.

Magnitudes which coincide with one another, or exactly fill the ſame ſpace, are equal.

IX.

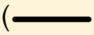
The whole is greater than its part.

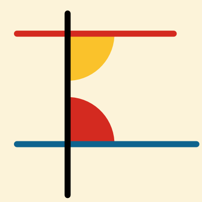
X.

Two ſtraight lines cannot include a ſpace.

XI.

All right angles are equal.

If two ſtraight lines () meet a third ſtraight line () ſo as to make the two interior angles ( and ) on the ſame ſide leſs than two right angles, theſe two ſtraight lines will meet if they be produced on that ſide on which the angles are leſs than two right angles.

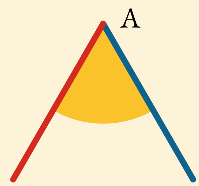


The fifth poſtulate may be expreſſed in any of the following ways:

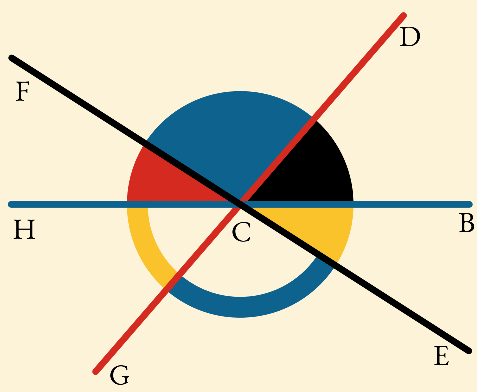
1. Two diverging ſtraight lines cannot be both parallel to the ſame ſtraight line.
2. If a ſtraight line interſect one of the two parallel ſtraight lines it muſt alſo interſect the other.
3. Only one ſtraight line can be drawn through a given point, parallel to a given ſtraight line.

Geometry has for its principal objects the expoſition and explanation of the properties of *figure*, and figure is defined to be the relation which ſubſiſts between the boundaries of ſpace. Space or magnitude is of three kinds, *linear*, *ſuperficial*, and *ſolid*.

Angles might properly be conſidered as a fourth ſpecies of magnitude. Angular magnitude evidently conſiſts of parts, and muſt therefore be admitted to be a ſpecies of quantity. The ſtudent muſt not ſuppoſe that the magnitude of an angle is affected by the length of the ſtraight lines which include it, and of whoſe mutual divergence it is the meaſure. The *vertex* of an angle is the point where the *ſides* or the *legs* of the angle meet, as A.



An angle is often deſignated by a ſingle letter when its legs are the only lines which meet together at its vertex. Thus the red and blue lines form the yellow angle, which in other ſyſtems would be called the angle A. But when more than two lines meet in the ſame point, it was neceſſary by former methods, in order to avoid confuſion, to employ three letters to deſignate an angle about that point, the letter which marked the vertex of the angle being always placed in the middle. Thus the black and red lines meeting together at C, form the blue angle, and has been uſually denominated the angle FCD or DCF. The lines FC and CD are the legs of the angle; the point C is its vertex. In like manner the black angle would be deſignated the angle DCB or BCD. The red and blue angles added together,



or the angle HCF added to FCD, make the angle HCD; and ſo of the other angles. When the legs of an angle are produced or prolonged beyond its vertex, the angles made by them on both ſides of the vertex are ſaid to be vertically oppoſite to each other: Thus the red and yellow angles are ſaid to be *vertically oppoſite* angles.

Superpoſition is the proceſs by which one magnitude may be conceived to be placed upon another, ſo as exactly to cover it, or ſo that every part of each ſhall exactly coincide.

A line is ſaid to be *produced*, when it is extended, prolonged, or has its length increaſed, and the increaſe of length which it receives is called its *produced part*, or its *production*.

The entire length of the line or lines which encloſe a figure, is called its *perimeter*. The firſt ſix books of Euclid treat of plane figures only. A line drawn from the centre of a circle to its circumference, is called a *radius*. The lines which include a figure are called its *ſides*. That ſide of a right angled triangle, which is oppoſite to the right angle, is called the *hypotenuſe*. An *oblong* is defined in the ſecond book, called a *rectangle*. All the lines which are conſidered in the firſt ſix books of the Elements are ſuppoſed to be in the ſame plane.

The *ſtraight-edge* and *compaſſes* are the only inſtruments, the uſe of which is permitted in Euclid, or plane Geometry. To declare this reſtriction is the object of the *poſtulates*.

The *Axioms* of geometry are certain general propoſitions, the truth of which is taken to be ſelf-evident and incapable of being eſtablished by demonſtration.

Propoſitions are thoſe reſults which are obtained in geometry by a proceſs of reaſoning. There are two ſpecies of propoſitions in geometry, *problems* and *theorems*.

A *Problem* is a propoſition in which ſomething is propoſed to be done; as a line to be drawn under ſome given conditions, a circle to be deſcribed, ſome figure to be conſtructed, &c.

The *ſolution* of the problem conſiſts in ſhowing how the thing required may be done by the aid of the rule or ſtraight-edge and compaſſes.

The *demonſtration* conſiſts in proving that the proceſs indicated in the ſolution really attains the required end.

A *Theorem* is a propoſition in which the truth of ſome principle is aſſerted. This principle muſt be deduced from the axioms and definitions, or other truths previously and independently eſtablished. To ſhow this is the object of demonſtration.

A *Problem* is analogous to a poſtulate.

A *Theorem* reſembles an axiom.

A *Poſtulate* is a problem, the ſolution of which is aſſumed.

An *Axiom* is a theorem, the truth of which is granted without demonſtration.

A *Corollary* is an inference deduced immediately from a propoſition.

A *Scholium* is a note or obſervation on a propoſition not containing an inference of ſufficient importance to entitle it to the name of a *corollary*.

A *Lemma* is a propoſition merely introduced for the purpoſe of eſtablishing ſome more important propoſition.

SYMBOLS AND ABBREVIATIONS.

∴ expreſſes the word *therefore*.

= . . . *equal*. This ſign of equality may be read *equal to*, or *is equal to*, or *are equal to*; but any diſcrepancy in regard to the introduction of the auxiliary verbs *is, are*, &c. cannot affect the geometrical rigour.

≠ means the ſame as if the words 'not equal' were written.

⊏ ſignifies *greater than*.

⊐ . . . *leſs than*.

\+ is read *plus* (*more*), the ſign of addition; when interpoſed between two or more magnitudes, ſignifies their ſum.

− is read *minus* (*leſs*), ſignifies ſubtraction; and when placed between two quantities denotes that the latter is to be taken from the former.

× this ſign expreſſes the product of two or more numbers when placed between them in arithmetic and algebra; but in geometry it is generally uſed to expreſs a *rectangle*, when placed between "two ſtraight lines which contain one of its right angles." A *rectangle* may alſo be repreſented by placing a point between two of its conterminous ſides.

: ∷ : expreſſes an analogy or proportion; thus, if A, B, C and D, repreſent four magnitudes, and A has to B the ſame ratio that C has to D, the propoſition is thus briefly written,

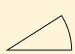
$$A : B :: C : D$$
$$A : B = C : D$$
$$\text{or } \frac{A}{B} = \frac{C}{D}.$$

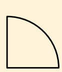
This equality or ſameneſs of ratio is read,

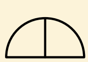
as A is to B, ſo is C to D;
or A is to B, as C is to D.

∥ ſignifies *parallel to.*

⊥ . . . *perpendicular to.*

 . . . *angle.*

 . . . *right angle.*

 . . . *two right angles.*

 or  briefly deſignates a *point.*

$\overline{\qquad}^{2}$ The ſquare deſcribed on a line is conciſely written thus.

$2\,\overline{\qquad}^{2}$ In the ſame manner twice the ſquare of, is expreſſed.

def. ſignifies *definition.*

poſt. . . . *poſtulate.*

ax. . . . *axiom*.

hyp. . . . *hypotheſis*. It may be neceſſary here to remark that the *hypotheſis* is the condition aſſumed or taken for granted. Thus, the hypotheſis of the propoſition given in the Introduction, is that the triangle is iſoſceles, or that its legs are equal.

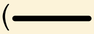
conſt. . . . *conſtruction*. The *conſtruction* is the change made in the original figure, by drawing lines, making angles, deſcribing circles, &c. in order to adapt it to the argument of the demonſtration or the ſolution of the problem. The conditions under which theſe changes are made, are indisputable as thoſe contained in the hypotheſis. For inſtance, if we make an angle equal to a given angle, theſe two angles are equal by conſtruction.

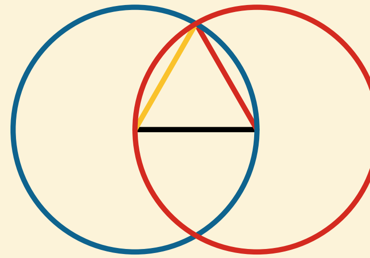
Q.E.F. . . . *Quod erat faciendum*.
. . . Which was to be done.

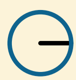
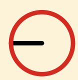
Q.E.D. . . . *Quod erat demonſtrandum*.
. . . Which was to be demonſtrated.

PROPOSITIONS.

PROPOSITION I. PROBLEM.

O*N a given finite ſtraight line* () *to deſcribe an equilateral triangle.*



Deſcribe  and  [poſtulate 3];

draw and [poſt. 1], then  will be equilateral.

For = [1. def. 15];

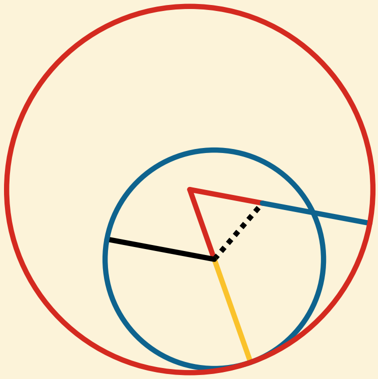
and = [1. def. 15],

∴ = [axiom. 1];

and therefore  is the equilateral triangle required.

Q.E.F.

PROPOSITION II. PROBLEM.



FROM *a given point* ( ), *to draw a ſtraight line equal to a given finite ſtraight line* ( ).

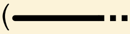
Draw [poſt. 1], deſcribe [I. 1], produce [poſt. 2], deſcribe [poſt. 3], and [poſt. 3]; produce [poſt. 2], then is the line required.

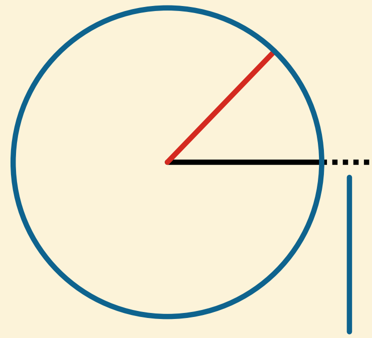
For = [I. def. 15], and = [conſt.], ∴ = [ax. 3], but [I. def. 15] = = ; ∴ drawn from the given point ( ), is equal the given line .

Q.E.F.

I. 19

PROPOSITION III. PROBLEM.

FROM *the greater* () *of two given ſtraight lines, to cut off a part equal to the leſs* ().



Draw = [I. 2]; deſcribe  [poſt. 3],

then = .

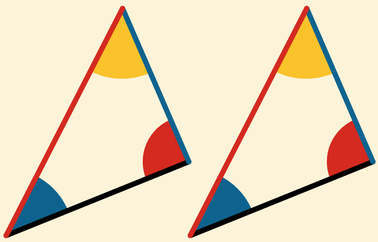
For = [I. def. 15],

and = [conſt.];

∴ = [ax. 1].

Q.E.F.

PROPOSITION IV. THEOREM.



IF *two triangles have two ſides of the one reſpectively equal to two ſides of the other,* (*to* *and* *to*) *and the angles* (*and*) *contained by thoſe equal ſides alſo equal; then their baſes or their ſides* (*and*) *are alſo equal: and the remaining and their remaining angles oppoſite to equal ſides are reſpectively equal* (= *and* =): *and the triangles are equal in every reſpect.*

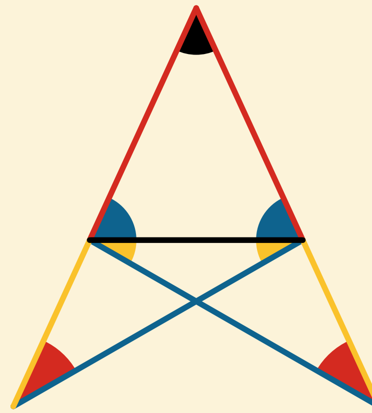
Let the two triangles be conceived, to be ſo placed, that the vertex of one of the equal angles, or ; ſhall fall upon that of the other, and to coincide with , then will coincide with if applied: conſequently will coincide with , or two ſtraight lines will encloſe a ſpace, which is impoſſible [ax. 10], therefore = , = and = , and as the triangles and coincide, when applied, they are equal in every reſpect.

Q.E.D.

I. 21

PROPOSITION V. THEOREM.

IN *any iſoſceles triangle* *if the equal ſides be produced, the external angles at the baſe are equal, and the internal angles at the baſe are alſo equal.*



Produce , and , [poſt. 2], take = , [I. 3]; draw and . Then in and we have,

= [conſt.], common to both, and

= [hyp.]

∴ = , = and =

[I. 4]. Again in and we have

= , = and

= , ∴ = and

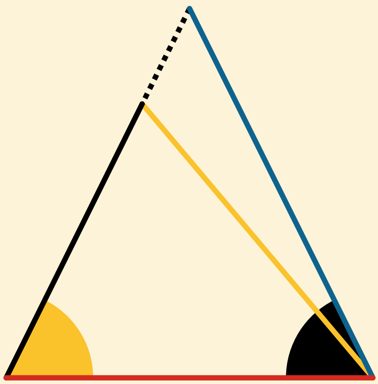
= [I. 4] but

= , ∴ = [ax. 3].

Q.E.D.

I. 22

PROPOSITION VI. THEOREM.



IN any triangle ( ) if two angles (and) are equal, the ſides (and) oppoſite to them are alſo equal.

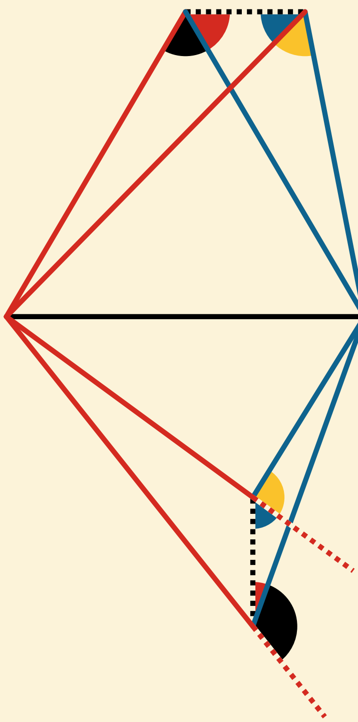
For if the ſides be not equal, let one of them be greater than the other , and from it cut off = [I. 3], draw .

Then and , = , [conſt.] = [hyp.] and common, ∴ the triangles are equal [I. 4] a part equal to the whole, which is abſurd; ∴ neither of the ſides or is greater than the other, ∴ hence they are are equal.

Q.E.D.

I. 23

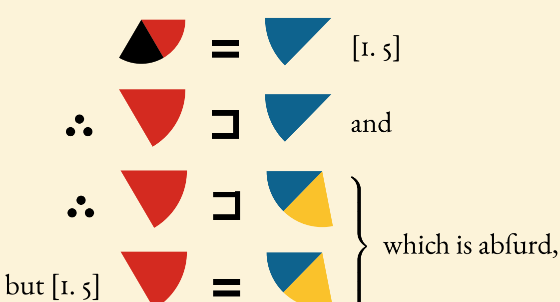
PROPOSITION VII. THEOREM.



ON *the ſame baſe* ( ) *and on the ſame ſide of it there cannot be two triangles having their conterminous ſides* (*and* , *and*) *at both extremities of the baſe, equal to each other.*

When two *triangles* ſtand on the ſame baſe, and on the ſame ſide of it, the vertex of one ſhall either fall outſide of the other triangle, or within it; or, laſtly, on one of its ſides. If it be poſſible let the two triangles be conſtructed ſo that

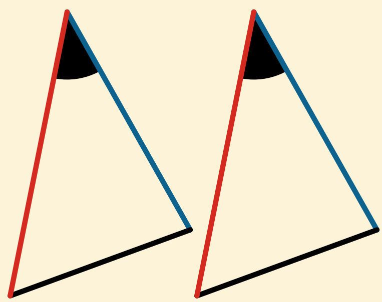
= , then draw and,
=



therefore the two triangles cannot have their conterminous ſides equal at both extremities of the baſe.

Q.E.D.

I. 24



I F *two triangles have two ſides of the one reſpectively equal to two ſides of the other* (= and =), *and alſo their baſes* (=), *equal; then the angles* (*and*) *contained by their equal ſides are alſo equal.*

If the equal baſes and be conceived to be placed upon the other, ſo that the triangles ſhall lie at the ſame ſide of them, and that the equal ſides and , and be conterminous, the vertex of the one muſt fall on the vertex of the other; for to ſuppoſe them not coincident would contradict the laſt propoſition.

Therefore the ſides and , being coincident with and ,

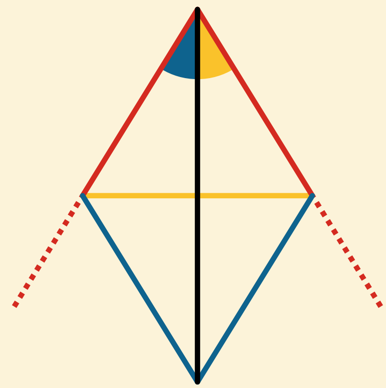
$\therefore$ = .

Q.E.D.

I. 25

PROPOSITION IX. PROBLEM.

TO *biſect a given rectilinear angle* ( ).



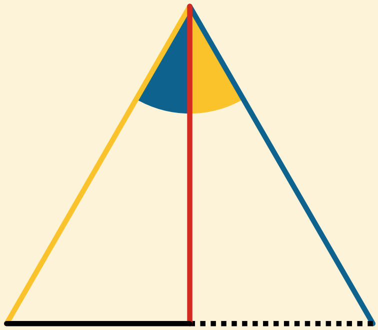
Take = [I. 3] draw , upon which deſcribe [I. 1], draw .

Becauſe = [conſt.] and common to the two triangles and = [conſt.], ∴ = [I. 8].

Q.E.F.

I. 26

PROPOSITION X. PROBLEM.



T*O biſect a given finite ſtraight line* ( ).

Conſtruct  [I. 1],

draw , making  =  [I. 9].

Then = by [I. 4],

for = [conſt.]

 =  and common to the two triangles.

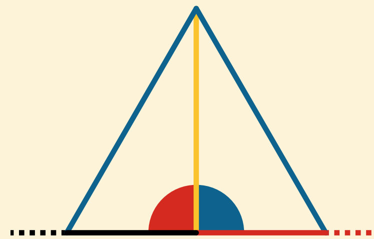
Therefore the given line is biſected.

Q.E.F.

I. 27

PROPOSITION XI. PROBLEM.

FROM *a given point* ( ), *in a given ſtraight line* ( ), *to draw a perpendicular.*



Take any point ( ) in the given line, cut off

= [I. 3], conſtruct [I. I],

draw and it ſhall be perpendicular to the given line.

For = [conſt.] = [conſt.]

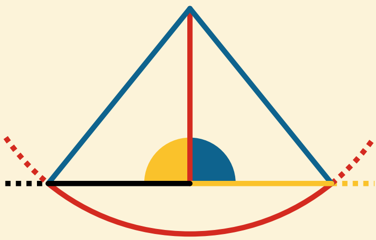
and common to the two triangles.

Therefore = [I. 8]

∴ ⊥ [I. def. 10].

Q.E.F.

PROPOSITION XII. PROBLEM.



TO *draw a ſtraight line perpendicular to a given indefinite ſtraight line* ( ) *from a given* (*point*), *without.*

With the given point ( ) as centre, at one ſide of the line, and any diſtance capable of extending to the other ſide, deſcribe .

Make = [I. 10]

draw , and .

Then ⊥ .

For [I. 8] ſince = [conſt.]

common to both,

and = [I. def. 15]

∴ = , and

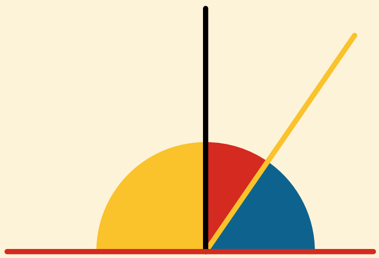
∴ ⊥ [I. def. 10].

Q.E.F.

I. 29

PROPOSITION XIII. THEOREM.

W*HEN a ſtraight line* ( ) *ſtanding upon another ſtraight line* ( ) *makes angles with it; they are either two right angles or together equal to two right angles.*



If be $\perp$ to then,

and $=$ [I. def. 10].

But if be not $\perp$ to ,

draw $\perp$; [I. 11]

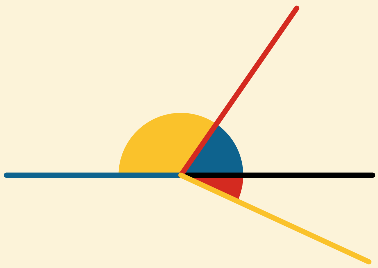
$+$ $=$ [conſt.],

$=$ $=$ $+$

$\therefore$ $+$ $=$ $+$ $+$ [ax. 2]

$=$ $+$ $=$.

Q.E.D.



IF *two ſtraight lines (and), meeting a third ſtraight line (), at the ſame point, and at oppoſite ſides of it, make with it adjacent angles (and) equal to two right angles; theſe ſtraight lines lie in one continuous ſtraight line.*

For, if poſſible, let , and not ,

be the continuation of ,

then + =

but by the hypotheſis + =

∴ = , [ax. 3]; which is abſurd [ax. 9].

∴ , is not the continuation of , and the like may be demonſtrated of any other ſtraight line except ,

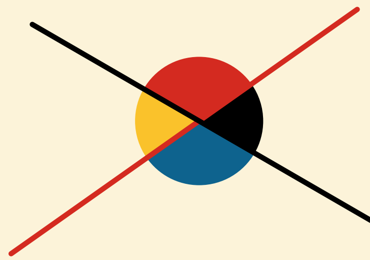
∴ is the continuation of .

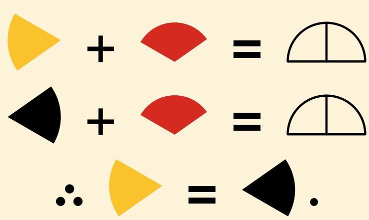
Q.E.D.

I. 31

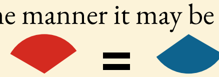
PROPOSITION XV. THEOREM.

IF *two ſtraight lines* (and) *interſect one another, the vertical angles* and , and *are equal.*





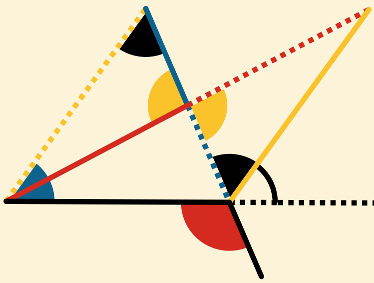
In the ſame manner it may be ſhown that



Q.E.D.

I. 32

PROPOSITION XVI. THEOREM.



IF a ſide of a triangle ( ) is produced, the external angle ( ) is greater than either of the internal remote angles (or).

Make = [I. 10].

Draw and produce it until

= ; draw .

In and ; =

= and = [conſt. I. 15],

∴ = [I. 4],

∴ ⊏ .

In like manner it can be ſhown, that if

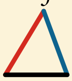
be produced, ⊏ , and therefore

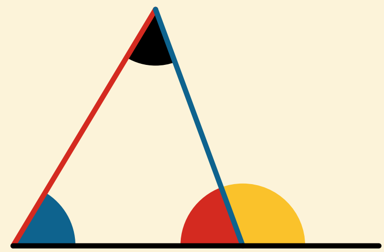
which is = is ⊏ .

Q.E.D.

I. 33

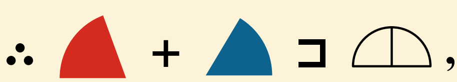
PROPOSITION XVII. THEOREM.

ANY *of two angles of a triangle*  *are together leſs than two right angles.*



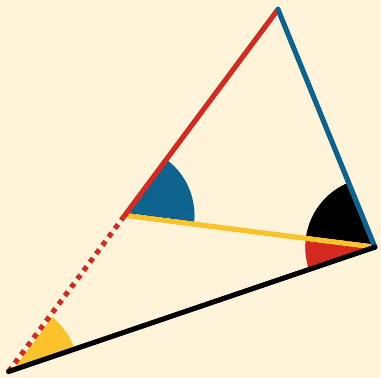
Produce ——— , then will

But  [I. 16]



and in the ſame manner it may be ſhown that any other two angles of the triangle taken together are leſs than two right angles.

Q.E.D.



IN *any triangle* *if one side* *be greater than another* , *the angle opposite to the greater side is greater than the angle to the opposite to the less. i.e.*

⊏ .

Make = [I. 3], draw .

Then will = [I. 5];

but ⊏ [I. 16]

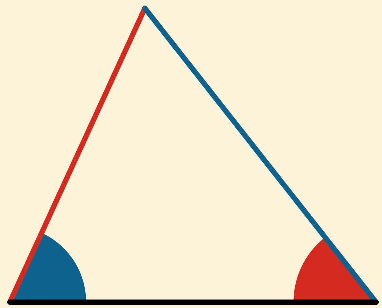
∴ ⊏ and much more

is ⊏ .

Q.E.D.

I. 35

PROPOSITION XIX. THEOREM.



I F *in any triangle* *one angle* *be greater than another* *the ſide* *which is oppoſite to the greater angle, is greater than the ſide* *oppoſite the leſs.*

If be not greater than then muſt

= or ⊐ .

If = then

⊐ [I. 5];

which is contrary to the hypotheſis.

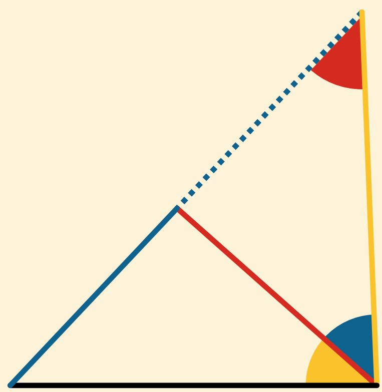
is not leſs than ; for if it were,

⊐ [I. 18]

which is contrary to the hypotheſis:

∴ ⊏ .

Q.E.D.



ANY *two ſides* and *of a triangle* *taken together are greater than the third ſide* ( ).

Produce , and

make = [I. 3];

draw .

Then becauſe = [conſt.],

= [I. 5]

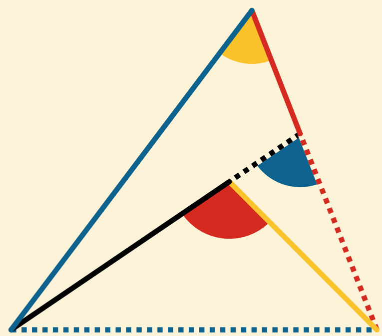
∴ ⊏ [ax. 9]

∴ + ⊏ [I. 19]

and ∴ + ⊏ .

Q.E.D.

PROPOSITION XXI. THEOREM.



I F *from any point* ( ) *within a triangle* *ſtraight lines be drawn to the extremities of one ſide* ( ), *theſe lines muſt be together leſs than the other two ſides, but muſt contain a greater angle.*

Produce ,

$+$ $\sqsubset$ [I. 20],

add to each,

$+$ $\sqsubset$ $+$ [ax. 4].

In the ſame manner it may be ſhown that

$+$ $\sqsubset$ $+$,

$\therefore$ $+$ $\sqsubset$ $+$,

which was to be proved.

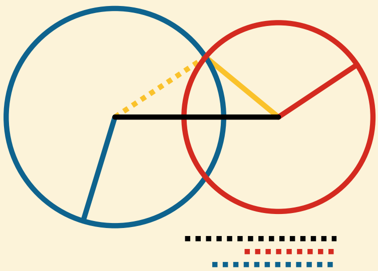
Again $\sqsubset$ [I. 16],

and also $\sqsubset$ [I. 16],

$\therefore$ $\sqsubset$.

Q.E.D.

PROPOSITION XXII. THEOREM.



GIVEN *three right lines { the ſum of any two greater than the third, to conſtruct a triangle whoſe ſides ſhall be reſpectively equal to the given lines.*

Aſſume = [I. 3].

Draw =

and = [I. 2].

With and as radii,

describe and [poſt. 3];

draw and ,

then will be the triangle required.

For = ,

= = ,

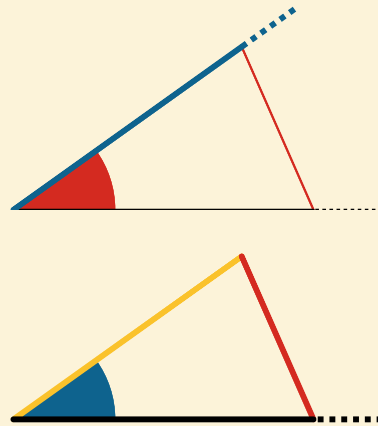
and = = . [conſt.]

Q.E.D.

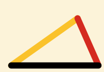
I. 39

PROPOSITION XXIII. THEOREM.

AT *a given point* () *in a given straight line* (), *to make an angle equal to a given rectilineal angle* ().



Draw between any two points in the legs of the given angle.

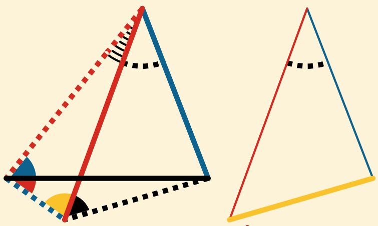
Construct  [I. 22], so that

 = , =

and = .

Then  =  [I. 8].

Q.E.D.



I*F two triangles have two ſides of one reſpectively equal to two ſides of the other (to and to), and if one of the angles ( ) contained by the equal ſides be greater than the other ( ), the ſide ( ) which is oppoſite to the greater angle is greater than the ſide ( ) which is oppoſite to the leſs angle.*

Make = [I. 23],

and = [I. 3],

draw and .

Becauſe = [ax. I. hyp. conſt.]

∴ = [I. 5]

but ⊐

and ∴ ⊐ ,

∴ ⊏ [I. 19]

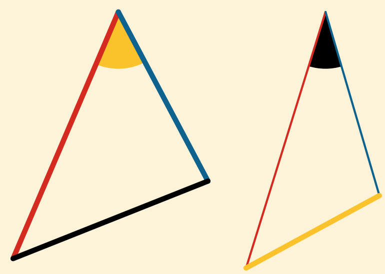
but = [I. 4]

∴ ⊏ .

Q.E.D.

I. 41

PROPOSITION XXV. THEOREM.



IF *two triangles have two ſides (and) of the one reſpectively equal to two ſides (and) of the other, but their baſes unequal, the angle ſubtended by the greater baſe ( ) of the one, muſt be greater than the angle ſubtended by the leſs baſe ( ) of the other.*

= , ⊏ or ⊐

is not equal to

for if = then = [I. 4]

which is contrary to the hypotheſis;

is not leſs than

for if ⊐

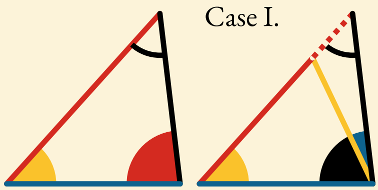
then ⊐ [I. 24],

which is alſo contrary to the hypotheſis:

∴ ⊏ .

Q.E.D.

IF *two triangles have two angles of the on refpectively equal to two angles of the other,* (= *and* =), *and a fide of the one equal to a fide of the other fimilarly placed with refpect to the equal angles, the remaining fides and angles are refpectively equal to one another.*



CASE I.

Let and which lie between the equal angles be equal, then

= .

For if it be poffible, let one of them be greater than the other; make

= , draw . In and we have

= , = , = ;

∴ = [I. 4]

but = [hyp.]

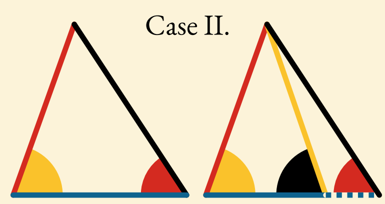
and therefore = , which is abfurd;

hence neither of the fides and is greater than the other;

and ∴ they are equal; ∴ = ,

and = , [I. 4].

CASE II.



Again, let = ,
which lie oppoſite the equal angles
and .

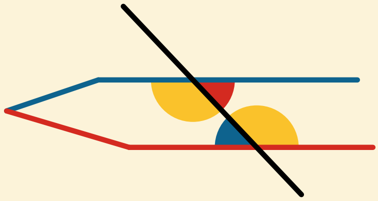
If it be poſſible, let ⊏ , then take = ,
draw .

Then in and we have = ,
= and = ,
∴ = [I. 4]
but = [hyp.]
∴ = which is abſurd [I. 16].

Conſequently, neither of the ſides or is greater than the other, hence they muſt be equal.
It follows [by I. 4] that the triangles are equal in all reſpects.

Q.E.D.

PROPOSITION XXVII. THEOREM.

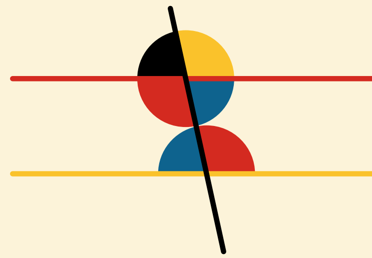


IF *a ſtraight line* ( ) *meeting two other ſtraight lines,* (*and*) *makes with them the alternate angles* (*and* ; *and*) *equal, theſe two ſtraight lines are parallel.*

If be not parallel to they ſhall meet when produced. If it be poſſible, let thoſe lines be not parallel, but meet when produce; then the external angle is greater than [I. 16], but they are alſo equal [hyp.], which is abſurd: in the ſame manner it may be ſhown that they cannot meet on the other ſide; ∴ they are parallel.

Q.E.D.

PROPOSITION XXVIII. THEOREM.



If a ſtraight line ( ), cutting two other ſtraight lines (and), makes the external equal the cutting line (namely, = or =), or if it makes the two internal angles at the ſame ſide (and , or and) together equal to two right angles, thoſe two ſtraight lines are parallel.

Firſt, if = , then = [I. 15],

∴ = ∴ ∥ [I. 27].

Secondly, if + = ,

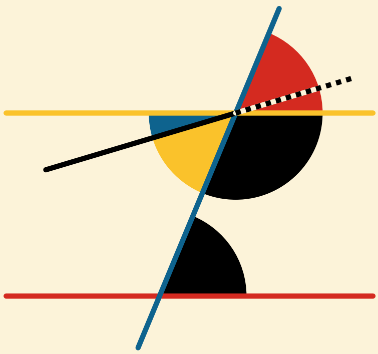
then + = [I. 13],

∴ + = + [ax. 3]

∴ =

∴ ∥ [I. 27].

Q.E.D.



A ſtraight line ( ) falling on two parallel ſtraight lines (and), makes the alternate angles equal to one another; and alſo the external equal to the internal and oppoſite angle on the ſame ſide; and the two internal angles on the ſame ſide together equal to two right angles.

For if the alternate angles and be not equal, draw ,
making = [I. 23]. Therefore ∥

[I. 27] and therefore two ſtraight lines which interſect are parallel to the ſame ſtraight line, which is impoſſible [ax. 12]. Hence the alternate angles and are not unequal, that is they are equal: =

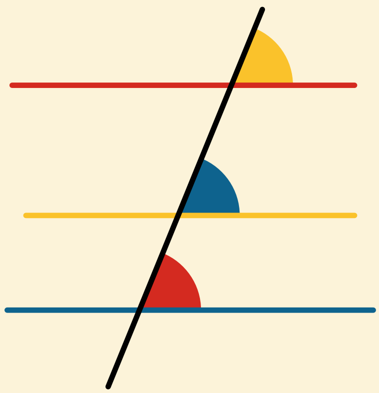
[I. 15]; ∴ = , the external angle equal to the internal and oppoſite on the ſame ſide: if be added to both, then

+ = = [I. 13]. That is to ſay, the two internal angles at the ſame ſide of the cutting line are equal to two right angles.

Q.E.D.

PROPOSITION XXX. THEOREM.

STRAIGHT *lines* ( ) *which are parallel to the ſame ſtraight line* ( ), *are parallel to one another.*



Let interſect ;

Then, = = [I. 29],

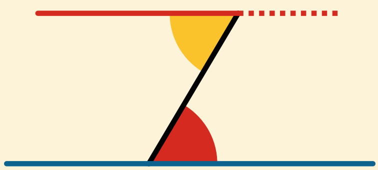
∴ =

∴ ∥ [I. 27].

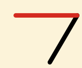
Q.E.D.

I. 48

PROPOSITION XXXI. PROBLEM.



FROM *a given point*  *to draw a ſtraight line parallel to a given ſtraight line* ().

Draw from the point 

to any point  in ,

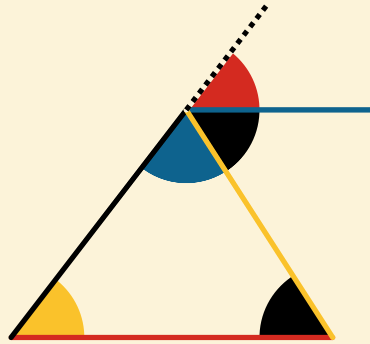
make  =  [I. 23],

then ∥ [I. 27].

Q.E.F.

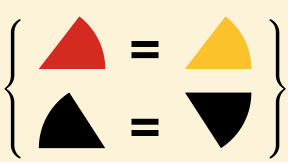
PROPOSITION XXXII. PROBLEM.

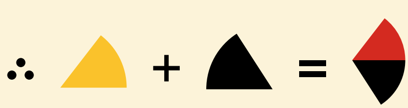
*I*F *any ſide ( ) of a triangle be produced, the external angle ( ) is equal to the ſum of the two internal and oppoſite angles (and), and the three internal angles of every triangle taken together are equal to two right angles.*

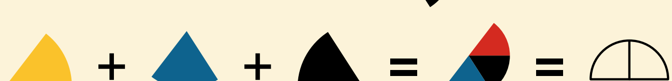


Through the point  draw

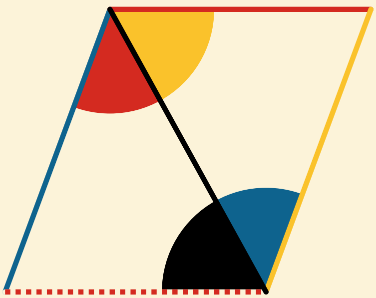
 [I. 31].

Then  [I. 29],

 [ax. 2], and therefore

 [I. 13].

Q.E.F.



STRAIGHT *lines* (*and*) *which join the adjacent extremities of two equal and parallel ſtraight lines* (*and*), *are themſelves equal and parallel.*

Draw the diagonal.

= [hyp.]

= [I. 29]

and common to the two triangles;

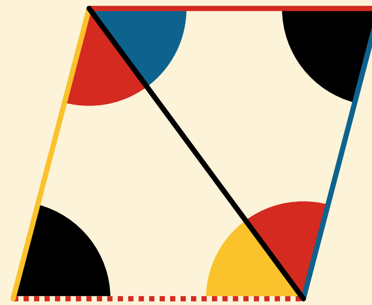
∴ = , and = [I. 4];

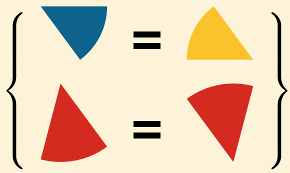
and ∴ ∥ [I. 27].

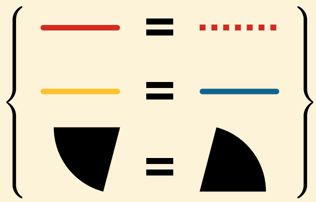
Q.E.D.

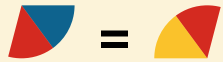
PROPOSITION XXXIV. THEOREM.

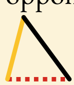
THE *oppoſite ſides and angles of any parallelogram are equal, and the diagonal ( ) divides it into two equal parts.*



Since  [I. 29] and common to the two triangles.

∴  [I. 26]

and  [ax. 2]:

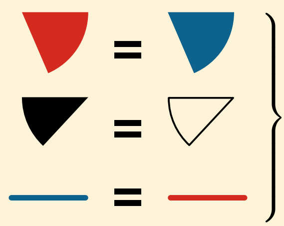
Therefore the oppoſite ſides and angles of the parallelogram are equal: and as the triangles  and  are equal in every reſpect [I. 4], the diagonal divides the parallelogram into two equal parts.

Q.E.D.

PROPOSITION XXXV. THEOREM.

PARALLELOGRAMS *on the ſame baſe, and between the ſame parallels, are* (*in area*) *equal.*

On account of the parallels,



[I. 29]
[I. 29]
[I. 34]

But,  =  [I. 8]

∴  minus  =  ,

and  minus  =  ;

∴  =  .

Q.E.D.

PROPOSITION XXXVI. THEOREM.

PARALLELOGRAMS (and) *on equal baſes, and between the ſame parallels, are equal.*

Draw and ,

= = , by [I. 34, and hyp.];

∴ = and ∥ ;

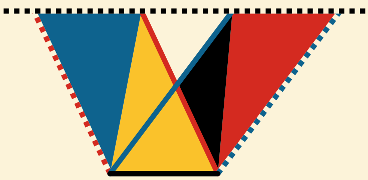
∴ = and ∥ [I. 33].

And therefore is a parallelogram:

but = = [I. 35]

∴ = [ax. 1].

Q.E.D.



TRIANGLES ( *and* ) *on the ſame baſe* ( ) *and between the ſame parallels are equal.*

Draw ⋯ ∥ —
⋯ ∥ — [I. 31]

Produce ⋯ .

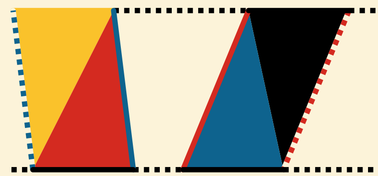
 and  are parallelograms on the ſame baſe, and between the ſame parallels, and therefore equal [I. 35].

∴  = twice 
 = twice  [I. 34]

∴  =  .

Q.E.D.

PROPOSITION XXXVIII. THEOREM.



TRIANGLES (*and*) *on equal baſes and between the ſame parallels are equal.*

Draw ‖ [I. 31]

‖

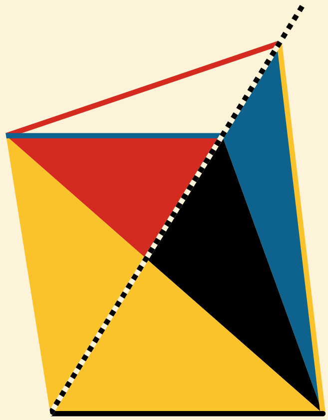
= [I. 36];

= twice [I. 34],

and = twice [I. 34],

∴ = [ax. 7].

Q.E.D.



PROPOSITION XXXIX. THEOREM.

EQUAL *triangles* (and) *on the ſame baſe* ( ) *and on the ſame ſide of it, are between the ſame parallels.*

If , which joins the vertices of the triangles, be not ∥ ,
draw ∥ [I. 31], meeting .

Draw . Becauſe ∥ [conſt.]

= [I. 37]:

but = [hyp.];

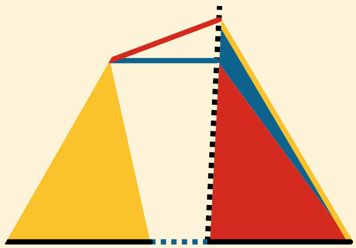
∴ = , a part equal to the whole, which is abſurd.

∴ ‡ ; and in the ſame manner it can be demonſtrated, that no other line except is ∥ ; ∴ ∥ .

Q.E.D.

PROPOSITION XL. THEOREM.

EQUAL *triangles* (*and*) *on equal baſes, and on the ſame ſide, are between the ſame parallels.*



If which joins the vertices of the triangles

be not $\parallel$,

draw $\parallel$ [I. 31],

meeting . Draw .

Becauſe $\parallel$ [conſt.]

$=$ but $=$

$\therefore$ $=$, a part equal to the whole,

which is abſurd.

$\therefore$ $\nparallel$; and in the ſame manner it can be demonſtrated, that no other line except

is $\parallel$; $\therefore$ $\parallel$.

Q.E.D.

I. 58

PROPOSITION XLI. THEOREM.

IF *a parallelogram*  *and a triangle*  *are upon the ſame baſe* ( ) *and between the ſame parallels* (and), *the parallelogram is double the triangle.*

Draw the diagonal;

Then  =  [I. 37]

 = twice  [I. 34]

∴  = twice  .

Q.E.D.

PROPOSITION XLII. THEOREM.

TO *construct a parallelogram equal to a given triangle*  *and having an angle equal to a given rectilinear angle*  .

Make = [I. 10].

Draw .

Make  =  [I. 23].

Draw ‖ [I. 31]

‖

 = twice  [I. 41]

but  =  [I. 38]

∴  =  .

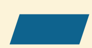
Q.E.D.

PROPOSITION XLIII. THEOREM.

THE *complements*  *and*  *of the parallelograms which are about the diagonal of a parallelogram are equal.*

 =  [I. 34]

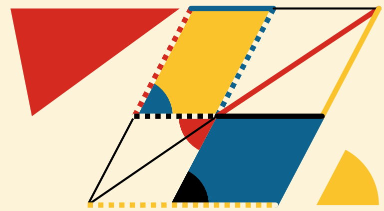
and  =  [I. 34]

∴  =  [ax. 3].

Q.E.D.

PROPOSITION XLIV. PROBLEM.

TO a given ſtraight line ( ) to apply a parallelogram equal to a given triangle ( ), and having an angle equal to a given rectilinear angle ( ).



Make = with = [I. 42]

and having one of its ſides conterminous with and in continuation of . Produce till it meets ‖ draw produce it till it meets continued; draw ‖ meeting produced, and produce .

= [I. 43]

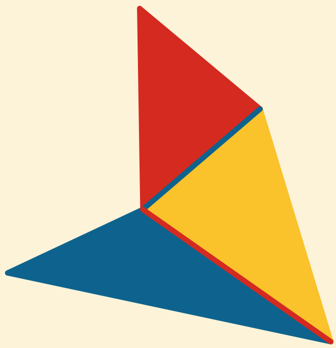
but = [conſt.]

∴ = , and

= = = [I. 29 and conſt.]

Q.E.F.

PROPOSITION XLV. PROBLEM.



TO conſtruct a parallelogram equal to a given rectilinear figure ( ) and having an angle equal to a given rectilinear angle ( ).

Draw and dividing the rectilinear figure into triangles.

Conſtruct = having = [I. 42]

to apply = having = [I. 44]

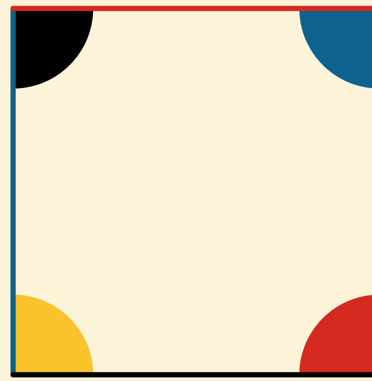
to apply = having = [I. 44]

∴ = and is a parallelogram [I. 29, 14, 30]

having = .

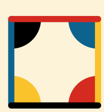
Q.E.F.

UPON *a given ſtraight line* (——) *to conſtruct a ſquare.*



Draw —— ⊥ and = —— [I. 11, 3].

Draw —— ∥ ——, and meeting —— drawn ∥ ——.

In  —— = —— [conſt.]

 = a right angle [conſt.]

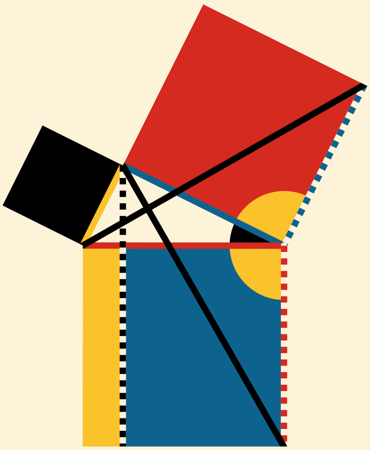
∴  =  = a right angle [I. 29],

and the remaining ſides and angles muſt be equal, [I. 34]

and ∴  is a ſquare [I. def. 27].

Q.E.F.

PROPOSITION XLVII. THEOREM.



IN *a right angled triangle* ( ) *the ſquare on the hypotenuſe* ( ) *is equal to the ſum of the ſquares of the ſides,* (*and*).

On , and deſcribe ſquares, [I. 46].

Draw [I. 31], alſo draw and .

= ,

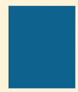
To each add ∴ = ,

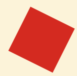
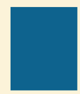
= and = ;

∴ = .

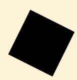
Again, because 

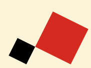
 = twice  ,

and  = twice  ;

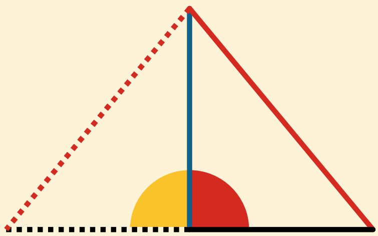
∴  =  .

In the ſame manner it may be ſhown

that  =  ;

hence  =  .

Q.E.D.



IF *the ſquare of one ſide ( ) of a triangle is equal to the ſquares of the other two ſides (and), the angle ( ) ſubtended by that ſide is a right angle.*

Draw $\perp$ and $=$ [I. 11, 3]

and draw alſo.

Since $=$ [conſt.]

$^2 = {}^2$;

$\therefore {}^2 + {}^2 = {}^2 + {}^2$,

but $^2 + {}^2 = {}^2$ [I. 47],

and $^2 + {}^2 = {}^2$ [hyp.]

$\therefore {}^2 = {}^2$,

$\therefore =$;

and $\therefore =$ [I. 8],

conſequently is a right angle.

Q.E.D.

BOOK II.

DEFINITIONS.

DEFINITION I.

A rectangle or a right angled parallelogram is ſaid to be contained by any two of its adjacent or conterminous ſides.

Thus: the right angled parallelogram is ſaid to be contained by the ſides and ; or it may be briefly deſignated by · .

If the adjacent ſides are equal; i.e. = , then · which is the expreſſion for the rectangle under and is a ſquare, and

is equal to { · or ^{2} ; · or ^{2} }

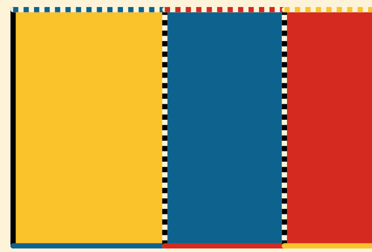
DEFINITION II.

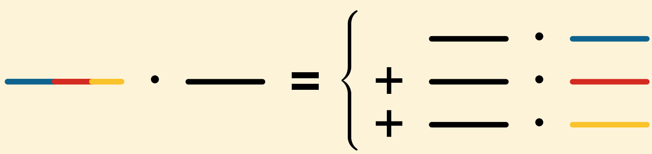
IN a parallelogram, the figure compoſed of one of the parallelograms about the diagonal, together with the two complements, is called a *Gnomon*.

Thus  and  are called Gnomons.

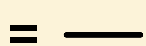
PROPOSITION I. PROBLEM.

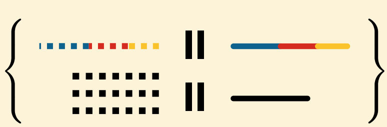
THE *rectangle contained by two ſtraight lines, one of which is divided into any number of parts,*

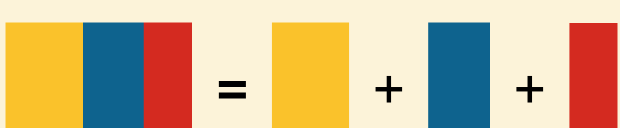


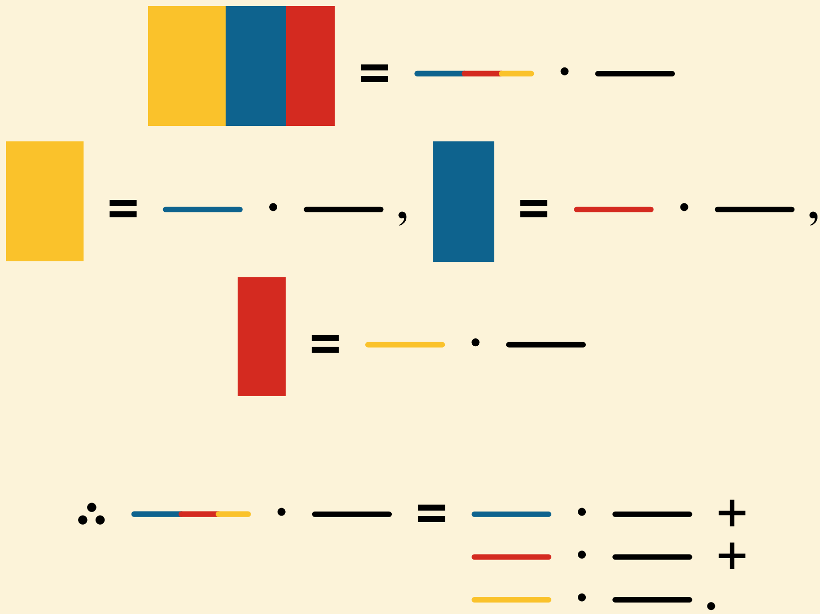


is equal to the ſum of the rectangles contained by the undivided line, and the ſeveral parts of the divided line.

Draw  and  [I. 2, 3]; complete parallelograms, that is to ſay,

Draw  [I. 31]

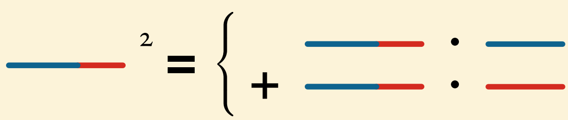




Q.E.F.

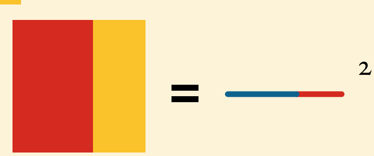
PROPOSITION II. THEOREM.

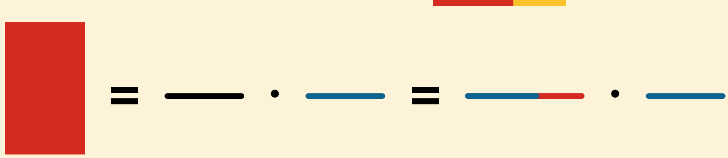
IF a ſtraight line be divided into any two parts , *the ſquare of the whole line is equal to the ſum of the rectangles contained by the whole line and each of its parts.*

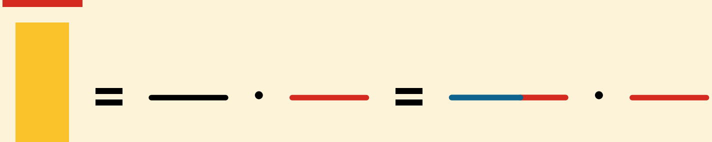


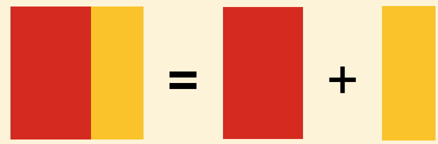
Deſcribe  [I. 46].

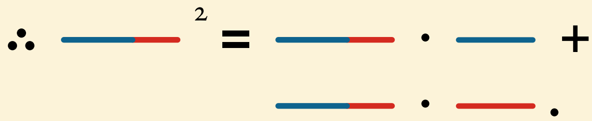
Draw parallel to [I. 31].









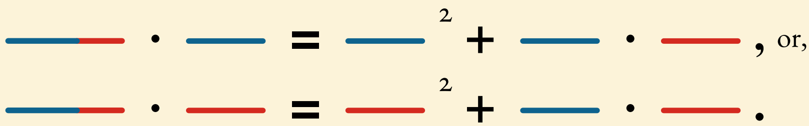


Q.E.D.

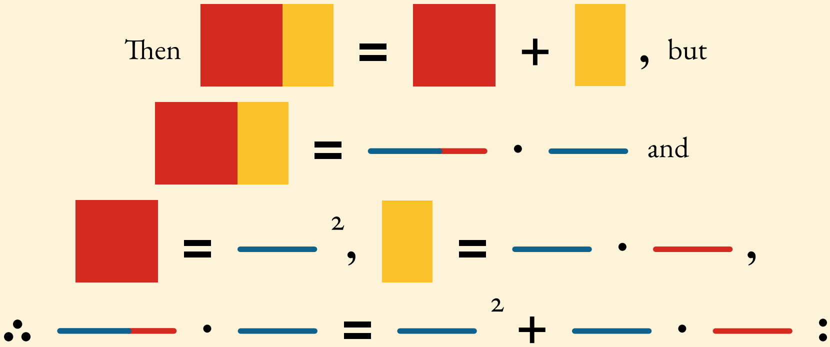
II. 73

PROPOSITION III. THEOREM.

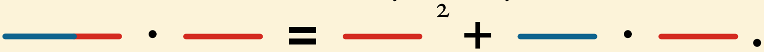
IF a ſtraight line be divided into any two parts *, the rectangle contained by the whole line and either of its parts, is equal to the ſquare of that part, together with the rectangle under the parts.*



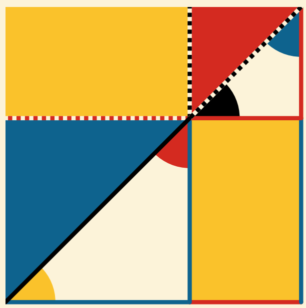
Deſcribe  [I. 46]. Complete  [I. 31].



In a ſimilar manner it may be readily ſhown that

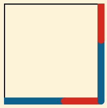


Q.E.D.

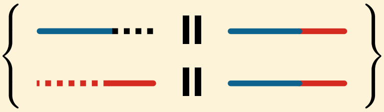


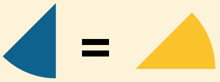
IF *a ſtraight line be divided into any two parts* , *the ſquare of the whole line is equal to the squares of the parts, together with twice the rectangle contained by the parts.*

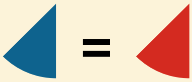
 ^{2} = ^{2} + ^{2} + twice · .

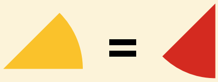
Deſcribe  [I. 46]

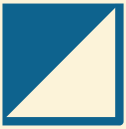
draw [poſt. 1],

and  [I. 31]

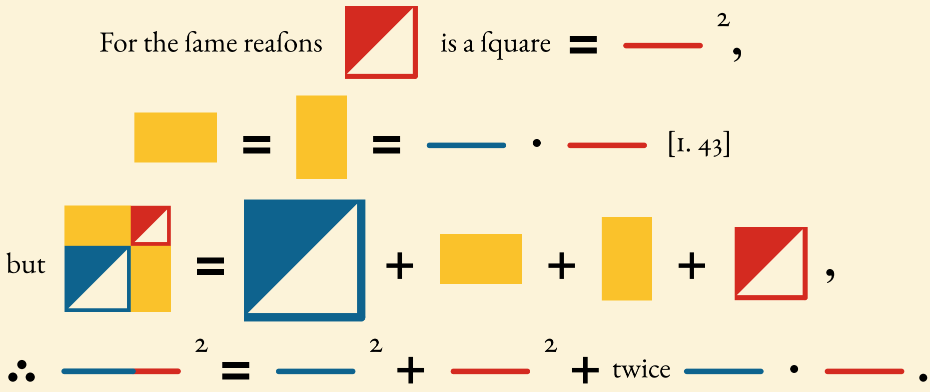
 [I. 5],

 [I. 29]

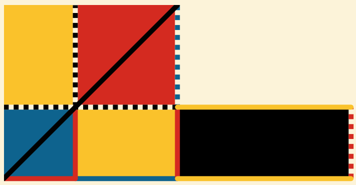
∴ 

∴ by [I. 6, 29, 34]  is a ſquare = ^{2}.

II. 75



Q.E.D.



IF a ſtraight line be divided into two equal parts and alſo into two unequal parts, the rectangle contained by the unequal parts, together with the ſquare of the line between the points of ſection, the ſquare of the line between the points of ſection, is equal to the ſquare of half that line

· + ^{2} = ^{2} = .

Deſcribe [I. 46], draw and

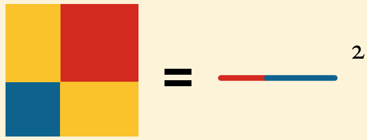
=
=
= [I. 31]

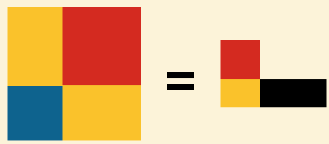
= [I. 36]

= [I. 43]

∴ [ax. 2] = = ·

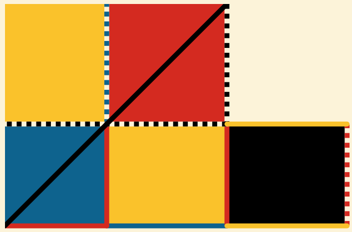
but = ^{2} [II. 4 cor.]

and  [conſt.]

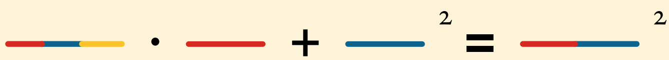
∴ [ax. 2] 

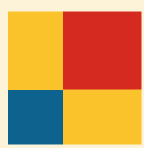
Q.E.D.

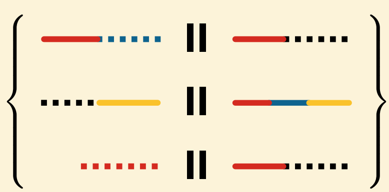
PROPOSITION VI. THEOREM.



IF a ſtraight line be biſected and produced to any point , the rectangle contained by the whole line ſo increaſed, and the part produced, together with the ſquare of half the line, is equal to the ſquare of the line made up of the half, and the produced part.

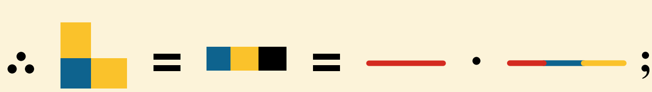


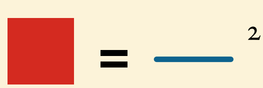
Deſcribe  [I. 46], draw and

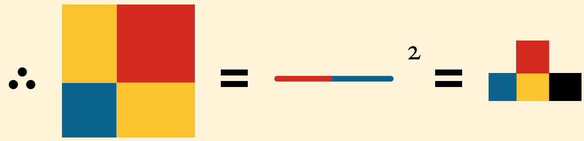


[I. 31]

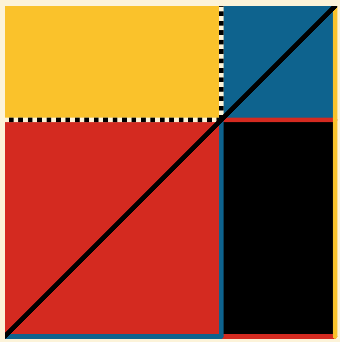
[I. 36, 43]



 [II. 4 cor.]

 [conſt. ax. 2]

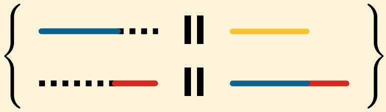
Q.E.D.

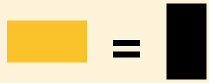


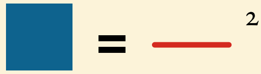
IF *a ſtraight line be divided into any two parts* , *the ſquares of the whole line and one of the parts are equal to twice the rectangle contained by the whole line and that part, together with the ſquare of the other parts.*

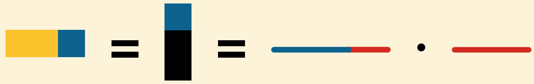
Deſcribe  , [I. 46].

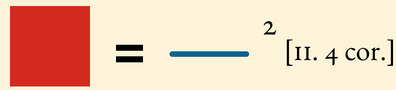
Draw [poſt. 1],

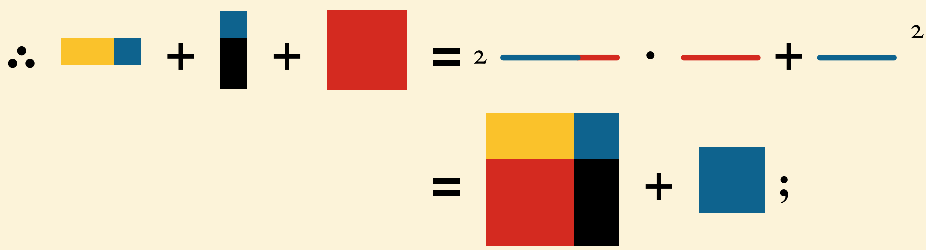
and  [I. 31].

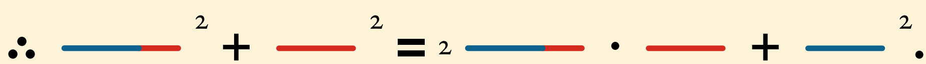
 [I. 43],

add  to both [II. 4 cor.]

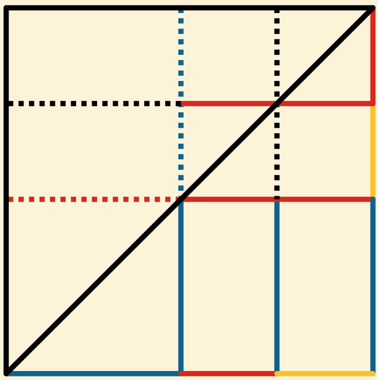








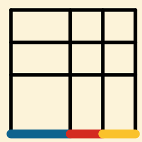
Q.E.D.

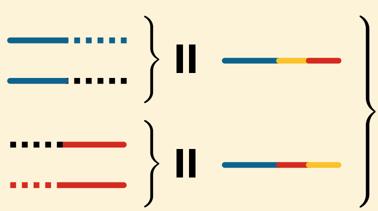


PROPOSITION VIII. THEOREM.

If a ſtraight line be divided into any two parts *, the ſquare of the ſum of the whole line and any one of its parts, is equal to four times the rectangle contained by the whole line, and that part together with the ſquare of the other part.*

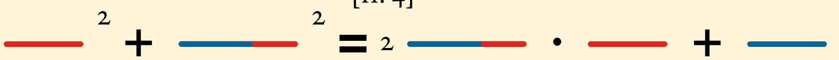
Produce and make =

Conſtruct  [I. 46]; draw ,

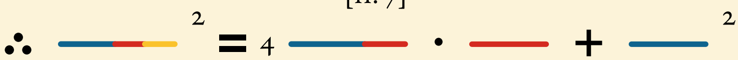


[I. 31]

[II. 4]

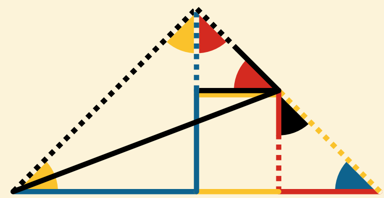
but 

[II. 7]



Q.E.D.

PROPOSITION IX. THEOREM.



If *a ſtraight line be divided into two equal parts* , *and alſo into two unequal parts* , *the ſquares of the unequal parts are together double the ſquares of half the line, and of the part between the points of ſection.*

$${}^{2} + {}^{2} = 2\,{}^{2} + 2\,{}^{2}.$$

Make $\perp$ and $=$ or ,

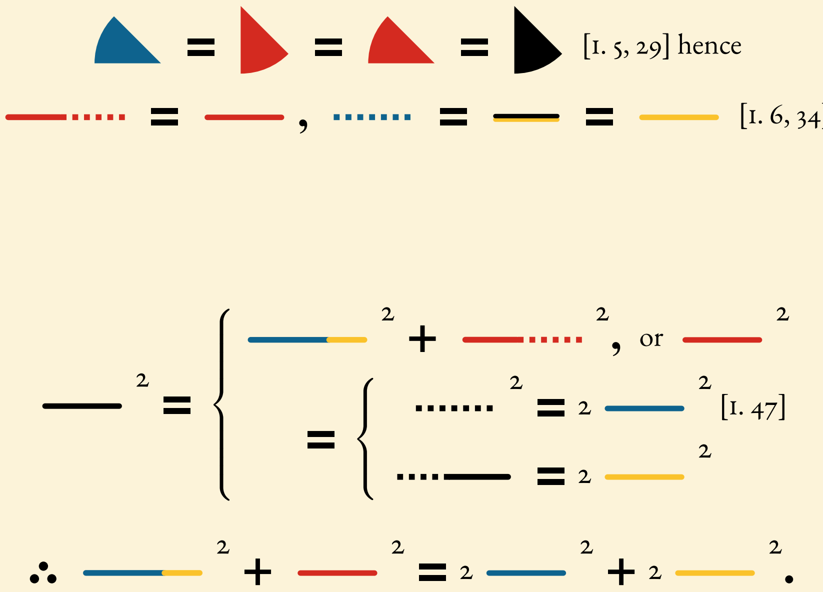
Draw and ,

$\parallel$, $\parallel$, and draw .

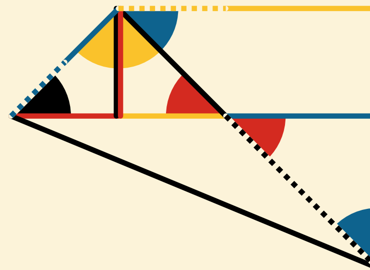
$=$ [I. 5] $=$ half a right angle. [I. 32 cor.]

$=$ [I. 5] $=$ half a right angle. [I. 32 cor.]

$\therefore$ $=$ a right angle.



Q.E.D.



IF a ſtraight line be biſected and produced to any point , the ſquares of the whole produced line, and of the produced part, are together double of the ſquares of the half line, and of the line made up of the half and produced part.

$$^2 + \ ^2 = 2 \ ^2 + 2 \ ^2.$$

Make $\perp$ and = or ,

draw and ,

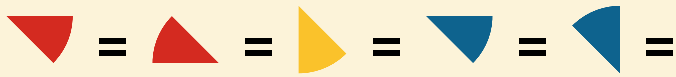
and { ∥ ; ∥ } [I. 31];

draw alſo.

= [I. 5] = half a right angle. [I. 32 cor.]

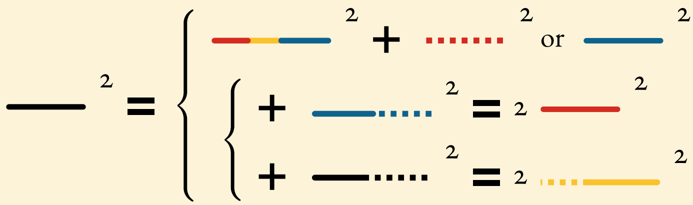
= [I. 5] = half a right angle. [I. 32 cor.]

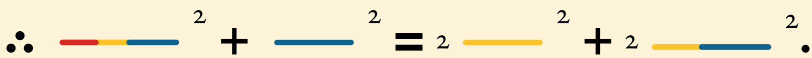
∴ = a right angle.



half a right angle [I. 5, 32, 29, 34], and

[I. 6, 34]. Hence by [I. 47]

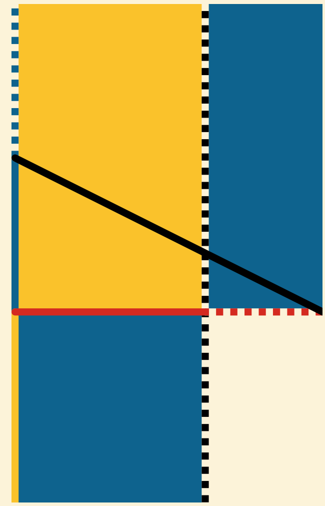




Q.E.D.

PROPOSITION XI. PROBLEM.

TO *divide a given ſtraight line* *in ſuch a manner, that the rectangle contained by the whole line and one of its parts may be equal to the ſquare of the other.*



Deſcribe  [I. 46],

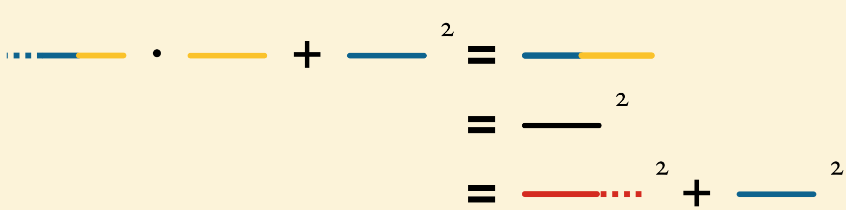
make = [I. 10], draw ,

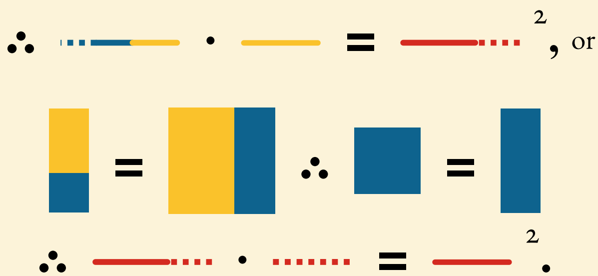
take = [I. 3],

on deſcribe  [I. 46].

Produce [poſt. 2].

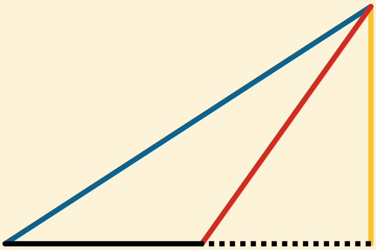
Then, [II. 6]



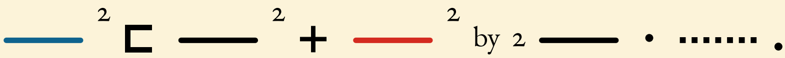


Q.E.F.

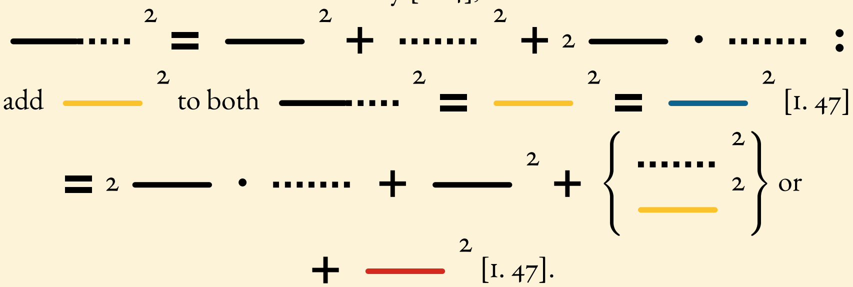
PROPOSITION XII. PROBLEM.



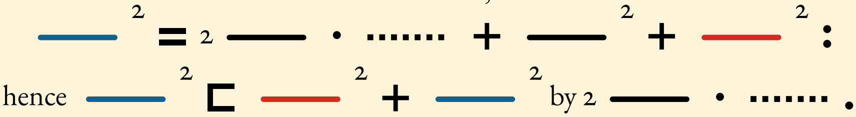
IN any obtuſe angled triangle, the ſquare of the ſide ſubtending the obtuſe angle exceeds the ſum of the ſquares of the ſides containing the obtuſe angle, by twice the rectangle contained by either of theſe ſides and the produced parts of the ſame from the obtuſe angle to the perpendicular let fall on it from the oppoſite acute angle.



By [II. 4],

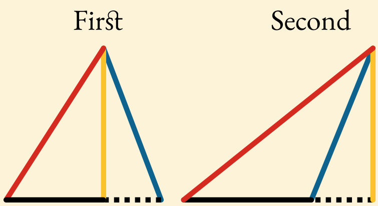


Therefore,



Q.E.F.

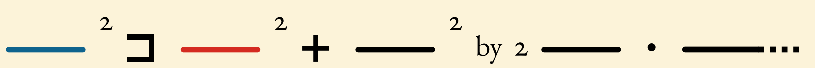
PROPOSITION XIII. PROBLEM.



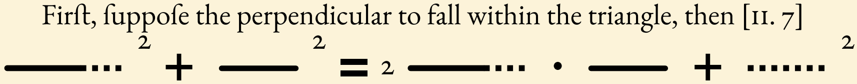
IN *any triangle, the ſquare of the ſide ſubtendeding an acute angle, is leſs than the ſum of the ſquares of the ſides containing that angle, by twice the rectangle contained by either of theſe ſides, and the part of it intercepted between the foot of the perpendicular let fall on it from the oppoſite angle, and the angular point of the acute angle.*

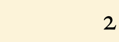
FIRST.

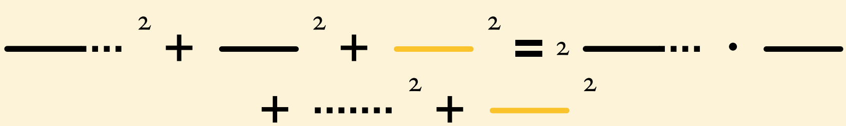
SECOND.



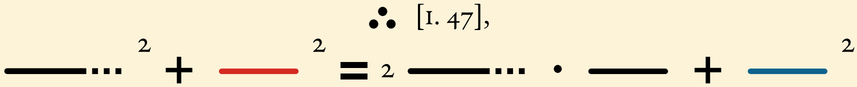
Firſt, ſuppoſe the perpendicular to fall within the triangle, then [II. 7]

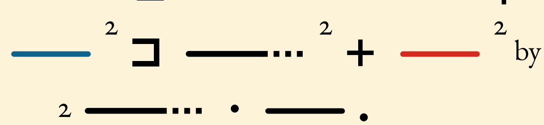


add to each  then,



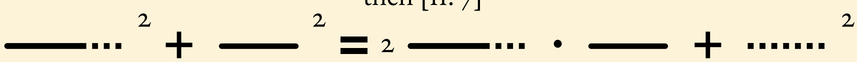
∴ [I. 47],



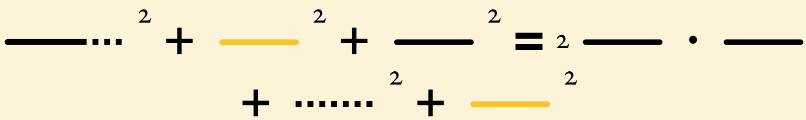
and ∴ 

Next suppose the perpendicular to fall without the triangle,

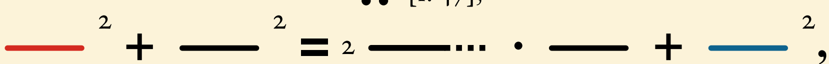
then [II. 7]



add to each  then,

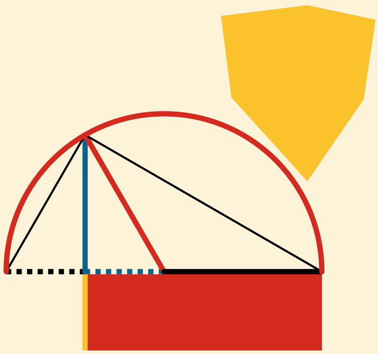


∴ [I. 47],



 by

Q.E.F.



PROPOSITION XIV. PROBLEM.

TO *draw a right line of which the ſquare ſhall be equal to a given rectilinear figure. To draw ſuch that*

^{2} =

Make = [I. 45],

produce until = ;

take = [I. 10].

Deſcribe [poſt. 3],

and produce to meet it: draw .

^{2} or ^{2} = · + ^{2}

[II. 5],

but ^{2} = ^{2} + ^{2} [I. 47];

∴ ^{2} + ^{2} = · + ^{2}

∴ ^{2} = · , and

∴ ^{2} = =

Q.E.F.

BOOK III.

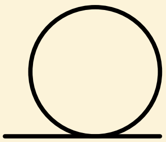
DEFINITIONS.

I.

EQUAL circles are thofe whofe diameters are equal.

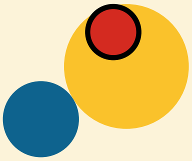
II.

A right line is said to touch a circle when it meets the circle, and being produced does not cut it.



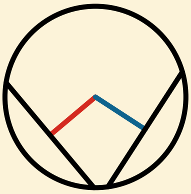
III.

Circles are faid to touch one another which meet but do not cut one another.



IV.

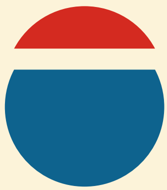
Right lines are faid to be equally diftant from the centre of a circle when the perpendiculars are drawn to them from the centre are equal.



V.

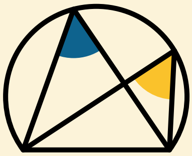
And the ſtraight line on which the greater perpendicular falls is ſaid to be farther from the centre.

VI.



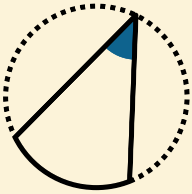
A ſegment of a circle is the figure contained by a ſtraight line and the part of the circumference it cuts off.

VII.



An angle in a ſegment is the angle contained by two ſtraight lines drawn from any in the circumference of the ſegment to the extremities of the ſtraight line which is the baſe of the ſegment.

VIII.



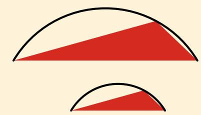
An angle is ſaid to ſtand on the part of the circumference, or the arch, intercepted between the right lines that contain the angle.

IX.

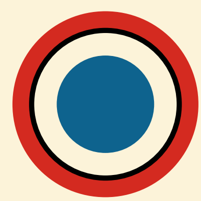
A ſector of a circle is the figure contained by two radii and the arch between them.

X.

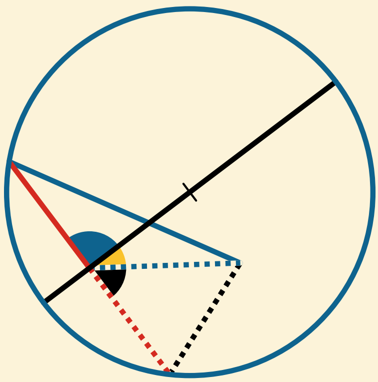
Similar ſegments of circles are thoſe which contain equal angles.



Circles which have the ſame centre are called *concentric circles*.



PROPOSITION I. PROBLEM.



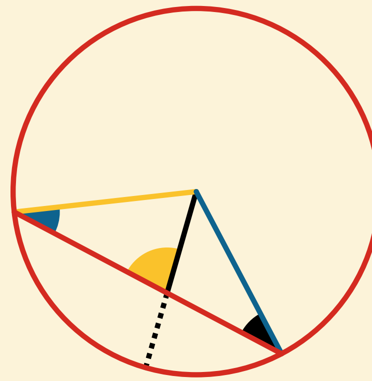
T*O find the centre of a given circle* .

Draw within the circle any ſtraight line , make = , draw ⊥ ; biſect , and the point of biſection is the centre. For, if it be poſſible, let any other point as the point of concourſe of , and be the centre. Becauſe in and = [hyp. and I. def. 15] = [conſt.] and common, = [I. 8], and are therefore right angles; but = [conſt.] = [ax. 11] which is abſurd; therefore the aſſumed point is not the centre of the circle; and in the ſame manner it can be proved that no other point which is not on is the centre, therefore the centre is in , and therefore the point where is biſected is the centre.

Q.E.F.

PROPOSITION II. THEOREM.

A straight line ( ) joining two points in the circumference of a circle , lies wholly within the circle.



Find the centre of [III. 1];

from the centre draw to any point in ,
meeting the circumference from the centre;
draw and .

Then = [I. 5]

but ⊏ or ⊏ [I. 16]

∴ ⊏ [I. 19]

but = ,

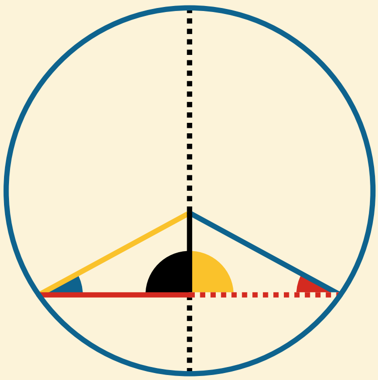
∴ = ⊏ ;

∴ ⊐ ;

∴ every point in lies within the circle.

Q.E.D.

PROPOSITION III. THEOREM.



I*F a ſtraight line* ( ) *drawn through the centre of a circle* *biſects a chord* ( ) *which does not paſs through the centre, it is perpendicular to it; or, if perpendicular to it, it biſects it.*

Draw and to the centre of the circle.

In and

= , common, and

= ∴ = [I. 8]

and ∴ ⊥ [I. def. 10].

Again let ⊥ .

Then in and

= [I. 5]

= [hyp.]

and =

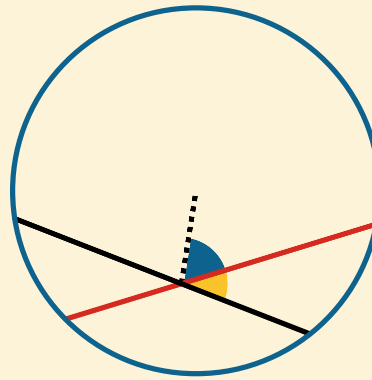
∴ = [I. 26]

and ∴ biſects .

Q.E.D.

PROPOSITION IV. THEOREM.

IF *in a circle two ſtraight lines cut one another, which do not both paſs through the centre, they do not biſect one another.*



If one of the lines paſs through the centre, it is evident that it cannot be biſected by the other, which does not paſs through the centre.

But if neither the lines or paſs through the centre, draw from the centre to their interſection.

If be biſected, $\perp$ to it [III. 3]

$\therefore$ $=$ and if be

biſected, $\perp$ [III. 3]

$\therefore$ $=$

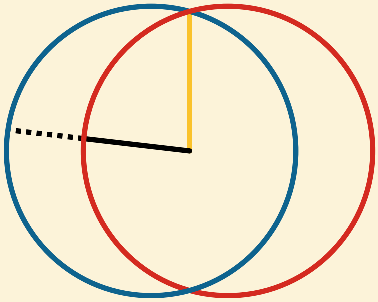
and $\therefore$ $=$; a part

equal to the whole, which is abſurd:

$\therefore$ and do not biſect one another.

Q.E.D.

PROPOSITION V. THEOREM.



IF *two circles* *interſect, they have not the ſame centre.*

Suppoſe it poſſible that two interſecting circles have a common centre; from ſuch ſuppoſed centre draw to the interſecting point, and meeting the circumferences of the circles.

Then = [1. def. 15]

and = [1. def. 15]

∴ = ;

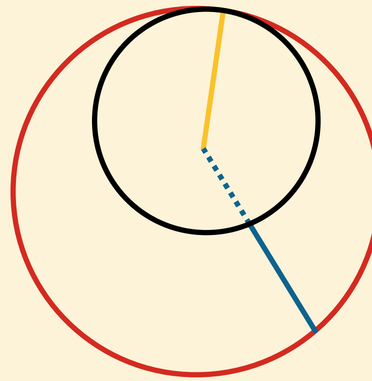
a part equal to the whole, which is abſurd:

∴ circles ſuppoſed to interſect in any point cannot have the ſame centre.

Q.E.D.

PROPOSITION VI. THEOREM.

IF *two circles* *touch one another internally they have not the ſame centre.*



For, if it be poſſible, let both circles have the ſame centre; from ſuch a ſuppoſed centre draw cutting both circles, and to the point of contact.

Then = [1. def. 15]

and = [1. def. 15]

∴ = ;

a part equal to the whole, which is abſurd:

therefore the aſſumed point is not the centre of both circles; and in the ſame manner it can be demonſtrated that no other point is.

Q.E.D.

PROPOSITION VII. THEOREM.

FIGURE I.

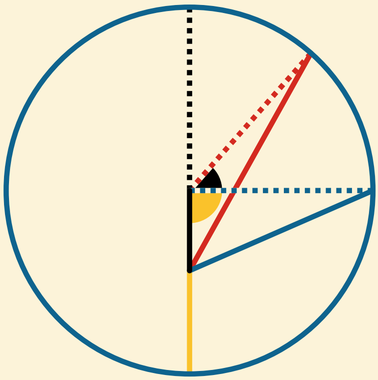
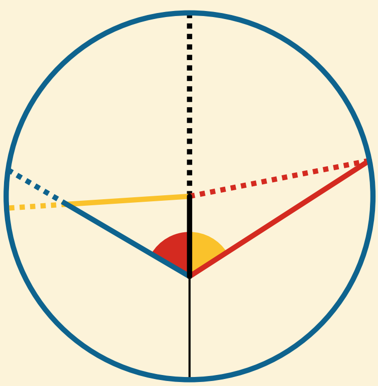


FIGURE II.



IF from any point within a circle which is not the centre, lines are drawn to the circumference; the greatest of those lines is that ( ) which passes through the centre, and the least is the remaining part ( ) of the diameter.

Of the others, that ( ) which is nearer to the line passing through the centre, is greater than that ( ) which is more remote.

Fig. 2 The two lines (and) which make equal angles with that passing through the centre, on opposite sides of it, are equal to each other; and there cannot be drawn a third line equal to them, from the same point to the circumference.

FIGURE I.

To the centre of the circle draw and ;

then = [I. def. 15]

= + ⊏ [I. 20] in like manner

III. 103

may be ſhewn to be greater than , or any other line drawn from the ſame point to the circumference. Again, by [I. 20] + ⊏ = + , take from both; ∴ ⊏ [ax. 3], and in like manner it may be ſhewn that is leſs than any other line drawn from the ſame point to the circumference. Again, in and , common, ⊏ , and = ∴ ⊏ [I. 24] and may in like manner be proved greater than any other line drawn from the ſame point to the circumference more remote from .

FIGURE II.

If = then = ,

if not take = draw ,

then in and , common, =

and = ∴ = [I. 4]

∴ = = a part equal to the whole, which is abſurd:

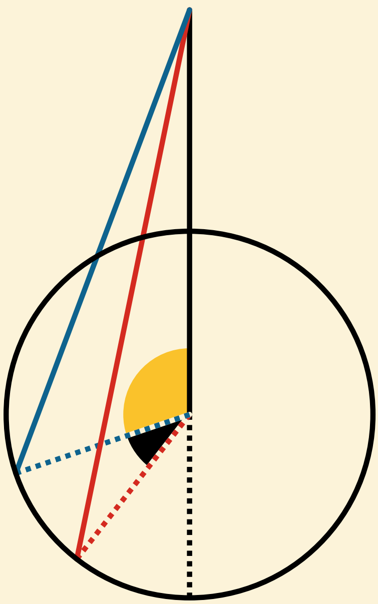
∴ = ; and no other line is equal to drawn from the ſame point to the circumference; for if it were nearer to the one paſſing through the centre it would be greater, and if it were more remote it would be leſs.

Q.E.D.

The original text of this propoſition is here divided into three parts.

PROPOSITION VIII. THEOREM.

I.



IF *from a point without a circle, ſtraight lines* { &c. } *are drawn to the circumference; of thoſe falling upon the concave circumference the greateſt is that* ( ) *which paſſes through the centre, and the line* ( ) *which is nearer the greateſt is greater than that* ( ) *which is more remote.*

Draw and to the centre.

Then, which paſſes through the centre, is greateſt; for ſince

= , if be added to both,

= + ; but ⊏ [I. 20]

∴ is greater than any other line drawn from the ſame point to the concave circumference.

Again in  and , = ,

III. 105

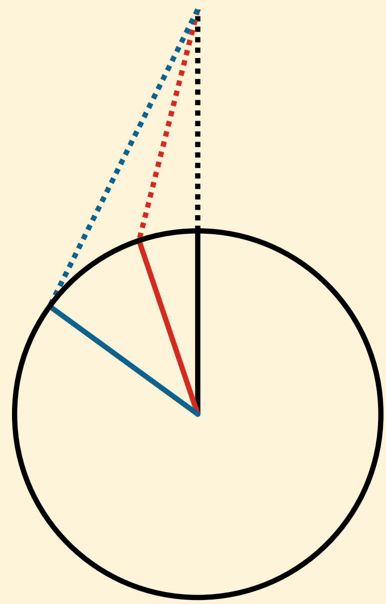
and common, but ⊏ ,

∴ ⊏ [I. 24];

and in like manner may be ſhewn ⊏ than any other line more remote from .

II.

Of thoſe lines falling on the convex circumference the leaſt is that ( ) which being produced would paſs through the centre, and the line which is nearer to the leaſt is leſs than that which is more remote.



For, ſince + ⊏ [I. 20]

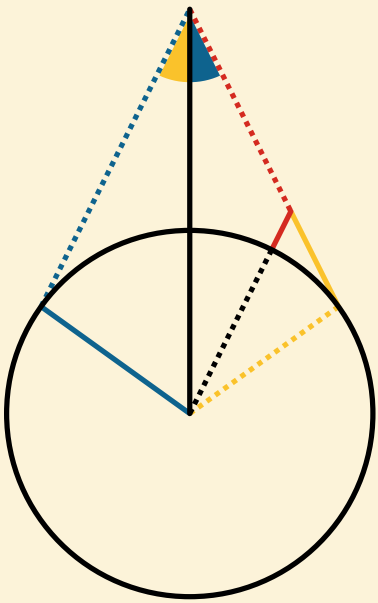
and = ,

∴ ⊏ [ax. 5].

And again, ſince + ⊏ +

[I. 21], and = ,

∴ ⊐ . And ſo of others.



III.

Alſo the lines making equal angles with that which paſſes through the centre are equal, whether falling on the concave or convex circumference; and no third line can be drawn equal to them from the ſame point to the circumference.

For if ⊏ , but making = ;

make = , and draw .

Then in and we have = ,

and common, and alſo = ,

∴ = [I. 4];

but = ;

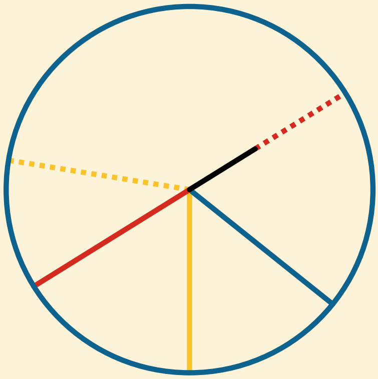
∴ = , which is abſurd.

∴ is not = , not to any part

of , ∴ is not ⊏ .
Neither is ⊏ , they are
∴ = to each other.

And any other line drawn from the ſame point to the circumference muſt lie at the ſame ſide with one of theſe lines, and be more or leſs remote than it from the line paſſing through the centre, and cannot therefore be equal to it.

Q.E.D.



IF a point be taken within a circle  *, from which more than two equal ſtraight lines* (, ,) *can be drawn tothe circumference, that point muſt be the centre of the circle.*

For if it be ſuppoſed that the point  in which more than two equal ſtraight lines meet is not the centre, ſome other point muſt be; join theſe two points by , and produce it both ways to the circumference.

Then ſince more than two equal ſtraight lines are drawn from a point which is not the centre, to the circumference, two of them at leaſt muſt lie at the ſame ſide of the diameter ; and ſince from a point  , which is not the centre, ſtraight lines are drawn to the circumference; the greateſt is , which paſſes through the centre: and which is nearer to ,  which is more remote [III. 8]; but = [hyp.] which is abſurd. The ſame may be demonſtrated of any other point, different from  , which muſt be the centre of the circle.

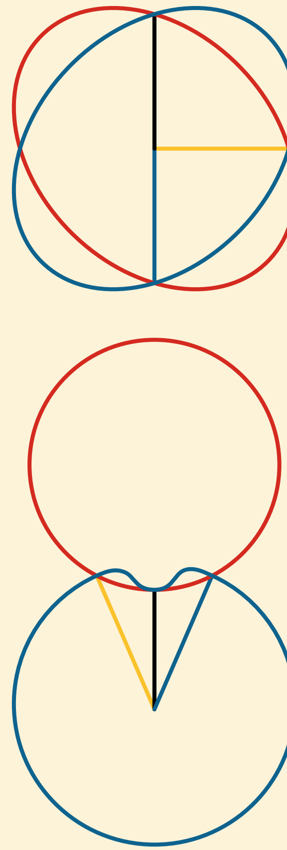
Q.E.D.

PROPOSITION X. THEOREM.

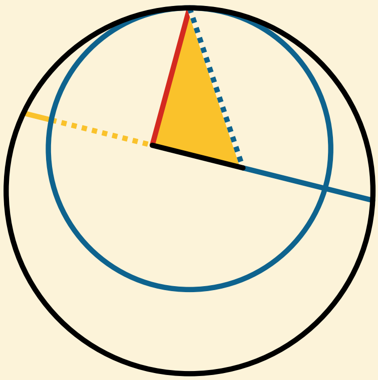
ONE *circle* *cannot interſect another* *in more points than two.*

For if it be poſſible, let it interſect in three points; from the centre of draw , and to the points of interſection;

∴ = = [I. def. 15], but as the circles interſect, they have not the ſame centre [III. 5]: ∴ the aſſumed point is not the centre of , and ∴ as , and are drawn from a point and not the centre, they are not equal [III. 7, 8]; but it was ſhewn before that they were equal, which is abſurd; the circles therefore do not interſect in three points.



Q.E.D.



I*F two circles* *and* *touch one another internally, the right line joining their centres, being produced, ſhall paſs through a point of contact.*

For if it be poſſible, let join their centres, and produce it both ways; from a point of contact draw to the centre of , and from the ſame point of contact draw to the centre of .

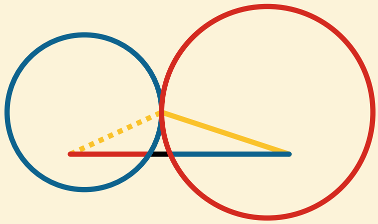
Becauſe in ; + ⊏ [I. 20],
and = as they are radii of ,
but + ⊏ ; take away which is common, and ⊏ ; but = ,
becauſe they are radii of , and ∴ ⊏ a part greater than the whole, which is abſurd.

The centres are not therefore ſo placed, that a line joining them can paſs through any point but a point of contact.

Q.E.D.

PROPOSITION XII. THEOREM.

IF *two circles* and *touch one another externally, the ſtraight line joining their centres paſſes through the point of contact.*



If it be poſſible, let join the centres, and not paſs through a point of contact; then from a point of contact draw and to the centres.

Becauſe $=$ $\sqsubset$ [I. 20],

and $=$ [I. def. 15],

and $=$ [I. def. 15],

$\therefore$ $+$ $\sqsubset$, a part greater than the whole, which is abſurd.

The centres are not therefore ſo placed, that the line joining them can paſs through any point but the point of contact.

Q.E.D.

ONE *circle cannot touch another, either externally or internally in more points than one.*

FIGURE I.

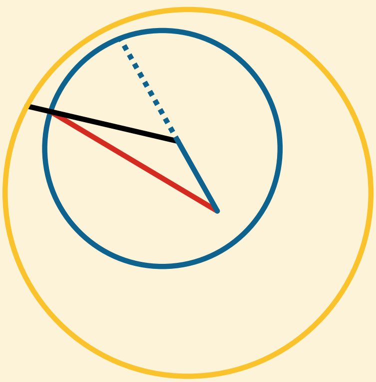


Fig. 1 For if it be possible, let and touch one another internally in two points; draw joining their centres, and produce it until it pass through one of the points of contact [III. 11]; draw and .

But = [I. def. 15],

∴ if be added to both,

= + ;

but = [I. def. 15],

and ∴ + = ; but

+ ⊏ [I. def. 20],

which is absurd.

Fig. 2 But if the points of contact be the extremities of the right line joining the centres, this ſtraight line muſt be biſected in two different points for the two centres; becauſe it is the diameter of both circle, which is abſurd.

FIGURE II.

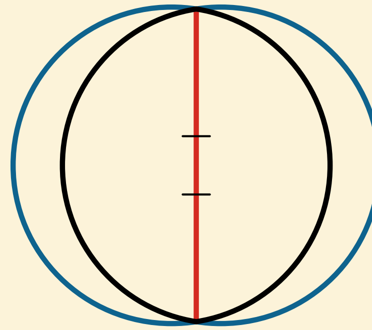
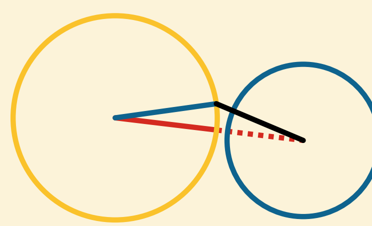


Fig. 3 Next, if it be poſſible, let and touch externally in two points; draw joining the centres of the circles, and paſſing through one of the points of contact, and draw and .

FIGURE III.



= [I. def. 15];

and = [I. def. 15]:

∴ + = ; but

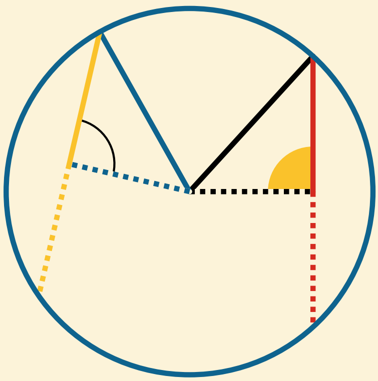
+ ⊏ [I. 20],

which is abſurd.

There is therefore no caſe in which two circles can touch one another in two points.

Q.E.D.

PROPOSITION XIV. THEOREM.



EQUAL *ſtraight lines* *inſcribed in a circle are equally diſtant from the centre; and alſo, ſtraight lines equally diſtant from the centre are equal.*

From the centre of draw

⊥ to and

⊥ , join and .

Then = half [III. 3]

and = $\frac{1}{2}$ [III. 3]

ſince = [hyp.]

∴ = ,

and = [I. def. 15]

∴ ^{2} = ^{2};

but ſince is a right angle

^{2} = ^{2} + ^{2} [I. 47]

and ^{2} = ^{2} + ^{2} for the ſame reaſon,

$$\therefore \;\; {}^2 + \;\; {}^2 = \;\; {}^2 + \;\; {}^2$$

$$\therefore \;\; {}^2 = \;\; {}^2,$$

$$\therefore \;\; = \;\; .$$

Alſo, if the lines and be equally diſtant from the centre; that is to ſay, if the perpendiculars and be given equal than $=$.

For, as in the preceding caſe,

$$\;\; {}^2 + \;\; {}^2 = \;\; {}^2 + \;\; {}^2;$$

but $\;\; {}^2 = \;\; {}^2:$

$\therefore \;\; {}^2 = \;\; {}^2$, and the doubles of theſe

and are alſo equal.

Q.E.D.

PROPOSITION XV. THEOREM.

THE *diameter is the greateſt ſtraight line in a circle: and, of all others, that which is neareſt to the centre is greater than the more remote.*

FIGURE I.

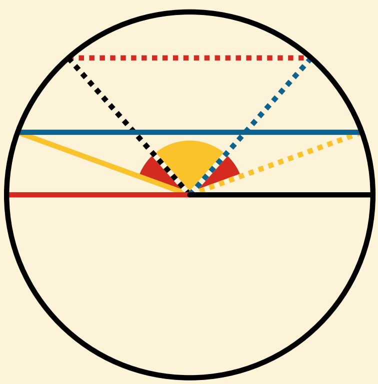


FIGURE I.

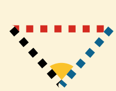
The diameter is ⊏ any line . For draw and . Then = and = ,

∴ + = but + ⊏ [I. 20] ∴ ⊏ .

Again, the line which is nearer the centre is greater than the one more remote.

Firſt, let the given lines be and , which are at the ſame ſide of the centre and do not interſect;

draw { , , , . }

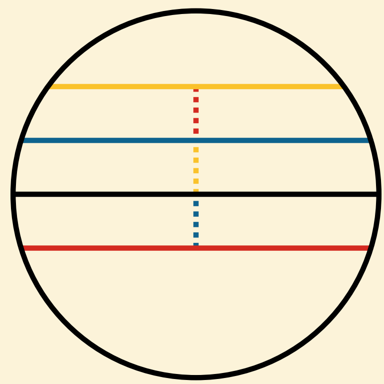
In  and  , and = and ;

⊏ ,

∴ ⊏ [I. 24]

FIGURE II.

FIGURE II.



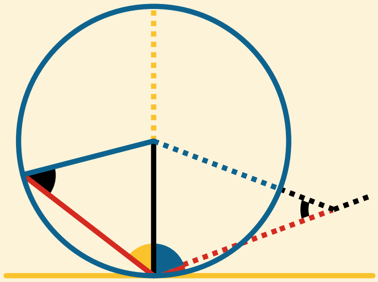
Let the given lines be and which either are at different ſides of the centre, or interſect; from the centre draw and ⊥ and , make = , and draw ⊥ .

Since and are equally diſtant from the centre, = [III. 14];

but ⊏ [Pt. 1; III. 15],

∴ ⊏ .

Q.E.D.



PROPOSITION XVI. THEOREM.

THE ſtraight line drawn from the extremity of the diameter of a circle perpendicular to it falls without the circle. And if any ſtraight line be drawn from a point within that perpendicular to the point of contact, it cuts the circle.

PART I.

If it be poſſible, let , which meets the circle again, be ⊥ , and draw .

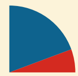
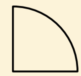
Then, becauſe = , = [I. 5],

and ∴ each of these angles is acute [I. 17]

but = [hyp.], which is abſurd, therefore drawn ⊥ does not meet the circle again.

PART II.

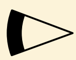
Let be ⊥ and let be drawn from a point between and the circle, which if it be poſſible, does not cut the circle.

Becaufe  =  ,

∴  is an acute angle; fuppofe

 ⊥ , drawn from the centre of the circle, it muft fall

at the fide of  the acute angle.

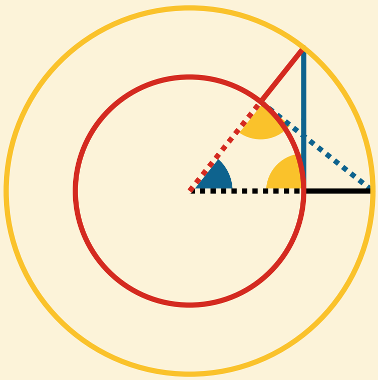
∴  which is fuppofed to be a right angle, is ⊏  ,

∴ ⊏ ;

but = ,

and ∴ ⊏ , a part greater than the whole, which is abfurd. Therefore the point does not fall outside the circle, and therefore the ftraight line cuts the circle.

Q.E.D.



TO *draw a tangent to a given circle from a given point, either in or outſide of its circumference.*

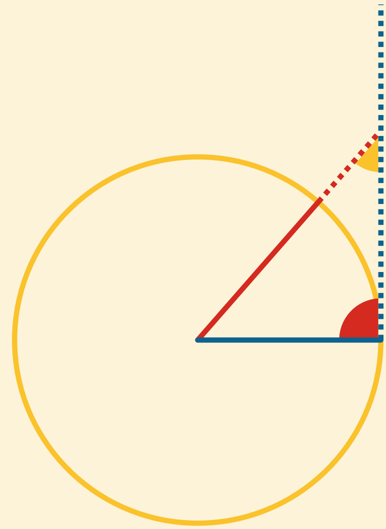
If the given point be in the circumference, as at , it is plain that the ſtraight line the radius, will be the required tangent [III. 16].

But if given the point be outſide of the circumference, draw from it to the centre, cutting ; and draw , deſcribe concentric with radius = , then will be the tangent required. For in and = , common, and = , ∴ [I. 4] = = a right angle, ∴ is a tangent to .

Q.E.D.

PROPOSITION XVIII. THEOREM.

IF *a right line* be *a tangent to a circle, the ſtraight line* drawn from the *centre to the point of contact, is perpendicular to it.*



For if it be poſſible, let be ⊥ ,

then becauſe = , is acute [I. 17]

∴ ⊏ [I. 19];

but = ,

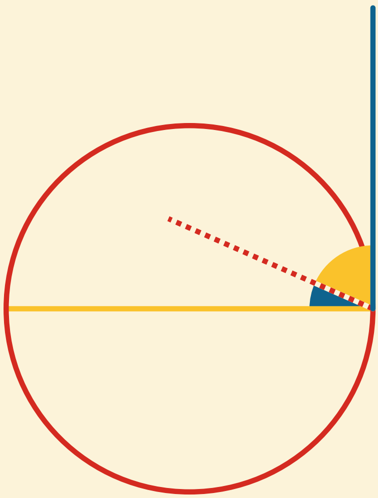
and ∴ ⊏ ,

a part greater than the whole, which is abſurd.

∴ is not ⊥ ; and in the ſame manner it can be demonſtrated, that no other line except is perpendicular to .

Q.E.D.

PROPOSITION XIX. THEOREM.



IF a ſtraight line be a tangent to a circle, the ſtraight line , drawn perpendicular to it from a point of the contact, paſſes through the centre of the circle.

For if it be poſſible, let the centre be without , and draw from the ſuppoſed centre to the point of contact.

Becauſe $\perp$ [III. 18]

$\therefore$ = , a right angle;

but = [hyp.], and $\therefore$ = ,

a part equal to the whole, which is abſurd.

Therefore the aſſumed point is not the centre; and in the ſame manner it can be demonſtrated, that no other point without is the centre.

Q.E.D.

PROPOSITION XX. THEOREM.

THE *angle at the centre of a circle, is double the angle at the circumference, when they have the ſame part of the circumference for their baſe.*

FIGURE I.

Let the centre of the circle be on a ſide of .

Becauſe = ,

= [I. 5].

But = + ,

= twice [I. 32].

FIGURE I.

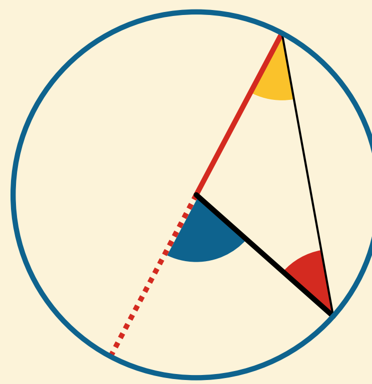


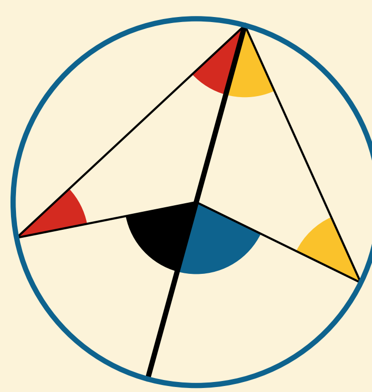
FIGURE II.

Let the centre be within , the angle at the circumference;

draw from the angular point through the centre of the circle;

then = ,

FIGURE II.



and = , becaufe of the equality of the fides [I. 5].

Hence + + + = twice .

But = + , and

= + ,

∴ = twice .

FIGURE III.

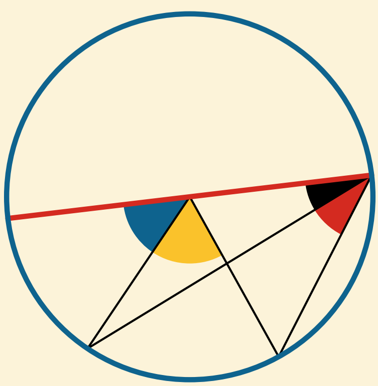


FIGURE III.

Let the centre be without and

draw , the diameter.

Becaufe = twice ;

and

= twice [cafe 1];

∴ = twice .

Q.E.D.

PROPOSITION XXI. THEOREM.

THE *angles* (,) *in the ſame ſegment of a circle are equal.*

FIGURE I.

Let the ſegment be greater than a ſemicircle, and draw and to the centre.

= twice or twice = [III. 20];

∴ = .

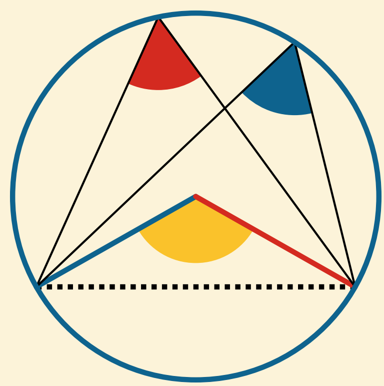


FIGURE I.

FIGURE II.

Let the ſegment be a ſemicircle, or leſs than a ſemicircle, draw the diameter, alſo draw .

= and = [caſe I];

∴ = .

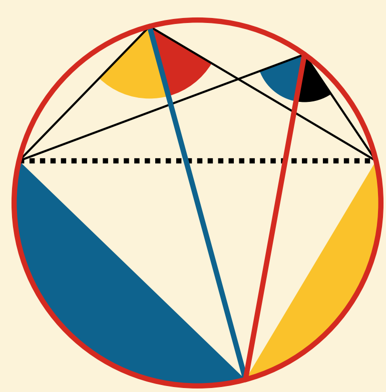
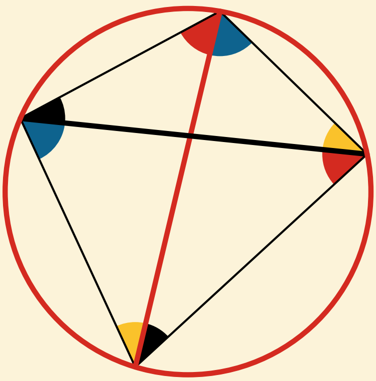


FIGURE II.

Q.E.D.

PROPOSITION XXII. THEOREM.



THE *oppoſite angles* and , and *of any quadrilateral figure inſcribed in a circle, are together equal to two right angles.*

Draw and the diagonals; and becauſe angles in the ſame ſegment are equal = ,

and = ;

add to both.

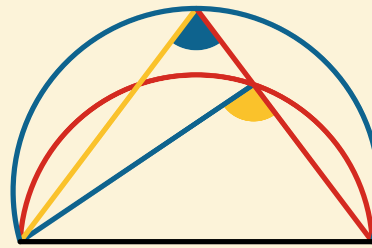
∴ + = + + =

two right angles [I. 32]. In like manner it may be ſhown that,

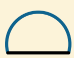
+ = .

Q.E.D.

PROPOSITION XXIII. THEOREM.



UPON *the ſame ſtraight line and upon the ſame ſide of it, two ſimilar ſegments of circles cannot be conſtructed which do not coincide.*

For if it be poſſible, let two ſimilar segments
 and  be conſtructed;

draw any right line cutting both the ſegments,

draw and .

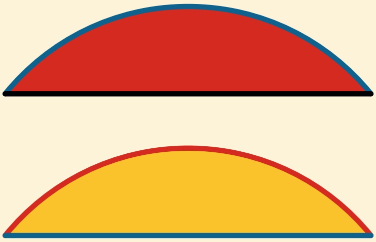
Becauſe the ſegments are ſimilar,

 =  [III. def. 10],

but  ⊏  [I. 16]

which is abſurd: therefore no point in either of the ſegments falls without the other, and therefore the ſegments coincide.

Q.E.D.



SIMILAR *ſegments*  *and* , *of circles upon equal ſtraight lines* (*and*) *are each equal to the other.*

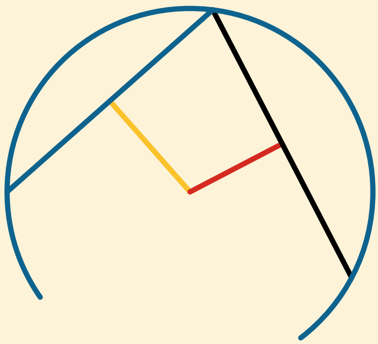
For, if  be ſo applied to ,
that may fall on , the extremities of
 may be on the extremities and
 at the ſame ſide as ;

becauſe = ,
 muſt wholly coincide with ;
and the ſimilar ſegments being then upon the ſame ſtraight line at the ſame ſide of it, muſt alſo coincide [III. 23], and are therefore equal.

Q.E.D.

PROPOSITION XXV. THEOREM.

A ſegment of a circle being given, to deſcribe the circle of which it is the ſegment.



From any point in the ſegment draw and bisect them, and from the points of biſection

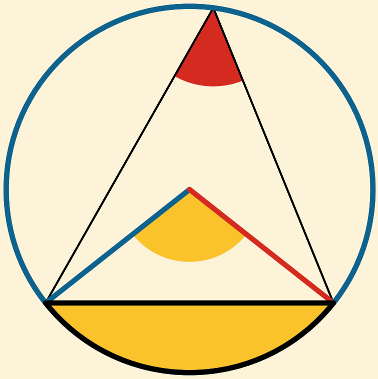
draw ⊥

and ⊥

where they meet is the centre of the circle.

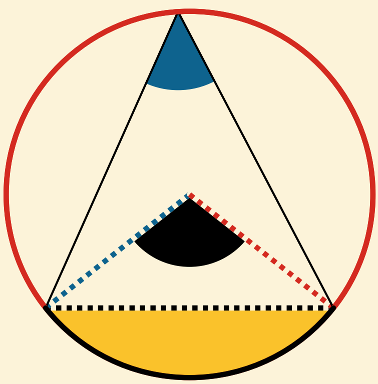
Becauſe terminated in the circle is biſected perpendicularly by , it paſſes through the centre [III. I], likewiſe paſſes through the centre, therefore the centre is in the interſection of theſe perpendiculars.

Q.E.D.



IN *equal circles* and , *the arcs* , *on which ſtand equal angles, whether at the centre or circumference, are equal.*

First, let = at the centre, draw and .



Then ſince = , and have = = = , and = ,

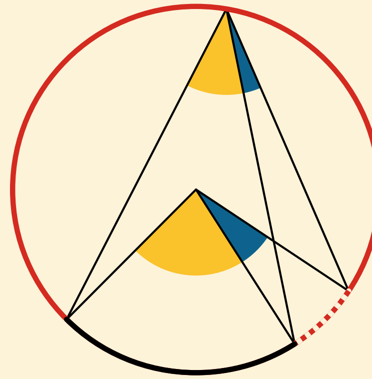
∴ = [I. 4]. But = [III. 20];

∴ and are ſimilar [III. def. 10]; they are alſo equal [III. 24]. If therefore the equal ſegments be taken from the equal circles, the remaining ſegments will be equal; hence = [ax. 3]; ∴ = . But if the given equal angles be at the circumference, it is evident that the angles at the centre, being double of thoſe at the circumference, are alſo equal, and therefore the arcs on which they ſtand are equal.

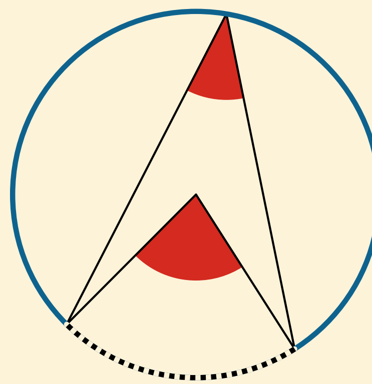
Q.E.D.

PROPOSITION XXVII. THEOREM.

I*N equal circles,* *and* *the angles* *and* *which ſtand upon equal arches are equal, whether they be at the centres or at the circumferences.*



For if it be poſſible, let one of them be greater than the other and make =



∴ = [III. 26]

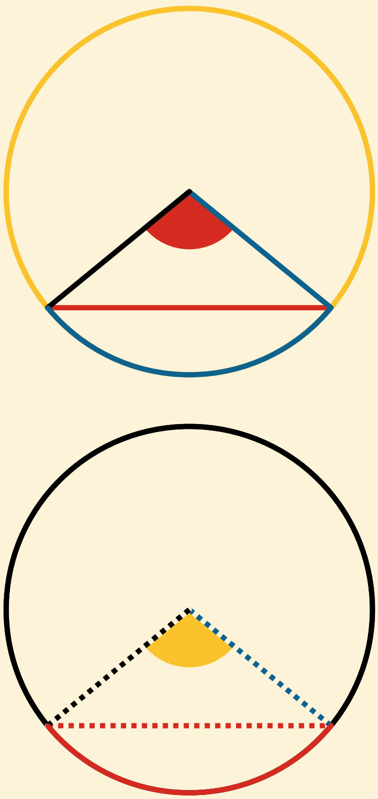
but = [hyp.]

∴ = a part equal

to the whole, which is abſurd; ∴ neither angle is greater than the other, and ∴ they are equal.

Q.E.D.

PROPOSITION XXVIII. THEOREM.



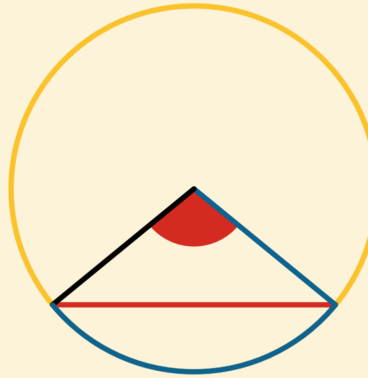
IN *equal circles* and , *equal chords* , *cut off equal arches.*

From the centres of the equal circles, draw , and , ; and becauſe = , = , , also = [hyp.] ∴ = ∴ = [III. 26] ∴ = [ax. 3].

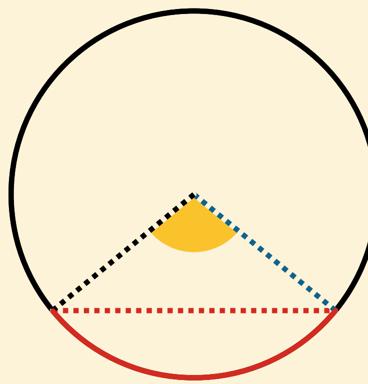
Q.E.D.

PROPOSITION XXIX. THEOREM.

IN *equal circles* and *the chords* *which ſubtend equal arcs are equal.*

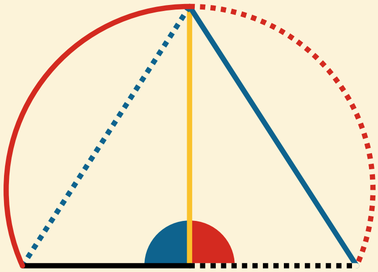


If the equal arcs be ſemicircles the propoſition is evident. But if not, let , and , be drawn to the centres; becauſe = [hyp.] and = [III. 27]; but and = and



∴ = [I. 4]; but theſe are the chords ſubtending the equal arcs.

Q.E.D.



TO *biſect a given arc* .

Draw ; make = , draw ⊥ , and it biſects the arc. Draw and . = [conſt.], is common, and = [conſt.] ∴ = [I. 4] = [I. 28], and therefore the given arc is biſected.

Q.E.F.

PROPOSITION XXXI. THEOREM.

IN *a circle the angle in a ſemicircle is a right angle, the angle in a ſegment greater than a ſemicircle is acute, and the angle in a ſegment leſs than a ſemicircle is obtuſe.*

FIGURE I.

The angle in a ſemicircle is a right angle. Draw and

= and

= [I. 5]

\+ = =

the half of two right angles = a right angle. [I. 32].

FIGURE I.

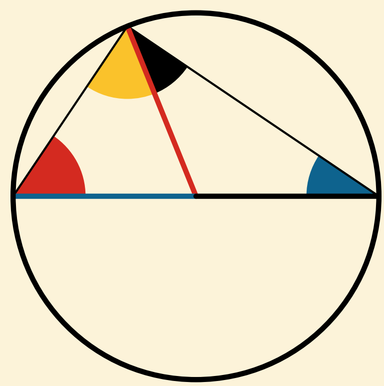


FIGURE II.

The angle in a ſegment greater than a ſemicircle is acute. Draw the diameter, and

∴ = a right angle

∴ is acute.

FIGURE II.

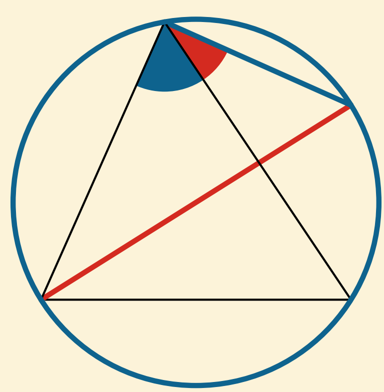


FIGURE III.

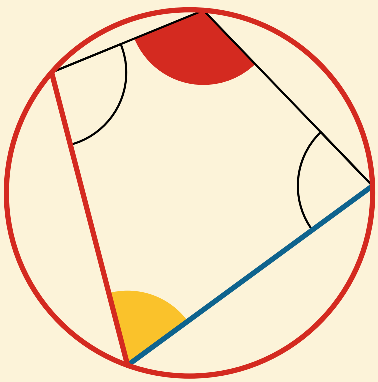


FIGURE III.

The angle in a ſegment leſs than ſemicircle is obtuſe. Take the oppoſite circumference any point, to which draw and .

Becauſe + =

[III. 22]

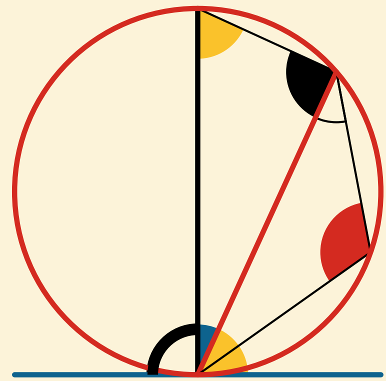
but ⊐ [part 2],

∴ is obtuſe.

Q.E.D.

PROPOSITION XXXII. THEOREM.

IF a right line be a tangent to a circle and from the point of contact a right line be drawn cutting the circle, the angle made by this line with the tangent is equal to the angle in the alternate ſegment of the circle.



If the chord ſhould paſs through the centre, it is evident the angles are equal, for each of them is a right angle [III. 16, 31].

But if not, draw $\perp$ from the point of contact, it muſt paſs through the centre of the circle [III. 19].

$\therefore$ $=$ [III. 31]

$+$ $=$ $=$ [III. 32]

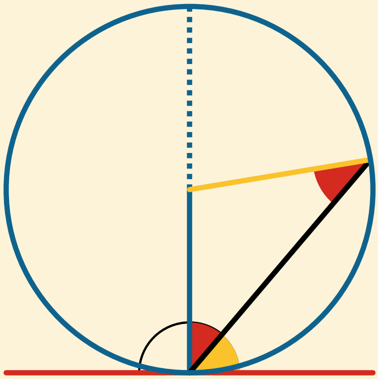
$\therefore$ $=$ [ax. 3].

Again $=$ $=$ $+$ [III. 22]

$\therefore$ $=$, [ax. 3], which is the angle in the alternate ſegment.

Q.E.D.

PROPOSITION XXXIII. PROBLEM.



ON *a given ſtraight line to deſcribe a ſegment of a circle that ſhall contain an angle equal to a given angle* , , .

If the given angle be a right angle, biſect the given line, and deſcribe a ſemicircle on it, this will evidently contain a right angle [III. 31].

If the given angle be acute or obtuſe, make with the given line, at its extremity,

= , draw ⊥ and make

= , deſcribe with or as radius,

for they are equal. is a tangent to [III. 16]

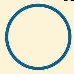
∴ divides the circle into two ſegments capable of containing

angles equal to and which were made reſpectively equal to

and [III. 32].

Q.E.F.

PROPOSITION XXXIV. PROBLEM.

T*O cut off from a given circle*  *a ſegment which ſhall contain an angle equal to a given angle*  .

Draw [III. 17], a tangent to the circle at any point; at the point of contact make

 =  the given angle;

and  contains an angle = the given angle.

Becauſe is a tangent,

and cuts it, the

angle in  =  [III. 32],

but  =  [conſt.].

Q.E.F.

PROPOSITION XXXV. THEOREM.

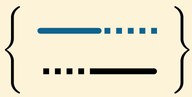
IF *two chords*  *in a circle interſect each other, the rectangle contained by the ſegments of the one is equal to the rectangle contained by the ſegments of the other.*

FIGURE I.

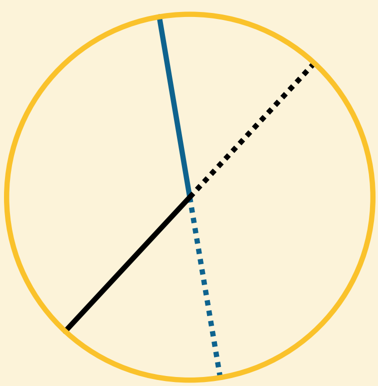


FIGURE I.

If the given right lines paſs through the centre, they are biſected in the point of interſection, hence the rectangles under their ſegments are the ſquares of their halves and are therefore equal.

FIGURE II.

Let pafs through the centre,

and not;

draw and .

Then × =

^{2} − ^{2} [II. 6],

or × =

^{2} − ^{2}.

∴ × =

× [II. 5].

FIGURE II.

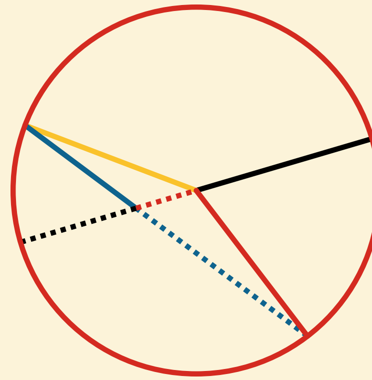


FIGURE III.

Let neither of the given lines pafs through the centre, draw through their interfection a diameter , and

× = ×

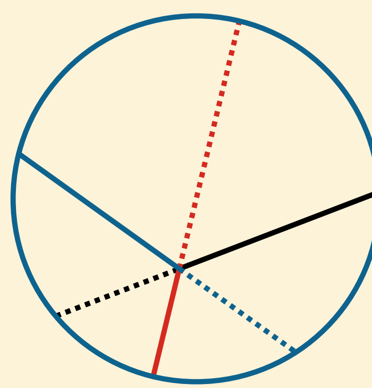
[part 2] alfo ×

= ×

[part 2]; ∴ ×

= × .

FIGURE III.



Q.E.D.

PROPOSITION XXXVI. THEOREM.

IF *from a point without a circle two ſtraight lines be drawn to it, one of which is a tangent to the circle, and the other cuts it; the rectangle under the whole cutting line and the external ſegment is equal to the ſquare of the tangent* .

FIGURE I.

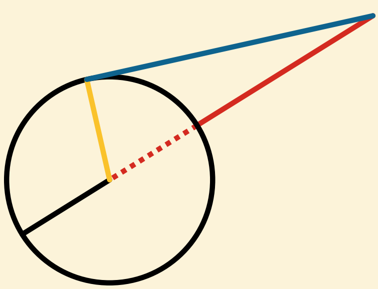


FIGURE I.

Let paſs through the centre; draw from the centre to the point of contact;

$^2 = {}^2$ minus ^{2} [I. 47], or

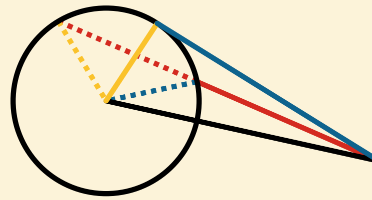
$^2 = {}^2$ minus ^{2},

$$\therefore \;{}^2 = \times \quad \text{[II. 6].}$$

FIGURE II.

If do not paſs through the centre, draw and .

FIGURE II.

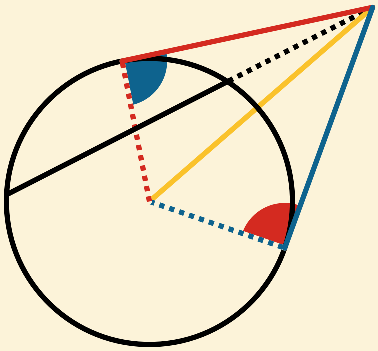


Then × = ^{2} minus ^{2} [II. 6],

that is, × = ^{2} minus ^{2},

∴ × = ^{2} [III. 18].

Q.E.D.



IF *from a point outſide a circle two ſtraight lines be drawn, the one cutting the circle, the other meeting it, and if the rectangle contained by the whole cutting line and its external ſegment be equal to the ſquare of the line meeting the circle, the latter is a tangent to the circle.*

Draw from the given point , a tangent to the circle, and draw from the centre , , and .

^{2} = × [III. 36] but

^{2} = × [hyp.],

∴ ^{2} = ^{2}, and ∴ = ;

Then in and and = and , and is common, ∴ = [I. 8]; but = a right angle [III. 18], ∴ = a right angle, and ∴ is a tangent to the circle [III. 16].

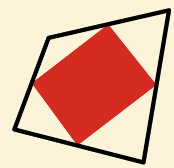
Q.E.F.

BOOK IV.

DEFINITIONS.

I.

A rectilinear figure is ſaid to be *inſcribed* in another, when all the angular points of the inſcribed figure are on the ſides of the figure in which it is ſaid to be inſcribed.

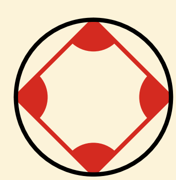


II.

A FIGURE is ſaid to be *deſcribed about* another figure, when all the ſides of the circumſcribed figure paſs through the angular points of the other figure.

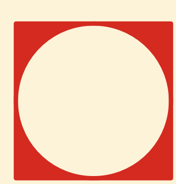
III.

A RECTILINEAR figure is ſaid to be *inſcribed in* a circle, when the vertex of each angle of the figure is in the circumference of the circle.

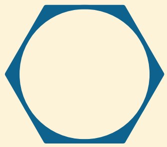


IV.

A RECTILINEAR figure is ſaid to be *circumſcribed about* a circle, when each of its ſides is a tangent to the circle.

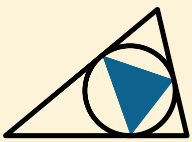


V.



A CIRCLE is ſaid to be inſcribed in a rectilinear figure, when each ſide of the figure is a tangent to the circle.

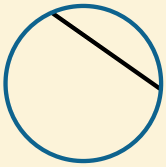
VI.



A CIRCLE is ſaid to be *circumſcribed about* a rectilinear figure, when the circumference paſſes through the vertex of each angle of the figure.

 is circumſcribed.

VII.



A STRAIGHT line is ſaid to be *inſcribe in* a circle, when its extremities are in the circumference.

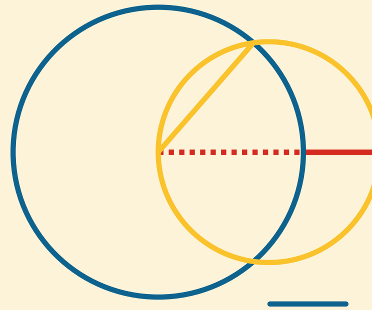
The Fourth Book of the Elements is devoted to the ſolution of problems, chiefly relating to the inſcription and circumſcription of regular polygons and circles.

A regular polygon is one whoſe angles and ſides are equal.

PROPOSITIONS.

PROPOSITION I. PROBLEM.

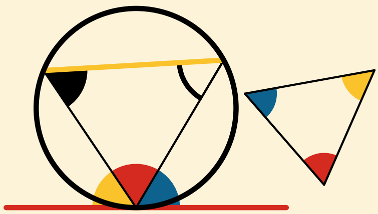
IN a given circle to place a ſtraight line, equal to a given ſtraight line (), not greater than the diameter of the circle.



Draw , the diameter of ;
and if = , then
the problem is ſolved.

But if be not equal to ,
⊏ [hyp.];
make = [I. 3] with
as radius,
deſcribe , cutting , and
draw , which the line is required.
For = = [I. def. 15, conſt.].

Q.E.F.



ON *a given circle* *to inſcribe a triangle equiangular to a given triangle.*

To any point of the given circle draw , a tangent [III. 17]; and at the point of contact

make = [I. 23]

and in like manner = , and draw .

Becauſe = [conſt.]

and = [III. 32]

∴ = ; alſo

= for the same reaſon.

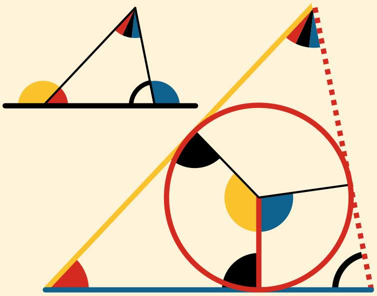
∴ = [I. 32],

and therefore the triangle inſcribed in the circle is equiangular to the given one.

Q.E.F.

PROPOSITION III. PROBLEM.

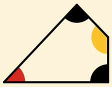
ABOUT *a given circle*  *to circumſcribe a triangle equiangular to a given triangle.*



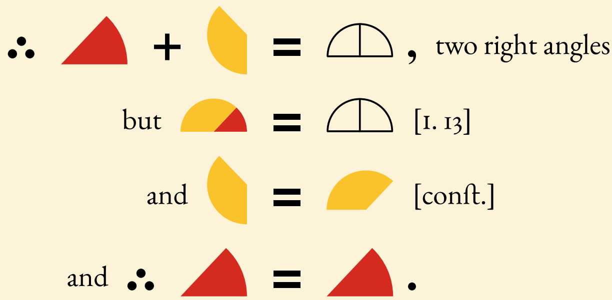
Produce any ſide , of the given triangle both ways; from the centre of the given circle draw , any radius.

Make  =  [I. 23] and  = .

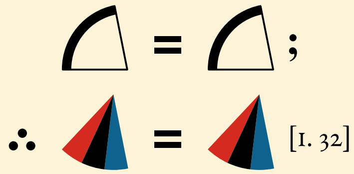
At the extremities of the three radii, draw , and , tangents to the given circle [III. 17].

The four angles , taken together, are equal to four right angles [I. 32];

but  and  are right angles [conſt.]



In the fame manner it can be demonftrated that

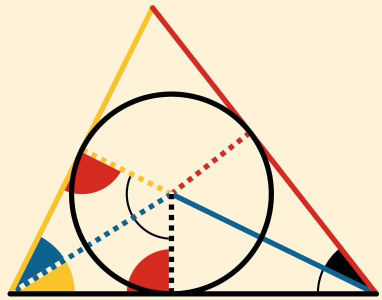


and therefore the triangle circumfcribed about the given circle is equiangular to the given triangle.

Q.E.F.

PROPOSITION IV. PROBLEM.

IN *a given triangle to inſcribe a circle.*



Biſect and [I. 9] by and ; from the point where theſe lines meet draw , and reſpectively perpendicular to , and .

In and

= , = and common,

∴ = [I. 4, 26].

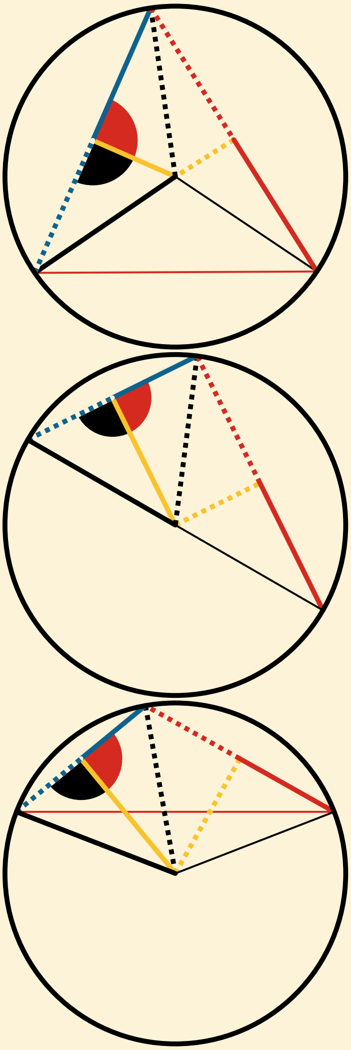
In like manner, it may be ſhown alſo

∴ = = ;

hence with any one of theſe lines as radius, deſcribe

and it will paſs through the extremities of the other two; and the ſides of the given triangle, being perpendicular to the three radii at their extremities, touch the circle [III. 16], which is therefore inſcribed in the given triangle.

Q.E.F.



PROPOSITION V. PROBLEM.

T*O describe a circle about a given triangle.*

Make = and = [I. 10]. From the points of bisection draw and ⊥ and respectively [I. 11], and from their point of concourse draw , and and describe a circle with any one of them, and it will be the circle required.

In and

= [const.],

common,

= [const.],

∴ = [I. 4].

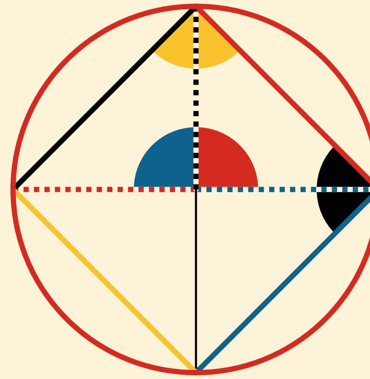
In like manner it may be shown that

= .

∴ = = ; and therefore a circle described from the concourse of these three lines with any one of them as a radius will circumscribe the given triangle.

Q.E.F.

PROPOSITION VI. PROBLEM.



IN *a given circle* *to inſcribe a ſquare.*

Draw the two diameters of the circle ⊥ to each other, and draw , , and

is a ſquare.

For ſince and are, each of them, in a ſemicircle, they are right angles [III. 31], ∴ ∥ [I. 28]:

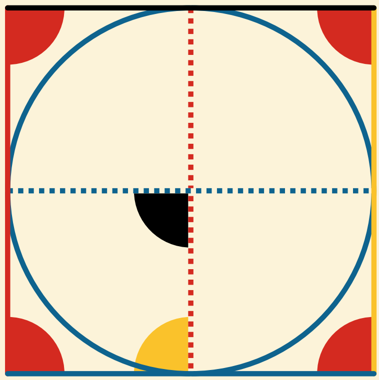
and in like manner ∥ .

And becauſe = [conſt.], and = = [I. def. 15]. ∴ = [I. 4];

and ſince the adjacent ſides and angles of the parallelogram are equal, they are all equal [I. 34]; and ∴ , inſcribed in the given circle, is a ſquare.

Q.E.F.

PROPOSITION VII. PROBLEM.



ABOUT *a given circle* *to circumſcribe a ſquare.*

Draw two diameters of the given circle perpendicular to each other, and through their extremities draw , , , and tangents to the circle; and is a ſquare.

= a right angle [III. 18], alſo = [conſt.],

∴ ‖ ; in the ſame manner it can be demonſtrated that ‖ , and alſo that and ‖ ; ∴ is a parallelogram, and

becauſe = = = = they are all right angles [I. 34]:

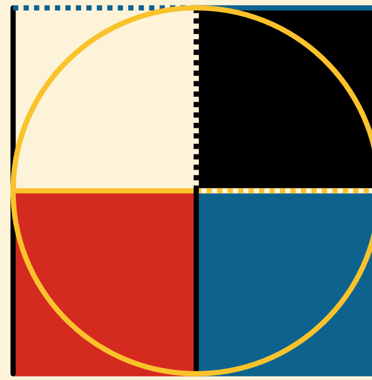
it is alſo evident that , , and are equal.

∴ is a ſquare.

Q.E.F.

PROPOSITION VIII. PROBLEM.

T*O inſcribe a circle in a given ſquare.*



Make = , and = , draw ∥ , and ∥ [I. 31].

∴ is a parallelogram; and ſince =

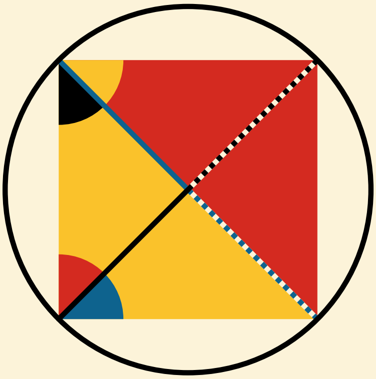
[hyp.] = ∴ is equilateral [I. 34].

In like manner, it can be ſhown that = are equilateral parallelograms;

∴ = = = ,

and therefore if a circle be deſcribed from the concourſe of theſe lines with any one of them as radius, it will be inſcribed in the given ſquare [I. 16].

Q.E.F.



TO *deſcribe a circle about a given ſquare* .

Draw the diagonals and interſecting each other; then,

becauſe and have their ſides equal, and the baſe common to both, = [I. 8], or is biſected: in like manner it can be ſhown that is biſected; but = ,

hence = their halves, ∴ = ; [I. 6]

and in like manner it can be proved that

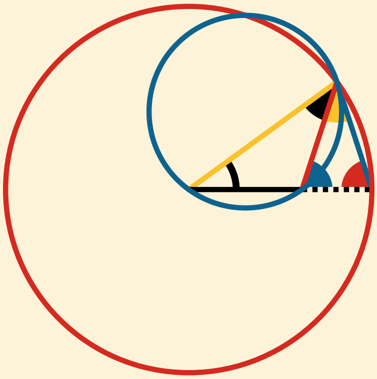
= = = .

If from the confluence of theſe lines with any one of them as radius, a circle can be deſcribed, it will circumſcribe the given ſquare.

Q.E.F.

PROPOSITION X. PROBLEM.

TO conſtruct an iſoſceles triangle, in which each of the angles at the baſe ſhall be double of the vertical angle.



Take any ſtraight line and divide it ſo that

× = ² [II. 11].

With as radius, deſcribe and place

in it from the extremity of the radius, = , [IV. 1];

draw .

Then is the required triangle.

For, draw and deſcribe

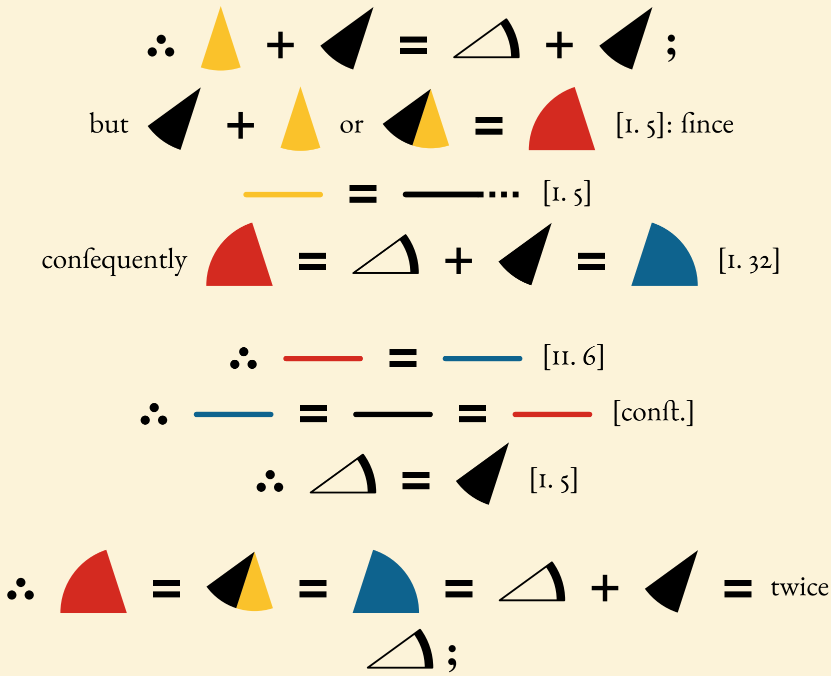
about [IV. 5].

Since × = ² = ²,

∴ is tangent to [III. 37].

∴ = [III. 32],

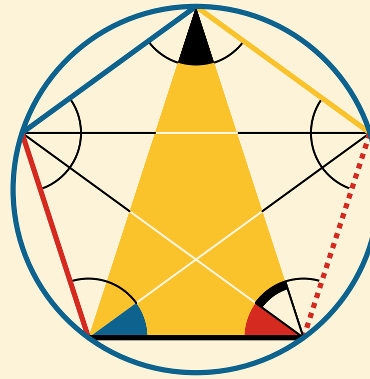
add to each,

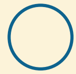


and conſequently each angle at the baſe is double of the vertical angle.

Q.E.F.

PROPOSITION XI. PROBLEM.



IN *a given circle*  *to inſcribe an equilateral and equiangular pentagon.*

Conſtruct an iſoſceles triangle, in which each of the angles at the baſe ſhall be double of the angle at the vertex, and inſcribe in the given circle a triangle  equiangular to it [IV. 2].

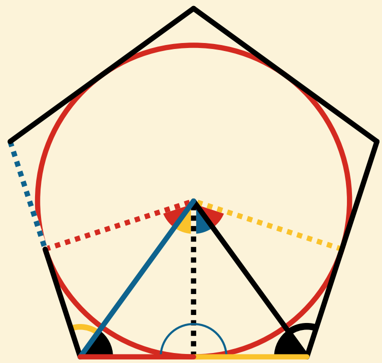
Biſect  and  [I. 9].

Draw , , and .

Becauſe each of the angles , , ,  and  are equal,

the arcs upon which they ſtand are equal [III. 26] and ∴ , , , and which ſubtend theſe arcs are equal [III. 29] and ∴ the pentagon is equilateral, it is alſo equiangular, as each of its angles ſtand upon equal arcs [III. 27].

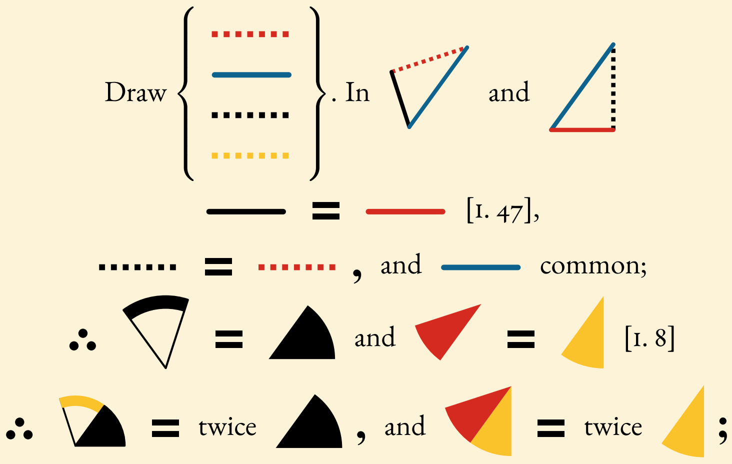
Q.E.F.



TO deſcribe an equilateral and equiangular pentagon about a given circle .

Draw five tangents through the vertices of the angles of any regular pentagon inſcribed in the given circle [III. 17].

Theſe five tangents will form the required pentagon.



In the ſame manner it can be demonſtrated that

= twice , and = twice ;

but = [III. 27],

∴ their halves = , alſo = , and

common;

∴ = and = ,

∴ = twice ;

In the ſame manner it can be demonſtrated that = twice ,

but =

∴ = ;

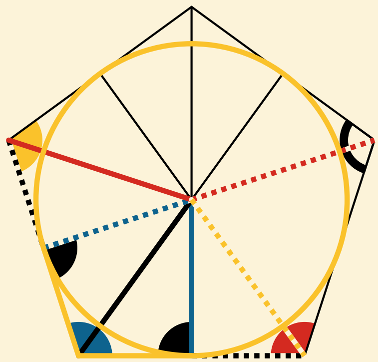
In the ſame manner it can be demonſtrated that the other ſides are equal, and therefore the pentagon is equilateral, it is alſo equiangular, for

= twice and = twice ,

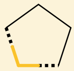
and therefore = ,

∴ = ; in the ſame manner it can be demonſtrated that the other angles of the deſcribed pentagon are equal.

Q.E.F.



TO *inſcribe a circle in a given equiangular and equilateral pentagon.*

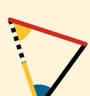
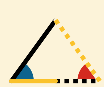
Let  be a given equiangular and equilateral pentagon; it is required to inſcribe a circle in it.

Make  = , and  =  [I. 9].

Draw , , , , &c.

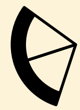
Becauſe = ,  = ,

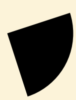
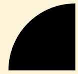
 common to the two triangles

 and ;

∴ = and  =  [I. 4].

And becaufe  =  twice 

∴ = twice , hence  is bifected by . In like manner it may be demonftrated that  is bifected by , and that the remaining angle of the polygon is bifected in a fimilar manner. Draw , , &c. perpendicular to the fides of the pentagon.

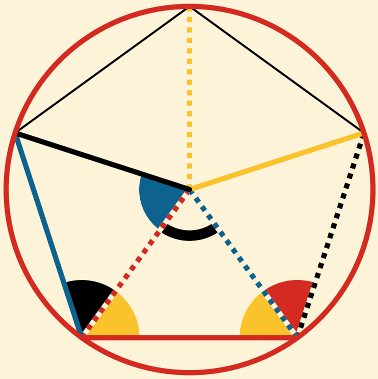
Then in the two triangles  and  we have  = , [conft.], common, and  =  = a right angle;

∴ = . [I. 26].

In the fame way it may be fhown that the five perpendiculars on the fides of the pentagon are equal to one another.

Defcribe  with any one of the perpendiculars as radius, and it will be the infcribed circle required. For if it does not touch the fides of the pentagon, but cut them, then a line drawn from the extremity at right angles to the diameter of a circle will fall within the circle, which has been fhown to be abfurd [III. 16].

Q.E.F.



TO *deſcribe a circle about a given equilateral and equiangular pentagon.*

Biſect and by and , and from the point of ſection, draw , , and .

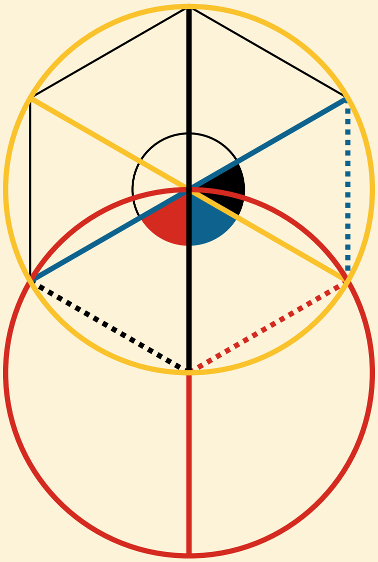
= , = , ∴ = [I. 6]; and ſince in and , = , and common, alſo = ; ∴ = [I. 4].

In like manner it may be proved that = = , and therefore = = = = :

Therefore if a circle be deſcribed from the point where theſe five lines meet, with any one of them as a radius, it will circumſcribe the given pentagon.

Q.E.F.

PROPOSITION XV. PROBLEM.



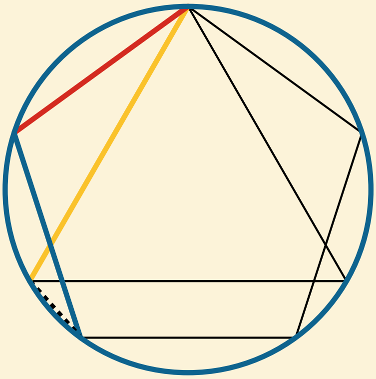
TO *inſcribe an equilateral and equiangular hexagon in a given circle* .

From any point in the circumference of the given circle deſcribe paſſing through its centre, and draw the diameters , and ; draw , , , &c. and the required hexagon is inſcribed in the given circle.

Since paſſes through the centres of the circles, and are equilateral triangles, hence = = one-third of two right angles [I. 32]; but = [I. 13];

∴ = = = one-third of [I. 32], and the angles vertically oppoſite to theſe are all equal to one another [I. 15], and ſtand on equal arches [III. 26], which are ſubtended by equal chords [III. 29]; and ſince each of the angles of the hexagon is double the angle of an equilateral triangle, it is alſo equiangular.

Q.E.F.



T*O inſcribe an equilateral and equiangular quindecagon in a given circle.*

Let and be the ſides of an equilateral pentagon inſcribed in the given circle, and the ſide of an inscribed equilateral triangle.

The arc ſubtended by and $= \frac{2}{5} = \frac{6}{15}$ of the whole circumference.

The arc ſubtended by $= \frac{1}{3} = \frac{5}{15}$ of the whole circumference.

Their difference is $= \frac{1}{15}$

$\therefore$ the arc ſubtended by $= \frac{1}{15}$ difference of the whole circumference. Hence if ſtraight lines equal to be placed in the circle [IV. I], and equilateral and equiangular quindecagon will be thus inſcribed in the circle.

Q.E.F.

BOOK V.

DEFINITIONS.

I.

A leſs magnitude is ſaid to be an aliquot part or ſubmultiple of a greater magnitude, when the leſs meaſures the greater; that is, when the leſs is contained a certain number of times exactly in the greater.

II.

A GREATER magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs; that is, when the greater contains the leſs a certain number of times exactly.

III.

RATIO is the relation which one quantity bears to another of the ſame kind, with reſpect to magnitude.

IV.

MAGNITUDES are ſaid to have a ratio to one another, when they are of the ſame kind and the one which is not the greater can be multiplied ſo as to exceed the other.

The other definitions will be given throughout the book where their aid is firſt required.

AXIOMS.

I.

EQUIMULTIPLES or equiſubmultiples of the ſame, or of equal magnitudes, are equal.

if A $=$ B, then

twice A $=$ twice B, that is,

2 A $=$ 2 B;

3 A $=$ 3 B;

4 A $=$ 4 B;

&c. &c.

and $\frac{1}{2}$ of A $=$ $\frac{1}{2}$ of B;

$\frac{1}{3}$ of A $=$ $\frac{1}{3}$ of B;

&c. &c.

II.

A MULTIPLE of a greater magnitude is greater than the ſame multiple of a leſs.

Let A $\sqsubset$ B, then

2 A $\sqsubset$ 2 B;

3 A $\sqsubset$ 3 B;

4 A $\sqsubset$ 4 B;

&c. &c.

III.

That magnitude, of which a multiple is greater than the ſame multiple of another, is greater than the other.

Let 2 A $\sqsubset$ 2 B, then

A $\sqsubset$ B;

or, let 3 A $\sqsubset$ 3 B, then

A $\sqsubset$ B

or, let *m* A $\sqsubset$ *m* B, then

A $\sqsubset$ B.

&c. &c.

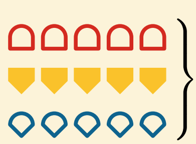
PROPOSITIONS.

PROPOSITION I. THEOREM.

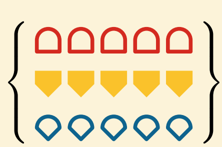
IF any number of magnitudes be equimultiples of as many others, each of each: what multiple soever any one of the firſt is of its part, the ſame multiple ſhall of the firſt magnitudes taken together be of all the others taken together.

Let  be the ſame multiple of ,
that  is of ,
that  is of .

Then it is evident that

 is the ſame multiple of 

which that  is of ;
becauſe there are as many magnitudes

in  = 

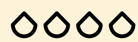
as there are in  = .

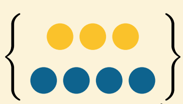
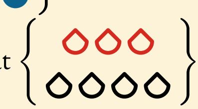
The ſame demonſtration holds in any number of magnitudes, which has here been applied to three.

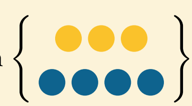
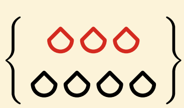
∴ If any number of magnitudes, &c.

PROPOSITION II. THEOREM.

IF *the firſt magnitude be the ſame multiple of the ſecond that the third is of the fourth, and the fifth the ſame multiple of the ſecond that the ſixth is of the fourth, then ſhall the firſt, together with the fifth, be the ſame multiple of the ſecond that the third, together with the ſixth, is of the fourth.*

Let , the firſt, be the ſame multiple of , the ſecond, that , the third, is of , the fourth; and let , the fifth, be the ſame multiple of , the ſecond, that , the ſixth, is of , the fourth.

The it is evident, that , the firſt and fifth together, is the ſame multiple of , the ſecond, that , the third and ſixth together, is of the ſame multiple of , the fourth;

becauſe there are as many magnitudes in  =  as there are in  = .

∴ If the firſt magnitude, &c.

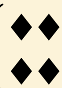
IF *the first four magnitudes be the ſame multiple of the ſecond that the third is of the fourth, and if any equimultiples whatever of the the firſt and third be taken, thoſe ſhall be equimultiples; one of the ſecond, and the other of the fourth.*

The First.

Let  be the ſame multiple of

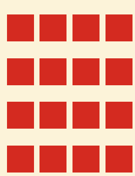
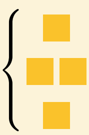
The Second.

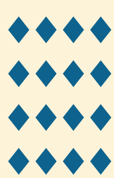
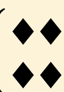
The Third.

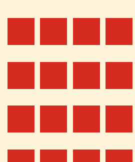
which  is of

The Fourth.

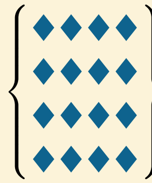
 ;

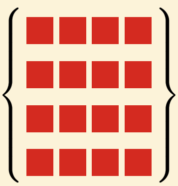
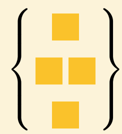
take  the ſame multiple of  ,

which  is of  .

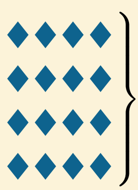
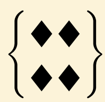
Then it is evident, that  is the ſame multiple of

The Second.

which  is of The Fourth.  ;

becaufe  contains  contains 

as many times as

 contains  contains  .

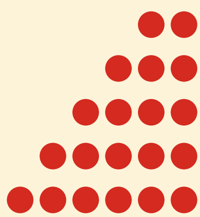
The fame reafoning is applicable in all cafes.

∴ If the firft four, &c.

DEFINITION V.

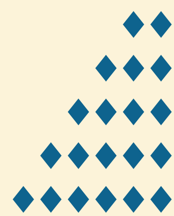
Four magnitudes , , , , are ſaid to be proportionals when every equimultiple of the firſt and third be taken, and every equimultiple of the ſecond and fourth, as,

of the firſt



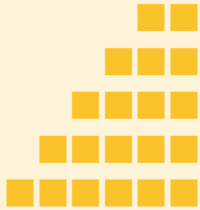
&c.

of the third



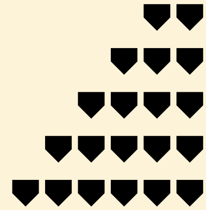
&c.

of the ſecond

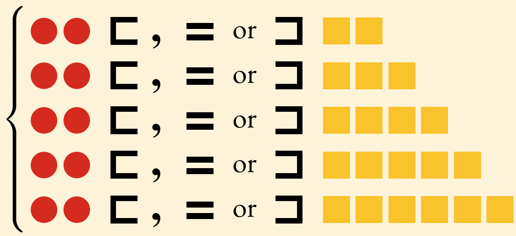


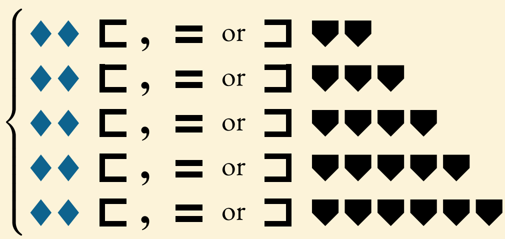
&c.

of the fourth

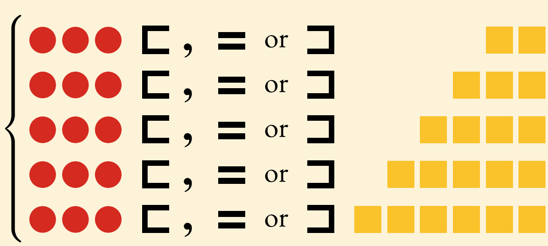


&c.

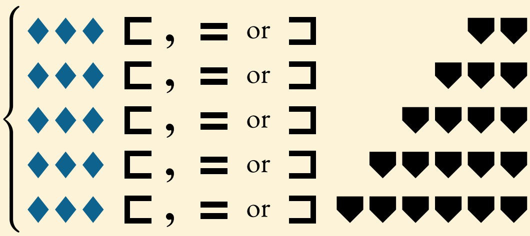
Then taking every pair of equimultiples of the firſt and third, and every pair of equimultiples of the ſecond and fourth, If 

then will 

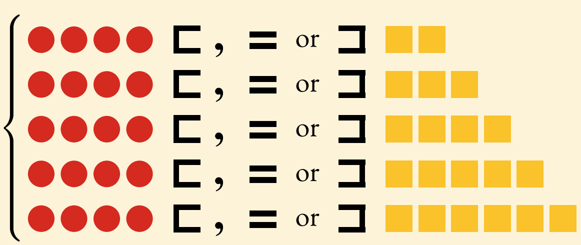
That is, if twice the firſt be greater, equal, or leſs than twice the ſecond, twice the third will be greater, equal, or leſs than twice the fourth; or, if twice the firſt be greater, equal, or leſs than three times the ſecond, twice the third will be greater, equal, or leſs than three times the fourth, and so on, as above expreſſed.

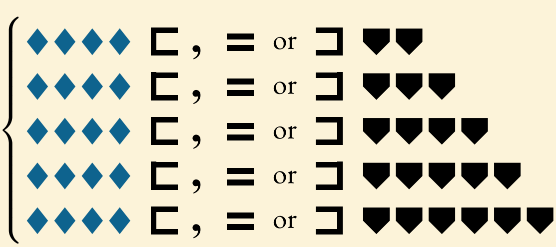
If 

v. 176

then will


In other terms, if three times the firſt be greater, equal, or leſs than twice the ſecond, three times the third will be greater, equal, or leſs than twice the fourth; or, if three times the firſt be greater, equal, or leſs than three times the ſecond, then will three times the third be greater, equal, or leſs than three times the fourth; or if three times the firſt be greater, equal, or leſs than four times the ſecond, then will three times the third be greater, equal, or leſs than four times the fourth, and so on. Again,

If


then will


And so on, with any other equimultiples of the four magnitudes, taken in the ſame manner.

Euclid expreſſes this definition as follows:—

The firſt of four magnitudes is ſaid to have the ſame ratio to the ſecond, which the third has to the fourth, when any equimultiples whatſoever of the firſt and third being taken, and any equimultiples whatſoever of the ſecond and fourth; if the multiple of the firſt be leſs than that of the second, the multiple of the third is alſo leſs than that of the fourth; or, if the multiple of the firſt be equal to that of the ſecond, the multiple of the third is alſo equal to that of the fourth; or, if the multiple of the firſt be greater than that of the ſecond, the multiple of the third is alſo greater than that of the fourth.

In future we ſhall expreſs this definition generally, thus:

If M ● ⊏, =, ⊐ *m* ■,
when M ◆ ⊏, =, ⊐ *m* ⛊.

Then we infer that ●, the firſt, has the ſame ratio to ■, the ſecond, which ◆, the third, has to ⛊ the fourth: expreſſed in the ſucceeding demonſtrations thus:

● : ■ :: ◆ : ⬟;

or thus, ● : ■ = ◆ : ⬟;

or thus, $\frac{●}{■} = \frac{◆}{⬟}$: and is read,

"as ● is to ■, so is ◆ to ⬟."

And if ● : ■ : ◆ : ⬟ we ſhall infer if

M ● ⊏, = or ⊐ *m* ■, then will

M ◆ ⊏, = or ⊐ *m* ⬟.

That is, if the firſt be to the second, as the third is to the fourth; then if M times the firſt be greater than, equal to, or leſs than *m* times the ſecond, then ſhall M times the third be greater than, equal to, or leſs than *m* times the fourth, in which M and *m* are not to be conſidered particular multiples, but every pair of multiples whatever; nor are ſuch marks as ●, ⬟, ■, &c. to be conſidered any more than repreſentatives of geometrical magnitudes.

The ſtudent ſhould thoroughly underſtand this definition before proceeding further.

PROPOSITION IV. THEOREM.

IF the firſt of four magnitudes have the ſame ratio to the ſecond, which the third has to the fourth, then any equimultiples whatever of the firſt and third ſhall have the ſame ratio to any equimultiples of the ſecond and fourth; viz., the equimultiple of the firſt ſhall have the ſame ratio to that of the ſecond, which the equimultiple of the third has to that of the fourth.

Let ● : ■ :: ♦ : ■, then 3 ● : 2 ■ :: 3 ♦ : 2 ⬟, every equimultiple of 3 ● and 3 ♦ are equimultiples of ● and ♦, and every equimultiple of 2 ■ and 2 ⬟, are equimultiples of ■ and ⬟ [v. 3].

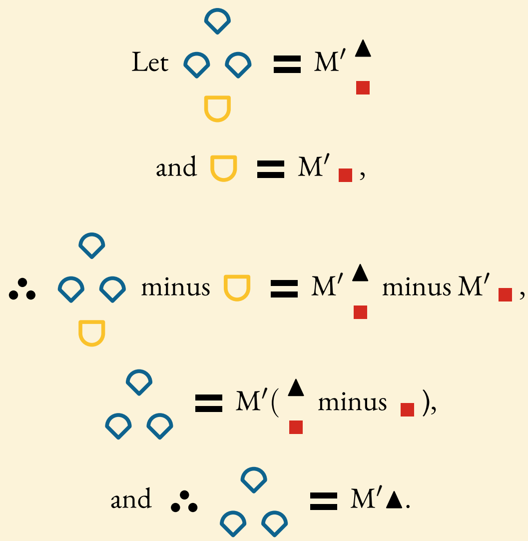
That is, M times 3 ● and M times 3 ♦ are equimultiples of ● and ♦, and *m* times 2 ■ and *m* 2 ⬟ are equimultiples of 2 ■ and 2 ⬟; but ● : ■ :: ♦ : ⬟ [hyp.]; ∴ if M 3 ● ⊏, = or ⊐ *m* 2 ■, then M 3 ♦ ⊏, = or ⊐ *m* 2 ⬟ [v. def. 5] and therefore 3 ● : 2 ■ :: 3 ♦ : 2 ⬟ [v. def. 5].

The ſame reaſoning holds good if any other equimultiple of the firſt and third be taken, any other equimultiple of the ſecond and fourth.

∴ If the firſt four magnitudes, &c.

PROPOSITION V. THEOREM.

IF *one magnitude be the ſame multiple of another, which a magnitude taken from the firſt is of a magnitude taken from the other the remainder ſhall be the ſame multiple of the remainder, that the whole is of the whole.*



∴ If one magnitude, &c.

PROPOSITION VI. THEOREM.

IF *two magnitudes be equimultiples of two others, and if equimultiples of theſe be taken from the firſt two, the remainders are either equal to theſe others, or equimultiples of them.*

Let ◇◇◇◇ = M′ ■ ; and ⊍⊍ = M′ ▲;

then ◇◇◇◇ minus m' ■ =

M′ ■ minus m' ■ = (M′ minus m') ■ ,
and ⊍⊍ minus m' ▲ = M′ ▲ minus m' ▲ = (M′ minus m') ▲.

Hence, (M′ minus m') ■ and (M′ minus m') ▲ are equimultiples of ■ and ▲,
and equal to ■
and ▲, when M′ minus m' = 1.

∴ If two magnitudes be equimultiples, &c.

I*F the firſt of the four magnitudes has the ſame ratio to the ſecond which the third has to the fourth, then if the firſt be greater than the ſecond, the third is alſo greater than the fourth; and if equal, equal; if leſs, leſs.*

Let ● : ■ :: ⬟ : ◆; therefore, by the fifth definition, if
●● ⊏ ■■, then will
⬟⬟ ⊏ ◆◆;
but if ● ⊏ ■, then ●● ⊏ ■■
and ⬟⬟ ⊏ ◆◆,
and ∴ ⬟ ⊏ ◆.

Similarly, if ● =, or ⊐ ■, then will ⬟ =, or ⊐ ◆.

∴ If the firſt of four, &c.

DEFINITION XIV.

Geometricians make uſe of the technical term "Invertendo," by inverſion, when there are four proportionals, and it is inferred, that the ſecond is to the firſt as the fourth to the third.

Let A : B :: C : D, then, by "invertendo" it is inferred B : A :: D : C.

PROPOSITION B. THEOREM.

IF *four magnitudes are proportionals, they are proportionals alſo when taken inverſely.*

Let : :: : ,
then inverſely, : :: : .

If M $\sqsupset$ *m* , then M $\sqsupset$ *m*
by the fifth definition.

Let M $\sqsupset$ *m* , that is, *m* $\sqsupset$ M ,
∴ M $\sqsupset$ *m* , or, *m* $\sqsubset$ M ;
∴ if *m* $\sqsubset$ M , then will *m* $\sqsubset$ M .

In the ſame manner it may be ſhown,

that if *m* $=$ or $\sqsupset$ M ,
then will *m* $=$, or $\sqsupset$ M ;
and therefore, by the fifth definition, we infer
that : :: : .

∴ If four magnitudes, &c.

PROPOSITION C. THEOREM.

IF *the firſt be the ſame multiple of the ſecond, or the ſame part of it, that the third is of the fourth; the firſt is to the ſecond, as the third is to the fourth.*

Let , be the firſt, the ſame multiple of , the ſecond,

that , the third, is of , the fourth.

Then : :: :

take M , *m* , M , *m* ;

becauſe is the ſame multiple of

that is of (according to the hypotheſis);

and M is taken the ſame multiple of

that M is of ,

∴ (according to the third propoſition),

M is the ſame multiple of

that M is of .

Therefore, if M be of a greater multiple than *m* is, then M is a greater multiple of than *m* is; that is, if M be greater than *m* , then M will be greater than *m* ; in the same manner it can be shewn, if M be equal *m* , then M will be equal *m* .

And, generally, if M ⊏ , = or ⊐ *m* then M will be ⊏ , = or ⊐ *m* ; ∴ by the fifth definition, : :: : .

Next, let be the same part of that is of .

In this case also : :: : .

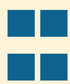
For, becauſe

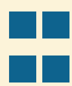
 is the ſame part of  that  is of ,

therefore  is the ſame multiple of 

that  is of .

Therefore, by the preceding caſe,

 :  ::  : ;

and ∴  :  ::  : ,

by propoſition B.

∴ If the firſt be the ſame multiple, &c.

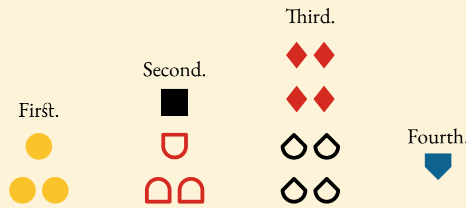
PROPOSITION D. THEOREM.

IF *the firſt be to the ſecond as the third to the fourth, and if the firſt be a multiple, or a part of the ſecond; the third is the ſame multiple, or the ſame part of the fourth.*

Let : :: : ;

and firſt, let be a multiple ;

ſhall be the ſame multiple of .



Take = .

Whatever multiple is of

take the ſame multiple of ,

then, becauſe : :: :

and of the ſecond and fourth, we have taken equimultiples,

and , therefore [v. 4],

: :: : , but [conſt.],

= ∴ [v. A] = and is the ſame multiple of

that is of .

Next, let : :: : ,

and alſo a part of ;

then ſhall be the ſame part of .

Inverſely [v. def. 14], : :: : ,

but is part of ;

that is, is a multiple of ;

∴ by the preceding caſe, is the ſame multiple of

that is, is the ſame part of

that is of .

∴ If the firſt be to the ſecond, &c.

PROPOSITION VII. THEOREM.

EQUAL *magnitudes have the ſame ratio to the ſame magnitude, and the ſame has the ſame ratio to equal magnitudes.*

Let ● = ♦ and ■ any other magnitude;
then ● : ■ = ♦ : ■ and ■ : ● = ■ : ♦.

Becauſe ● = ♦,
∴ M ● = M ♦;

∴ if M ● ⊏, = or ⊐ *m* ■, then
M ♦ ⊏, = or ⊐ *m* ■,
and ∴ ● : ■ = ♦ : ■ [v. def. 5].

From the foregoing reaſoning it is evident that,
if *m* ■ ⊏, = or ⊐ M ●, then
m ■ ⊏, = or ⊐ M ♦
∴ ■ : ● = ■ : ♦ [v. def. 5].

∴ Equal magnitudes, &c.

DEFINITION VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the firſt is greater than that of the ſecond, but the multiple of the third is not greater than the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is ſaid to have the fourth a leſs ratio than the firſt has to the ſecond.

If, among the equimultiples of four magnitudes, compared as the in the fifth definition, we ſhould find ●●●●● ⊏ ■■■■, but ♦♦♦♦♦ = or ⊐ ■■■■, or if we ſhould find any particular multiple M′ of the firſt and third, and a particular multiple *m*′ of the ſecond and fourth, ſuch, that M′ times the firſt is ⊏ *m*′ times the ſecond, the M′ times the third is not ⊏ *m*′ times the fourth, i.e. = or ⊐ *m*′ times the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third has to the fourth; or the third has to the fourth, under ſuch circumſtances, a leſs ratio than the firſt has to the second: although ſeveral other equimultiples may tend to ſhow that the four magnitudes are proportionals.
This definition will in future be expreſſed thus:—

If M′ ▼ ⊏ *m*′ ∪, but M′ ■ = or ⊐ *m*′ ♦,
then ▼ : ∪ ⊏ ■ : ♦.

In the above general expreſſion, M′ and *m*′ are to be conſidered particular multiples, not like the multiples M and *m* introduced in the fifth definition, which are in that definition conſidered to be every pair of multiples that can be taken. It muſt alſo be here obſerved, that ▼, ∪, ■, and the like ſymbols are to be conſidered merely the representatives of geometrical magnitudes.

In a partial arithmetical way, this may be ſet forth as follows:

Let us take four numbers, 8, 7, 10, and 9.

Firſt. 8	*Second.* 7	*Third.* 10	*Fourth.* 9
16	14	20	18
24	21	30	27
32	28	40	36
40	35	50	45
48	42	60	54
56	49	70	63
64	56	80	72
72	63	90	81
80	70	100	90
88	77	110	99
96	84	120	108
104	91	130	117
112	98	140	126
&c;	&c;	&c;	&c;

Among the above multiples we find 16 ⊏ 14 and 20 ⊏ 18; that is, twice the firſt is greater than twice the ſecond, and twice the third is greater than twice the fourth; and 16 ⊏ 21 and 20 ⊐ 27; that is, twice the firſt is leſs than three times the ſecond, and twice the third is leſs than three times the fourth; and among the ſame multiples we can find 72 ⊏ 56 and 90 ⊏ 72; that is 9 times the firſt is greater than 8 times the ſecond, and 9 times the third is greater than 8 times the fourth. Many other equimultiples might be selected, which would tend to ſhow that the numbers 8, 7, 10, 9, were proportionals, but they are not, for we can find a multiple of the firſt ⊏ a multiple of the ſecond, but the ſame multiple of the third that has been taken of the firſt not ⊏ than the ſame multiple of the fourth which has been taken of the ſecond; for inſtance, 9 times the firſt is ⊏ 10 times the ſecond, but 9 times the third is not ⊏ 10 times the fourth, that is, 72 ⊏ 70, but 90 not ⊏ 90, or 8 times the firſt we find ⊏ 9 times the ſecond, but 8 times the third is not greater than 9 times the fourth, that is 64 ⊏ 63, but 80 is not ⊏ 81. When any ſuch multiples as theſe can be found, the firſt (8) is ſaid to have the ſecond (7) a greater ratio than the third (10) has to the fourth (9), and on the contrary the third (10) is ſaid to have the fourth (9) a leſs ratio than the firſt (8) has to the ſecond (7).

PROPOSITION VIII. THEOREM.

IF *unequal magnitudes the greater has a greater ratio to the ſame than the leſs has: and the ſame magnitude has a greater ratio to the leſs than it has to the greater.*

Let ▲🟥 and 🟨 be two unequal magnitudes, and 🔵 and other.

We ſhall firſt prove that ▲🟥 which is the greater of the two unequal magnitudes, has a greater ratio to 🔵 than 🟨, the leſs, has to 🔵;

that is, ▲🟥 : 🔵 ⊏ 🟨 : 🔵;

take M′ ▲🟥, m' 🔵, M′ 🟨, and m' 🔵;

ſuch, that M′ ▲ and M′ 🟥 ſhall be each ⊏ 🔵;

alſo take m' 🔵 the leaſt multiple of 🔵,

which will make m' 🔵 ⊏ M′ 🟨 = M′ 🟥;

∴ M′ 🟨 is not ⊏ m' 🔵,

but M′ ▲🟥 is ⊏ m' 🔵, for,

as m' 🔵 is the firſt multiple which firſt becomes ⊏ M′ 🟥, than (m' minus 1) 🔵 or m' 🔵

minus 🔵 is not ⊏ M′ 🟥, and 🔵 is not ⊏ M′ ▲,

∴ m' minus 🔵 + 🔵 muſt be ⊐ M′ 🟥 + M′ ▲;

that is, m' 🔵 muſt be ⊐ M′ 🟥;

∴ M′ ▲🟥 is ⊏ m' 🔵; but it has been ſhown above that

M′ 🟨 is not ⊏ m' 🔵, therefore, by the ſeventh definition,

▲🟥 has to 🔵 a greater ratio than 🟨 : 🔵.

Next we ſhall prove that 🔵 has a greater ratio to 🟨, the leſs than it has to

▲🟥, the greater;

or, 🔵 : 🟨 ⊏ 🔵 : ▲🟥.

Take m' 🔵, M′ 🟨, m' 🔵, and M′ ▲🟥,

the ſame as in the firſt caſe, ſuch that

M′▲ and M′ 🟥 will be each ⊏ 🔵, and m' 🔵 the leaſt multiple of

🔵, which firſt becomes

greater than M′ 🟥 = M′ 🟨.

∴ m' 🔵 minus 🔵 is not ⊏ M′ 🟥,

and 🔵 is not ⊏ M′▲; conſequently

m' 🔵 minus 🔵 + 🔵 is ⊐ M′ 🟥 + M′ ▲;

∴ m' is ⊐ M′ ▲🟥, and ∴ by the ſeventh definition,

🔵 has to 🟨 a greater ratio than 🔵 has to ▲🟥.

∴ Of unequal magnitudes, &c.

The contrivance employed in this propoſition for finding among multiples taken, as in the fifth definition, a multiple of the firſt greater than the multiple of the ſecond, but the ſame multiple of the third which has been taken of the firſt, not greater than the ſame multiple of the fourth which has been taken of the ſecond, may be illuſtrated numerically as follows:—

The number 9 has a greater ratio to 7 than 8 has to 7: that is, 9 : 7 ⊏ 8 : 7 ⊏ 8 : 7; or, 8 + 1 : 7 ⊏ 8 : 7.

The multiple of 1, which firſt becomes greater than 7, is 8 times, therefore we may multiply the firſt and third by 8, 9, 10, or any other greater number; in this caſe, let us multiply the firſt and third by 8, and we have 64 + 8 and 64: again, the firſt multiple of 7 which becomes greater than 64 is 10 times; then, by multiplying the ſecond and fourth by 10, we ſhall have 70 and 70; then, arranging theſe multiples, we have—

8 times the firſt.	10 times the second.	8 times the third.	10 times the fourth.
64 + 8	70	64	70

Conſequently, 64 + 8, or 72, is greater than 70, but 64 is not greater than 70, ∴ by the ſeventh definition, 9 has a greater ratio to 7 than 8 has to 7.

The above is merely illuſtrative of the foregoing demonſtration, for this property could be ſhown of theſe or other numbers very readily in the following manner; becauſe if an antecedent contains it conſequent a greater number of times than another antecedent contains its conſequent, or when a fraction is formed of an antecedent for the numerator, and its conſequent for the denominator be greater than another fraction which is formed of another antecedent for the numerator and its conſequent for the denominator, the ratio of the firſt antecedent to its conſequent is greater than the ratio of the laſt antecedent to its conſequent.

Thus, the number 9 has a greater ratio to 7, than 8 has to 7, for $\frac{9}{7}$ is greater than $\frac{8}{7}$.

Again, 17 : 19 is a greater ratio than 13 : 15, becaufe $\frac{17}{19} = \frac{17 \times 15}{19 \times 15} = \frac{255}{285}$, and $\frac{13}{15} = \frac{13 \times 19}{15 \times 19} = \frac{247}{285}$, hence it is evident that $\frac{255}{285}$ is greater than $\frac{247}{285}$, $\therefore$ $\frac{17}{19}$ is greater than $\frac{13}{15}$, and, according to what has been above fhown, 17 has to 19 a greater ratio than 13 has to 15.

So that the general terms upon which a greater, equal, or lefs ratio exifts are as follows:—

If $\frac{A}{B}$ be greater than $\frac{C}{D}$, A is faid to have to B a greater ratio than C has to D; if $\frac{A}{B}$ equal to $\frac{C}{D}$, then A has to B the fame ratio which C has to D; and if $\frac{A}{B}$ be lefs than $\frac{C}{D}$, A is faid to have to B a lefs ratio than C has to D.

The ftudent fhould underftand all up to this propofition perfectly before proceeding further, in order to fully comprehend the following propositions in of this book. We therefore ftrongly recommend the learner to commence again, and read up to this flowly, and carefully reafon at each ftep, as he proceeds, particularly guarding againft the mifchievous fyftem of depending wholly on the memory. By following thefe inftructions, he will find that the parts which ufually prefent confiderable difficulties will prefent no difficulties whatever, in profecuting the ftudy of this important book.

PROPOSITION IX. THEOREM.

MAGNITUDES *which have the ſame ratio to the ſame magnitude are equal to one another; and thoſe to which the ſame magnitude has the ſame ratio are equal to one another.*

Let ◆ : ■ :: ● : ■, then ◆ = ●.

For, if not, let ◆ ⊏ ●, then will
◆ : ■ ⊏ ● : ■ [v. 8],
which is abſurd according to the hypotheſis.
∴ ◆ is not ⊏ ●.

In the ſame manner it may be ſhown, that
● is not ⊏ ◆,
∴ ◆ = ●.

Again, let ■ : ◆ :: ■ : ●, then will ◆ = ●.

For [invert.] ◆ : ■ :: ● : ■,
therefore, by the firſt caſe, ◆ = ●.

∴ Magnitudes which have the ſame ratio, &c.

Let A : B = A : C, then B = C, for as the fraction $\frac{A}{B}$ = the fraction $\frac{A}{C}$, and the numerator of one equal to the numerator of the other, therefore the denominator of theſe fractions are equal, that is B = C.

Again, if B : A = C, B = C. For, as $\frac{B}{A} = \frac{C}{A}$, B muſt = C.

PROPOSITION X. THEOREM.

THAT *magnitude which has a greater ratio than another has unto the ſame magnitude, is the greater of the two: and that magnitude to which the ſame has a greater ratio than it has unto another magnitude, is the leſs of the two.*

Let : ⊏ : , then ⊏ .

For if not, let = or ⊐ ;
then, : = : [v. 7] or
: : ⊐ : [v. 8] and [invert.],
which is abſurd according to the hypotheſis.

∴ is not = or ⊐ , and
∴ muſt be ⊏ .

Again, let : ⊏ : ,
then, ⊐ .

For if not, muſt be ⊏ or = ,
then : ⊐ : [v. 8] and [invert.];
or : = : [v. 7], which is abſurd [hyp.];
∴ is not ⊏ or = ,
and ∴ muſt be ⊐ .

∴ That magnitude which has, &c.

RATIOS *that are the ſame to the ſame ratio, are the ſame to each other.*

Let ◆ : ■ = 🔴 : ⬟ and 🔴 : ⬟ = ▲ : ●,
then will ◆ : ■ = ▲ : ●.

For if M ◆ ⊏, =, or ⊐ *m* ■,
then M 🔴 ⊏, =, or ⊐ *m* ⬟,
and if M 🔴 ⊏, =, or ⊐ *m* ⬟,

then M ▲ ⊏, = or ⊐ *m* ●, [v. def. 5];

∴ if M ◆ ⊏, = or ⊐ *m* ■, M ▲ ⊏, = or ⊐ *m* ●,
and ∴ [v. def. 5] ◆ : ■ = ▲ : ●.

∴ Ratios that are the ſame, &c.

PROPOSITION XII. THEOREM.

IF *any number of magnitudes be proportionals as one of the antecedents is to its conſequent, ſo ſhall all the antecedents taken together be to all the conſequents.*

Let ■ : ● = ⊍ : ⬠ = ◆ : ⬟ = ● : ▼ = ▲ : ●;
then will ■ : ● =
■ + ⊍ + ◆ + ● + ▲ : ● + ⬠ + ⬟ + ▼ + ●.

For if M ■ ⊏ *m* ●, then M ⊍ ⊏ *m* ⊍, and M ◆ ⊏ *m* ⬟ M
● ⊏ *m* ▼,
alſo M ▲ ⊏ *m* ●. [v. def. 5].

Therefore, if M ■ + M ⊍ + M ⬟ + M ● + M ▲,
of M (■ + ⊍ + ◆ + ● + ▲) be greater
than *m* ● + *m* ⊍ + *m* ⬟ + *m* ▼ + *m* ●,
or *m* (● + ⊍ + ⬟ + ▼ + ●).

In the ſame way it may be ſhown, if M times one of the antecedents be equal or leſs than m times one of the conſequents, M times all the antecedents taken together, will be equal to or leſs than m times all the conſequents taken together. Therefore, by the fifth definition, as one of the antecedents is to its conſequent, ſo are all the antecedents taken together to all the conſequents taken together.

∴ If any number of magnitudes, &c.

IF *the firſt has to the ſecond the ſame ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth to the ſixth.*

Let : = : , but : ⊏ : ,
then : ⊏ : .

For becauſe : ⊏ : , there are ſome multiples (M′ and *m*′)
of and , and
of and , ſuch that M′ ⊏ *m*′ ,
but M′ not ⊏ *m*′ , by the ſeventh definition.

Let theſe multiples be taken, and take the ſame multiples of and .

∴ [v. def. 5] if M′ ⊏ , = , or ⊐ *m*′ ;
then will M′ ⊏ , = , ⊐ *m*′ ,
but M′ ⊏ *m*′ (conſtruction);

∴ M′ ⊏ *m*′ ,
but M′ is not ⊏ *m*′ (conſtruction);
and therefore by the ſeventh definition,
: ⊏ : .

∴ If the firſt has to the ſecond, &c.

PROPOSITION XIV. THEOREM.

I*F the firſt has the ſame ratio to the ſecond which the third has to the fourth; then, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; and if leſs, leſs.*

Let ▼ : ∪ :: ■ : ◆, and firſt ſuppoſe
▼ ⊏ ■, then will ∪ ⊏ ◆.

For ▼ : ∪ ⊏ ■ : ∪ [v. 8], and by the
hypotheſis, ▼ : ∪ = ■ : ◆;
∴ ■ : ◆ ⊏ ■ : ∪ [v. 13],
∴ ◆ ⊐ ∪ [v. 10], or ∪ ⊏ ◆.

Secondly, let ▼ = ■, then will ∪ = ◆.

For ▼ : ∪ = ■ : ∪ [v. 7],
and ▼ : ∪ = ■ : ◆ [hyp.];
∴ ■ : ∪ = ■ : ◆ [v. 11],
and ∴ ∪ = ◆ [v. 9].

Thirdly, if ▼ ⊐ ■, then will ∪ ⊐ ◆;
becauſe ■ ⊏ ▼ and ■ : ◆ = ▼ : ∪;
∴ ◆ ⊏ ∪, by the firſt caſe,
that is, ∪ ⊐ ◆.

∴ If the firſt has the ſame ratio, &c.

PROPOSITION XV. THEOREM.

MAGNITUDES *have the ſame ratio to one another which their equimultiples have.*

Let ● and ■ be two magnitudes;
then ● : ■ :: M′ ● : M′ ■.

For ● : ■ = ● : ■
= ● : ■
= ● : ■
∴ ● : ■ :: 4 ● : 4 ■ [V. 12].

An the ſame reaſoning is generally applicable, we have

● : ■ :: M′ ● : M′ ■.

∴ Magnitudes have the ſame ratio, &c.

DEFINITION XIII.

THE technical term permutando or alternando, by permutation or alternately , is uſed when there are four proportionals, and it is inferred that the firſt has the ſame ratio to the third which the ſecond has to the fourth; or that the firſt is to the third as the ſecond is to the fourth: as it ſhown in the following proposition:—

Let ● : ◆ :: ▼ : ■,
by "permutando" or "alternando" it is
inferred ● : ▼ :: ◆ : ■.

It may be neceſſary here to remark that the magnitudes ●, ◆, ▼, ■, muſt be homogeneous, that is, of the ſame nature or ſimilitude of kind; we muſt therefore, in ſuch caſes, compare lines with lines, ſurfaces with ſurfaces, ſolids with ſolids, &c. Hence the ſtudent will readily perceive that a line and a ſurface, a ſurface and a ſolid, or other heterogeneous magnitudes, can never ſtand in the relation of antecedent and conſequent.

PROPOSITION XVI. THEOREM.

IF *four magnitudes of the ſame kind be proportionals, they are alſo proportionals when taken alternately.*

Let : :: : , then : :: : .

For M : M :: : [v. 15],

and M : M :: : [hyp.] and [v. 11];

alſo *m* : *m* :: : [v. 15];

∴ M : M :: *m* : *m* [v. 14],

and ∴ if M ⊏, =, or ⊐ *m* ,

then will M ⊏, =, or ⊐ *m* [v. 14];

therefore by the fifth definition,

: :: : .

∴ If four magnitudes of the ſame kind, &c.

DEFINITION XVI.

Dividendo, by diviſion, when there are four proportionals, and it is inferred, that the exceſs of the firſt above the ſecond is to the ſecond, as the exceſs of the third above the fourth, is to the fourth.

Let A : B :: C : D;
by "dividendo" it is inferred
A minus B : B :: C minus D : D.

According to the above, A is ſuppoſed to be greater than B, and C greater than D; if this be not the caſe, but to have B greater than A, and D greater than C, B and D can be made to ſtand as antecedents, and A and C as conſequents, by "invertion"

B : A :: D : C;
then, by "dividendo," we infer
B minus A : A :: D minus C : C.

PROPOSITION XVII. THEOREM.

IF *magnitudes, taken jointly, be proportionals, they ſhall alſo be proportionals when taken ſeparately: that is, if two magnitudes together have to one of them the ſame ratio which two others have to one of theſe, the remaining one of the firſt two ſhall have to the other the ſame ratio which the remaining one of the laſt two has to the other of theſe.*

Let ▼ + ∪ : ∪ :: ■ + ◆ : ◆,
then will ▼ : ∪ :: ■ : ◆.

Take M▼ ⊏ m∪ to each add M∪,
then we have M▼ + M∪ ⊏ m∪ + M∪,
or M(▼ + ∪) ⊏ (m + M)∪ :
but becauſe ▼ + ∪ : ∪ :: ■ + ◆ : ◆ [hyp.],
and M(▼ + ∪) ⊏ (m + M)∪;
∴ M(■ + ◆) ⊏ (m + M)◆ [v. def. 5];
∴ M■ + M◆ ⊏ m◆ + M◆;
∴ M■ ⊏ m◆, by taking M◆ from both ſides:
that is, when M▼ ⊏ m∪, then M■ ⊏ m◆.

In the ſame manner it may be proved, that if
M▼ = or ⊐ m∪, then will M■ = or ⊐ m◆;
and ∴ ▼ : ∪ :: ■ : ◆ [v. def. 5].

∴ If magnitudes taken jointly, &c.

DEFINITION XV.

The, term componendo, by compoſition, is uſed when there are four proportionals; and it is inferred that the firſt together with the ſecond is to the ſecond as the third together with the fourth is to the fourth.

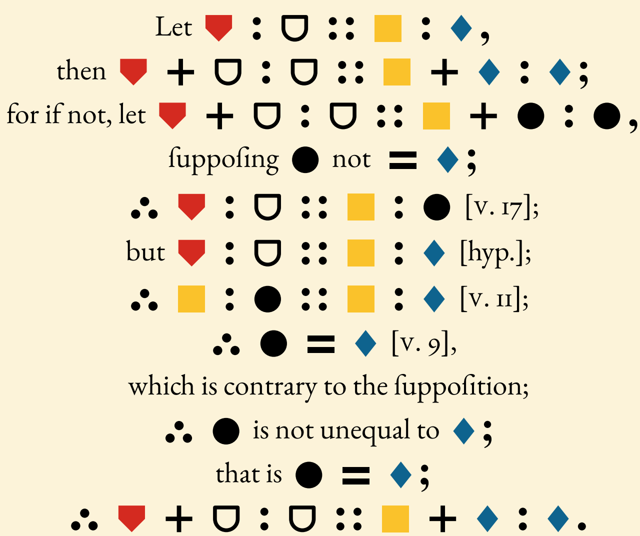
Let A : B :: C : D;
then, by the term "componendo," it is inferred that
A + B : B :: C + D : D.

By "invertion" B and D may become the firſt and third, A and C the ſecond and fourth as

B : A :: D : C;
then, by "componendo," we infer that
B + A : A :: D + C : C.

PROPOSITION XVIII. THEOREM.

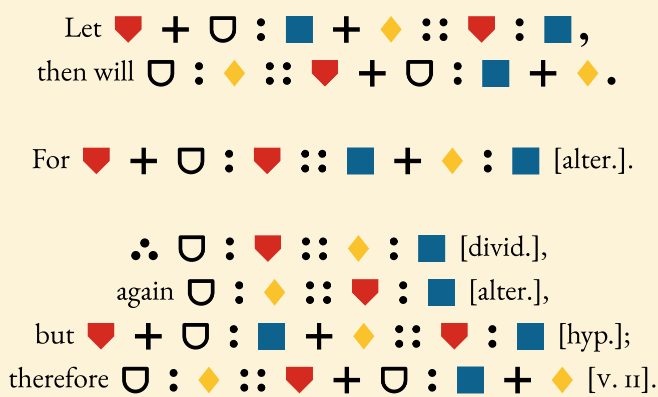
IF *magnitudes, taken ſeparately, be proportionals, they ſhall alſo be proportionals when taken jointly: that is, if the firſt be to the ſecond as the third is to the fourth, the firſt and ſecond together ſhall be to the ſecond as the third and fourth together is to the fourth.*



∴ If magnitudes, taken ſeparately, &c.

PROPOSITION XIX. THEOREM.

IF *a whole magnitude be to a whole, as a magnitude taken from the firſt, is to a magnitude taken from the other; the remainder ſhall be to the remainder, as the whole to the whole.*



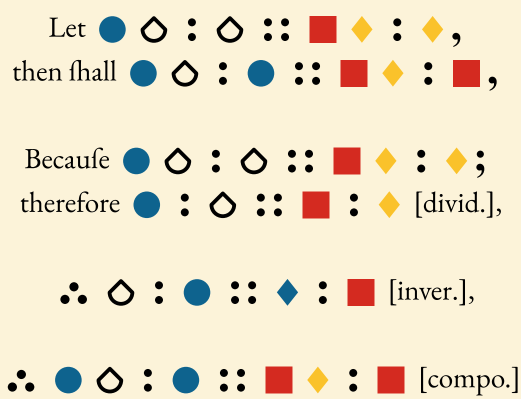
∴ If a whole magnitude be to a whole, &c.

DEFINITION XVII.

The term "convertendo," by conversion, is made use of by geometricians, when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third is to its excess above the fourth. See the following proposition:—

PROPOSITION E. THEOREM.

IF *four magnitudes be proportionals, they are alſo proportionals by converſion: that is, the firſt is to its exceſs above the ſecond, as the third is to its exceſs above the fourth.*



∴ If four magnitudes, &c.

DEFINITION XVIII.

"Ex æquali" (ſc. diſtantiâ), or ex æquo from equality of diſtance: when there is any number of magnitudes more than two, and as many others, ſuch that they are proportionals when taken two and two of each rank, and it is inferred that the firſt is to the laſt of the firſt rank of magnitudes, as the firſt is to the laſt of the others: "of this there are the two following kinds, which ariſe from the different order in which the magnitudes are taken, two and two."

DEFINITION XIX.

"Ex æquali," from equality. This term is uſed ſimply by itſelf, when the firſt magnitude is to the ſecond of the firſt rank, as the firſt to the ſecond of the other rank; and as the ſecond is to the third of the firſt rank, ſo is the ſecond to the third of the other; and ſo on in order: and in the inference is as mentioned in the preceding definition; whence this is called ordinate proposition. It is demonſtrated in Book 5, pr. 22.

Thus, if there be two ranks of magnitudes,

A, B, C, D, E, F, the firſt rank,
and L, M, N, O, P, Q, the ſecond,
ſuch that A : B :: L : M, B : C :: M : N,
C : D :: N : O, D : E :: O : P, E : F :: P : Q;
we infer by the term "ex æquali" that
A : F :: L : Q.

DEFINITION XX.

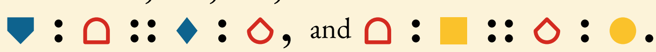
"Ex æquali in proportione perturbatâ ſeu inordinatâ," from equality in perturbate, or diſorderly proportion. This term is uſed when the firſt magnitude is to the ſecond of the firſt rank as the laſt but one is to the laſt of the ſecond rank; and as the ſecond is to the third of the firſt rank, ſo is the laſt but two to the laſt but one of the ſecond rank; and as the third is to the fourth of the firſt rank, ſo is the third from the laſt to the laſt but two of the ſecond rank; and ſo on in croſs order: and the inference is in the 18th definition. It is demonſtrated in B. v, pr. 23.

Thus, if there be two ranks of magnitudes,

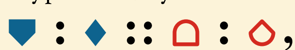
A, B, C, D, E, F, the firſt rank,
and L, M, N, O, P, Q, the ſecond,
ſuch that A : B :: P : Q, B : C :: O : P,
C : D :: N : O, D : E :: M : N, E : F :: L : M;
the term "ex æquali in proportione perturbatâ ſeu inordinatâ" infers that
A : F :: L : Q.

PROPOSITION XX. THEOREM.

IF there be three magnitudes, and other three, which, taken two and two, have the ſame ratio; then, if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

Let  be the firſt three magnitudes,
and  be the other three,
ſuch that 

Then, if  ⊏, =, or ⊐ , then will  ⊏, =, or ⊐ .

From the hypotheſis, by alternando, we have

and 

∴  [v. 11];

∴ if  ⊏, =, or ⊐ , then will  ⊏, =, or ⊐  [v. 14].

∴ If there be three magnitudes, &c.

PROPOSITION XXI. THEOREM.

IF *there be three magnitudes, and the other three which have the ſame ratio, taken two and two, but in a croſs order; then if the firſt magnitude be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.*

Let , , , be the firſt three magnitudes,

and , , , the other three,

ſuch that : :: : , and : :: : .

Then, if ⊏ , = , or ⊐ , then

will ⊏ , = , or ⊐ .

Firſt, let be ⊏ :

then, becauſe is any other magnitude,

: ⊏ : [v. 8];

but : :: : [hyp.];

∴ : ⊏ : [v. 13];

and becauſe : :: : [hyp.];

∴ : :: : [inv.],

and it was ſhown that : ⊏ : ,

∴ : ⊏ : [v. 13];

∴ ⊐ ,

that is ⊏ .

Secondly, let = ; then ſhall = .

For becauſe = ,

: = : [v. 7];

but : = : [hyp.],

and ■ : ⬟ = ♢ : ◆ [hyp. and inv.],
∴ ♢ : ● = ♢ : ◆ [v. 11],
∴ ◆ = ● [v. 9].

Next, let ⛉ be ⊐ ■, then ◆ ſhall be ⊐ ●;
for ■ ⊏ ●,
and it has been ſhown that ■ : ⬟ = ♢ : ◆,
and ⬟ : ⛉ = ● : ♢;
∴ by the firſt caſe ● is ⊏ ◆,
that is, ◆ ⊐ ●.

∴ If there be three, &c.

v. 220

PROPOSITION XXII. THEOREM.

IF there be any number of magnitudes, and as many others, which, taken two and two in order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt of the ſame.

N.B. — *This is uſually cited by the words "ex æquali," or "ex æquo."*

Firſt, let there be magnitudes ⬟, ◆, ■,
and as many others ♦, ◇, ●, ſuch that
⬟ : ◆ :: ♦ : ◇,
and ◆ : ■ :: ◇ : ●;
then ſhall ⬟ : ■ :: ♦ : ●.

Let theſe magnitudes, as well as any equimultiples whatever of the antecedents and conſequents of the ratios, ſtand as follows:—

⬟, ◆, ■, ♦, ◇, ●, and
M ⬟, *m* ◆, N ■, M ♦, *m* ◇, N ●,
becauſe ⬟ : ◆ :: ♦ : ◇;
∴ M ⬟ : *m* ◆ :: M ♦ : *m* ◇ [v. 4].

For the ſame reaſon
m ◆ : N ■ :: *m* ◇ : N ●;
and becauſe there are three magnitudes,
M ⬟, *m* ◆, N ■,
and other three M ♦, *m* ◇, N ●,
which, taken two and two, have the ſame ratio;

∴ if M ⊏ , = , ⊐ N

then will M ⊏ , = , ⊐ N , by [v. 20];

and ∴ : :: : [v. 5].

Next, let there be four magnitudes, , , , ,

and other four , , , ,

which, taken two and two, have the ſame ratio,

that is to ſay, : :: : ,

: :: : ,

and : :: : ,

then ſhall : :: : ;

for, becauſe , , , are three magnitudes,

and , , other three,

which, taken two and two, have the ſame ratio;

therefore, by the foregoing caſe, : :: : ,

but : :: : ;

therefore again, by the firſt caſe, : :: : ;

and ſo on, whatever the number of magnitudes be.

∴ If there be any number, &c.

PROPOSITION XXIII. THEOREM.

IF *there be any number of magnitudes, and as many others, which, taken two and two in a croſs order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt of the ſame.*

N.B. — *This is uſually cited by the words "ex æquali in proportione perturbatâ;" or "ex æquo perturbato."*

Firſt, let there be three magnitudes , , ,
and other three, , , ,
which, taken two and two in a croſs order, have the ſame ratio;
that is, : :: : ,
and : :: : ,
then ſhall : :: : .

Let theſe magnitudes and their reſpective equimultiples be arranged as follows:—

, , , , , ,
M , M , *m* , M , *m* , *m* ,
then : :: M : M [v. 15];
and for the ſame reaſon
: :: *m* : *m* ;
but : :: : [hyp.],
∴ M : M :: : [v. 11];
and becauſe : :: : [hyp.],
∴ M : *m* :: M : *m* [v. 4];
then becauſe there are three magnitudes,

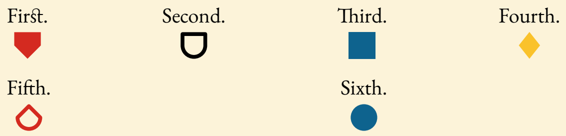
M , M , *m* ,
and other three, M , *m* , *m* ,
which, taken two and two in a croſs order, have the ſame ratio;
therefore, if M ⊏ , = , or ⊐ *m* ,
then will M ⊏ , = , or ⊐ *m* [v. 21],
and ∴ : :: : [v. def. 5].

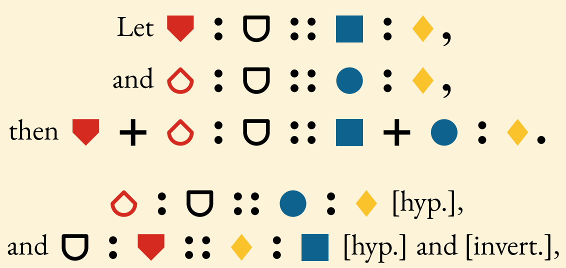
Next, let there be four magnitudes,
, , , ,
and other four, , , , ,
which, when taken two and two in a croſs order, have the ſame ratio; namely,
: :: : ,
: :: : ,
and : :: : .
then ſhall : :: : .
For, becauſe , , are three magnitudes,
and , , , other three,
which,taken two and two in a croſs order, have the ſame ratio,
therefore, by the firſt caſe, : :: : ,
but : :: : ,
therefore again, by the firſt caſe, : :: : ;
and ſo on, whatever be the number of ſuch magnitudes.

∴ If there be any number, &c.

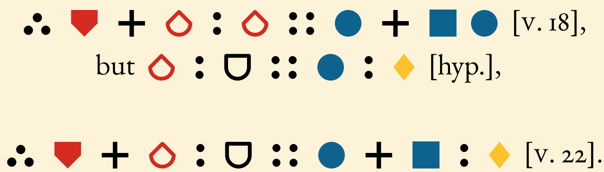
PROPOSITION XXIV. THEOREM.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth, and the fifth to the ſecond the ſame which the ſixth has to the fourth, the firſt and fifth together ſhall have to the ſecond the ſame ratio which the third and ſixth together have to the fourth.





and, becauſe theſe magnitudes are proportionals, they are proportionals when taken jointly,



∴ If the firſt, &c.

PROPOSITION XXV. THEOREM.

IF *four magnitudes of the ſame kind are proportionals, the greateſt and leaſt of them together are greater than the other two together.*

Let four magnitudes ⬟ + ∪, ■ + ◆, ∪, and ◆,
of the ſame kind, be proportionals, that is to ſay,

⬟ + ∪ : ■ + ◆ ∷ ∪ ◆,
and let ⬟ + ∪ be the greateſt of the four, and
conſequently by pr. A and 14 of Book 5, ◆ is the leaſt;
then will ⬟ + ∪ + ◆ be ⊏ ■ + ◆ + ∪;
becauſe ⬟ + ∪ : ■ + ◆ ∷ ∪ : ◆,

∴ ⬟ : ■ ∷ ⬟ + ∪ : ■ + ◆ [v. 19],
but ⬟ + ∪ ⊏ ■ + ◆ [hyp.],

∴ ⬟ ⊏ ■ [v. A];
to each of theſe add ∪ + ◆,
∴ ⬟ + ∪ + ◆ ⊏ ■ + ∪ + ◆.

∴ If four magnitudes, &c.

DEFINITION X.

When three magnitudes are proportionals, the firſt is ſaid to have to the third the duplicate ratio of that which it has to the ſecond.

For example, if A, B, C, be continued proportionals, that is A : B :: B : C, A is ſaid to have to C the duplicate ratio of A : B;

$$\text{or } \frac{\text{A}}{\text{C}} = \text{the ſquare of } \frac{\text{A}}{\text{B}}.$$

This property will be more readily ſeen of the quantities

$$ar^2,\ ar,\ a,\ \text{for } ar^2 : ar :: ar : a;$$

$$\text{and } \frac{ar^2}{a} = r^2 = \text{the ſquare of } \frac{ar^2}{ar} = \text{r},$$

$$\text{or of } a,\ ar,\ ar^2;$$

$$\text{for } \frac{a}{ar^2} = \frac{1}{r^2} = \text{the ſquare of } \frac{a}{ar} = \frac{1}{r}.$$

DEFINITION XI.

When four magnitudes are continual proportionals, the firſt is ſaid to have to the fourth the triplicate ratio of that which it has to the ſecond; and ſo on, quadruplicate, &c. increaſing the denomination ſtill by unity, in any number of proportionals.

For example, let A, B, C, D, be four continued proportionals, that is, A : B :: B: C :: C : D; A ſaid to have to D, the triplicate ratio of A to B;

$$\text{or } \frac{A}{D} = \text{ the cube of } \frac{A}{B}.$$

This definition will be better underſtood and applied to a greater number of magnitudes than four that are continued proportionals, as follows:—

Let ar^3, ar^2, ar, a, be four magnitudes in continued proportion,

that is, $ar^3 : ar^2 :: ar^2 : ar :: ar : a$,

$$\text{then } \frac{ar^3}{a} = r^3 = \text{ the cube of } \frac{ar^3}{ar^3} = r.$$

Or, let ar^5, ar^4, ar^3, ar^2, ar, a, be ſix magnitudes in proportion, that is

$$ar^5 : ar^4 :: ar^4 : ar^3 :: ar^3 : ar^2 :: ar^2 : ar :: ar : a,$$

$$\text{then the ratio } \frac{ar^5}{a} = r^5 \text{ the fifth power of } \frac{ar^5}{ar^4} = r.$$

Or, let a, ar, ar^2, ar^3, ar^4, be five magnitudes in continued proportion; then $\frac{a}{ar^4} = \frac{1}{r^4} = $ the fourth power of $\frac{a}{ar} = \frac{1}{r}$.

DEFINITION A.

To know a compound ratio:—

When there are any number of magnitudes of the ſame kind, the firſt is ſaid to have to the laſt of them the ratio compounded of the ratio which the firſt has to the ſecond, and of the ratio which the ſecond has to the third, and of the ratio which the third has to the fourth; and ſo on, unto the laſt magnitude.

For example, if A, B, C, D, be four magnitudes of the ſame kind, the firſt A is ſaid to have to the laſt D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is ſaid to be compounded of the ratios of A to B, B to C, and C to D.

A B C D
E F G H K L
M N

And if A has to B the ſame ratio which E has to F, and B to C the ſame ratio that G has to H, and C to D the ſame that K has to L; then by this definition, A is ſaid to have to D the ratio compounded of ratios which are the ſame with the ratios of E to F, G to H, and K to L. And the ſame thing is to be underſtood when it is more briefly expreſſed by ſaying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the ſame things being ſuppoſed; if M has to N the ſame ratio which A has to D, then for ſhortneſs ſake, M is ſaid to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

This definition may be better underſtood from an arithmetical or algebraical illuſtration; for, in fact, a ratio compounded of ſeveral other ratios, is nothing more than a ratio which has four its antecedent the continued product of all the antecedents of the ratios compounded, and for its conſequent the continued product of all the conſequents of the ratios compounded.

Thus, the ratio compounded of the ratios of

$2 : 3,\ 4 : 7,\ 6 : 11,\ 2 : 5,$

is the ratio of $2 \times 4 \times 6 \times 2 : 3 \times 7 \times 11 \times 5$,

or the ratio of $96 : 1155$, or $32 : 385$.

And of the magnitudes A, B, C, D, E, F, of the ſame kind, A : F is the ratio compounded of the ratios of

$$A : B,\ B : C,\ C : D,\ D : E,\ E : F;$$

for $A \times B \times C \times D \times E : B \times C \times D \times E \times F$,

or $\dfrac{A \times B \times C \times D \times E}{B \times C \times D \times E \times F} = \dfrac{A}{F}$ or the ratio of A : F.

PROPOSITION F. THEOREM.

RATIOS *which are compounded of the ſame ratios are the ſame to one another.*

Let A : B :: F : G,

B : C :: G : H,

C : D :: H : K,

and D : E :: K : L.

A B C D E
F G H K L

Then, the ratio which is compounded of the ratios of A : B, B : C, C : D, D : E, or the ratio of A : E, is the ſame as the ratio compounded of the ratios of F : G, G : H, H : K, K : L, or the ratio of F : L.

$$\text{For } \frac{A}{B} = \frac{F}{G},$$

$$\frac{B}{C} = \frac{G}{H},$$

$$\frac{C}{D} = \frac{H}{K},$$

$$\frac{D}{E} = \frac{K}{L};$$

$$\therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{F \times G \times H \times K}{G \times H \times K \times L}$$

$$\text{and } \therefore \frac{A}{E} = \frac{F}{L},$$

or the ratio of A : E is the ſame as the ratio of F : L.

The ſame may be demonſtrated of any number of ratios ſo circumſtanced.

Next, let $A : B :: K : L$,

$B : C :: H : K$,

$C : D :: G : H$,

$D : E :: F : G$.

Then the ratio which is compounded of the ratios of $A : B$, $B : C$, $C : D$, $D : E$, or the ratio of $A : E$, is the ſame as the ratio compounded of the ratios of $K : L$, $H : K$, $G : H$, $F : G$, or the ratio of $F : L$.

For $\frac{A}{B} = \frac{K}{L}$,

$\frac{B}{C} = \frac{H}{K}$,

$\frac{C}{D} = \frac{G}{H}$,

$\frac{D}{E} = \frac{F}{G}$;

$$\therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{K \times H \times G \times F}{L \times K \times H \times G}$$

and $\therefore \frac{A}{E} = \frac{F}{L}$,

or the ratio of $A : E$ is the ſame as the ratio of $F : L$.

$\therefore$ Ratios which are compounded, &c.

PROPOSITION G. THEOREM.

IF ſeveral ratios be the ſame to ſeveral ratios, each to each, the ratio which is compounded of ratios which are the ſame to the firſt ratios, each to each, ſhall be the ſame to the ratio compounded of ratios which are the ſame to the other ratios, each to each.

A B C D E F G H	P Q R S T
a b c d e f g h	V W X Y Z

If A : B :: a : b	and A : B :: P : Q	a : b :: V : W
C : D :: c : d	C : D :: Q : R	c : d :: W : X
E : F :: e : f	E : F :: R : S	e : f :: X : Y
and G : H :: g : h	and G : H :: S : T	g : h :: Y : Z

then P : T = V : Z.

$$\text{For } \frac{P}{Q} = \frac{A}{B} = \frac{a}{b} = \frac{V}{W},$$

$$\frac{Q}{R} = \frac{C}{D} = \frac{c}{d} = \frac{W}{X},$$

$$\frac{R}{S} = \frac{E}{F} = \frac{e}{f} = \frac{X}{Y},$$

$$\frac{S}{T} = \frac{F}{H} = \frac{g}{h} = \frac{Y}{Z};$$

$$\text{and } \therefore \frac{P \times Q \times R \times S}{Q \times R \times S \times T} = \frac{V \times W \times X \times Y}{W \times X \times Y \times Z},$$

and $\therefore \frac{P}{T} = \frac{V}{Z}$, or the ratio of P : T = V : Z.

$\therefore$ If ſeveral ratios, &c.

PROPOSITION H. THEOREM.

If a ratio which is compounded of ſeveral ratios be the ſame to a ratio which is compounded of ſeveral other ratios; and if one of the firſt ratios, or the ratio which is compounded of ſeveral of them, be the ſame to one of the laſt ratios, or to the ratio which is compounded of ſeveral of them; then the remaining ratio of the firſt, or, if there be more than one, the ratio compounded of the remaining ratios, ſhall be the ſame to the remaining ratio of the laſt, or if there be more than one, to the ratio compounded of theſe remaining ratios.

A B C D E F G H
P Q R S T X

Let A : B, B : C, C : D, D : E, E : F, F : G, G : H, be the firſt ratios, and P : Q, Q : R, R : S, S : T, T : X, the other ratios; alſo, let A : H, which is compounded of the firſt ratios, be the ſame as the ratio of P : X, which is the ratio compounded of the other ratios; and let the ratio of A : E, which is compounded of the ratios of A : B, B : C, C : D, D : E, be the ſame as the ratio of P : R, which is compounded of the ratios P : Q, Q : R.

Then the ratio which is compounded of the remaining firſt ratios, that is, the ratio compounded of the ratios E : F, F : G, G : H, that is the ratio of E : H, ſhall be the ſame as the ratio of R : X, which is compounded of the ratios of R : S, S : T, T : X, the remaining other ratios.

$$\text{Becauſe } \frac{A \times B \times C \times D \times E \times F \times G}{B \times C \times D \times E \times F \times G \times H} = \frac{P \times Q \times R \times S \times T}{Q \times R \times S \times T \times X},$$

$$\text{or } \frac{A \times B \times C \times D}{B \times C \times D \times E} \times \frac{E \times F \times G}{F \times G \times H} = \frac{P \times Q}{Q \times R} \times \frac{R \times S \times T}{S \times T \times X},$$

$$\text{and } \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{P \times Q}{Q \times R},$$

$$\therefore \frac{E \times F \times G}{F \times G \times H} = \frac{R \times S \times T}{S \times T \times X},$$

$$\therefore \frac{E}{H} = \frac{R}{X},$$

$$\therefore E : H = R : X.$$

$\therefore$ If a ratio which, &c.

PROPOSITION K. THEOREM.

IF *there be any number of ratios, and any number of other ratios, ſuch that the ratio which is compounded of ratios, which are the ſame to the firſt ratios, each to each, is the ſame to the ratio which is compounded of ratios, which are the ſame, each to each, to the laſt ratios—and if one of the firſt ratios, or the ratio which is compounded of ratios, which are the ſame to ſeveral of the firſt ratios, each to each, be the ſame to one of the laſt ratios, or to the ratio which is compounded of ratios, which are the ſame, each to each, to ſeveral of the laſt ratios—then the remaining ratio of the firſt; or, if there be more than one, the ratio which is compounded of ratios, which are the ſame, each to each, to the remaining ratios of the firſt, ſhall be the ſame to the remaining ratio of the laſt; or, if there be more than one, to the ratio which is compounded of ratios, which are the ſame, each to each, to theſe remaining ratios.*

h k m n s	
A B, C D, E F, G H, K L, M N,	*a b c d e f g*
O P, Q R, S T, V W, X Y,	*h k l m n p*
a b c d e f g	

Let A : B, C : D, E : F, G : H, K : L, M : N, be the firſt ratios, and O : P, Q : R, S : T, V : W, X : Y, the other ratios;

and let A : B $=$ $a : b$,

C : D $=$ $b : c$,

E : F $=$ $c : d$,

G : H $=$ $d : e$,

K : L $=$ $e : f$,

M : N $=$ $f : g$.

Then, by the definition of a compound ratio, the ratio of $a : g$ is compounded of the ratios of $a : b$, $b : c$, $c : d$, $d : e$, $e : f$, $f : g$, which are the ſame as the ratio of A : B, C : D, E : F, G : H, K : L, M : N, each to each.

$$\text{Alſo, } \mathrm{O} : \mathrm{P} = h : k,$$
$$\mathrm{Q} : \mathrm{R} = k : l,$$
$$\mathrm{S} : \mathrm{T} = l : m,$$
$$\mathrm{V} : \mathrm{W} = m : n,$$
$$\mathrm{X} : \mathrm{Y} = n : p.$$

Then will the ratio of $h : p$ be the ratio compounded of the ratios $h : k$, $k : l$, $l : m$, $m : n$, $n : p$, which are the ſame ratios of O : P, Q : R, S : T, V : W, X : Y, each to each.

$$\therefore \text{ by the hypotheſis, } a : g = h : p.$$

Alſo, let the ratio which is compounded of the ratios of A : B, C : D, two of the firſt ratios (or the ratios of $a : c$, for A : B $=$ $a : b$, and C : D $=$ $b : c$), be the ſame as the ratio of a : d, which is compounded of the ratios a : b, b : c, c : d, which are the ſame as the ratios of O : P, Q : R, S : T, three of the other ratios.

And let the ratios of h : s, which is compounded of the ratios h : k, k : m, m : n, n : s, which are the ſame as the remaining firſt ratios, namely, E: F, G: H, K: L, M: N; alſo, let the ratio of e : g, be that which is compounded of the ratios e : f, f : g, which are the ſame, each to each, to the remaining other ratios, namely, V: W, X: Y. Then the ratio of h : s ſhall be the ſame as the ratio of e : g; or h : s $=$ e : g.

$$\text{For } \frac{\mathrm{A} \times \mathrm{C} \times \mathrm{E} \times \mathrm{G} \times \mathrm{K} \times \mathrm{M}}{\mathrm{B} \times \mathrm{D} \times \mathrm{F} \times \mathrm{H} \times \mathrm{L} \times \mathrm{N}} = \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g},$$
$$\text{and } \frac{\mathrm{O} \times \mathrm{Q} \times \mathrm{S} \times \mathrm{V} \times \mathrm{X}}{\mathrm{P} \times \mathrm{R} \times \mathrm{T} \times \mathrm{W} \times \mathrm{Y}} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p},$$

by the compoſition of the ratios;

$$\therefore \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p}, \text{ [hyp.]},$$

$$\text{or } \frac{a \times b}{b \times c} \times \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{h \times k \times l}{k \times l \times m} \times \frac{m \times n}{n \times p},$$

$$\text{but } \frac{a \times b}{b \times c} = \frac{\mathrm{A} \times \mathrm{C}}{\mathrm{B} \times \mathrm{D}} = \frac{\mathrm{O} \times \mathrm{Q} \times \mathrm{S}}{\mathrm{P} \times \mathrm{R} \times \mathrm{T}} = \frac{\mathrm{a} \times \mathrm{b} \times \mathrm{c}}{\mathrm{b} \times \mathrm{c} \times \mathrm{d}} = \frac{h \times k \times l}{k \times l \times m};$$

$$\therefore \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{m \times n}{n \times p} \text{ [hyp.]},$$

$$\text{And } \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{\mathrm{h} \times \mathrm{k} \times \mathrm{m} \times \mathrm{n}}{\mathrm{k} \times \mathrm{m} \times \mathrm{n} \times \mathrm{s}} \text{ [hyp.]},$$

$$\text{and } \frac{m \times n}{n \times p} = \frac{\mathrm{e} \times \mathrm{f}}{\mathrm{f} \times \mathrm{g}} \text{ [hyp.]},$$

$$\therefore \frac{\mathrm{h} \times \mathrm{k} \times \mathrm{m} \times \mathrm{n}}{\mathrm{k} \times \mathrm{m} \times \mathrm{n} \times \mathrm{s}} = \frac{\mathrm{ef}}{\mathrm{fg}},$$

$$\therefore \frac{\mathrm{h}}{\mathrm{s}} = \frac{\mathrm{e}}{\mathrm{g}},$$

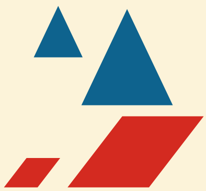
$$\therefore \mathrm{h} : \mathrm{s} = \mathrm{e} : \mathrm{g}.$$

∴ If there be any number, &c.

BOOK VI.

DEFINITIONS.

I.



RECTILINEAR figures are ſaid to be ſimilar, when they have their ſeveral angles equal, each to each, and the ſides about the equal angles proportional.

II.

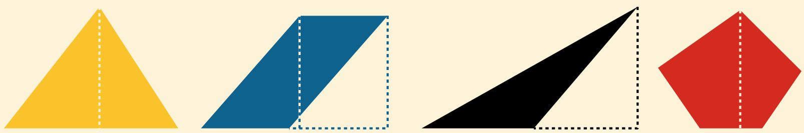
Two ſides of one figure are ſaid to be reciprocally proportional to two ſides of another figure when one of the ſides of the firſt is to the ſecond, as the remaining ſide of the ſecond is to the remaining ſide of the firſt.

III.

A straight line is ſaid to be cut in extreme and mean ratio, when the whole is to the greater ſegment, as the greater ſegment is to the leſs.

IV.

THE altitude of any figure is the ſtraight line drawn from its vertex perpendicular to its baſe, or the baſe produced.



PROPOSITIONS.

PROPOSITION I. THEOREM.

TRIANGLES *and parallelograms having the ſame altitude are to one another as their baſes.*

Let the triangles  and  have a common vertex, and their baſes, and in the ſame ſtraight line.

Produce both ways, take ſucceſſively on produced lines equal to it; and on produced lines ſucceſſively equal to it; and draw lines from the common vertex to their extremities.

The triangles  thus formed are all equal to one another, ſince their baſes are equal [I. 38].

∴  and its baſe are reſpectively equimultiples of  and the baſe .

VI. 241

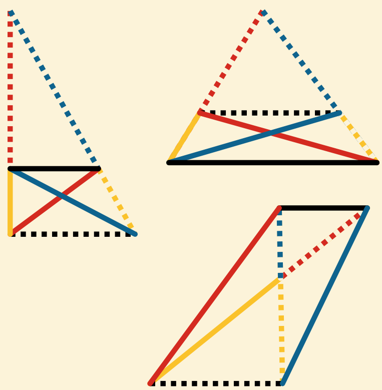
In like manner  and its baſe are reſpectively equimultiples of  and the baſe .

∴ if *m* or 6 times  ⊏ = or ⊐ *n* or 5 times  then *m* of 6 times ⊏ = ⊐ *n* or 5 times , *m* and *n* ſtand for every multiple taken as in the fifth definition of the Fifth Book. Although we have only ſhown that this property exiſts when *m* equal 6, and *n* equal 5, yet it is evident that the property holds good for every multiple value that may be given to *m*, and to *n*.

∴  :  :: : [v. def. 5]

Parallelograms having the ſame altitude are the doubles of triangles, on their baſes, and are proportional to them (Part 1), and hence their doubles, the parallelograms, are as their baſes [v. 15].

Q.E.D.



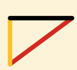
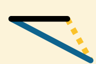
IF a ſtraight line ——— be drawn parallel to any ſide ······· of a triangle, it ſhall cut the other ſides, or thoſe ſides produced, into proportional ſegments. And if any ſtraight line ——— divide the ſides of a triangle or thoſe ſides produced, into proportional ſegments, it is parallel to the remaining ſide ······· .

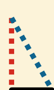
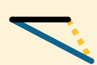
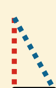
PART I.

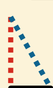
Let ——— ∥ ······· , then ſhall

——— : ······· :: ······· : ······· .

Draw ——— and ——— ,

and  =  [I. 37];

∴  :  ::  :  [v. 7];

but  :  :: ——— : ······· [vi. 1],

∴ ——— : ······· :: ······· : ······· . [v. 11].

PART II.

Let : :: : ,

then ‖ .

Let the ſame conſtruction remain,

becauſe : :: : [VI. I]

and : :: :

but : :: : [hyp.],

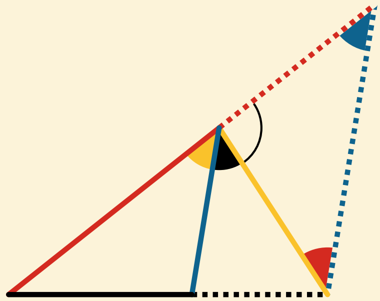
∴ : :: : [V. II]

∴ = [V. 9];

but they are on the ſame baſe , and at the ſame ſide of it, and

∴ ‖ [I. 39].

Q.E.D.



A right line ( ) biſecting the angle of a triangle, divides the oppoſite ſide into ſegments (,) proportional to the conterminous ſides (,). And if a ſtraight line ( ) drawn from any angle of a triangle divide the oppoſite ſide ( ) into ſegments (,) proportional to the conterminous ſides (,), it biſects the angle.

PART I.

Draw ∥ , to meet ;

then, = [I. 29],

∴ = ; but

= , ∴ = ,

∴ = [I. 6];

and becauſe ∥ ,

: :: : [VI. 2]

but = ;

∴ : :: : [V. 7].

PART II.

Let the same construction remain,

and : :: : [VI. 2];

but : :: : [hyp.]

∴ : :: : [V. 11],

and ∴ = [V. 9],

and ∴ = [V. 5]; but since

∥ ; = ,

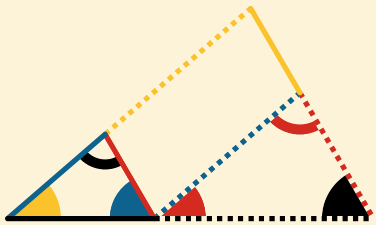
and = [I. 29];

∴ = , and = ,

and ∴ bisects .

Q.E.D.

PROPOSITION IV. THEOREM.



IN *equiangular triangles* (*and*) *the ſides about the equal angles are proportional, and the ſides which are oppoſite to the equal angles are homologous.*

Let the equiangular triangles be ſo placed that two ſides , oppoſite to equal angles and may be conterminous and in the ſame ſtraight line; and that the triangles lying at the ſame ſide of that ſtraight line, may have the equal angles not conterminous,

i.e. oppoſite to , and to .

Draw and . Then, becauſe

= , ∥ [I. 28];

and for a like reaſon, ∥ ,

∴ is a parallelogram.

But : :: : [VI. 2];

and ſince = [I. 34],

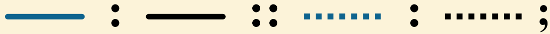
: :: : ; and by

alternation, : :: : [V. 16].

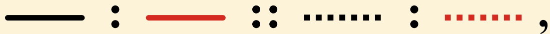
In like manner it may be ſhown, that

: :: : ;

and by alternation, that



but it has already been proved that

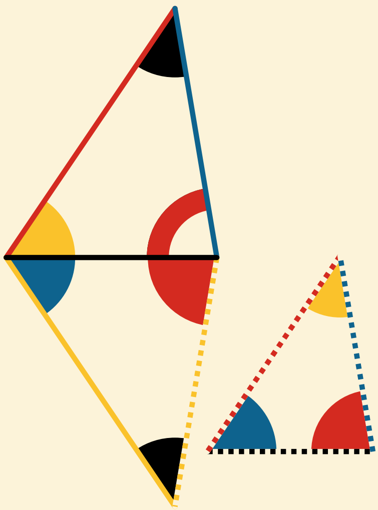


and therefore, ex æquali,

 [v. 22],

therefore the ſides about the equal angles are proportional, and thoſe which are oppoſite to the equal angles are homologous.

Q.E.D.



IF *two triangles have their ſides proportional*

(: :: :) *and* (: :: :) *they are equiangular, and the equal angles are ſubtended by the homologous ſides.*

From the extremities of , draw and ,

making = , = [I. 23];

and conſequently = [I. 32],

and ſince the triangles are equiangular,

: :: : [VI. 4];

but : :: : [hyp.];

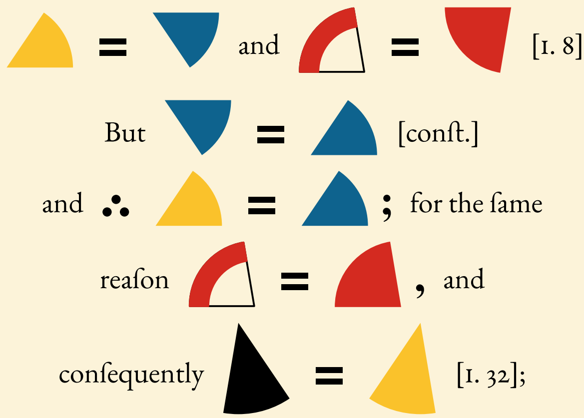
∴ : :: : ,

and conſequently = [V. 9].

In the like manner it may be ſhown that

= .

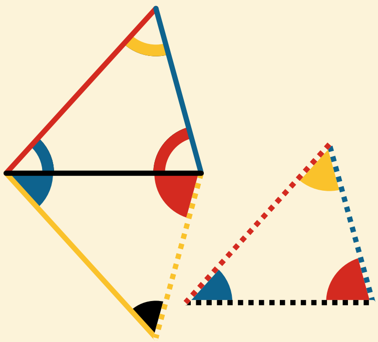
Therefore, the two triangles having a common bafe ——— , and their fides equal, have alfo equal angles oppofite to equal fides, i.e.



and therefore the triangles are equiangular, and it is evident that the homologous fides fubtend the equal angles.

Q.E.D.

PROPOSITION VI. THEOREM.



If two triangles (and) have one angle ( ) of the one, equal to one angle ( ) of the other, and the ſides about the equal angles proportional, the triangles ſhall be equiangular, and have thoſe angles equal which the homologous ſides ſubtend.

From the extremities of , one of the ſides of , about , draw and , making = , and = ; then = [I. 32], and the two triangles being equiangular,

 : :: : [VI. 4];

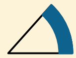
but : :: : [hyp.];

∴ : :: : [V. 11],

and conſequently = [V. 9];

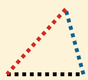
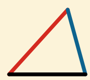
∴ = in every reſpect [I. 4].

But  =  [conſt.],

and ∴  = ; and

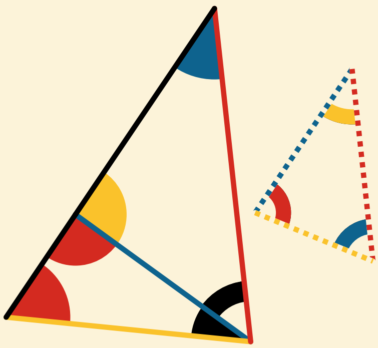
ſince also  = ,

 =  [I. 32];

and ∴  and  are equiangular, with their equal angles oppoſite to homologous ſides.

Q.E.D.

PROPOSITION VII. THEOREM.



IF *two triangles* (*and*) *have one angle in each equal* (*equal to*), *the sides about two other angles proportional* (: :: :) *and each of the remaining angles* (*and*) *either less or not less than a right angle, the triangles are equiangular, and those angles are equal about which the sides are proportional.*

First let it be assumed that the angles and are each less than a right angle: then if it be supposed that and contained by the proportional sides are not equal, let be the greater, and make = .

Because = [hyp.], and = [const.]

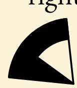
∴  =  [I. 32];

∴ : :: : [VI. 4],
but : :: : [hyp.]

∴ : :: : ;
 = [V. 9],
and ∴  =  [I. 5].

But  is leſs than a right angle [hyp.]

∴  is leſs than a right angle; and ∴  muſt be greater than a right angle [I. 13], but it has been proved =  and therefore leſs than a right angle, which is abſurd.

∴  and  are not unequal;

∴ they are equal, and ſince  =  [hyp.]

∴  =  [I. 32], and therefore the triangles are equiangular.

But if  and  be aſſumed to be each not leſs than a right angle, it may be proved as before that the triangles are equiangular, and have the ſides about the equal angles proportional [VI. 4].

Q.E.D.

IN a right angled triangle ( *), if a perpendicular (——) be drawn from the right angle to the oppoſite ſide, the triangles (*  ,  *) on each ſide of it are ſimilar to the whole triangle and to each other.*

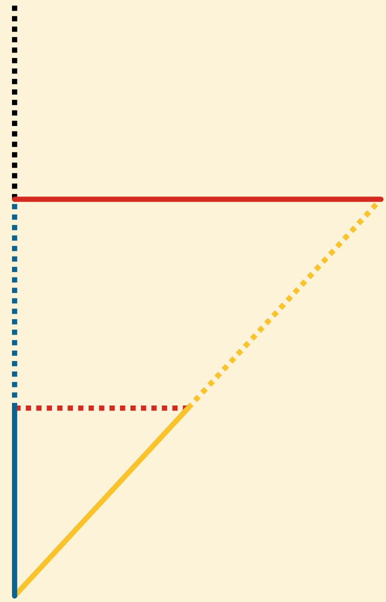
Becauſe  =  [ax. 11], and  common to  and  ;  =  [I. 32];

∴  and  are equiangular; and conſequently have their ſides about the equal angles proportional [VI. 4], and are therefore ſimilar [VI. def. 1].

In like manner it may be proved that  is ſimilar to  ; but  has been ſhewn to be ſimilar to  ; ∴  and  are ſimilar to the whole and to each other.

Q.E.D.

PROPOSITION IX. PROBLEM.



FROM *a given ſtraight line* ( ) *to cut off any required part.*

From either extremity of the given line draw making any angle with ; and produce till the whole produced line contains as often as contains the required part.

Draw , and draw ‖ . is the required part of .

For ſince ‖

: ∷ : [VI. 2], and by compoſition [V. 18];

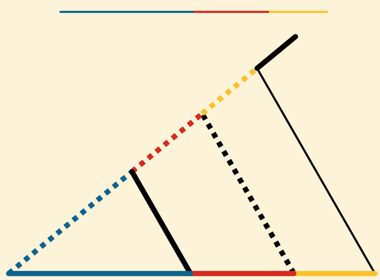
: ∷ : ;

but contains as often as contains the required part [conſt.];

∴ is the the required part.

Q.E.F.

PROPOSITION X. PROBLEM.



TO *divide a ſtraight line* ( ) *ſimilarly to a given divided line* ( ).

From either extremity of the given line draw making any angle;

take , and equal to , and respectively [I. 2];

draw , and draw and ∥ to it.

Since { } are ∥

: :: : [VI. 2],

or : :: : [conſt.],

and : :: : [VI. 2],

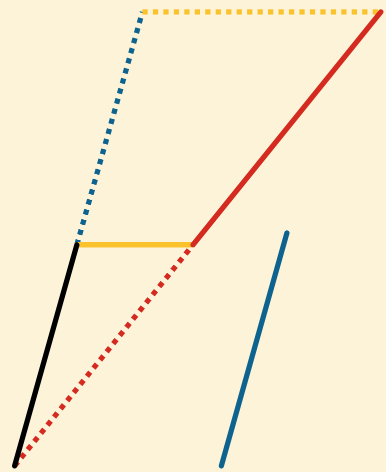
: :: : [conſt.], and

∴ the given line is divided ſimilarly to .

Q.E.F.

PROPOSITION XI. PROBLEM.

TO *find a third proportional to two given ſtraight lines* (*and*).



At either extremity of the given line

draw making an angle;

take = , and draw ;

make = ,

and draw ∥ ; [I. 31]

is the third proportional to and .

For ſince ∥ ,

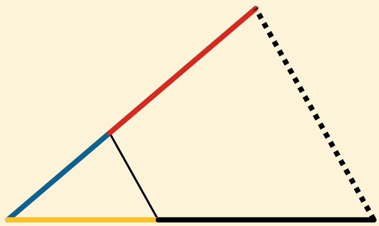
∴ : ∷ : [VI. 2];

but = = [conſt.];

∴ : ∷ : [V. 7].

Q.E.F.

PROPOSITION XII. PROBLEM.



TO *find a fourth proportional to three given lines* $\left\{\right\}$.

Draw

and making any angle;

take $=$,

and $=$,

alſo $=$,

draw ,

and $\parallel$; [I. 31];

is the fourth proportional.

On account of the parallels,

$:$ $::$ $:$ [VI. 2];

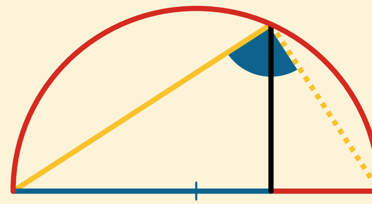
but $\left\{\right\} = \left\{\right\}$ [conſt.];

$\therefore$ $:$ $::$ $:$ [V. 7].

Q.E.F.

PROPOSITION XIII. PROBLEM.

T*O find a mean proportional between two given ſtraight lines* { }.



Draw any ſtraight line , make = , and = ; biſect ; and from the point of biſection as a centre, and half the line as a radius, deſcribe a ſemicircle , draw ⊥ : is the mean proportional required.

Draw and .

Since is a right angle [III. 31], and is ⊥ from it upon the oppoſite ſide, ∴ is a mean proportional between and [VI. 8], and ∴ between and [conſt.].

Q.E.F.

PROPOSITION XIV. THEOREM.

I.

EQUAL parallelograms and , which have one angle in each equal, have the ſides about the equal angles reciprocally proportional

(: :: :)

II.

And parallelograms which have one angle in each equal, and the ſides about them reciprocally proportional, are equal.

Let and ; and and , be ſo placed that and may be continued right lines. It is evident that they may aſſume this poſition [I. 13, 14, 15].

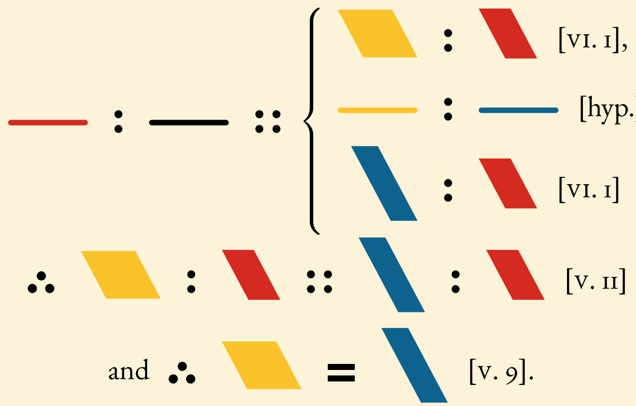
Complete .

Since = ;

∴ : :: : [V. 7]

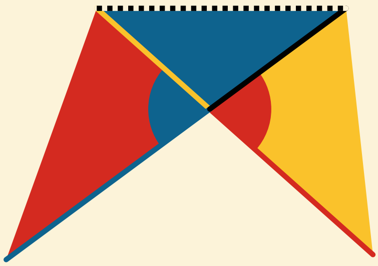
∴ : :: : [VI. 1].

The ſame conſtruction remaining:



Q.E.D.

PROPOSITION XV. THEOREM.



I.

EQUAL *triangles, which have one angle in each equal* (=), *have the ſides about the equal angles reciprocally proportional*

(: :: :)

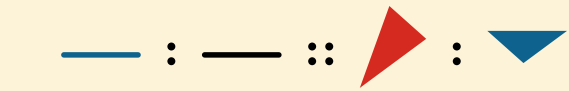
II.

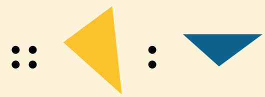
And two triangles which have an angle of the one equal to an angle of the other, and the ſides about the equal angles reciprocally proportional, are equal.

I.

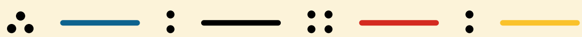
Let the triangles be ſo placed that the equal angles and may be vertically oppoſite, that is to ſay, ſo that and may be in the ſame ſtraight line. Whence alſo and muſt be in the ſame ſtraight line [I. 14].

Draw , then

 [vi. 1]

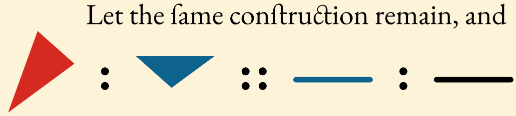
 [v. 7]

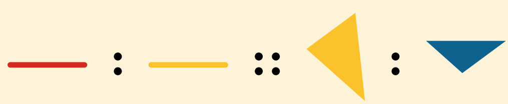
 [vi. 1]

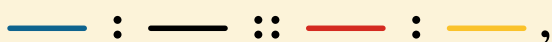
 [v. 11].

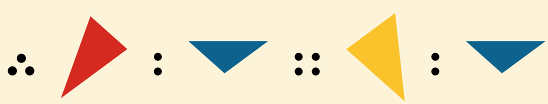
II.

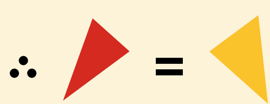
Let the ſame conſtruction remain, and

 [vi. 1]

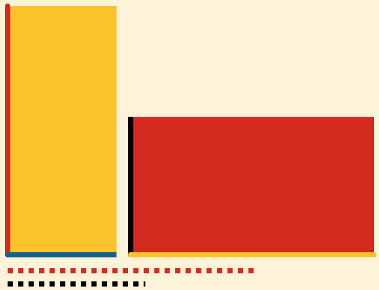
and  [vi. 1].

But  [hyp.]

 [v. 11];

 [v. 9]

Q.E.D.



PROPOSITION XVI. THEOREM.

PART I.

I*F four ſtraight lines be proportional*

(: :: :),

the rectangle (×) *contained by the extremes, is equal to the rectangle* (×) *contained by the means.*

PART II.

And if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four ſtraight lines are proportional.

PART I.

From the extremities and draw and ⊥ to them and = and reſpectively: complete the parallelograms:

and .

And since,

: ∷ : [hyp.]

∴ : ∷ : [const.]

∴ = [VI. 14],

that is, the rectangle contained by the extremes, equal to the rectangle contained by the means.

PART II.

Let the same construction remain; because

= , =

and = ,

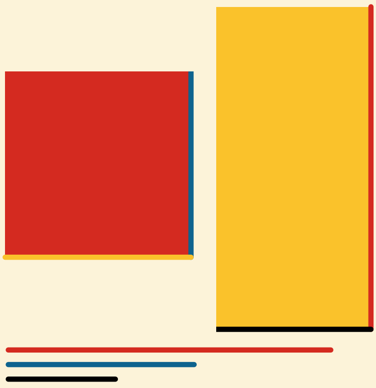
∴ : ∷ : [VI. 14].

But = ,

and = [const.]

∴ : ∷ : [V. 7].

Q.E.D.



PART I.

IF *three ſtraight lines be proportional* (: :: :) *the rectangle under the extremes is equal to the ſquare of the mean.*

PART II.

And if the rectangle under the extremes be equal to the ſquare of the mean, the three ſtraight lines are proportional.

PART I.

Aſſume = , and

ſince : :: : ,

then : :: : ,

∴ × = × [VI. 16].

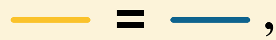
But = ,

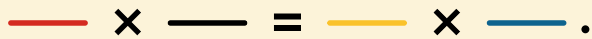
∴ × = × , or

= ^{2};

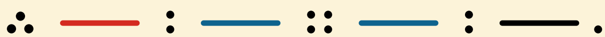
therefore, if the three ſtraight lines are proportional, the rectangle contained by the extremes is equal to the ſquare of the mean.

PART II.

Aſſume  then

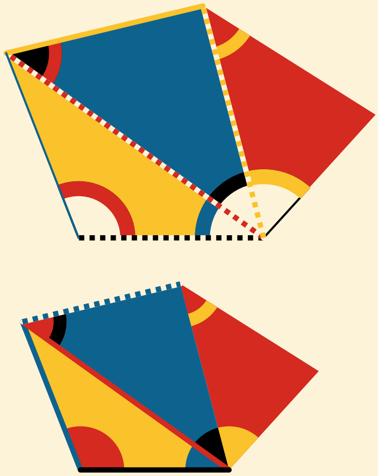


 [VI. 16], and



Q.E.D.

PROPOSITION XVIII. THEOREM.



O N *a given ſtraight line ( ) to conſtruct a rectilinear figure ſimilar to a given one ( ) and ſimilarly placed.*

Reſolve the given figure into triangles by drawing the lines and .

At the extremities of make = and = : again at the extremities of make = and = : in the like manner make = and = .

Then is ſimilar to .

It is evident from the conſtruction and [I. 32] that the figures are equiangular; and ſince the triangles

 and  are equiangular; then by [VI. 4],

: :: :

and : :: : .

Again, becauſe  and  are equiangular,

: :: : ;

∴ ex æquali,

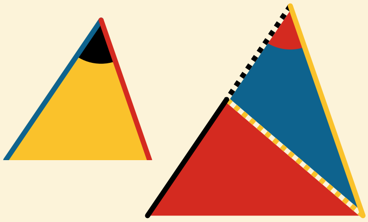
: :: : [VI. 22].

In like manner it may be ſhown that the remaining ſides of the two figures are proportional.

∴ by [VI. def. 1]  is ſimilar 

and ſimilarly ſituated; and on the given line .

Q.E.D.



SIMILAR *triangles* (*and*) *are to one another in the duplicate ratio of their homologous ſides.*

Let and be equal angles, and and let homologous ſides of the ſimilar triangles and and on the greater of theſe lines take a third proportional, ſo that

: :: : ;

draw .

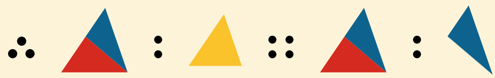
: :: : [VI. 4];

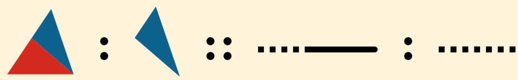
∴ : :: : [V. 16, alt],

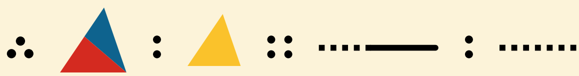
but : :: : [conſt.],

∴ : :: :

conſequently = for they have the ſides about the equal angles and reciprocally proportional [VI. 15];

 [v. 7];

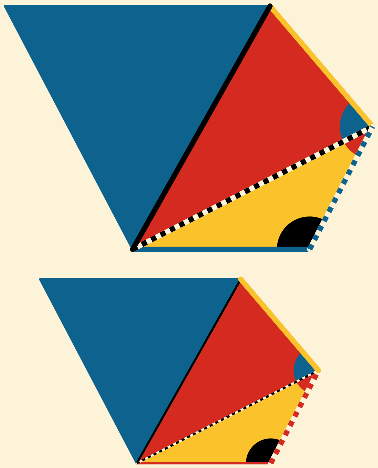
but  [vi. 1],

 ,

that is to ſay, the triangles are to one another in the duplicate ratio of their homologous ſides

 and [v. def. 11].

Q.E.D.



PROPOSITION XX. THEOREM.

SIMILAR *polygons may be divided into the ſame number of ſimilar triangles, each ſimilar pair of which are proportional to the polygons; and the polygons are to each other in the duplicate ratio of their homologous ſides.*

Draw and , and and , reſolving the polygons into triangles. Then becauſe the polygons are ſimilar,

= , and : :: :

∴ and are ſimilar, and = [vi. 6];

but = becauſe they are angles of ſimilar polygons;

therefore the remainders and are equal;

hence : :: : ,

on account of the ſimilar triangles,

and : :: : ,

on account of the ſimilar polygons,

∴ : :: : ,

ex æquali [v. 22], and as theſe proportional ſides contain equal angles, the triangles and are ſimilar [vi. 6].

In like manner it may be ſhown that the

triangles and are ſimilar.

But is to in the duplicate ratio of

to [VI. 19], and

is to in like manner, in the duplicate

ratio of to ;

∴ : :: : [V. 11].

Again is to in the duplicate ratio of

to , and is to in

the duplicate ratio of to ,

∴ : :: : :: : ;

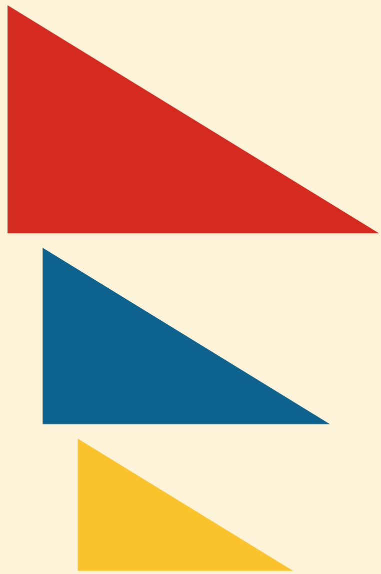
and as one of the antecedents is to one of the conſequents, ſo is the ſum of all the antecedents to the ſum of all the conſequents; that is to ſay, the ſimilar triangles have to one another the ſame ratio as the polygons [V. 12].

But is to in the duplicate ratio of to ;

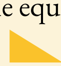
∴ is to in the duplicate ratio of to .

Q.E.D.

PROPOSITION XXI. THEOREM.



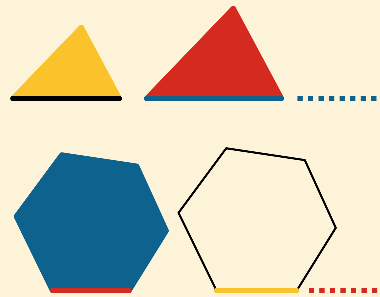
RECTILINEAR *figures* ( *and* ) *which are ſimilar to the ſame figure* () *are ſimilar alſo to each other.*

Since  and  are ſimilar, they are equiangular, and have the ſides about the equal angles proportional [VI. def. 1]; and ſince the figures  and  are alſo ſimilar, they are equiangular, and have the ſides about the equal angles proportional; therefore  and  are alſo equiangular, and have the ſides about the equal angles proportional [V. 11], and are therefore ſimilar.

Q.E.D.

PROPOSITION XXII. THEOREM.

PART I.



IF *four ſtraight line be proportional* (: :: :), *the ſimilar rectilinear figures ſimilarly deſcribed on them are alſo proportional.*

PART II.

And if four ſimilar rectilinear figures, ſimilarly deſcribed on four ſtraight lines, be proportional, the ſtraight lines are alſo proportional.

PART I.

Take a third proportional to

and , and a third proportional

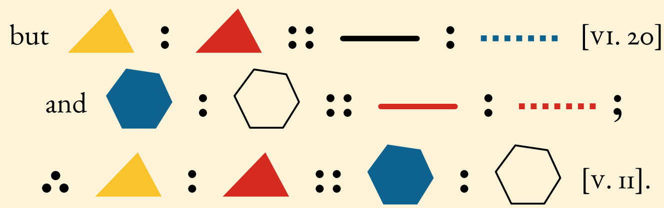
to and [vi. 11];

ſince : :: : [hyp.],

: :: : [conſt.]

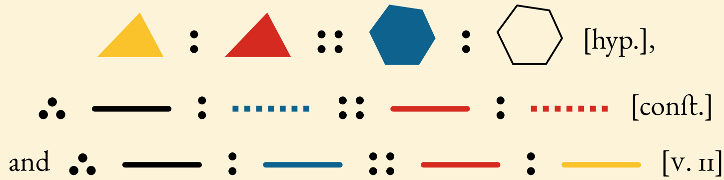
∴ ex æquali,

: :: : ;



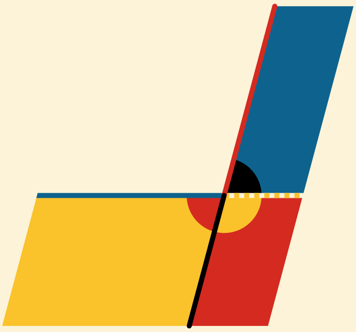
PART II.

Let the ſame conſtruction remain:



Q.E.D.

PROPOSITION XXIII. THEOREM.



EQUIANGULAR *parallelograms* (*and*) *are to one another in a ratio compounded of the ratios of their ſides.*

Let two of the ſides and about the equal angles be placed ſo that they may form one ſtraight line.

Since + = ,

and = [hyp.],

+ = ,

and ∴ and form one ſtraight line [I. 14];

complete .

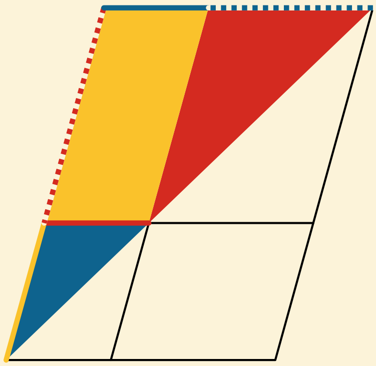
Since : :: : [VI. 1],

and : :: : [VI. 1],

has to a ratio compounded of the ratios of

to , and of to .

Q.E.D.



PROPOSITION XXIV. THEOREM.

IN *any parallelograms ( ) the parallelograms (and) which are about the diagonal are ſimilar to the whole, and to each other.*

As and have a common angle they are equiangular; but becauſe ‖

and are ſimilar [VI. 4],

∴ : :: : ;

and the remaining oppoſite ſides are equal to thoſe,

∴ and have the ſides about the equal angles proportional, and are therefore ſimilar.

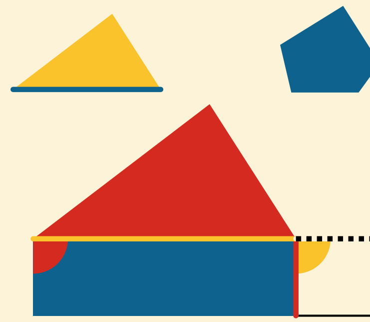
In the ſame manner it can be demonſtrated that the parallelograms and are are ſimilar.

Since, therefore, each of the parallelograms and is ſimilar to ,

they are ſimilar to each other.

Q.E.D.

PROPOSITION XXV. PROBLEM.



TO *deſcribe a rectilinear figure which ſhall be ſimilar to a given rectilinear figure* ( ), *and equal to another* ( ).

Upon deſcribe = ,

and upon deſcribe = ,

and having = [I. 45], and then

and will lie in the ſame ſtraight line [I. 29, 14].

Between and find a mean proportional

[VI. 13], and upon

deſcribe , ſimilar to , and ſimilarly ſituated.

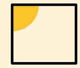
Then = .

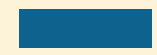
For ſince and are ſimilar, and

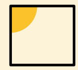
: :: : [conſt.],

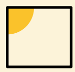
: :: : [VI. 20];

but : :: : [VI. 1];

∴  :  ::  :  [v. 11];

but  =  [conſt.],

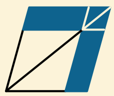
and ∴  =  [v. 14];

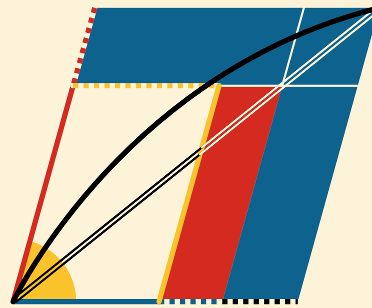
and  =  [conſt.]; conſequently,

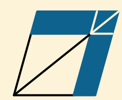
 which is ſimilar to  is alſo =  .

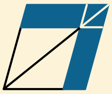
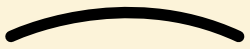
Q.E.F.

PROPOSITION XXVI. THEOREM.

IF ſimilar and ſimilarly poſited parallelograms ( *and*  *) have a common angle, they are about the ſame diagonal.*



For, if it be poſſible, let  be the diagonal of  and draw ∥ [I. 31].

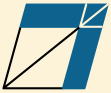
Since  and  are about the ſame diagonal , and have  common, they are ſimilar [VI. 24];

∴ : ∷ : ;

but : ∷ : [hyp.],

∴ : ∷ : ,

and ∴ = [V. 9], which is abſurd.

∴  is not the diagonal of 

in the ſame manner it can be demonſtrated that no other line is except .

Q.E.D.

OF *all the rectangles contained by the ſegments of a given ſtraight line, the greateſt is the ſquare which is deſcribed on half the line.*

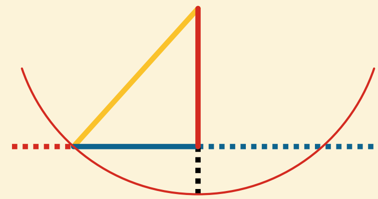
Let be the given line, and unequal ſegments, and and equal ſegments;

then  ⊏ .

For it has been demonſtrated already [II. 5], that the ſquare of half the line is equal to the rectangle contained by any unequal ſegments together with the ſquare of the part intermediate between the middle point and the point of unequal ſection. The ſquare deſcribed on half the line exceeds therefore the rectangle contained by any unequal ſegments of the line.

Q.E.D.

PROPOSITION XXVIII. PROBLEM.



TO divide a given ſtraight line ( ) ſo that the rectangle contained by its ſegments may be equal to a given area, not exceeding the ſquare of half the line.

Let the given area be $= \;{}^2$. Biſect , or make

$=$; and if ${}^2 = \;{}^2$,

the problem is ſolved. But if ${}^2 \neq \;{}^2$,

then muſt $\sqsubset$ [hyp.].

Draw $\perp$ $=$; make

$=$ or ; with

as radius deſcribe a circle cutting the given line; draw .

Then $\times$ $+$ ${}^2 = \;{}^2$

[II. 5] $= \;{}^2$.

But ${}^2 = \;{}^2 + \;{}^2$ [I. 47];

$\therefore$ $\times$ $+$ ${}^2 = \;{}^2 + \;{}^2$,

from both, take ^{2}, and

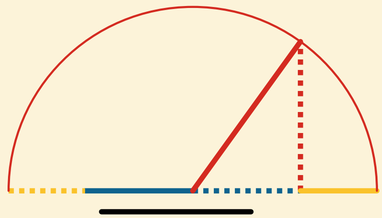
$\times$ $= \;{}^2$.

But $=$ [conſt.], and $\therefore$ is ſo divided that

$\times$ $= \;{}^2$.

Q.E.F.

PROPOSITION XXIX. PROBLEM.



TO *produce a given ſtraight line ( ), ſo that the rectangle contained by the ſegments between the extremities of the given line and the point to which it is produced, may be equal to a given area, i.e. equal to the ſquare on .*

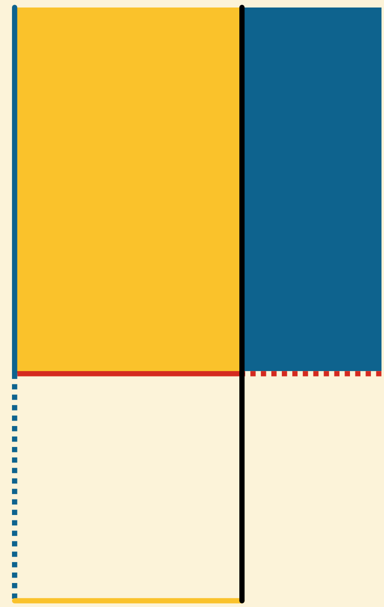
Make = , and
draw ⊥ = ;
draw ; and with the radius ,
deſcribe a circle meeting produced.

Then × + ^{2} =
^{2} [II. 6] = ^{2}.

But ^{2} = ^{2} + ^{2} [I. 47]
∴ × + ^{2} =
^{2} + ^{2}, from both take ^{2},
and × = ^{2};
but = , ∴ ^{2} = the given area.

Q.E.F.

PROPOSITION XXX. PROBLEM.



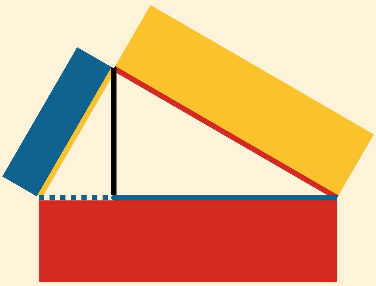
TO *cut a given finite ſtraight line* ( ), *in extreme and mean ratio.*

On describe the ſquare [I. 46]; and produce , ſo that × = ^{2} [VI. 29]; take = , and draw ∥ , meeting ∥ [I. 31].

Then = × , and is ∴ = ; and if from both theſe equals be taken the common part , , which is the ſquare of , will be = , which is = × ; that is ^{2} = × ; ∴ : :: : , and is divided in extreme and mean ratio [VI. def. 3].

Q.E.F.

PROPOSITION XXXI. THEOREM.



IF *any ſimilar rectilinear figures be ſimilarly deſcribed on the ſides of a right angled triangle* ( ), *the figure deſcribed on the ſide* ( ) *ſubtending the right angle is equal to the ſum of the figures on the other ſides.*

From the right angle draw perpendicular to ;

then : ∷ : [vi. 8].

∴ : ∷ : [vi. 20].

but : ∷ : [vi. 20].

Hence + :

∷ + : ;

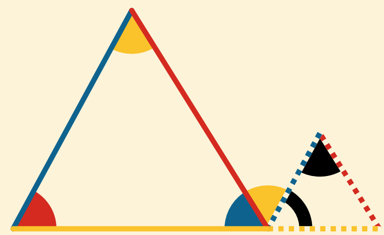
but + = ;

and ∴ + = .

Q.E.D.

PROPOSITION XXXII. THEOREM.

IF *two triangles* (*and*), *have two ſides proportional* (: :: :), *and be ſo placed at an angle that the homologous ſides are parallel, the remaining ſides* (*and*) *form one right line.*



Since ‖ , = [I. 29];

and alſo ſince ‖ , = [I. 29];

∴ = ;

and ſince : :: : [hyp.],

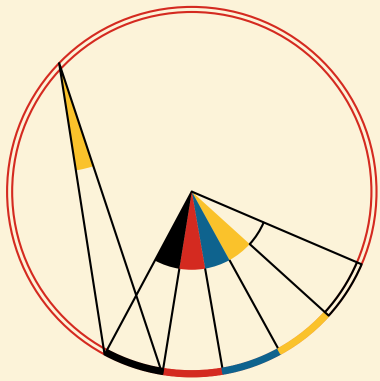
the triangles are equiangular [VI. 6];

∴ = : but = ;

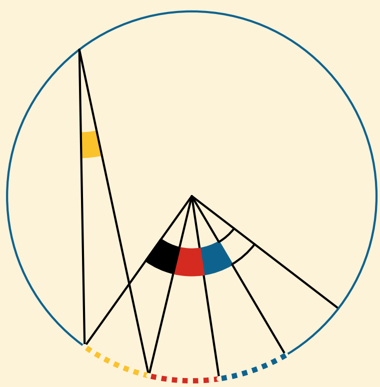
∴ + + = + + = [I. 32],

and ∴ and lie in the ſame ſtraight line [I. 14].

Q.E.D.

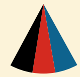


IN *equal circles* (,), *angles, whether at the centre or circumference, are in the ſame ratio to one another as the arcs on which they ſtand* (: :: :); *ſo alſo are ſectors.*



Take in the circumference of any number of arcs , , &c. each = , and alſo in the circumference of take any number of arcs , , &c. each = , draw the radii to the extremities of the equal arcs.

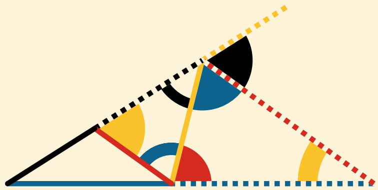
Then ſince the arcs , , , &c. are all equal, the angles , , , &c. are alſo equal [III. 27]; ∴ is the ſame multiple of which the arc is of ; and in the ſame manner is the ſame multiple of , which the arc is of the arc .

Then it is evident [III. 27], if  (or if *m* times )
⊏ , = , ⊐  (or *n* times ) then  (or *m* times
) ⊏ , = , ⊐  (or *n* times);

∴  :  :: : , [V. def. 5], or the angles at the centre are as the arcs on which they ſtand; but the angles at the circumference being halves of the angles at the centre [III. 20] are in the ſame ratio [V. 15], and therefore are as the arcs on which they ſtand.

It is evident that ſectors in equal circles, and on equal arcs are equal [I. 4; III. 24, 27, and def 9]. Hence, if the ſectors be ſubſtituted for the angles in the above demonſtration, the ſecond part of the propoſition will be eſtabliſhed, that is, in equal circles the ſectors have the ſame ratio to one another as the arcs on which they ſtand.

Q.E.D.



IF *the right line* ( ) *biſecting an external angle* *of the triangle* *meet the oppoſite ſide* ( ) *produced, that whole produced ſide* ( ), *and its external ſegment* ( ) *will be proportional to the ſides* (*and*), *which contain the angle adjacent to the external biſected angle.*

For if be drawn ∥ ,

then = , [I. 29];

= , [hyp.],

= , [I. 29];

and ∴ = , [I. 6],

and : :: : [V. 7];

But alſo, : :: : [VI. 2];

and therefore : :: : [V. 11].

Q.E.D.

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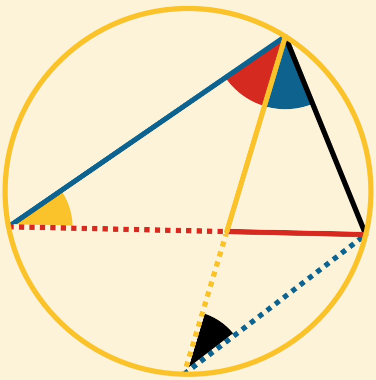
PROPOSITION B. THEOREM.

IF *an angle of a triangle be biſected by a ſtraight line, which likewiſe cuts the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the ſegments of the baſe, together with the ſquare of the ſtraight line which biſects the angle.*



Let be drawn, making = ; then ſhall

× = × + ^{2}.

About deſcribe [IV. 5],

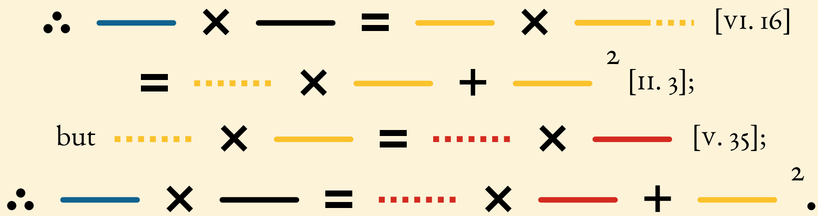
produce to meet the circle, and draw .

Since = [hyp.],

and = [III. 21],

∴ and are equiangular [I. 32];

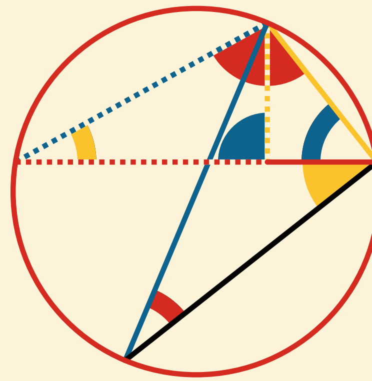
∴ : :: : [VI. 4];



Q.E.D.

PROPOSITION C. THEOREM.

IF *from any angle of a triangle a ſtraight line be drawn perpendicular to the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle deſcribed about the triangle.*



From of draw ⊥ ;

then ſhall × = × the diameter of the deſcribed circle. Deſcribe [IV. 5], draw its diameter , and draw ; then becauſe = [conſt. and III. 31];

and = [III. 21];

∴ is equiangular to [VI. 4];

∴ : ∷ : ;

and ∴ × = × [VI. 16].

Q.E.D.

PROPOSITION D. THEOREM.

THE *rectangle contained by the diagonals of a quadrilateral figure inſcribed in a circle, is equal to both the rectangles contained by oppoſite ſides.*

Let be any quadrilateral figure inſcribed in ;

and draw and ; then

× =

× + × .

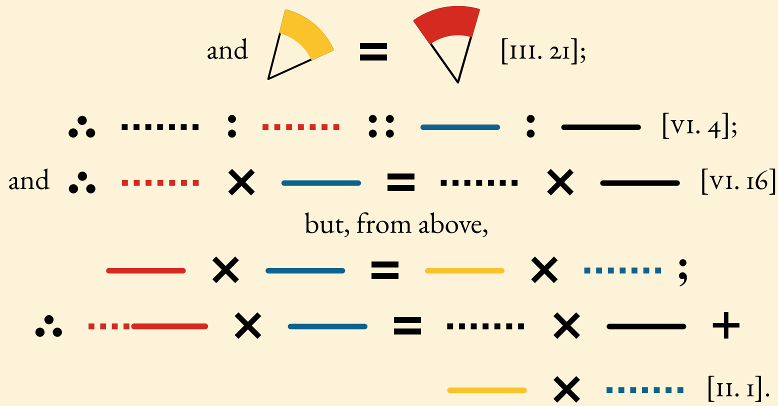
Make = [I. 23],

∴ = ; and = [III. 21];

∴ : :: : [VI. 4];

and ∴ × = × [VI. 16];

again, becauſe = [conſt.],



Q.E.D.

THE END.