Universal Algebra Week 3

James Newman Winter 2019

Definition. An algebra $\langle A, F\rangle$ is the reduct of an algebra $\langle A, F^*\rangle$ to $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{F}^*$, and F is the reduction of F^* to $\mathcal{F}$.

Theorem. Given $n \geq 1$, find equations Σ for semigroups such that Σ will hold in a semigroup $\langle S, \cdot\rangle$ iff $\langle S, \cdot\rangle$ is a reduct of a group $\langle S, \cdot, ^{-1}, 1\rangle$ of exponent n (i.e. every element of S is such that its order divides n).

Proof. Assume $n \geq 1$ and $\langle S, \cdot\rangle$ is a semigroup which satisfies the equations in Σ where Σ contains

$$x^{n+1} = x \text{ and,}$$

$$x^n = y^n$$

for all $x, y \in S$. Define $x^{-1} = x^{n-1}$ for all $x \in S$, and $1 = c^n$ for an arbitrary $c \in S$. Then for any $x, y \in S$,

(i) G1 holds because it is a semigroup.

(ii) $1 \cdot x = c^n \cdot x = x^n \cdot x = x^{n+1} = x = x^{n+1} = x \cdot x^n = x \cdot c^n = x \cdot 1$. Thus $1 \cdot x = x = x \cdot 1$, that is G2 holds.

(iii) $x \cdot x^{-1} = x \cdot x^{n-1} = x^n = c^n = 1 = c^n = x^n = x^{n-1} \cdot x = x^{-1} \cdot x$. Thus $x \cdot x^{-1} = 1 = x^{-1} \cdot x$, that is G3 holds.

Furthermore is easy to see that since $x^n = 1$, the order of x must divide n. Since G1-3 hold, the semigroup $\langle S, \cdot\rangle$ is a reduct of the group $\langle S, \cdot, ^{-1}, 1\rangle$.

In the other direction, assume $\langle S, \cdot, ^{-1}, 1\rangle$ is a group where the order of every element divides n. Then for any x, the order of x is i where $ij = n$ for some j. So $x^n = x^{ij} = (x^i)^j = 1^j = 1$. Thus for any $x, y \in S$,

(i) $x^{n+1} = x^n \cdot x = 1 \cdot x = x$, and

(ii) $x^n = 1 = y^n$.

Thus the reduct semigroup $\langle S, \cdot\rangle$ satisfies Σ. $\square$

Definition. Two elements a, b of a bounded lattice $\langle L, \wedge, \vee, 0, 1\rangle$ are complements if $a \vee b = 1$ and $a \wedge b = 0$. In this case, each of a, b is the complement of the other. A complemented lattice is a bounded lattice in which every element has a complement.

Theorem. In a bounded distributive lattice any element has at most one complement.

Proof. Assume $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice. Furthermore take any elements $a, b, c \in L$ such that both b and c are complements of a. That is $a \wedge b = 0$, $a \vee b = 1$, $a \wedge c = 0$, and $a \vee c = 0$. Then

$$
\begin{aligned}
b &= b \vee (b \wedge a) && \text{Lattice absorbtion}\\
b &= b \vee (a \wedge b) && \text{Lattice commutativity}\\
&= b \vee 0 && a \wedge b = 0\\
&= b \vee (a \wedge c) && a \wedge c = 0\\
&= (b \vee a) \wedge (b \vee c) && \text{Distributive Lattice}\\
&= (a \vee b) \wedge (b \vee c) && \text{Lattice commutativity}\\
&= 1 \wedge (b \vee c) && a \vee b = 1\\
&= (a \vee c) \wedge (b \vee c) && a \vee c = 1\\
&= (c \vee a) \wedge (c \vee b) && \text{Lattice commutativity}\\
&= c \vee (a \wedge b) && \text{Distributive Lattice}\\
&= c \vee 0 && a \wedge b = 0\\
&= c \vee (c \wedge 0) && \text{Bounded Lattice}\\
&= c && \text{Lattice absorbtion}
\end{aligned}
$$

Therefore $b = c$ and thus there is at most 1 complement of any element a in a bounded distributive lattice. □

Before continuing let us prove some useful facts about Boolean algebras.

Theorem. If $\langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra then $x \vee 0 = x$ and $x \wedge 1 = x$ for all $x \in B$.

Proof. Assume $\langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra and take any $x \in B$. Then

$$
\begin{aligned}
x &= x \vee (x \wedge 0) && \text{Lattice absorbtion}\\
&= x \vee 0 && \text{Boolean algebra}
\end{aligned}
$$

Dually,

$$
\begin{aligned}
x &= x \wedge (x \vee 1) && \text{Lattice absorbtion}\\
&= x \wedge 1 && \text{Boolean algebra}
\end{aligned}
$$

□

Theorem. If $\langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra then $(x \vee y)' = (x' \wedge y')$ and $(x \wedge y)' = (x' \vee y')$ for all $x, y \in B$.

Proof. Assume $\langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra. Then

$$\begin{aligned}
(x \vee y) \wedge (x' \wedge y') &= ((x \vee y) \wedge x') \wedge y' \\
&= (x' \wedge (x \vee y)) \wedge y' \\
&= ((x' \wedge x) \vee (x' \wedge y)) \wedge y' \\
&= (0 \vee (x' \wedge y)) \wedge y' \\
&= (x' \wedge y) \wedge y' \\
&= x' \wedge (y \wedge y') \\
&= x' \wedge 0 \\
&= 0
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(x \vee y) \vee (x' \wedge y') &= x \vee (y \vee (x' \wedge y')) \\
&= x \vee ((y \vee x') \wedge (y \vee y')) \\
&= x \vee ((y \vee x') \wedge 1) \\
&= x \vee (y \vee x') \\
&= x \vee (x' \vee y) \\
&= (x \vee x') \vee y \\
&= 1 \vee y \\
&= 1
\end{aligned}$$

Therefore $x' \wedge y'$ is a complement of $x \vee y$ and since there is at most one complement for an element of a boolean algebra, $x' \wedge y' = (x \vee y)'$. A similar proof shows that $x' \vee y' = (x \wedge y)'$ □

Theorem. If $\langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra then $(x')' = x$.

Proof.

$$\begin{aligned}
(x')' &= 1 \wedge (x')' \\
&= (x \vee x') \wedge (x')' \\
&= (x')' \wedge (x \vee x') \\
&= ((x')' \wedge x) \vee ((x')' \wedge x') \\
&= ((x')' \wedge x) \vee 0 \\
&= (x')' \wedge x
\end{aligned}$$

So $(x')' \leq x$. Next,

$$\begin{aligned} x &= x \wedge 1 \\ &= x \wedge (x' \vee (x')') \\ &= (x \wedge x') \vee (x \wedge (x')') \\ &= 0 \vee (x \wedge (x')') \\ &= x \wedge (x')' \end{aligned}$$

So $x \leq (x')'$. Therefore $x = (x')'$ by anti-symmetry because $(x')' \leq x$ and $x \leq (x')'$. $\square$

Theorem. If $\langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra and $a, b \in B$, define $a \to b$ to be $a' \vee b$. Then $\langle B, \wedge, \vee, \to, 0, 1\rangle$ is a Heyting algebra.

Proof. Assume $\langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra and define $a \to b$ as $a' \vee b$ for all $a, b \in B$. Then for any $x, y, z \in B$

(H1) Follows from B1.

(H2) Follows from B2.

(H3) $x \to x = x' \vee x = 1$

(H4)

$$\begin{aligned} (x \to y) \wedge y &= (x' \vee y) \wedge y && x \to y = x' \vee y \\ &= y \wedge (x' \vee y) && \text{Lattice commutativity} \\ &= y \wedge (y \vee x') && \text{Lattice commutativity} \\ &= y && \text{Lattice absorbtion} \end{aligned}$$

Furthermore

$$\begin{aligned} x \wedge (x \to y) &= x \wedge (x' \vee y) \\ &= (x \wedge x') \vee (x \wedge y) \\ &= 0 \vee (x \wedge y) \\ &= x \wedge y \end{aligned}$$

(H5)

$$\begin{aligned} x \to (y \wedge z) &= x' \vee (y \wedge z) \\ &= (x' \vee y) \wedge (x' \vee z) \\ &= (x \to y) \wedge (x \to z) \end{aligned}$$

And

$$\begin{aligned}(x \vee y) \rightarrow z &= ((x \vee y)' \vee z) \\ &= ((x' \wedge y') \vee z) \\ &= (x' \vee z) \wedge (y' \vee z) \\ &= (x \rightarrow z) \wedge (y \rightarrow z)\end{aligned}$$

Therefore the $\langle B, \wedge, \vee, \rightarrow, 0, 1 \rangle$ satisfies H1-5 and is a Heyting algebra. $\square$