Universal Algebra Week 2

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Theorem. *If L is a lattice and $A \subseteq L$, let $u(A) = \{b \in L \mid a \leq b$ for all $a \in A\}$, the set of upper bounds of A, and let $l(A) = \{b \in L \mid b \leq a$ for all $a \in A\}$, the set of lower bounds of A. Show that $C(A) = l \circ u(A)$ is a closure operator on A, and that the map $\alpha : a \mapsto C(\{a\})$ gives an embedding of L into the complete lattice L_C.*

Proof. Take any $X, Y \subseteq L$, we must prove that $X \subseteq C(X)$, $C^2(X) = C(X)$, and $C(X) \subseteq C(Y)$ when $X \subseteq Y$.

Take any $x \in X$ and $a \in u(X)$. Then $x \leq a$ by definition of upper bound. So $x \in l \circ u(X)$. Therefore $X \subseteq l \circ u(X) = C(X)$, proving C is extensive. A similar proof shows that $X \subseteq u \circ l(X)$.

Also, let's prove that $l = l \circ u \circ l$. Because C is extensive, $l(X) \subseteq C(l(X)) = l \circ u \circ l(X)$. Next, take any $a \in l \circ u \circ l(X)$ and $x \in X$. So $x \in u \circ l(X)$ because $X \subseteq u \circ l(X)$. $a \leq x$ because a is a lower bound of $u \circ l(X)$ and $x \in u \circ l(X)$. Then $a \in l(X)$ because $a \leq x$ for all $x \in X$. Therefore $l \circ u \circ l(X) \subseteq l(X)$. Finally $l = l \circ u \circ l$ because $l \circ u \circ l(X) \subseteq l(X)$ and $l(X) \subseteq l \circ u \circ l(X)$. Then $C^2 = l \circ u \circ l \circ u = l \circ u = C$, so C is idempotent.

Another detour: if $X \subseteq Y$, then $u(Y) \subseteq u(X)$ and $l(Y) \subseteq l(X)$. Assume $X \subseteq Y$. Assume $a \in l(Y)$ and take any $b \in X$. Then $b \in Y$ because $X \subseteq Y$. So $a \leq b$ because a is a lower bound of Y and $b \in Y$. Since $a \leq b$ for all $b \in X$, $a \in l(Y)$. Therefore $l(Y) \subseteq l(X)$ because $a \in l(Y)$ when $a \in l(X)$. A similar proof shows $u(Y) \subset u(X)$.

Take any $X, Y \subseteq L$ such that $X \subseteq Y$. Take any $a \in C(X) = l \circ u(X)$. Next, take any $b \in u(Y)$. Then $b \in u(X)$ because $u(Y) \subseteq u(X)$ by the above lemma. So $a \leq b$ because a is a lower bound of $u(X)$. Therefore $a \in l \circ u(Y) = C(Y)$ because $a \leq b$ for all $b \in u(Y)$. This is two say that $C(X) \subseteq C(Y)$ and C is isotone.

Therefore C is a closure operator as it is extensive, idempotent, and isotone.

Next the mapping $\alpha : a \mapsto C(\{a\})$ is order preserving. Take any $x, y \in L$ such that $x \leq y$. First let us prove that $u(\{y\}) \subseteq u(\{x\})$. Take any $a \in u(\{y\})$, which is to say $y \leq a$. Then since $x \leq y$, by transitivity, $x \leq a$. So $a \in u(\{x\})$, and therefore $u(\{y\}) \subseteq u(\{x\})$. Next, by the above detour, $l \circ u(\{x\}) \subseteq l \circ u(\{y\})$ since $u(\{y\}) \subseteq u(\{x\})$. Therefore α is order preserving because $\alpha(x) \subseteq \alpha(y)$ when $x \leq y$ for any $x, y \in L$.

Obviously α is surjective if we consider its codomain its image.

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