Universal Algebra Week 4

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Theorem. $\mathrm{Sg}(X) = X \cup E(X) \cup E^2(X) \cup \dots$

Proof. First by induction, $E^k(X) \subseteq \mathrm{Sg}(X)$ for all $k \in \mathbb{N}$.

(i) The base case when $k = 0$ holds: $E^0(X) = X \subseteq \mathrm{Sg}(X)$ because $\mathrm{Sg}(X)$ is the intersection of supersets of X.

(ii) Assume that $E^k(X) \subseteq \mathrm{Sg}(X)$. Then take any $a \in E^{k+1}(X) = E(E^k(X)) = E^k(X) \cup \{f(a_1, \dots, a_n) \mid f$ is a fundamental n-ary operation on A and $a_1, \dots, a_n \in E^k(X)\}$. In the first case $a \in E^k(X)$, which case clearly $a \in \mathrm{Sg}(X)$ as $E^k(X) \subseteq \mathrm{Sg}(X)$. Otherwise $a \in \{f(a_1, \dots, a_n)\}$ So $a = f(a_1, \dots, a_n)$ for $a_1, \dots, a_n \in E^k(X)$. Then $a_1, \dots, a_n \in \mathrm{Sg}(X)$ because $E^k(X) \subseteq \mathrm{Sg}(X)$. Take any subuniverse B such that $X \subseteq B$. Then $a_1, \dots, a_n \in B$ because $a_1, \dots, a_n \in \mathrm{Sg}(X)$. Thus $f(a_1, \dots, a_n) \in B$ because B is a subuniverse and is closed. Therefore $a = f(a_1, \dots, a_n) \in \mathrm{Sg}(X)$ because it is in every subuniverse that contains X. In either case, $a \in \mathrm{Sg}(X)$, thus $E^{k+1}(X) \subseteq \mathrm{Sg}(X)$ since if $a \in E^{k+1}(X)$ then $a \in \mathrm{Sg}(X)$.

Thus by induction $E^k(X) \subseteq \mathrm{Sg}(X)$ for all $k \in \mathbb{N}$. Then it is clear that $X \cup E(X) \cup E^2(X) \cup \dots \subseteq \mathrm{Sg}(X)$.

Next, $X \cup E(X) \cup E^2(X) \cup \dots$ is a subuniverse. Take any n-ary operation f on A and $a_1, \dots, a_n \in X \cup E(X) \cup E^2(X) \cup \dots$. Since $\{a_1, \dots, a_n\}$ is finite, each a_n is in some $E^k(X)$ and $X \subseteq E(X) \subseteq E^2(X) \subseteq \dots$, there is some $E^k(X)$ such that $a_1, \dots, a_n \in E^k(X)$. Then by definition $f(a_1, \dots, a_n) \in E^{k+1}(X)$, and so $f(a_1, \dots, a_n) \in X \cup E(X) \cup E^2(X) \cup \dots$ because $E^{k+1} \subseteq X \cup \dots \cup E^K(X) \cup E^{k+1}(X) \cup \dots$. Therefore it is closed and contains X so is a subuniverse.

Assume $x \notin X \cup E(X) \cup E^2(X) \cup \dots$. Then since it is a subuniverse containing X and $\mathrm{Sg}(X)$ is the intersection of all such subuniverses, $x \notin \mathrm{Sg}(X)$. Then by contrapositive, if $x \in \mathrm{Sg}(X)$ then $x \in X \cup E(X) \cup E^2(X) \cup \dots$. That is $\mathrm{Sg}(X) \subseteq X \cup E(X) \cup E^2(X) \cup \dots$.

With both directions of the inclusion demonstrated, they are equal. $\square$

Theorem. If $\sim \in \mathrm{Con}\, \mathbf{G}$ then $1/\sim$ is the universe of a normal subgroup of $\mathbf{G}$, and for $a, b \in G$ we have $a \sim b$ if and only if $a \cdot b^{-1} \in 1/\sim$.

Proof. Assume $\sim \in \mathrm{Con}\, \mathbf{G}$. Let $N = 1/\sim$

For any $g \in G$, assume $g \cdot a \cdot g^{-1} \in gNg^{-1}$. Then $a \in N$ and thus $1 \sim a$. Then by compatibility,

$$\begin{aligned} g \cdot 1 &\sim g \cdot a \\ g &\sim g \cdot a \\ g \cdot g^{-1} &\sim g \cdot a \cdot g^{-1} \\ 1 &\sim g \cdot a \cdot g^{-1} \end{aligned}$$

Then $g \cdot a \cdot g^{-1} \in N$. So $gNg^{-1} \subseteq N$ for all $g \in G$ and thus $N = 1/\sim$ is the universe of normal subgroup of $\mathbf{G}$.

Next, for $a, b \in G$,

$$\begin{aligned} a &\sim b && \text{iff} \\ a \cdot b^{-1} &\sim b \cdot b^{-1} && \text{iff} \\ a \cdot b^{-1} &\sim 1 && \text{iff} \\ a \cdot b^{-1} &\in N \end{aligned}$$

by compatibility and properties of groups. □

Theorem. If $\mathbf{N}$ is a normal subgroup of a group $\mathbf{G}$, then the relationship $\sim$ defined as

$$a \sim b \text{ iff } a \cdot b^{-1} \in N$$

is a congruence on $\mathbf{G}$ with $1/\sim = N$.

Proof. Assume $\mathbf{N}$ is a normal subgroup of a group $\mathbf{G}$ and define $\sim$ as above. First we confirm $\sim$ is an equivalence relation:

1. For any $a \in G$ we have that $a \sim a$ because $a \cdot a^{-1} = 1 \in N$.

2. For any $a, b \in G$ assume $a \sim b$. Then $a \cdot b^{-1} \in N$ so its inverse $b \cdot a^{-1} \in N$, and thus $b \sim a$.

3. For any $a, b, c \in G$ assume $a \sim b$ and $b \sim c$. That is, $a \cdot b^{-1} \in N$ and $b \cdot c^{-1} \in N$. Then since $\mathbf{N}$ is closed, $a \cdot b^{-1} \cdot b \cdot c^{-1} = a \cdot c^{-1} \in N$. Thus $a \sim c$.

Therefore $\sim$ is an equivalence relation.

Next we check the compatibility property.

1. Take any $a, b \in G$ such that $a \sim b$. Then $b \sim a$, i.e. $b \cdot a^{-1} \in N$. Then because N is normal, $a^{-1}ba^{-1}a = a^{-1}b = a^{-1}(b^{-1})^{-1} \in N$. Therefore $a^{-1} \sim b^{-1}$.

2. Next take any $a_1, a_2, b_1, b_2 \in G$ such that $a_1 \sim b_1$ and $a_2 \sim b_2$. Then $a_1 b_1^{-1} \in N$ and $a_2 b_2^{-1} \in N$. Because N is normal and $a_1 b_1^{-1} \in N$ we have that $b_1^{-1} a_1 b_1^{-1} b_1 = b_1^{-1} a_1 \in N$. Next since $b_1^{-1} a_1 \in N$ and $a_2 b_2^{-1} \in N$ their product $b_1^{-1} a_1 a_2 b_2^{-1} \in N$. Again since N is normal, $b_1 \cdot b_1^{-1} \cdot a_1 \cdot a_2 \cdot b_2^{-1} \cdot b_1^{-1} = a_1 \cdot a_2 \cdot b_2^{-1} \cdot b_1^{-1} = (a_1 \cdot a_2) \cdot (b_1 \cdot b_2)^{-1} \in N$. Therefore $a_1 \cdot a_2 \sim b_1 \cdot b_2$.

Finally, since $\sim$ is an equivalence relationship that satisfies the compatibility property it is a congruence on $\mathbf{G}$. □

Theorem. The normal subgroups of a group form an algebraic lattice which is modular.

Proof. Take any group $\mathbf{G}$. By Exercise 1 the lattice of normal subgroups of $\mathbf{G}$ is isomorphic to the lattice of congruences on $\mathbf{G}$. By theorem 5.5, the lattice of congruences on $\mathbf{G}$ is algebraic. By theorem 5.10, if $\mathbf{G}$ is congruence permutable, then $\mathbf{G}$ is congruence modular. Then it is sufficient to show that $\mathbf{G}$ is congruence permutable to show that the lattice of normal subgroups of $\mathbf{G}$ is algebraic and modular.

Take any $\sim_1, \sim_2 \in \text{Con } \mathbf{G}$. Take any $a, b \in G$ such that $a \sim_1 \circ \sim_2 b$. Then there is a $c \in G$ such that $a \sim_1 c$ and $c \sim_2 b$. Then by compatbility

$$\begin{array}{lllll} c & \sim_2 b & \qquad & a & \sim_1 c \\ 1 & \sim_2 c^{-1}b & & ac^{-1} & \sim_1 1 \\ a & \sim_2 ac^{-1}b & & ac^{-1}b & \sim_1 b \end{array}$$

Then $a \sim_2 \circ \sim_1 b$. Thus $\sim_1 \circ \sim_2 \subseteq \sim_2 \circ \sim_1$, and by theorem 5.9, $\sim_1 \circ \sim_2 = \sim_2 \circ \sim_1$. That is, $\mathbf{G}$ is congruence permutable and thus also congruence modular.

Therefore the lattice of normal subgroups of a group form an algebraic and modular lattice. ☐

Theorem. If $\mathbf{A}$ is a unary algebra and B is a subuniverse then the relation $\sim$ defined by $a \sim b$ if and only if $a = b$ or $a, b \in B$ is a congruence on $\mathbf{A}$.

Proof. Assume $\mathbf{A}$ is a unary algebra and B is a subuniverse. Defined the realtion $\sim$ by $a \sim b$ if and only if $a = b$ or $a, b \in B$. Then

1. For all $a \in A$, $a \sim a$ because $a = a$.
2. For all $a, b \in A$, if $a \sim b$ then $b \sim a$ because if $a = b$ or $a, b \in B$ then $b = a$ or $b, a \in B$.
3. Take any $a, b, c \in A$ such that $a \sim b$ and $b \sim c$. Then by cases
   (a) $a, b \in B$ and $b, c \in B$ in which case $a, c \in B$ and $a \sim c$.
   (b) $a, b \in B$ and $b = c$ in which case $a, c \in B$ and $a \sim c$.
   (c) $a = b$ and $b = c$ in which case $a = c$ and $a \sim c$.

Thus $\sim$ is an equivalence relation. To show it is compatible, take any operation f of $\mathbf{A}$ and any $a, b \in A$ such that $a \sim b$. Since $\mathbf{A}$ is a unary algebra, f is unary. Then either

1. $a = b$ so $f(a) = f(b)$ and thus $f(a) \sim f(b)$.
2. $a, b \in B$ then because B is a subuniverse and is closed under f, $f(a), f(b) \in B$, so $f(a) \sim f(b)$.

Therefore $\sim$ is a compatible equivalence relation on A and thus a congruence of $\mathbf{A}$. ☐