Universal Algebra Week 4

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Theorem. $\text{Sg}(X) = X \cup E(X) \cup E^2(X) \cup \dots$

Proof. First by induction, $E^k(X) \subseteq \text{Sg}(X)$ for all $k \in \mathbb{N}$. □

Theorem. If $\sim \in \text{Con } \mathbf{G}$ then $1/\sim$ is the universe of a normal subgroup of $\mathbf{G}$, and for $a, b \in G$ we have $a \sim b$ if and only if $a \cdot b^{-1} \in 1/\sim$.

Proof. Assume $\sim \in \text{Con } \mathbf{G}$.

For any $g \in G$, take any $g \cdot a \cdot g^{-1} \in g(1/\sim)g^{-1}$. Then $a \in 1/\sim$ and thus $1 \sim a$. Then by compatibility,

$$\begin{aligned} g \cdot 1 &\sim g \cdot a \\ g &\sim g \cdot a \\ g \cdot g^{-1} &\sim g \cdot a \cdot g^{-1} \\ 1 &\sim g \cdot a \cdot g^{-1} \end{aligned}$$

Then $g \cdot a \cdot g^{-1} \in 1/\sim$. So $g(1/\sim)g^{-1} \subseteq (1/\sim)$ for all $g \in G$ and thus $1/\sim$ is the universe of normal subgroup of $\mathbf{G}$.

Next, for $a, b \in G$,

$$\begin{aligned} & a \sim b && \text{iff} \\ & a \cdot b^{-1} \sim b \cdot b^{-1} && \text{iff} \\ & a \cdot b^{-1} \sim 1 && \text{iff} \\ & a \cdot b^{-1} \in 1/\sim \end{aligned}$$

by compatibility and properties of groups. □

Theorem. If $\mathbf{N}$ is a normal subgroup of a group $\mathbf{G}$, then the relationship $\sim$ defined as

$$a \sim b \text{ iff } a \cdot b^{-1} \in N$$

is a congruence on $\mathbf{G}$ with $1/\sim = N$.

Proof. Assume $\mathbf{N}$ is a normal subgroup of a group $\mathbf{G}$ and define $\sim$ as above. First we confirm $\sim$ is an equivalence relation:

1. For any $a \in G$ we have that $a \sim a$ because $a \cdot a^{-1} = 1 \in N$.

2. For any $a, b \in G$ assume $a \sim b$. Then $a \cdot b^{-1} \in N$ so its inverse $b \cdot a^{-1} \in N$, and thus $b \sim a$.

3. For any $a, b, c \in G$ assume $a \sim b$ and $b \sim c$. That is, $a \cdot b^{-1} \in N$ and $b \cdot c^{-1} \in N$. Then since $\mathbf{N}$ is closed, $a \cdot b^{-1} \cdot b \cdot c^{-1} = a \cdot c^{-1} \in N$. Thus $a \sim c$.

Therefore $\sim$ is an equivalence relation.

Next we check the compatibility property.

1. Take any $a, b \in G$ such that $a \sim b$. Then $b \sim a$, i.e. $b \cdot a^{-1} \in N$. Then because N is normal, $a^{-1}ba^{-1}a = a^{-1}b = a^{-1}(b^{-1})^{-1} \in N$. Therefore $a^{-1} \sim b^{-1}$.

2. Next take any $a_1, a_2, b_1, b_2 \in G$ such that $a_1 \sim b_1$ and $a_2 \sim b_2$. Then $a_1b_1^{-1} \in N$ and $a_2b_2^{-1} \in N$. Because N is normal and $a_1b_1^{-1} \in N$ we have that $b_1^{-1}a_1b_1^{-1}b_1 = b_1^{-1}a_1 \in N$. Next since $b_1^{-1}a_1 \in N$ and $a_2b_2^{-1} \in N$ their product $b_1^{-1}a_1a_2b_2^{-1} \in N$. Again since N is normal, $b_1 \cdot b_1^{-1} \cdot a_1 \cdot a_2 \cdot b_2^{-1} \cdot b_1^{-1} = a_1 \cdot a_2 \cdot b_2^{-1} \cdot b_1^{-1} = (a_1 \cdot a_2) \cdot (b_1 \cdot b_2)^{-1} \in N$. Therefore $a_1 \cdot a_2 \sim b_1 \cdot b_2$.

Finally, since $\sim$ is an equivalence relationship that satisfies the compatibility property it is a congruence on $\mathbf{G}$. $\square$

Theorem. The normal subgroups of a group form an algebraic lattice which is modular.

Proof. Take any group $\mathbf{G}$. By Exercise 1 the lattice of normal subgroups of $\mathbf{G}$ is isomorphic to the lattice of congruences on $\mathbf{G}$. By theorem 5.5, the lattice of congruences on $\mathbf{G}$ is algebraic. By theorem 5.10, if $\mathbf{G}$ is congruence permutable, then $\mathbf{G}$ is congruence modular. Then it is sufficient to show that $\mathbf{G}$ is congruence permutable to show that the lattice of normal subgroups of $\mathbf{G}$ is algebraic and modular.

Take any $\sim_1, \sim_2 \in \text{Con } \mathbf{G}$. Take any $a, b \in G$ such that $a \sim_1 \circ \sim_2 b$. Then there is a $c \in G$ such that $a \sim_1 c$ and $c \sim_2 b$. Then by compatbility

$$\begin{array}{lll@{\qquad}lll} c & \sim_2 & b & a & \sim_2 & c \\ 1 & \sim_2 & c^{-1}b & ac^{-1} & \sim_2 & 1 \\ a & \sim_2 & ac^{-1}b & ac^{-1}b & \sim_2 & b \end{array}$$

Then $a \sim_2 \circ \sim_1 b$. Thus $\sim_1 \circ \sim_2 \subseteq \sim_2 \circ \sim_1$, and by theorem 5.9, $\sim_1 \circ \sim_2 = \sim_2 \circ \sim_1$. That is, $\mathbf{G}$ is congruence permutable and thus also congruence modular.

Therefore the lattice of normal subgroups of a group form an algebraic and modular lattice. $\square$