Universal Algebra Week 1

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Theorem. *If L is a distributive lattice then the set of ideals of L is a distributive lattice.*

Proof. Assume L is a distributive lattice. Recall that an ideal of the lattice L is a non-empty, lower set of L, closed under $\vee$. We know from a previous exercise that the set ideals of L form a lattice with $X \wedge Y = X \cap Y$ and $X \vee Y = \{a \in L \mid a \leq x \vee y \text{ for some } x \in X \text{ and } y \in Y\}$, and that $X \leq Y$ means $X \subseteq Y$ for all ideals X and Y of L. Take any ideals X, Y, and Z of L. To prove the lattice of ideals of L is distributive, it is sufficient to prove the inequality $X \wedge (Y \vee X) \leq (X \wedge Y) \vee (X \wedge Z)$. Assume $x \in X \wedge (Y \vee Z)$. Then $x \in X$ and $x \in Y \vee Z$. So by definition of $Y \vee Z$, $x \leq y \vee z$ for some $y \in Y$ and $z \in Z$. Then $x = x \wedge (y \vee z)$ by using the lattice definition of $\leq$. Then $x = (x \wedge y) \vee (x \wedge z)$ because L is a distributive lattice. So clearly $x \leq (x \wedge y) \vee (x \wedge z)$. Since $x \wedge y \in L$, $x \in X$, $x \wedge y \leq x$, and X is a lower set, it must be that $x \wedge y \in X$. Similarly $x \wedge y \in Y$, and thus $x \wedge y \in X \wedge Y$. The same argument shows that $x \wedge z \in X \wedge Z$. Then we have that $x \leq (x \wedge y) \vee (x \wedge z)$ where $x \wedge y \in X \wedge Y$ and $x \wedge z \in X \wedge Z$. So by definition, $x \in (X \wedge Y) \vee (X \wedge Z)$. Therefore $X \wedge (Y \vee Z) \subseteq (X \wedge Y) \vee (X \wedge Z)$, that is the inequality is true and the lattice of ideals of the distributive lattice L is itself distributive. $\square$