Unit 2: Conditioning and Independence

- Conditioning: Revising a model based on new information.
  - Divide-and-conquer tools

- Independence
  Probability of occurrence of one event does not affect another.

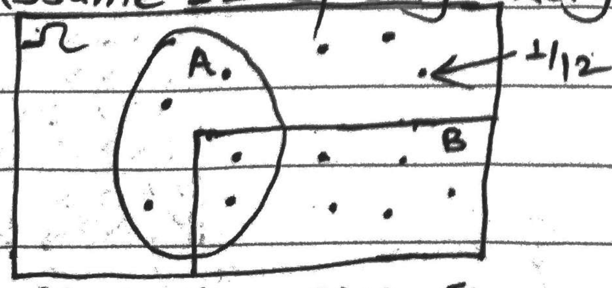
Conditional Probability (Baye's Rule)

- condition probability
  > Three important tools
  * Multiplication rule * Total probability theorem
  * Baye's rule (Inference)

Conditional Probabilities

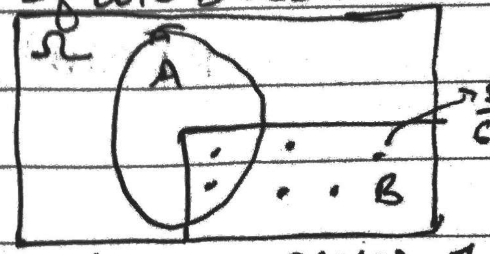
The idea of conditioning → use new information to revise a model

Assume 12 equally likely outcomes

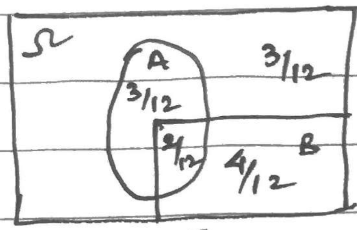


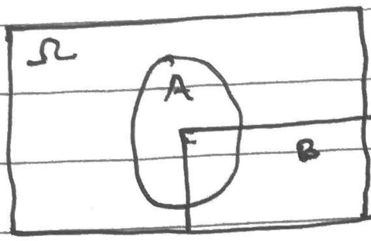
$P(A) = 5/12$ $P(B) = 5/12$

If told B occurred:



$P(A|B) = \frac{2}{6} = \frac{1}{3}$ $P(B|B) = 1$



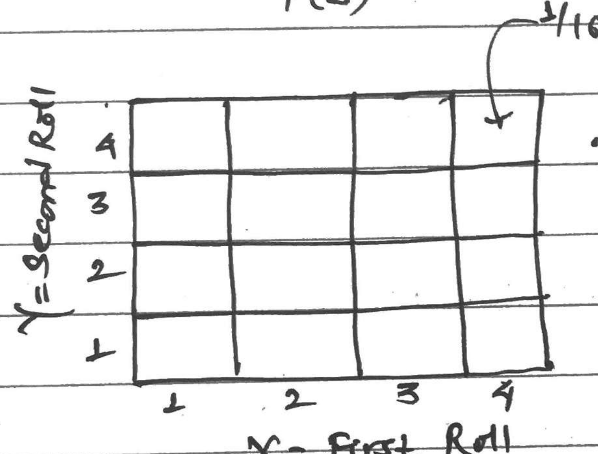


$P(A) = \frac{1}{3}$

$P(A) = 5/12, \; P(B) = 6/12$

- $P(A|B)$ = Prob. of A given that B occurred

$= \dfrac{P(A \cap B)}{P(B)}$ only when $P(B) > 0$.



- Let B be event: $\min(X, Y) = 2$

$M = \max(X, Y)$

$P(M = 1 | B) = 0$ (Impossible outcome)

$P(M = 3 | B) = \dfrac{P(M = 3 \cap B)}{P(B)}$

$= \dfrac{2/16}{5/16} = \dfrac{2}{5}$

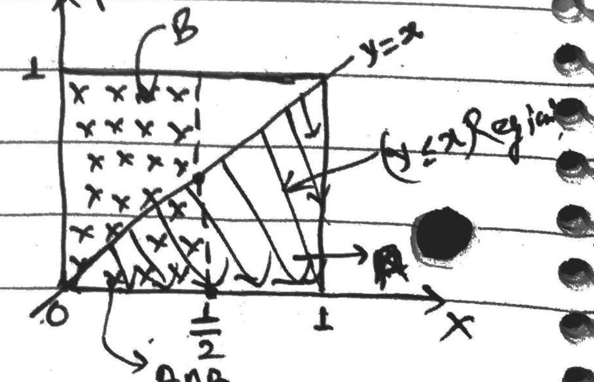
Exercise: Unit square $\rightarrow \Omega = [0,1]^2$ (sample space)

$A = \{(x, y) \in [0,1]^2 : y \leq x\}$

$B = \{x : x \leq 1/2\}$

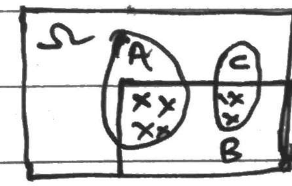
$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot 1 \cdot 1} = \dfrac{1}{4}$



Conditional Probabilities obey the same axioms,

- $P(A|B) \geq 0$ assuming $P(B) > 0$
- $P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$ (B becomes our new sample space)
- $P(B|B) = \frac{P(B \cap B)}{P(B)} = 1$



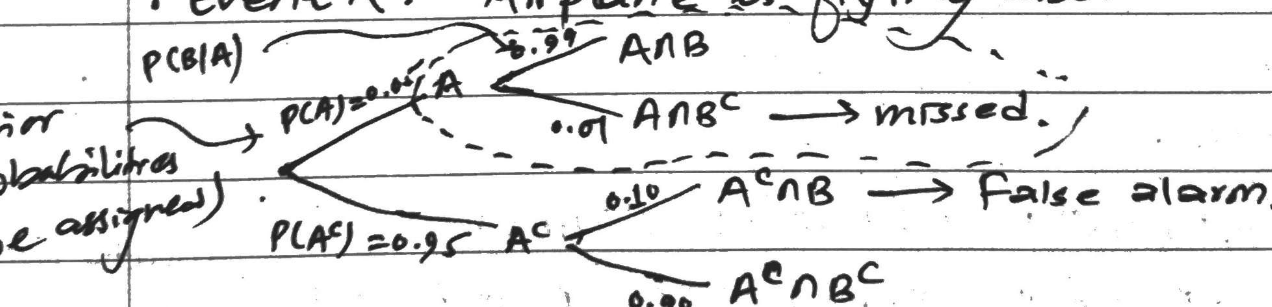
- If $A \cap C = \phi$, $P(A \cup C|B) = P(A|B) + P(C|B)$ (disjoint events)

$$= \frac{P((A \cup C) \cap B)}{P(B)} = \frac{P((A \cap B) \cup (C \cap B))}{P(B)}$$

$$= \frac{P(A \cap B) + P(C \cap B)}{P(B)} = P(A|B) + P(C|A)$$

A radar example: models based on conditional probabilities and three basic tools

- Event A: Airplane is flying above



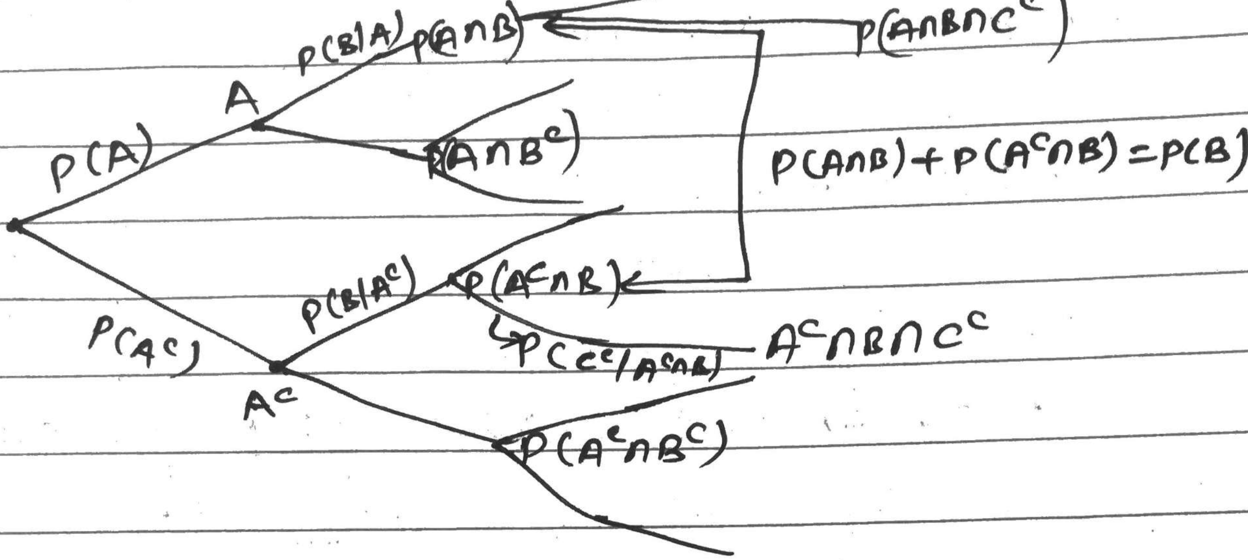
Event B: Something registers on screen

- $P(A \cap B) = P(A|B)\,P(B) = 0.05 * 0.99 = 0.0495$
- $P(B) = P(A|B)P(B) + P(B|A)P(A) = 0.05 * 0.99 + 0.95 * 0.1 = .1495$
- $P(A|B) = \frac{0.05 * 0.99}{0.195} = 0.34$

(A)

The Multiplication Rule

$$P(A\cap B) = P(A|B)\,P(B) = P(A|B)\,P(A)$$



$$P(A^c\cap B\cap C^c) = P\left((A^c\cap B)\cap C^c\right)$$
$$= P(A^c\cap B)\,P(C^c|A^c\cap B)$$
$$= P(A^c)\cdot P(B|A^c)\,P(C^c|A^c\cap B)$$

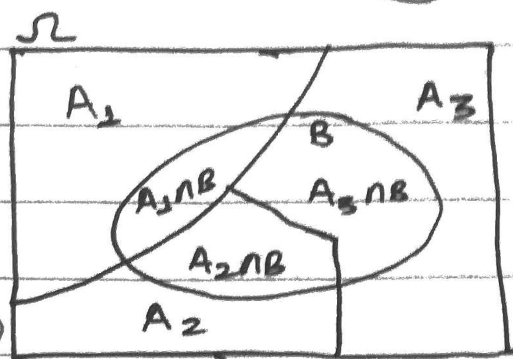
$$\boxed{P(A_1\cap A_2\cap\cdots\cap A_n) = P(A_1)\prod_{i=2}^{n} P(A_i|A_1\cap\cdots\cap A_{i-1})}$$

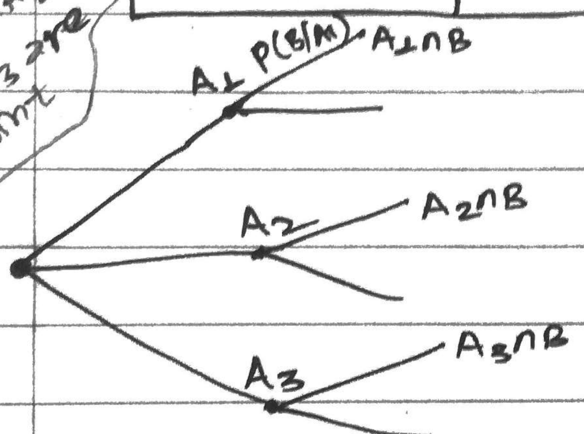
Exercise:

- $P(A\cap B\cap C^c) = P(A\cap B)\,P(C^c|A\cap B)$
- $P(A\cap B\cap C^c) = P(A)\,P(C^c|A)\,P(B|A\cap C^c)$
- $P(A\cap B\cap C^c) = P(A)\,P(C^c\cap A|A)\,P(B|A\cap C^c)$
- $P(A\cap B|C) = P(A|C)\,P(B|A\cap C)$

Note intersections are commutative and associative.

Total Probability Theorem





A_1, $B \cap A_2$, $B \cap A_3$ are disjoint

- Partition of sample space into $A_1, A_2, A_3, \ldots$ (three events or scenarios)
- Have $P(A_i)\ \forall\ i$
- Have $P(B|A_i)$, for every i.

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3)$$

$$= P(A_1)\,P(B|A_1) + \cdots + \cdots \quad \text{(weights)}$$

$$\Rightarrow \boxed{P(B) = \sum_i P(A_i)\,P(B|A_i)} \qquad \sum_i P(A_i) = 1$$

weighted average of conditional prob. of B $P(B|A_i)$

(Total Prob.)

Exercise: Total probability Theorem

Infinite collection of biased coins, indexed by the positive integers. Coin $i \xrightarrow[\text{being selected}]{\text{prob. of}} 2^{-i}$

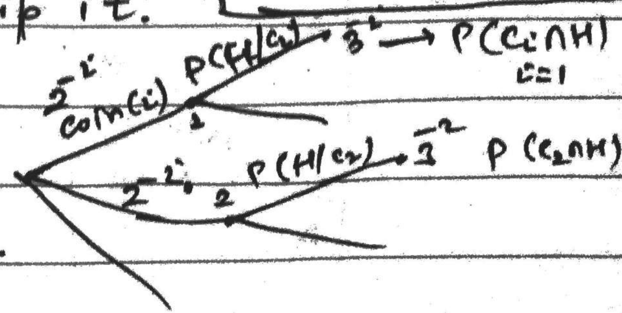
Flip of coin i $\xrightarrow[\text{in H w/}]{\text{results}}$ 3^{-i} $p(H) =$

$$\boxed{\sum_{i=1}^{\infty} \alpha^i = \frac{\alpha}{1-\alpha}, \ |\alpha| < 1}$$

We select a coin and flip it.

$P(H) = ?$

We think of the selection of coin i as scenario/event.



By the Total Probability Theorem,

$$P(\text{Heads}) = \sum_{i=1}^{\infty} P(H \cap A_i)$$

$$= \sum_{i=1}^{\infty} P(A_i)\, P(H/A_i)$$

$$P(\text{Heads}) = \sum_{i=1}^{\infty} 2^{-i}\, 3^{-i} = \frac{1}{6} \sum_{i=0}^{\infty} 6^{-i} = \frac{1}{6} \frac{1}{1-1/6} = \frac{1}{5}$$

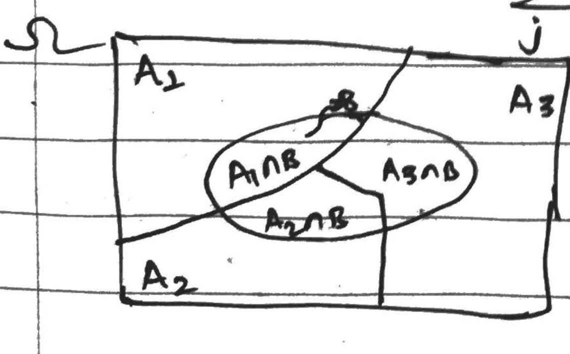
Baye's Rule

- Partition of sample space into $\overbrace{A_1, A_2, A_3}^{\text{(disjoint)}}$
- Have $P(A_i)\ \forall i$ ("initial beliefs")
- Have $P(B/A_i),\ \forall i$

"revised beliefs" given that B occurred.

$$P(A_i/B) = \frac{P(A_i \cap B)}{P(B)}$$

$$P(A_i/B) = \frac{P(A_i)\, P(B/A_i)}{\underbrace{\sum_j P(A_j)\, P(B/A_j)}_{P(B)}}$$



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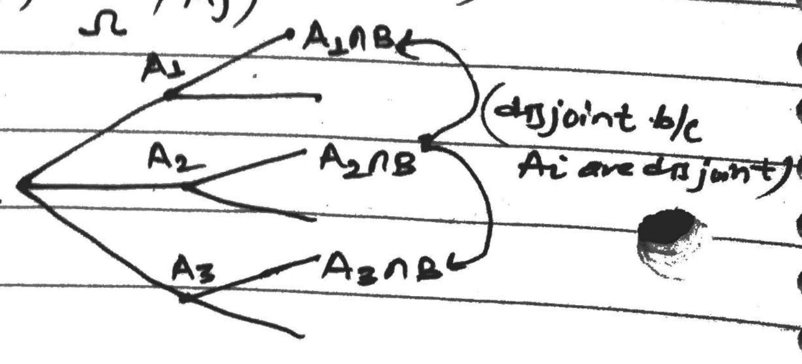


System approach for incorporating new evidence.

- **Bayesian Inference**
  - Initial beliefs $P(A_i)$ on possible causes of an observed event B
  - Model of the world under each A_i : $P(B/A_i)$

$$A_i \xrightarrow[P(B/A_i)]{\text{model}} B$$

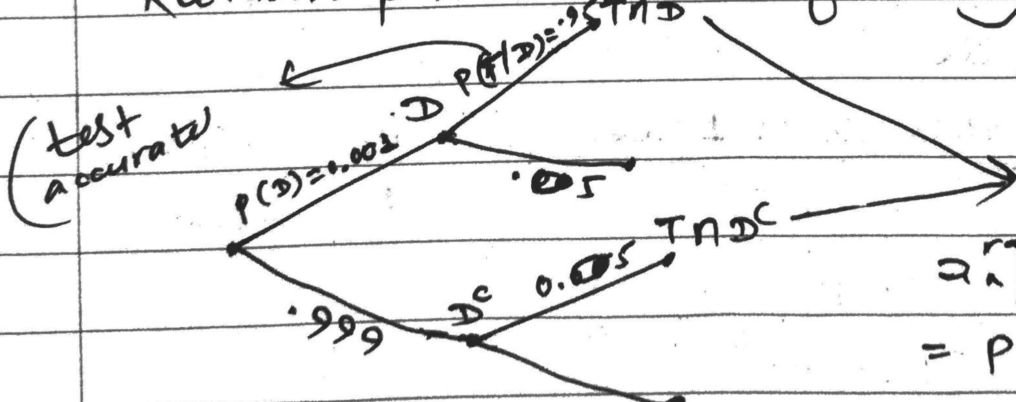
  - draw conclusions about causes

$$B \xrightarrow[P(A_i/B)]{\text{inference}} A_i$$

Exercise: Baye's Rule and the False-positive Puzzle.

Test → 95% accurate → event T

Random person chance of having a disease → 0.001
→ event D



$P(T)$ = Prob. that a random person tests positive

$= P(T \cap D) + P(T \cap D^c)$

$= .001 * .95 + .999 * .05$

$= .0509$

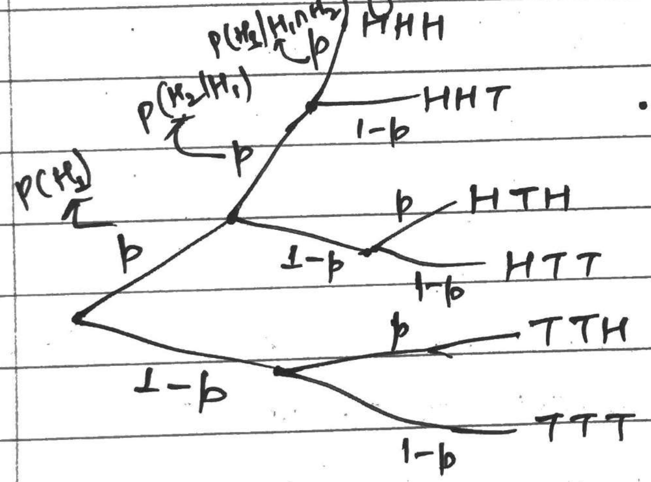
$$P(D|T) = \frac{P(T \cap D)}{P(T)} = \frac{.001 * .95}{.0509} = .0187$$

Independence

- Independence of two events
- conditional Independence
- Independence of a collection of events
- Pairwise Independence

A model based on conditional probabilities

- 3 tosses of a biased coin: $P(H) = p$, $P(T) = 1-p$



- Multiplication rule

$$P(THT) = (1-p)\,p\,(1-p)$$

- Total probability

$$P(1\ \text{head}) = P(HTT) + P(THT) + P(TTH)$$
$$= 3p(1-p)^2$$

- Baye's Rule

$$P(\text{first toss in } H \mid 1\ \text{head}) = \frac{P(H_1 \cap 1\ \text{Head})}{P(1\ \text{Head})}$$

~~out new~~ (our new sample space)

$$= \frac{p(1-p)^2}{3p(1-p)^2} = \frac{1}{3}$$

NB: $P(H_2 \mid H_1) = P(H_2 \mid T_1) \rightarrow$ what occurs in first trial does not affect its second trial outcome

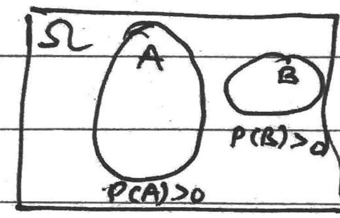
$\therefore$, $P(H_2) = P(H_1)\,P(H_2 \mid H_1) + P(T_1)\,P(H_2 \mid T_1) = p$

Independence of two events

- Intuitive "definition": $P(B|A) = P(B)$
  - occurrence of A provides no new information about B $\longrightarrow$ $P(A \cap B) = P(A) \cdot P(B|A) = P(A) \cdot P(B)$

Definition of Independence: $P(A \cap B) = P(A) \cdot P(B)$

- symmetric w.r.to A and B
- implies $P(B|A) = P(A)$
- applies even if $P(A) = 0$



Independent? $P(A \cap B) = 0$, $P(A) \cdot P(B) > 0$ $\Rightarrow$ Not independent but disjoint (dependent events)

Q. Let A be an event, a subset of the sample space Ω. Are A and Ω independent?

$\rightarrow$ Yes, because $P(A \cap \Omega) = P(A) = P(A) \cdot 1 = P(A) \cdot P(\Omega)$

Intuitively, $P(A)$ represents our beliefs about the likelihood that A will occur. If we are told that Ω occurred, this does not give us any new information; we already know that Ω is certain to occur. For this reason, $P(A|\Omega) = P(A)$
$= P(A|A)$.

Q. An event A is independent of itself if $P(A) = 0$ or 1. i.e. $P(A \cap A) = P(A) \cdot P(A)$

$P(A) = P(A) \cdot P(A)$

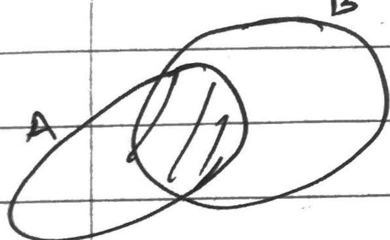
$P(A)[1 - P(A)] = 0$ $\Rightarrow$ $\boxed{P(A) = 1 \text{ or } 0.}$

Independence of Event Complements,

- If A and B are independent, then A and B^c are independent.

– Intuitive argument

→ if Occurrence of A does not tell anything bout B then nor about B^c.



(disjoint)

$$\rightarrow A = (A \cap B) \cup (A \cap B^c)$$

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

$$\boxed{P(A \cap B^c) = P(A) \cdot P(B^c)}$$

symmetry $\rightarrow P(A^c \cap B) = P(A^c) \cdot P(B)$

Q. Also if A and B are independent then A^c and B^c are.

$$\rightarrow P(A^c \cap B^c) = [P(A \cup B)]^c \quad \text{[De-Morgan's Law]}$$

$$= 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

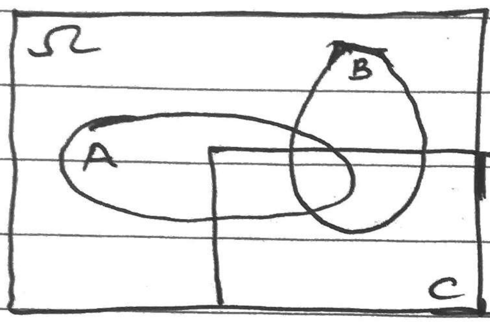
$$= [1 - P(A)] - P(B)\,[1 - P(A)]$$

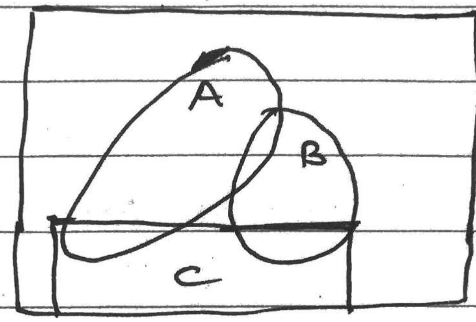
$$\boxed{P(A^c \cap B^c) = P(A^c) \cdot P(B^c)}$$

Conditional Independence

→ Conditional independence, given C, is defined as independence under the probability law $P(\cdot|C)$

Assume A and B are indep.





$P(A \cap B|C) = P(A|C)\,P(B|C)$

Iff A and B are independent under condition C has occurred

. If we're told that C occurred, are A and B indep.?

⟶ NO → if A occurs this tells us that B did not occur (don't for~~ge~~get, C has occurred ⇒ new sample space is C and in C, A and B are disjoint ⇒ if one A (or B) occurs, the other does not occur).

Q. Suppose that A and B are conditionally independent given C. Suppose that $P(C) > 0$ and $P(C^c) > 0$.

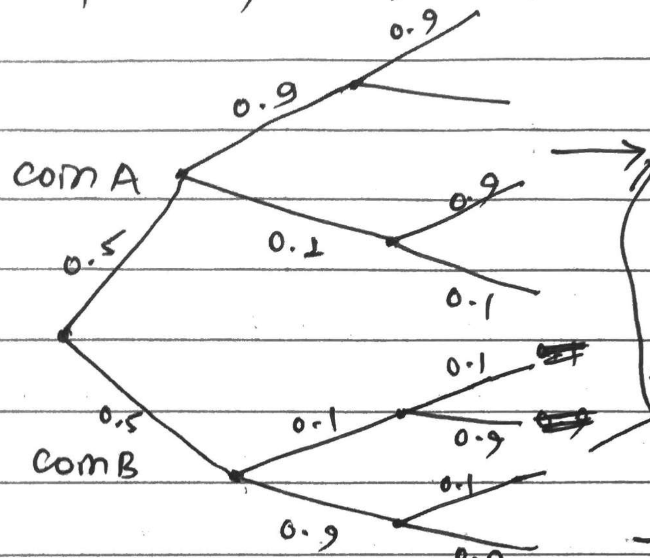
1. A and B^c are guaranteed to be conditionally independent given C. (just like in prob model)
2. A and B are (not guaranteed) conditionally dependent given C^c. (only true if A and B both have zero probability)

Independence vs Conditional Independence

Conditioning may affect independence

- Two unfair coins A and B:

$P(H|\text{coin } A) = 0.9$, $P(H|\text{coin } B) = 0.1$



given a coin (either A or B): we've independent tosses! (Conditionally independent)

⇒ Q. Are coin tosses independent? (here)

Compare:

$P(\text{toss } 11 = H) = P(A)\,P(H_{11}|A) + P(B)\,P(H_{11}|B)$

$= .5 * .9 + .5 * .1 = 0.5$ → unconditional probability.

$P(\text{toss}11 = H \mid \text{first } 10 \text{ tosses are heads})$

$\approx P(H_{11}|A) = 0.9$ → conditional probability

(we're extremely certain that we're dealing w/ coin A)

⇒ conditional prob. $= 0.5 \neq$ unconditional prob. $= 0.9$ ⇒ no conditional independence of coin tosses!

Independence of a collection of events,

- Intuitive "definition": Information on some of the events does not change probabilities related to the remaining events

$A_1, A_2, ----$ independent if $P(A_3 \cap A_4^c) = P(A_3 \cap A_4^c \,/\, A_1 \cup (A_2 \cap A_5^c))$

$P(A_3) = P(A_3 | A_1 \cap A_2) = P(A_3 | A_1 \cap A_2^c) = P(A_3 | A_1^c \cap A_2)$

these indices vs these indices should be different

Definition: Events $A_1, A_2, ---, A_n$ are called independent if $P(A_i \cap A_j \cap --- \cap A_m)$ $= P(A_i)\, P(A_j) --- P(A_m)$ for any distinct indices $i, j, --m$.

$n = 3$: a)
$$\left.\begin{aligned} P(A_1 \cap A_2) &= P(A_1) \cdot P(A_2) \\ P(A_1 \cap A_3) &= P(A_1) \cdot P(A_3) \\ P(A_2 \cap A_3) &= P(A_2) \cdot P(A_3) \end{aligned}\right\} \text{pairwise independence}$$

$\not\Downarrow$ (in general)

b) $P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$

ⓐ and ⓑ both condition suffice that A_1, A_2 & A_3 are independent.

Ex: Suppose that A, B, C and D are independent. Use intuitive reasoning to answer the following:

1. Is it guaranteed that $A \cap C$ is indep. from $B^c \cap D$?

→ The occurrence of event $A \cap C$ contains information about A and C, but provides no information on the occurrence of B, D, or for that matter, ~~$B^c \cap D$~~ $B^c \cap D$. Hence, we have independence.

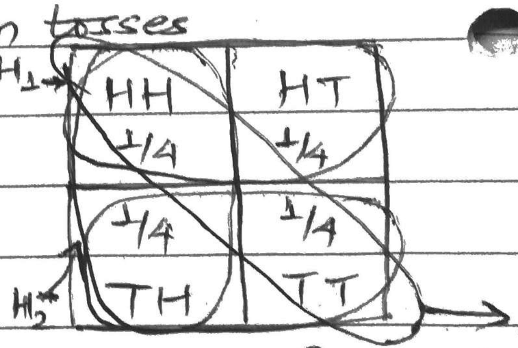
2. Is it guaranteed that $A \cap B^c \cap D$ is independent from $B^c \cup D^c$.

→ Event D influences both the events $A \cap B^c \cap D$ and $B^c \cup D^c$, and therefore introduces a dependence between them. For a more concrete argument, if we are told that event $A \cap B^c \cap D$ occurs, we know that D occurred. Therefore, D^c did not occur, and this generally reduces the probability of event $B^c \cup D^c$.

Independence vs Pairwise Independence.

- Two independent fair coin tosses
  - H_1: First toss is H
  - H_2: Second toss is H

  $P(H_1) = P(H_2) = \frac{1}{2}$



- C: The two tosses had the same result $= \{HH, TT\}$

(define new event) $P(H_1 \cap C) = P(H_1 \cap H_2) = 1/4$ ⇒ H_1 and C are independent.

and $P(H_1)P(C) = \frac{1}{2} \cdot \frac{1}{2}$

Also, H_1, C : independent ; H_2, C : independent

$P(H_1 \cap H_2 \cap C) = P(HH) = 1/4$ → H_1, H_2 and C are pairwise independent, but not independent

$P(H_1)P(H_2)P(C) = \frac{1}{8}$

15

Another way to understand independence,

$P(C|H_1) = P(H_2|H_1) = P(H_2) = \frac{1}{2} = P(C)$

$\Rightarrow H_1$ and H_2

$P(C|H_1 \cap H_2) = 1 \neq P(C) = \frac{1}{2}$ carry information relevant to C and hence H_1, H_2 and C are not independent.

Reliability

p_i: probability that unit i is "up"

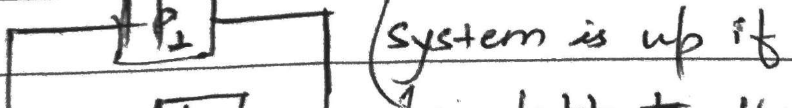
independent units

U_i: ith unit up

$U_1, U_2, \ldots, U_n$ are independent

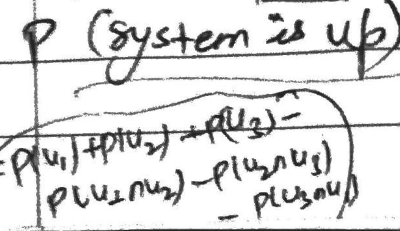
F_i: ith unit is down.

$\Rightarrow F_i$ independent. (if two events are independent then their complements are also independent)



(system is up if there is a line from left to right)

$P(\text{system up}) = P(U_1)P(U_2)P(U_3) = p_1 p_2 p_3$



(system is up if there exists a path from left to the right that consists of units that are "up").

$P(\text{system is up}) = P(U_1 \cup U_2 \cup U_3)$ (De-Morgan's law)

$= 1 - P(F_1 \cap F_2 \cap F_3) = 1 - P(F_1).P(F_2).P(F_3)$

$= 1 - (1-p_1)(1-p_2)(1-p_3)$

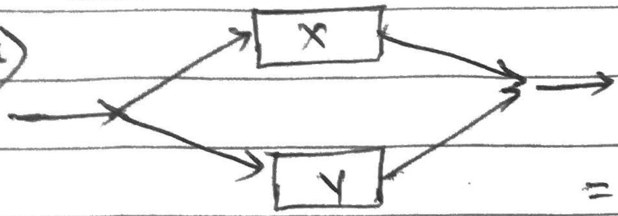
$= P(U_1) + P(U_2) + P(U_3) - P(U_1 \cap U_2) - P(U_2 \cap U_3) - P(U_3 \cap U_1)$

sample space

(16)

Independent Units (Failurs are also ind.)

<u>NB</u>: 1)



$P(X \cup Y) = P(\text{Either } X \text{ OR } Y)$

$= P(X) + P(Y) - P(X \cap Y)$

or $P(X \cup Y) = 1 - P[(X \cup Y)]^c = 1 - P(X^c \cap Y^c)$

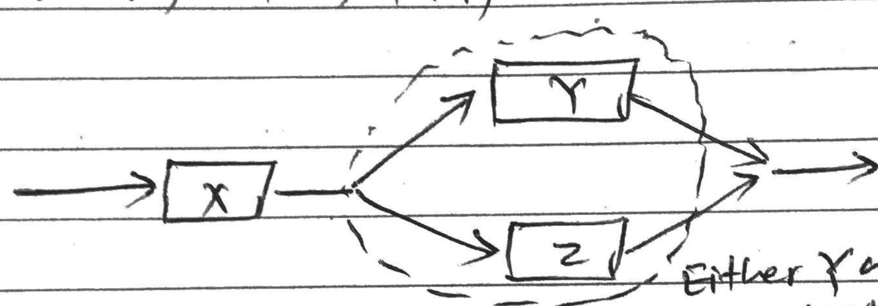
$= 1 - P(X^c) \cdot P(Y^c)$

$\equiv \to \boxed{X} \to \boxed{Y} \to (S_1)$

2) $\longrightarrow \boxed{X} \longrightarrow \boxed{Y} \longrightarrow$

$P(X \text{ AND } Y) = P(X) \cdot P(Y)$

3)



Either Y or Z works

$\Downarrow$

S_1

$\equiv \longrightarrow \boxed{X} \longrightarrow \boxed{S_1} \longrightarrow$

$= P(X \cap S_1) = P(X) \cdot P(S_1) = P(X)\left[1 - P(Y^c) \cdot P(Z^c)\right]$

The King's sibling.

The king comes from a family of two children. What is the probability that his sibling is female?

Assume a) Boys have a precedence
b) $P(B) = P(G) = 1/2$
c) independent
d) they have exactly two children

BB $1/4$	BG $1/4$
GB $1/4$	GG $1/4$

$\xrightarrow{\text{we know}}$ king in Boy (our new universe)

In new sample space the three events are also independent as they were independent in unconditional (event) probability. Now, each outcome has equal probability $1/3$.

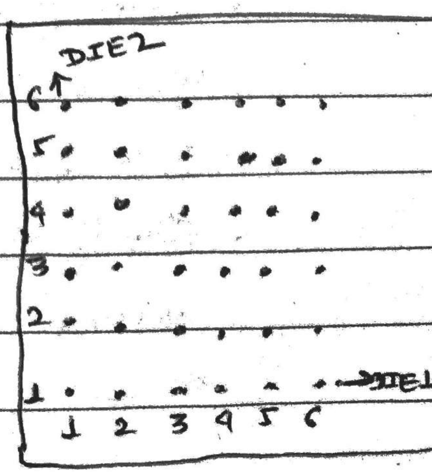
$P(\text{King sibling is female}) = 2 \cdot \frac{1}{3} = \frac{2}{3}$

If we modify our assumptions:

a) Royal family decided to have child until a boy is born $\Rightarrow P(G) = 1$ (the first was girl).

b) Till 2 boys, $P(G) = 0$

Conditional Probability Example.



Discrete Uniform Law: $P(A) = \frac{|A|}{|\Omega|}$

1. $P(\text{Doubles}) = P(D) = \frac{|D|}{36} = \frac{1}{6}$

2. $P(D \mid \text{sum} \le 4) = \frac{P(D \cap \text{sum} \le 4)}{P(\text{sum} \le 4)} = \frac{2}{6}$ (better way) or $\frac{2/36}{6/36}$

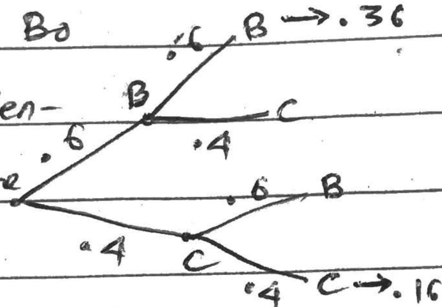
(new sample space)

3. $P(\text{at least one is } 6) = \frac{11}{36}$
S

4. $P(S \mid \text{lands in diff. number}) = \frac{10}{30} = \frac{1}{3}$
(new sample space)

Chess Tournament Problem

This year's Belmont chess champion is to be selected by the following procedure. Bo and Ci, the leading challengers, first play a two-game match. If one of them wins both games, he gets to play a two-game second round with Al, the current champion. Al retains his championship unless a second is required and the challengers beats Al in both games. If Al wins the initial game of the second round, no more games are played.



Furthermore, we know the following:

- The probability that Bo will beat Ci in any particular game is .6.
- The prob. that Al will beat Bo in any particular game is .5.
- The prob. that Al will beat Ci in any particular game is .3

Assume no tie games are possible and all games are independent.

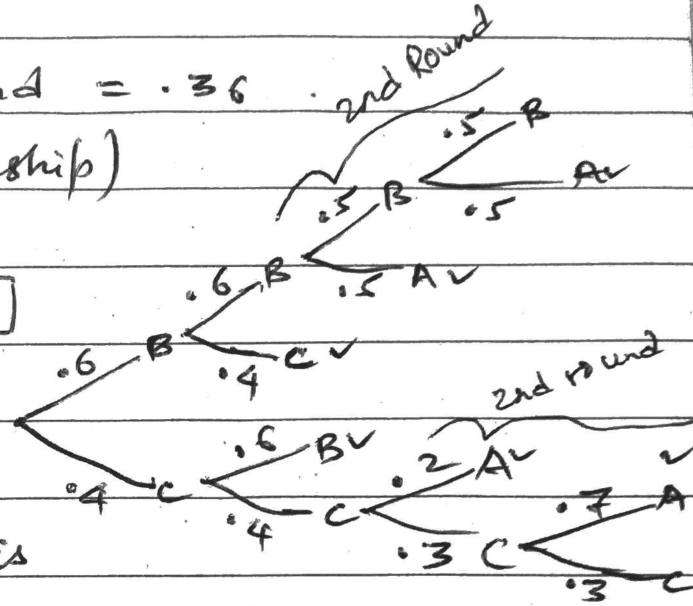
(19)

1. Determine the a priori probabilities tha

a) the second round will be required $= R_2$

$P(R_2) = .36 + .16 = .52$

b) Bo wins the first round $= .36$



c) P(Al wins the championship)

$= 1 - P(A^c)$

$= 1 - [.6^2(.5)^2 + (.4)^3(.3)^2]$

$= .8956$

2. Given the second round is required, determine the conditional probabilities that

a) Bo is surviving the Challenger

$$P(B|R_2) = \frac{P(B \cap R_2)}{P(R_2)} = \frac{.36}{.52} = .6923$$

b) $$P(A|R_2) = \frac{.6^2(.5) + .6^2 .5^2 + .4^2(.7) + (.4)^2(.3)(.7)}{.52}$$

$= .7992$

3) Given that the second round is required and that it comprised only one game, what is the conditional probability that it was Bo who won the first round?

$I = \{$ 2nd round was 1 game only $\}$

$B = \{$ Bo won 1st round $\}$

$$P(B|\underbrace{R_2 \cap I}_{W}) = P(B|W) = \frac{P(B \cap W)}{P(W)}$$

$$= \frac{P(B \cap R_2 \cap I)}{P(R_2 \cap I)} = \frac{(.6^2).5}{.4^2(.7)+(.6^2)(.5)} = .6164$$

Coin Toss Puzzle

A: 1st toss is head $\Rightarrow A \cap B$

B: 2nd toss is head

$P(A \cap B \mid A)$

$\boxed{\geq}$? $\longrightarrow$ ~~check if this is True~~

$P(A \cap B \mid A \cup B)$

a) $\dfrac{P((A \cap B) \cap A)}{P(A)} = \dfrac{P(A \cap B)}{P(A)}$ $[\because A \cap B \subseteq A]$

b) $\dfrac{P(A \cap B \mid A \cup B)}{P(A \cup B)} = \dfrac{P(A \cap B)}{P(A \cup B)}$ $[\because A \cap B \subset A \cup B]$

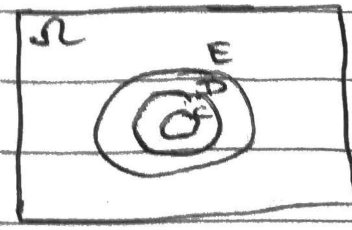
Since $P(A) \leq P(A \cup B)$

$\Rightarrow \boxed{P(A \cap B \mid A) \geq P(A \cap B \mid A \cup B)} \rightarrow$ does not matter if coin is fair or not.

Events: C, D, E

1) $D \subset E$

2) $C \cap D = C \cap E$



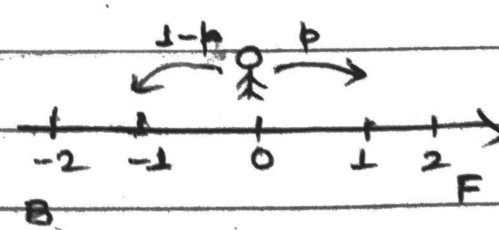
$P(C|D) \geq P(C|E)$

* If $C = A \cap B$, $D = A$ and $E = A \cup B$ → then we recover the earlier result (for coin)

A Random Walker.

Imagine a drunk tightrope walker, who manages to keep his balance, but takes a step forward with probability p and takes a step back w/ prob. $(1-p)$

a) What is the probability that after two steps, the tightrope walker will be at the same place on the rope as where he started?



soln → FB or BF → $\{FB, BF\} = A$

$P(A) = P(\{FB, BF\})$

$= P(\{FB\}) + P(\{BF\})$

$= p(1-p) + (1-p)p = 2p(1-p)$

b) What is the probability that after three steps, the tightrope walker will be one step forward from where he started?

soln → $\{FFB, FBF, BFF\} = C$

$P(C) = 3p^2(1-p)$

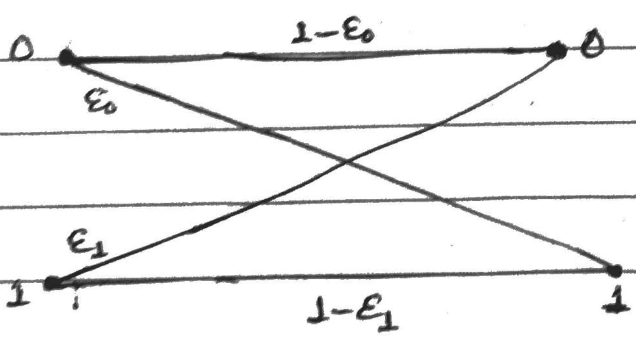
c) Given that after three steps he has managed to move ahead one step, what is the probability that the first step he took was a step forward?

Soln → $D = \{FFB, FBF\}$

$$P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{P(D)}{P(C)} \quad [\because D \subset C] = \frac{2}{3}$$

Communication Over A Noisy Channel

A source transmits a message (a string of symbols) over a noisy communication channel. Each symbol is 0 or 1 with probability p and $1-p$, respectively and is received incorrectly with probability ε_0 and ε_1, respectively (see the figure below). Errors in different symbol transmissions are independent.



Assume:
a) Multiplication
b) Total probability
c) Independence
d) Baye's Rule

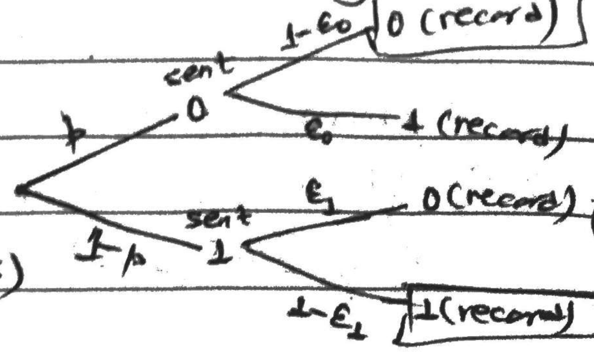
a) What is the probability that the kth symbol is received ~~in~~correctly?

Soln → P(kth symbol correctly sent)

$= p(1-\varepsilon_0) + (1-p)(1-\varepsilon_1)$

$= P(0)\,P(\text{success}) + P(1)\,P(\text{success})$

$= P(\text{success})$



b) What is the probability that the string of symbols 1011 is received correctly? $\rightarrow P(1011 \rightarrow 1011)$

Soln $\rightarrow$ $P(1\rightarrow 1 \cap 0 \rightarrow 0 \cap 1 \rightarrow 1 \cap 1 \rightarrow 1)$

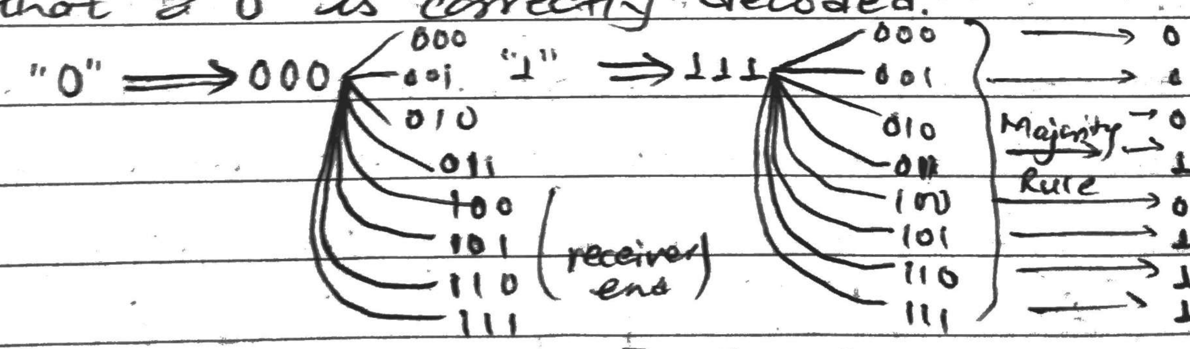
(Independence)

$= P(1\rightarrow 1) \cdot P(0\rightarrow 0) \cdot P(1\rightarrow 1) \cdot P(1\rightarrow 1)$

$= (1-\varepsilon_0)(1-\varepsilon_1)^3$

c) In an effort to improve reliability, each symbol is transmitted three times and the received string is decoded by majority rule. In other words, a 0 (or 1) is transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0's (or 1's respectively). What is the probability that a 0 is correctly decoded?

Soln $\rightarrow$



$P(000 \rightarrow 000) = (1-\varepsilon_0)^3 + 3(1-\varepsilon_0)^2\varepsilon_0$

d) For what value of ε_0 is there an improvement in the probability of correct decoding of a 0 when the scheme of part (c) is used?

Soln $\rightarrow$ $P(\text{"0"} \mid 101) = \dfrac{P(\text{"0"})\,P(101 \mid \text{"0"})}{P(101)}$

$P(101) \rightarrow P(\text{"0"})\,P(101 \mid \text{"0"}) + P(\text{"1"})\,P(101 \mid \text{"1"})$

$P(\text{"0"}) = p \qquad p(\text{"1"}) = 1-p$

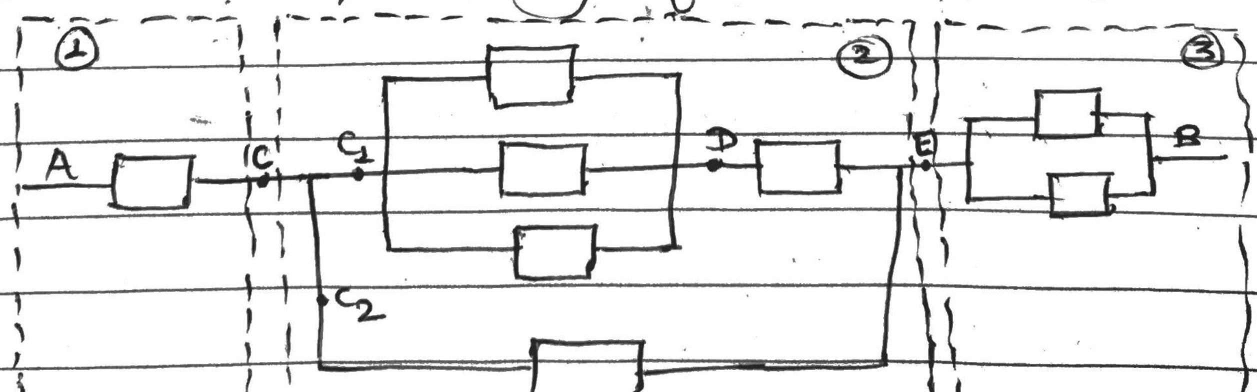
$P(101 | \text{"0"}) = (1-\varepsilon_0)\varepsilon_0^2$

$P(101 | \text{"1"}) = (1-\varepsilon_0)^2\varepsilon_1$

$$P(\text{"0"} | 101) = \frac{p(1-\varepsilon_0)\varepsilon_0^2}{p(1-\varepsilon_0)\varepsilon_0^2 + (1-p)(1-\varepsilon_1)^2\varepsilon_1}$$

Network Reliability.

An electrical system consists of identical components, each of which is operational w/ prob. p, independent of other components. The components are connected in three subsystems, as shown in the figure. The figure is operational if there ~~are~~ is a path that starts at point A, ends at point B, and consists of operational components. What is the probability of this happening?



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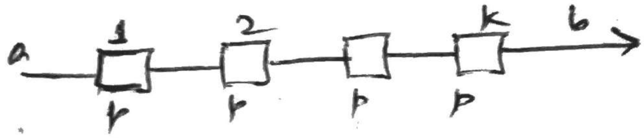
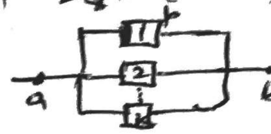


$p^k = P(a\to b)$



$P(a\to b) = 1-(1-p)^K$

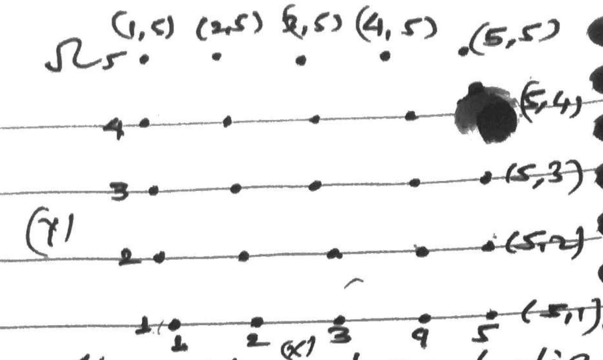
$P(A\to B) = ? = \underbrace{P(A\to C)\,P(C\to E)\,P(E\to B)}_{\text{(Independent subsystems)}}$

$= p\cdot[1-(1-p)^2]\,P(C\to E)$

$P(C\to E) = 1-(1-P(C_1\to E))\,(1-P(C_2\to E))$

$= 1-\{1-(1-(1-p)^3 p)\}\{1-p\}.$

$P(A\to B) = p[1-(1-p)^2]\{1-(1-p)[1-p[1-(1-p)^3]]\}$



Problem Set: 2

1. Two five-sided dice

You roll two five-sided dice. The sides of each die are numbered from 1 to 5. The dice are "fair" (all sides are equally likely), and the two die rolls are independent.

Part (a): 1. Event A is independent of the event "at least one of the dice resulted in a 5"?

→ NO (Dependent)

Event $A = \{sum = 10\} = \{(5,5)\} \rightarrow P(\text{Event } A) = \frac{1}{25}$

Event B = at least one die resulted in $5 = \{(1,5), \ldots (5,5), \ldots, (5,1)\}$

$P(\text{Event } B) = \frac{9}{25}$

$P(\text{sum} = 10 \cap \text{at least one } 5) = P\{(5,5)\} = 1/25$ $\boxed{A \subset B}$

$P(A \cap B) = \frac{1}{25} \neq P(A) \cdot P(B) = \frac{1}{25} \cdot \frac{9}{25}$

Also, $P(\text{sum} = 10 \mid \text{at least one 5 occurred}) = \frac{1}{9} \neq P(\text{sum}=10) = \frac{1}{25}$

2. Is event A independent of the event "at least one of the die resulted in a 1"?

→ NO (Dependent)

Since $B = \{\text{at least one } 1\} = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (3,1), (4,1), (5,1)\} \Rightarrow$ Event A cannot occur b/c $A = \{(5,5)\}$.

Occurrence of event B tells us that event A cannot occur, i.e. they're dependent (mutually exclusive events are always dependent!)

Part(b): Event B is "the total is 8".

1. Is event B independent of getting "doubles" (i.e, both dice resulting in the same number)?

$B = \{(3,5),(5,3),(4,4)\} \rightarrow$ total = 8

$A = \{(1,1),(2,2),(3,3),(4,4),(5,5)\} \rightarrow$ doubles.

$P(\text{total}=8 \cap \text{doubles}) = P(A \cap B) = P\{(4,4)\} = \frac{1}{25}$

$P(\text{total}=8) = P(B) = \frac{3}{25}$

$\Rightarrow P(A) \cdot P(B) = \frac{3}{125}$

$P(\text{doubles}) = P(A) = \frac{5}{25} = \frac{1}{5}$

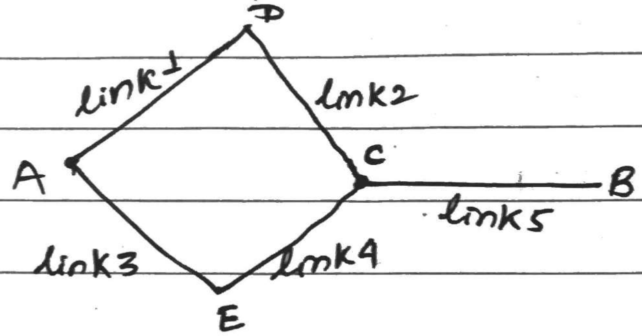
Since $P(A \cap B) = \frac{1}{25} \neq P(A) \cdot P(A) = \frac{3}{125} \Rightarrow$ dependent.

Can we tell if B occurs then A does not occur? $\longrightarrow$ NO $\rightarrow$ Not mutually exclusive $\rightarrow$ Independent?

2. $P(\text{at least one die results in a } 3 \mid \text{total} = 8) = \frac{2}{3}$

2. A Reliability Problem

Consider the communication network shown in the figure below and suppose that each link can fail w/ prob. p. Assume that failures of different links are independent.



1. Assume that $p = 1/3$. Find the probability that there exists a path from A to B along which no link has failed.

soln → P(either ADC works or AEC works $\cap$ CB works)

$$= P(A\to D\to C \cup A\to E\to C \cap C\to B)$$

$$= P(A\to D\to C \cup A\to E\to C) \cap P(C\to B)$$

$$P(A\to D\to C \cup A\to E\to C) = P((A\to D)\cap(D\to C) \cup (A\to E)\cap(E\to C))$$

$$= P(A\to D \cap D\to C \cancel{A\to E \cap E\to C}) + P(A\to E \cap E\to C) - P(A\to D \cap D\to C \cap A\to E \cap E\to C)$$

$$= 2(1-p)^2 - (1-p)^4$$

$$P(\underline{\qquad}) = \underline{\qquad} \cdot P(C\to B)$$

$$= 2(1-p)^3 - (1-p)^5$$

$$= \frac{112}{243}$$

2. Given that exactly one link in the network has failed, find the probability that there exists a path from A to B along which no link has failed?

Soln → P(path exists from A→B | at least one link has failed)

$= \frac{4}{5}$

Total links = 5

In How many ways a link can fail? → $^5C_1 = 5$
(either link1, ----, link5 → one of them can fail)

How many paths exist w/ one link failed?
link1,2 fails → link3, link4, link5 (A→B) → 2 paths
link 3,4 fails → link1, link2, link5 (A→B) → 2 paths

Problem-3: Oscar's lost dog in the forest

Oscar has lost his dog in either forest A (with probability 0.4) or in forest B (with probability 0.6). If the dog is in forest A and Oscar depends a day searching for it in forest A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in forest B and Oscar spends a day looking for it there, he will find the dog that day with probability 0.15.

The dog cannot go from one forest to the another.

Oscar can search only in the daytime, and he can travel from one forest to the other only overnight.

The dog is alive during day 0, when Oscar loses it, and during day 1, when Oscar starts searching. It is alive during day 2 with probability 2/3. In general, for $n \geq 1$, if the dog is alive during day $n-1$, then the probability it is alive during day n is $2/n+1$. The dog can only die overnight. Oscar stops searching as soon as he finds his dog either alive or dead.

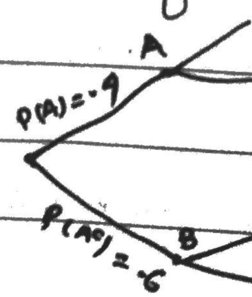
a) In which forest should Oscar look on the first day of the search to maximize the probability that he finds his dog that day?

Set up: Let A be the event that the dog was lost in forest A and A^c be the event that the dog was lost in forest B. Let D_n be the event that the dog dies on the nth day. Let F_n be the event that the dog is found on nth day. Let S_n be the event that Oscar searches forest A on nth day and S_n^c be the event that he searches forest B on day n.

Given $P(A) = .4$, $P(A^c) = .6$

$$P(D_{n+1} \mid D_n^c \cap F_n^c) = \frac{n}{n+2}$$

prob. that dog will die on the (n+1)th night given it was alive till nth day & was not found



$P(F_n \mid A \cap S_n \cap F_{n-1}^c) = .25$

prob. of finding on n^{th} day given dog was lost in forest A and was searched (forest A) on nth day and was not found until (n-1)th day.

$P(F_n \mid B \cap S_n^c \cap F_{n-1}^c) = .15$

(searched on nth day in forest B)

Soln: $P(F_1 / S_1) \overset{?}{=} P(F_1 / S_1^c)$

$P(F_1/S_1) = P(F_1 \mid S_1 \cap A) P(A) + \underbrace{P(F_1 / S_1 \cap A^c)}_{0} \cdot P(A^c)$

$= .25 * .4 = .1$

Ily, $P(F_1 \mid S_1^c) = P(F_1 \mid S_1^c \cap A) P(A) + \underbrace{P(F_1 \mid S_1^c \cap A^c)}_{0} P(A^c)$

$= .15 * .6 = .09$

$\Rightarrow P(F_1/S_1) > P(F_1 \mid S_1^c) \Rightarrow$ should search in A.

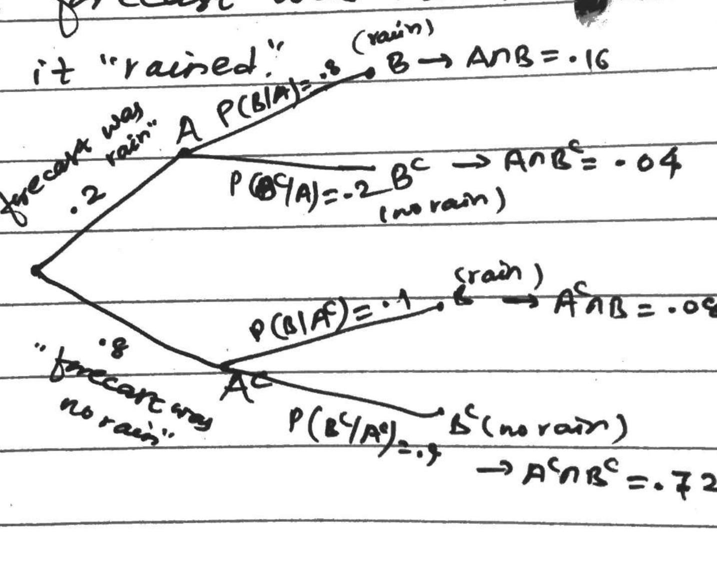
Problem-4: Serap and her umbrella

Before leaving for work, Serap checks weather report in order to decide whether to carry an umbrella. On any given day, with probability .2 the forecast is "rain" and with probability .8 the forecast is "no rain", the probability of actually having rain on that day is .8. On the other hand, if the forecast is "no rain," the probability of actually raining is .1.

1. One day, Serap missed the forecast and it rained. What is the probability that forecast was "rain"?

Soln → Let A be the event that forecast was "rain".

Let B be the event that it "rained."

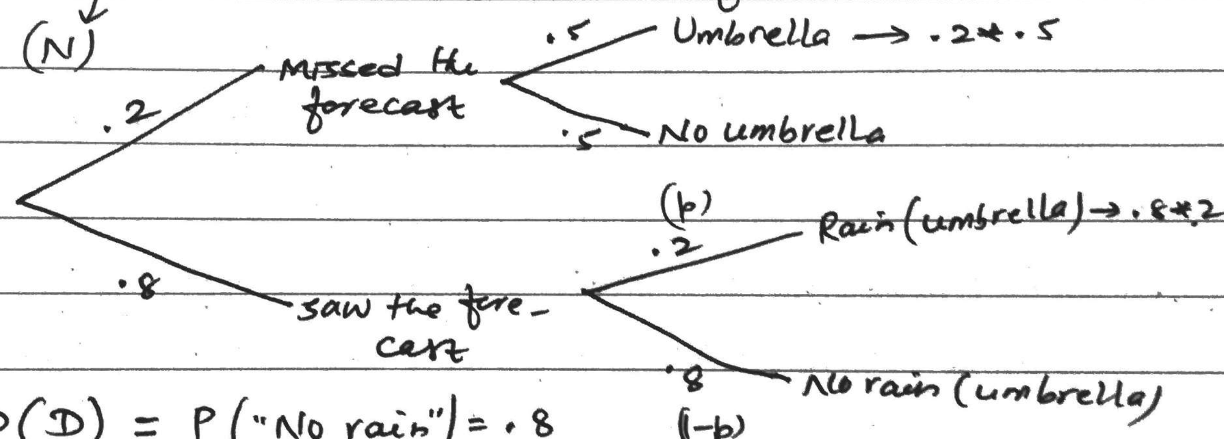


$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{.16}{.16 + .08} = \frac{2}{3}$$

b) Serap misses the morning forecast with prob. 0.2 on any day in the year. If she misses the forecast, Serap will flip a fair coin to decide whether to carry an umbrella. (We assume that the result of the coin flip is independent from the forecast and the weather). On any day she sees the forecast, if it says "rain" she will carry an umbrella, and if it says "no rain" she will not carry an umbrella. Let U be the event that "Serap is carrying an umbrella", and let N be the event that the forecast is "no rain". Are events U and N independent?

Soln → Let C (U) be the event that Serap is carrying an umbrella.
Let D (N) be the event that the forecast is "no rain".



$P(D) = P(\text{"No rain"}) = .8$

$P(C) = P(\text{"carry an umbrella"}) = .2*.5 + .2*.8 = .26$

$$P(C/D) = \frac{P(C\cap D)}{P(D)} = \frac{\cancel{P(C).P(D)}}{\cancel{P(D)}} \cancel{= .26} \neq P(C) \text{ unless } p=0.$$

⇒ N and U can never be independent.

3. Serap is carrying an umbrella and it is not raining. What is the probability that she saw the forecast?

(34)

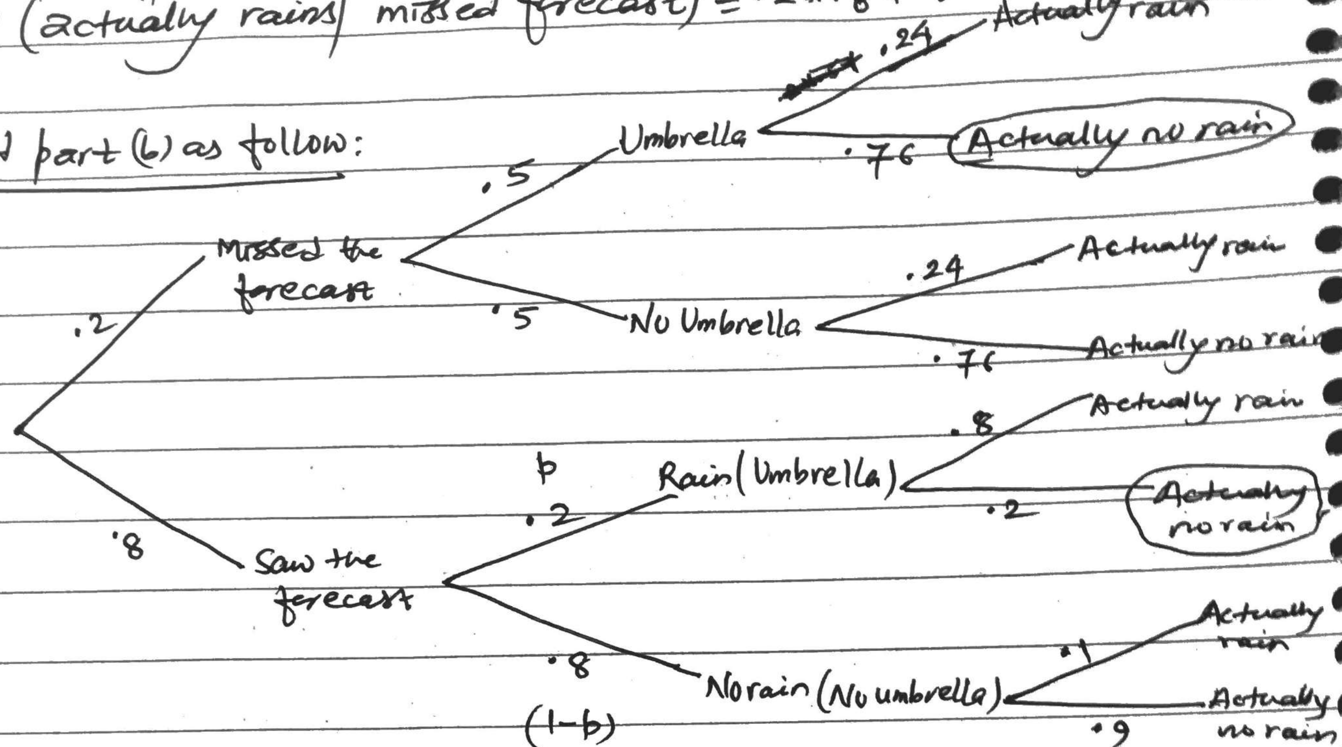
(does not tell anything about if it rains or not)

part (a)

(rains)

$P(\text{actually rains} \mid \text{missed forecast}) = .2 * .8 + .8 * .1 = .24$

Extend part (b) as follow:



Therefore, given that Serap is carrying an umbrella and it is not raining, we are looking at two circled cases.

$$P(\text{saw forecast} \mid \text{umbrella and not raining}) = \frac{P(\text{saw forecast} \cap \text{umbrella} \cap \text{not raining})}{P(\text{umbrella} \cap \text{not raining})}$$

$$= \frac{.8 * .2 * .2}{.8 * .2 * .2 + .2 * .5 * .76}$$

$$= \frac{8}{27}$$