IDSS x 6.431x – Probability: The Science of Uncertainty and Data.

Probability Models and Axioms

- Sample space

- Probability laws
  > Axioms
  > Properties that follow from the axioms

- Examples
  > discrete
  > continuous

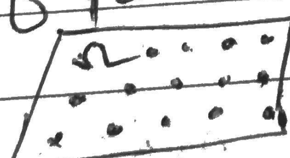
- Interpretations of Probabilities

Sample Space

- Two steps
  > describe possible outcomes
  > describe beliefs about likelihood of outcomes.

List of possible outcomes, $\Omega \rightarrow$ sample space

Coin $\rightarrow \Omega \rightarrow$ [H T]



(all possible outcomes)

Must be → Mutually exclusive; Collectively exhaustive;

at the "right" granularity (able to tell at the end which one was the outcome / which sample space Ω is "right" depends on what question you want to answer)

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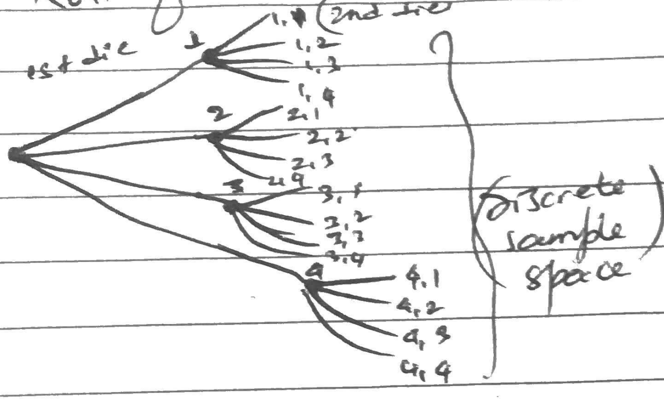
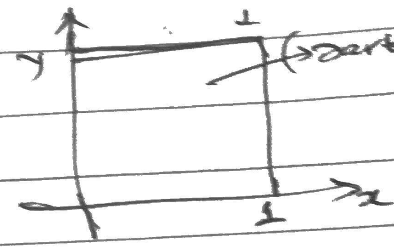
Sequential description (Tree)

Roll of two tetrahedral die



(sample space)

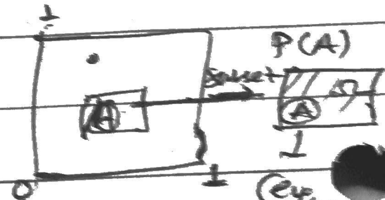
; (x, y) s.t. $0 \le x, y \le 1$



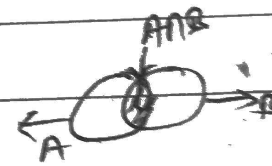
$P(x,y) = 0$ for a fixed value of x and y.

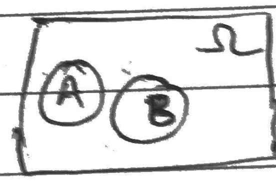
Probability Axioms,

- Event is a subset of the sample space
- Probability is assigned to events



- **Axioms:**
  - Nonnegativity: $P(A) \geq 0$
  - Normalization: $P(\Omega) = 1$
  - (Finite) additivity:
    If $A \cap B = \phi$; $P(A \cup B) = P(A) + P(B)$





Some simple consequences of the axioms,

- $P(A) \le 1$
- $P(\phi) = 0$
- $P(A) + P(A^c) = 1$. where $\Omega = A \cup A^c$; $A \cap A^c = \phi$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C)$ → similar for K disjoint ev

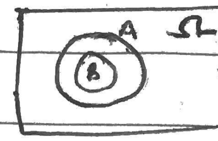
$P(\{s_1, s_2, ---, s_k\}) = P(\{s_1\} + ---+ \{s_k\})$

~~$= P(\{s_1\}) + P(\{s_2\}) + ------ + P(\{s_k\})$~~

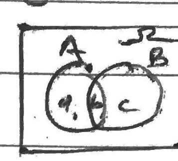
(Probability of a finite set) $= P(s_1) + P(s_2) + ---- + P(s_k)$

More Properties of Probabilities

- If $A \subset B$, $P(A) \le P(B)$

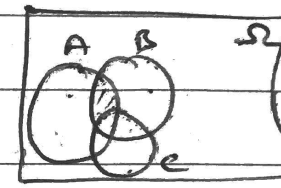


- If A and B are not two disjoint subset of a sample space, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$a = P(A \cap B^c)$, $b = P(A \cap B)$, $c = P(B \cap A^c)$

- $P(A \cup B) \le P(A) + P(B)$ (union bound)
- $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$



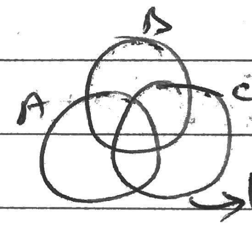
Exercise # $A, B, C \subseteq \Omega$, not necessarily disjoint.

$P((A \cap B) \cup (C \cap A^c))$

$= P(A \cap B) + P(C \cap A^c)$

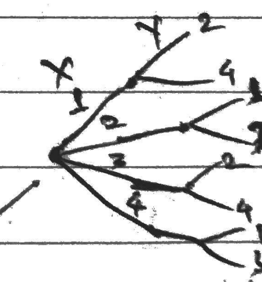
$\le P(A \cup B \cup C)$

Since $(A \cap B)$ and $(C \cap A^c)$ are subset of $(A \cup B \cup C)$.



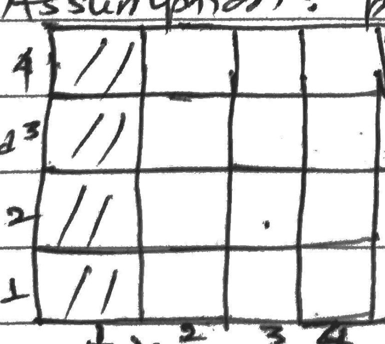
$\rightarrow P(C) + P(C^c \cap A) + P(C^c \cap A^c \cap B)$

Discrete example



- Two rolls of a tetrahedral die.

Assumption: probability of every outcome is equal to $1/16$.



Y = second roll

X = First Roll

- $P(X+Y = odd) = P(X = odd \cap Y = even) + P(X = even \cap Y = odd)$
- $P(X = 1) = 4 \cdot 1/16 = 1/4$

Let $Z = \min(X, Y)$

if $X = 2, Y = 3, Z = 2$.

- $P(Z = 4) = 1/16$
- $P(Z = 2) = 5 \cdot 1/16$ → $X = 2, Y = 2, 3, 4$; $Y = 2, X = 3, 4$

X=1, Y=1,3
X=3, Y=1,3
Y=1, X=3
Y=3, X=1
(X,Y) = (2,2) ...

$P(X > Y) = 6 \cdot 1/16$

Y = 1, X = 2, 3, 4
Y = 2, X = 3, 4.
Y = 3, X = 4

$P(X+Y = even) = P(X = odd \cap Y = odd) + P(X = even \cap Y = even)$

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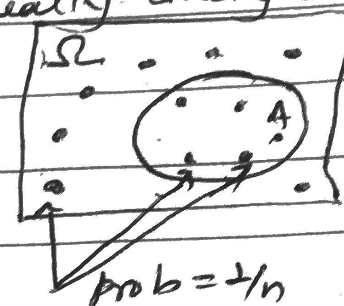
Discrete Uniform Law

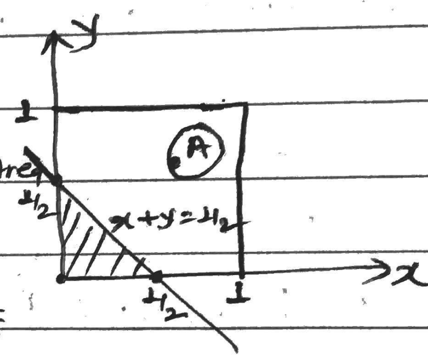
- Assume Ω (finite) consists of n equally likely elements
- Assume A consists of k elements



$$P(A) = k \cdot \frac{1}{n}$$

Continuous example

- (x, y) such that $0 \le x, y \le 1$
- Uniform probability law: Probability = Area



$$P\left(\{(x,y) \mid x+y \le \tfrac{1}{2}\}\right) = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{} = \frac{1}{8}$$

(Area of shaded triangle = $\frac{1}{2} \cdot$ base $\cdot$ height)

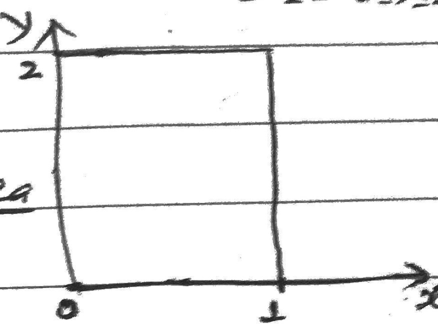
$$P(\underbrace{\{(0.5, 0.3)\}}_{\text{(single element)}}) = \text{Area of a single point } (0.5, 0.3) = 0$$

Probability calculation steps:

- Specify the sample space
- Specify a probability law
- Identify an event of interest
- Calculate ...

Exercise: sample space → rectangular region $[0,1] \times [0,2]$
$0 \le x \le 1 \quad 0 \le y \le 2$

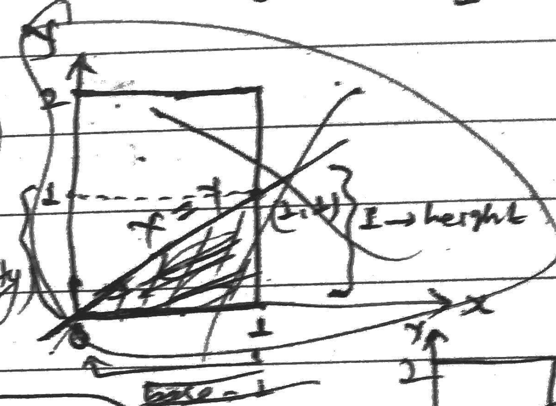
Uniform probability law: probability of an event is the half of the area of the event.

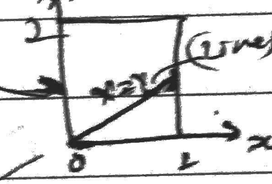


$P(X=Y) = \dfrac{\frac{1}{2}\cdot 0 \cdot 0}{2} = \dfrac{\frac{0}{2}}{2} = 0$

What is the area of a line?

from the definition of event → probability of area



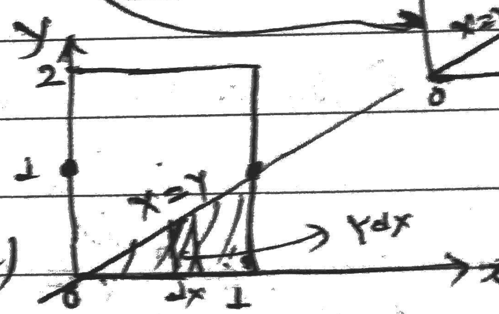


$P(X>Y) = P(Y<X)$

Inequality: Step I: $Y = X$ (always put Y 1st)

step II: $\geq$ or $>$ (top part)

$<$ (below part)

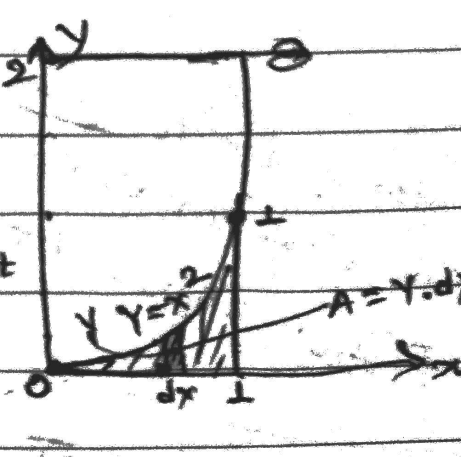


$P(X>Y) = \dfrac{\frac{1}{2}\cdot 1 \cdot 1}{2} = \dfrac{1}{4}$ or $\dfrac{\int_0^1 Y\,dx}{2} = \dfrac{1}{4}$

$P(X^2 \geq Y) = \dfrac{\int_0^1 Y\,dx}{2} = \int_0^1 \dfrac{x^2\,dx}{2} = \dfrac{1}{6}$

Inequality: $Y = X^2$ → step I

$Y \leq X^2$ → down part



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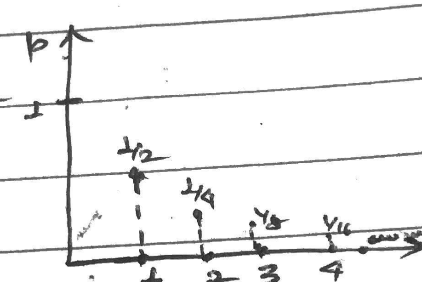
Countable additivity

- discrete but infinite sample space

> sample space: $\{1, 2, ---\}$

(set of integers which are +ve)

– We are given $P(n) = \frac{1}{2^n}$; $n = 1, 2, ---$



Check: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n}$ ⟶ (infinite geometric series)

$= \frac{1}{2} \frac{1}{1 - 1/2} = 1$ ⟶ This probability law is legitimate.

- $P(\text{outcome is even}) = P(\{2, 4, 6, ---\})$

$= P(\{2\} \cup \{4\} \cup \{6\} \cup ---)$

$= P(2) + P(4) + P(6) + ----$

$= \cancel{\frac{1}{2^n}} (\cancel{1} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + ---$

$= \frac{1}{4} \left(1 + \frac{1}{4} + \frac{1}{4^2} + ---\right)$

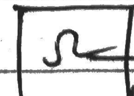
$= \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{3}$

This axiom is defined only for finite set.

What about infinite set?

for infinite sequence of disjoint events,

$P(A_1 \cup A_2 \cup A_3 \cup ---) = P(A_1) + P(A_2) + ---$



Ω ⟶ square $P(\Omega) = P(\cup \{x, y\})$

$\overset{?}{=} \sum P(\{x, y\})$

$= \sum 0 = 0 \, ! = 1$

⟹ Looks like paradox

⟶ This law does not apply to an uncountable set.

Additivity applies only on infinite sequence of disjoint events

NB: • Addititivity only holds for "countable" sequence of events

• The unit square (similarly, real Line, etc) is not countable (its elements cannot be arranged in sequence)

• "Area" is a legitimate probability law on the unit square, as long as we do not try to assign probabilities/areas to "very strange" sets.

Ex: $P(n) = \frac{1}{2^n}$ for $n = 1, 2, ---$

Probability on the sequence of multiple of 3.

$$P(\{3, 6, 9, ---\}) = P(\{3\} \cup \{6\} \cup \{9\} \cup ---)$$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + ---$$

$$= \frac{1}{2^3}\left(1 + \frac{1}{2^3} + \frac{1}{2^6} + ---\right)$$

$$= \frac{1}{2^3} \sum_{n=0}^{\infty} 1 \cdot \frac{1}{2^{3n}}$$

$$= \frac{1}{2^3} \sum_{n=0}^{\infty} 1 \cdot \frac{1}{8^n} = \frac{1}{8}\left(\frac{1}{1 - 1/8}\right) = \frac{1}{7}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

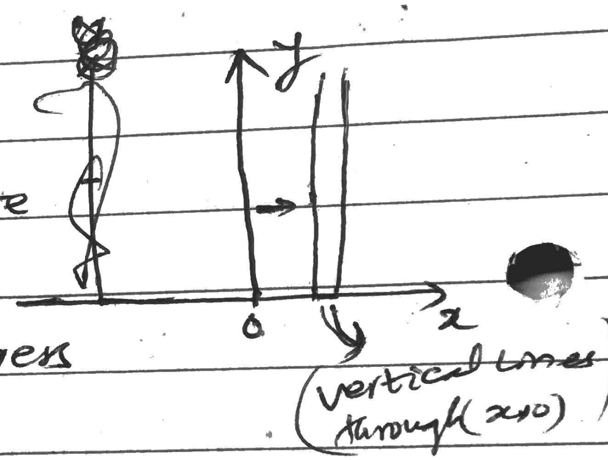
$$t_n = a + (n-1)d$$

Ex: Let the sample space be the set of all positive integers. Is it possible to have a "uniform" probability law, that is, a probability law that assigns the same probability c to each positive integer?

④

→ NO! Suppose $c=0$ then by countable additivity,
$1 = P(\Omega) = P(\{1\} \cup \{2\} \cup \{3\} \cup \cdots) = 0+0+\cdots = 0$
which is a contradiction.

Suppose $c>0$, there exists k such integers such that
$kc>1$. By additivity $P(\{1,2,\ldots,k\}) = kc > 1$ → contradiction of normalization axiom.



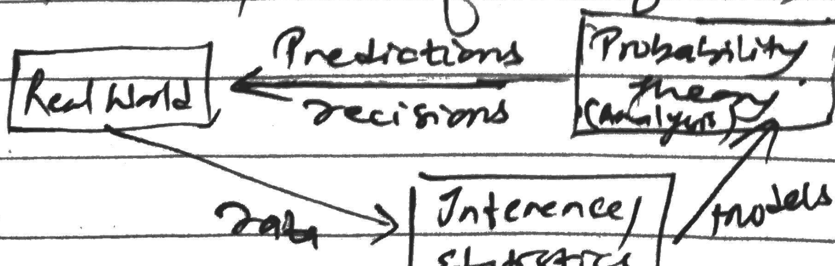
Ex: $A_x = \{(x,y) ; y \in \mathbb{R}e\}$
$P(\cup\{A_x\}) \neq \sum_x P(A_x)$ → infinite x value

If x are only positive integers
(countable sequence)
$P(\cup\{A_x\}) = \sum_x P(A_x)$

Interpretation and uses of probabilities

- Axioms ⇒ theorems "Thm": "Frequency" of event A "is" P(A)

- Probabilities are often interpreted as:
  > description of beliefs > Betting preferences

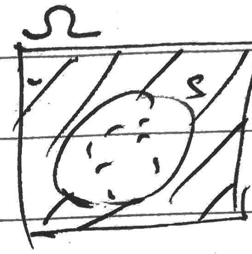


Sets

- Collection of distinct elements $\rightarrow \{\ \}$
  - $>$ finite set
  - $>$ infinite set

$x \in S$
$x \notin S$

$\{x \in \mathbb{R} : \cos(x) > \frac{1}{2}\} \longrightarrow$ a set



$\Omega \longrightarrow$ Universal set

$x \in S^c$ if $x \in \Omega$, $x \notin S$ $\qquad \Omega^c = \phi$

$S \subseteq T \Leftrightarrow x \in S \Rightarrow x \in T$

Unions and Intersections

$x \in S \cup T \Leftrightarrow x \in S$ or $x \in T$

$x \in S \cap T \Leftrightarrow x \in S$ and $x \in T$.

$S_n ; n = 1, 2, ---$

$x \in \bigcup_n S_n \Leftrightarrow x \in S_n$, for some n.

$x \in \bigcap_n S_n \Leftrightarrow x \in S_n \ \forall n$.

De-Morgan's law

- $\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$
- $\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$

$$\boxed{\begin{array}{l}(S \cap T)^c = S^c \cup T^c \\ (S \cup T)^c = S^c \cap T^c\end{array}}$$

Sequences and their limits,

sequence $a_i : \{a_i\}$ $\quad i \in \mathbb{N} = \{1, 2, 3, \ldots\}$

$a_i \in S \quad S = \mathbb{R}, \mathbb{R}^n$

function: $f : \mathbb{N} \to S \quad f(i) = a_i$

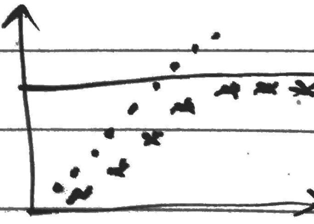
$a_i \xrightarrow[i \to \infty]{} a$

For any $\varepsilon > 0$, there exist i_0,
s.t. if $i \geq i_0$, then $|a_i - a| < \varepsilon$.

$$\left.\begin{array}{l} a_i \to a \\ b_i \to b \end{array}\right\} \Rightarrow a_i + b_i \to a + b$$

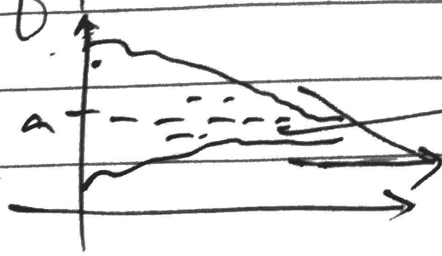
g: continuous function $\to g(a_i) \to g(a)$
If $a_i \to a$ then $a_i^2 \to a^2$

When does a sequence converge?



- If $a_i \leq a_{i+1}$ $\forall i$, then either:
  - the sequence "converges to ∞"
  - the sequence converges to some real # a.

- If $|a_i - a| \leq b_i$ $\forall i$ and $b_i \to 0$, then $a_i \to a$:



(our sequence)
(Two diff. sequence converge to a)

Infinite series

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \quad \text{provided limit exists.}$$

- if $a_i \geq 0$; limit exists.

- if terms a_i do not all have the same sign;
  > limit need not exist
  > limit may exist but be different if we sum in a different order

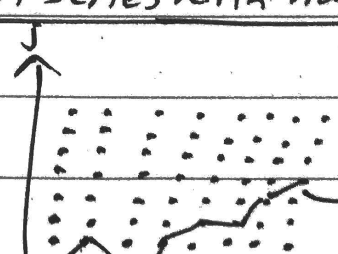
- Fact: limit exists and independent of order of summation if $\sum_{i=1}^{\infty} |a_i| < \infty$.

The geometric series

$$S = \sum_{i=0}^{\infty} \alpha^i \;;\; |\alpha| < 1 \longrightarrow S = \frac{1}{1-\alpha}$$

About the order of summation in series with multiple indices

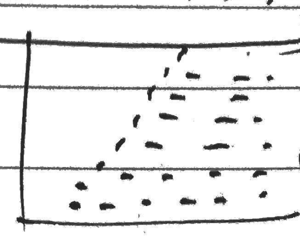
- $\sum_{i\geq 1, j\geq 1} a_{ij} \xrightarrow[\text{way}]{\text{one}} \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right)$



if $\sum_{i\geq 1, j\geq 1} |a_{ij}| < \infty \longrightarrow$ the order in which carry out summation does not matter

- $\sum_{(i,j): j\leq i} a_{ij} \xrightarrow[\text{way}]{\text{one}} \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} a_{ij}\right)$

  $\xrightarrow[\text{way}]{\text{another}} \sum_{j\geq 1}^{\infty}\left(\sum_{i\geq j}^{\infty} a_{ij}\right)$



will be same as long as $\sum_{(i,j): j\leq i} |a_{ij}| < \infty$

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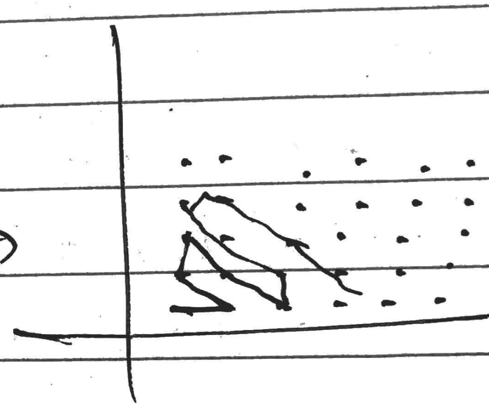
Countable and Uncountable sets

(discrete)

can be put in 1-1 correspondence w/ the integers

$\{a_1, a_2, \_\_\_\} = \Omega$

eg. positive integers, integers, pairs of positive integers, rational



numbers w/ q, $0 < q < 1$.

$1/2, 1/3, 2/3, 1/4, \cancel{2/4}, 3/4, 1/5, ---$

(continuous)

not countable

- the interval $[0,1]$
- the reals, the plane

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Unit 1: Probability Models and Axioms

The probability of the symmetric difference of two events.

$$P\left((A \cap B^c) \cup (A^c \cap B) \right) = P(A) + P(B) - 2P(A \cap B)$$

(exactly one of A or B occurs)

Bonferroni Inequality

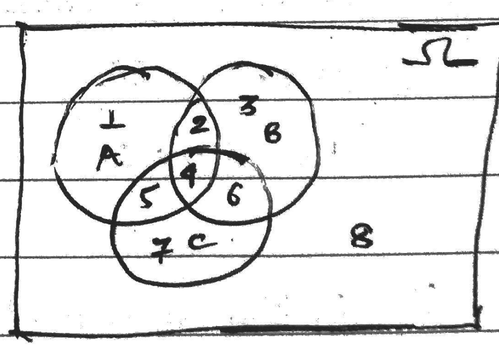
$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1 \longrightarrow$ can generalize to k events.

Problem Set 1

due Sep 8, 2020 19:59 EDT

Problem-1 Venn diagrams

eg. $1 \longrightarrow A \cap B^c \cap C^c \longrightarrow$ only A.



1. At least two of the events A, B, C occur.

Event E1: $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

$\cup \rightarrow$ either or at least

Regions: 2 4 5 6

2. At most two of the events A, B, C occur.

Event E2: $(A \cap B \cap C)^c \longrightarrow$ Not all three events occur

Regions: 1 2 3 5 6 7 8

$\longrightarrow$

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3. None of the events A, B, C occurs.

Event 5: $A^c \cap B^n \cap C^c$

Region: 8

4. All three events A, B, C occur.

Event E1: $A \cap B \cap C$

Region: 4

5. Exactly one of the events A, B, C occurs.

Event: E7

Regions: 1 3 7

6. Events A and B occur, but C does not occur.

Event: E3

Region: 2

7. Either (i) event B occurs, or (ii) neither B or C occurs.

Event: E4

Regions: 1 2 3 4 6 8

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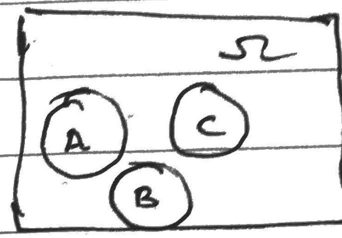
Problem 2: Set operations and probabilities

Find the value of $P(A\cup(B^c\cup C^c)^c)$ for each of the following cases:

1. The events A, B, C are disjoint events and $P(A) = 2/5$.

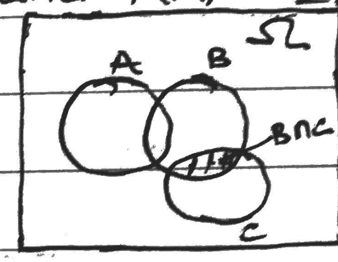
$$P(A\cup(B^c\cup C^c)^c) = P(A\cup(B\cap C))$$

$$= P(\underbrace{A\cup\phi}_{\text{~~Mutually~~ (disjoint sets)}})$$



$$= P(A)+P(\phi) = P(A) = 2/5$$

2. The events A and C are disjoint, and $P(A) = 1/2$ and $P(B\cap C) = 1/4$.



(De-Morgan's law)

$$P(A\cup(B^c\cup C^c)^c) = P(\underbrace{A\cup(B\cap C)}_{\text{disjoint sets}})$$

$$= P(A)+P(B\cap C) = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$$

3. $P(A^c\cap(B^c\cup C^c)) = 0.7$

$$P(A\cup(B^c\cup C^c)^c) = P(A\cup(B\cap C))$$

De-Morgan's law

$$= 1 - P([A\cup(B\cap C)]^c)$$

$$= 1 - P(A^c\cap(B\cap C)^c)$$

$$= 1 - P(A^c\cap(B^c\cup C^c))$$

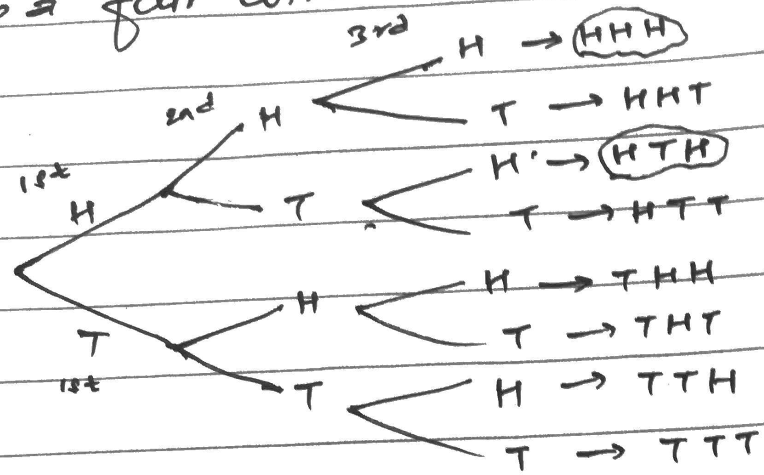
$$= 1 - 0.7$$

$$= 0.3$$

(14)

Problem 3: Three tosses of a fair coin. → $P(H) = P(T) = \frac{1}{2}$

Flip a fair coin three times → $(2^3 = 8)$



1. $P(\{HHH\}) = \frac{1}{8} = P(H) \cdot P(H) \cdot P(H)$

2. $P(\{HTH\}) = P(H) \cdot P(T) \cdot P(H) = \frac{1}{8}$

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

(# of ways HTH can occur)

3. $P(\text{Two heads \& one Tail}) = \frac{3}{8}$

$3 \times \frac{1}{2} \times \frac{1}{2}$

(three ways a head can occur)

4. $P(\text{\# of heads is greater than tails}) = \frac{4}{8} = \frac{1}{2}$

Problem 4 - Parking lot Problem

Mary and Tom park their cars in an empty parking lot with $n \geq 2$ consecutive parking spaces (i.e, n spaces in a row, where only one car fits in each space). Mary and Tom pick parking spaces at random; of course, they must each choose a different space. (All pairs of distinct parking spaces are equally likely). What is the probability that there is at most one empty parking space between them?

→ Order of parking matters → Permutation

Total possible pairs = ${}^{n}P_2 = n(n-1)$

1	2	-	-	-	-	n

If we park them w/o leaving space,
1st(1) & n (nth) parking space → can be parked car only on one side (1st → right & nth → left); all other (n-2) parking space → can be parked on both side

$= 2*1 + (n-2)*2 = 2n-2$

If we leave a space between them, 1st 2 & last 2, can leave space only one space on one side and for (n-4) spaces, can leave a space on both sides

$$P(\text{at most one space}) = \frac{4n-6}{n(n-1)}$$

Problem-5, Probabilities on a continuous sample space

Alice and Bob each choose at random a real number zero and one. We assume that the pair of numbers is chosen according to the uniform probability law on the unit square, so the probability of an event is equal to its area. We define the following events:

$A = \{$The magnitude of the difference (for any two real numbers x and y, the value $|x-y|$) of the two numbers is greater than $1/3\}$.

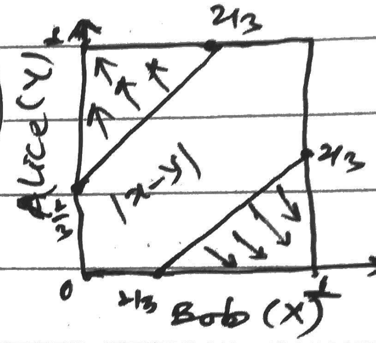
$B = \{$At least one of the numbers is greater than $1/4\}$

$C = \{$The sum of the two numbers is $1\}$

$D = \{$Alice's number is greater than $1/4\}$

Find the following probabilities:

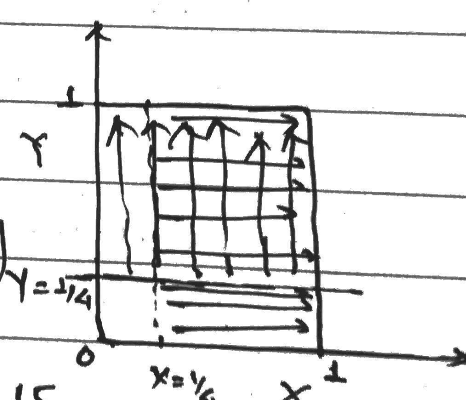
1. $P(A) = P\left(\{x,y\} \mid |x-y| > \frac{1}{3},\ 0 \le x \le 1,\ 0 \le y \le 1\right)$



$= 1 - P\left(\{x,y\} \mid |x-y| \le \frac{1}{3}\right)$

$= 1 - \left[1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot 2\right] = \frac{4}{9}$

2. $P(B) = P\left(\{x,y\} \mid \underbrace{x > \frac{1}{4} \text{ or } y > \frac{1}{4}}_{\text{(both)}}\right)$

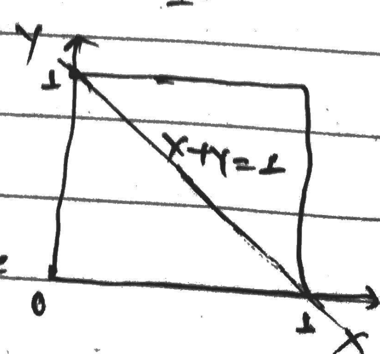


$= \underset{(X\text{ only})}{P(X > 1/4)} + \underset{(Y\text{ only})}{P(Y > 1/4)} + P\left(X \cap Y > \frac{1}{4}\right)$

$= \frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{15}{16}$

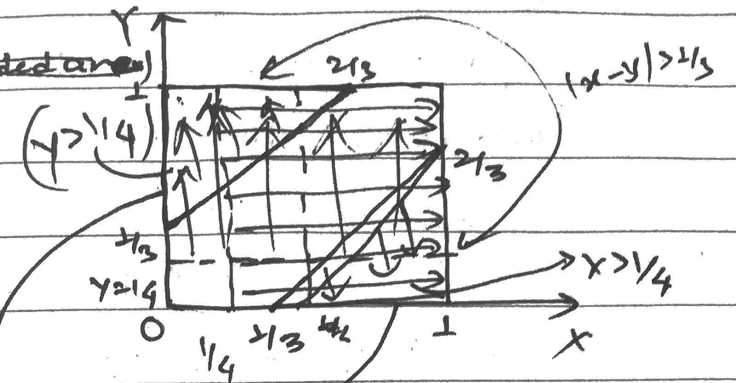
3. $P(C) = P(X+Y=1)$

$= 0 \longrightarrow$ Area of a line is zero.



4. $P(A \cap B) = P\left(\{x,y\} \mid |x-y| > \frac{1}{3} \text{ and } x > \frac{1}{4} \text{ or } y > \frac{1}{4} \text{ or } x \text{ and } y > \frac{1}{4}\right)$

$= \cancel{1 - P(\text{unshaded area})}$



$= \cancel{1 - \frac{1}{4} \cdot \frac{1}{4}}$

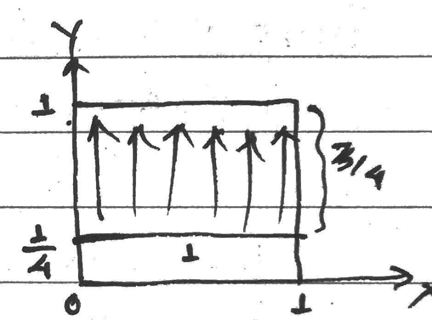
(area of two shared triangles)

$= 2 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3}$

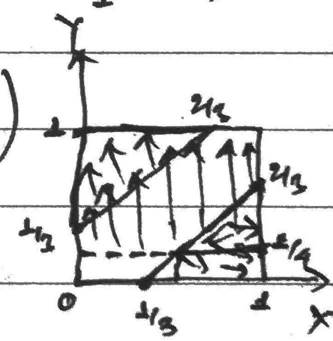
(these two triangles are common)

$= \frac{4}{9}$

5. $P(D) = P\left(Y > \frac{1}{4}\right) = \frac{3}{4} \cdot 1$



6. $P(A \cap D) = P\left(\{x,y\} \mid |x-y| > \frac{1}{3} \cap y > \frac{1}{4}\right)$



= sum of two shared triangles

$= \frac{11}{36}$

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Problem-5: Upper and lower bounds on the probability of intersection,

Given two events A, B with $P(A) = 3/4$ and $P(B) = 1/3$,

What is the smallest possible value of $P(A \cap B)$? The largest? That is, find a and b such that,

$a \leq P(A \cap B) \leq b$, holds and any value in the closed interval $[a,b]$ is possible.

$\longrightarrow P(A) = 3/4$

$P(B) = 1/3$

Bonferroni Inequality,

$P(A \cap B) \geq P(A) + P(B) - 1 = \frac{1}{12}$ → lowest value = a

The largest value $A \cap B$ can take is when $B \subseteq A$.

$P(A \cap B)_{max} = P(B) = \frac{1}{3}$ → maximum value = b

$$\boxed{\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}}$$