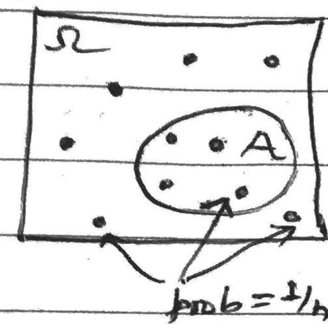
Unit 3: Counting

Discrete Uniform law

- Assume Ω consists of n equally likely events
- Assume A cosists of K elements

Then: $P(A) = \dfrac{\#\text{ of elements of } A}{\#\text{ of elements of } \Omega} = \dfrac{K}{n}$

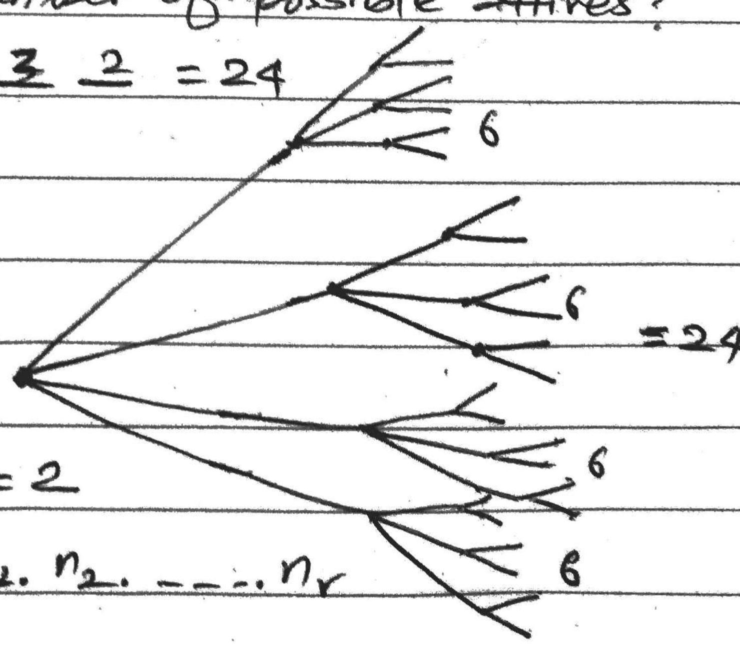


- Basic Counting Principle.
- Applications

permutations — number of students
combinations — binomial probabilities
partitions

* Basic Counting Principles.

4 shirts — Q. Number of possible attires?
3 ties — $\underline{4}\ \underline{3}\ \underline{2} = 24$
2 jackets



- r stages
- n_i choices at stage i

eg. $r = 3$

$n_1 = 4,\ n_2 = 3,\ n_2 = 2$

Number of choices $= n_1 . n_2 . \ldots . n_r$

Basic Counting Principle Example

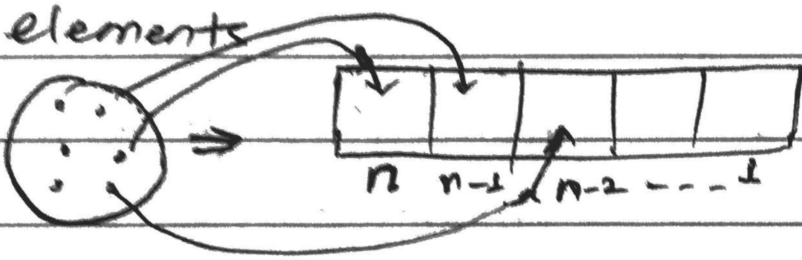
- Number of license plates with 2 letters followed by 3 digits:

$\underline{26}\ \underline{26}\ \underline{10}\ \underline{10}\ \underline{10} = 26 * 26 * 10 * 10 * 10$

** → if repetition is prohibited:

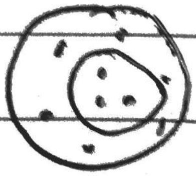
$\underline{26}\ \underline{25}\ \underline{10}\ \underline{9}\ \underline{8} = 26 * 25 * 10 * 9 * 8$

- Permutations: Number of ways of ordering n elements



$= n.(n-1)(n-2) - - - .1 = n!$

* Number of subsets of $\{1, ---, n\}$.



1st element choice

$\underset{1}{\underline{2}}\ \underset{2}{\underline{2}}\ - - - \underline{2} = 2^n$

(either in or outside subset)

eg. $n = \{1\}$

1 subset

$\{1\}, \phi \to 2^1 = 2 \to$ subsets

Exercise: Counting.

* You are given the set of letters $\{A, B, C, D, E\}$.

1. How many three-letter strings (i.e. sequence of 3 letters) can be made out of these letters if each letter can be used only once?

Soln. → $\underset{\text{(1st choice)}}{\underline{5}} * \underset{\text{(2nd choice)}}{\underline{4}} * \underset{\text{(3rd choice)}}{\underline{3}}$ = # of ways of Permuting (ordering) 3 elements out of 5 = ${}^5P_3 = 60$

(no repetition / order matters)

2. How many subsets does the set $\{A, B, C, D, E\}$ have?

Soln. → # of Subsets $= \underline{2} * \underline{2} * \underline{2} * \underline{2} * \underline{2} = 2^5 = 32$

3. How many five letter strings can be made if we require that each letter appears exactly once and the letters A and B are next to each other, as either "AB" or "BA"?

Soln. → Here, we can think of "AB" (or "BA") as a single unit and can assume we have four slots and four choices to make.

$2 * \underline{4} * \underline{3} * \underline{2} * \underline{1} = 48$

(AB or BA are different)
(choose whether order will be AB or BA)

$\{AB, C, D, E\} \rightarrow 4.3.2.1$
or
$\{BA, C, D, E\} \rightarrow 4.3.2.1$

(4)

Die Roll Example

- Find the probability that six rolls of a (six-sided) die $\underbrace{\text{all give different numbers}}_{\text{(event A)}}$.

(Assume all outcomes equally likely)

$$P(A) = \frac{\#\text{ in } A}{\text{Total \# of possible outcomes}} = \frac{6!}{6^6} =$$

Typical outcome → $(2,3,4,3,6,2) \to P(2,3,4,3,6,2) = \frac{1}{6^6}$

$= P(2) \cdot P(3) P(4) P(3) P(6) P(2)$

Typical element of A → $(3,2,4,1,6,5)$

How many ways can we order six elements? → $6!$

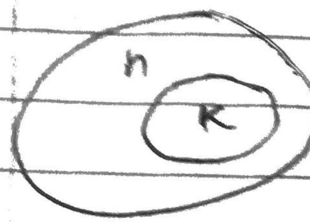
Exercise: Use counting to calculate probabilities

You are given the set of letters $\{A, B, C, D, E\}$. What is the probability that in a random five-letter string (in which each letter appears exactly once, and with all such strings equally likely) the letters A and B are next to each other? The answer to previous exercise may be useful here.

Soln. Total # of possible outcomes $= 5! =$

P(letters A and B are next to each other in five-letter string) $= \frac{2 * 4!}{} = \frac{2}{5}$

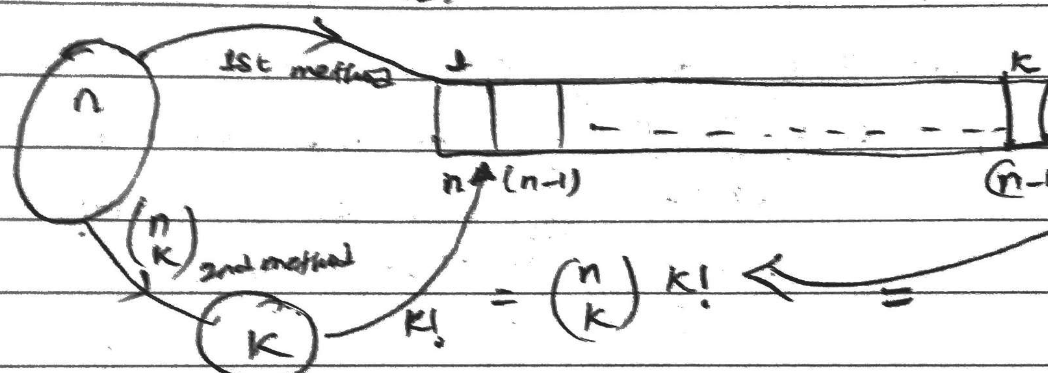
Combinations:



Pick k elements of original set with n elements.

$\binom{n}{k}$: # of k-element subsets of a given n-element set

* Two ways of constructing an ordered sequence of k distinct items:



$$n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!} = \binom{n}{k} k!$$

$$\Rightarrow \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \quad ; \; n = 0, 1, 2, \ldots \quad k = 0, 1, \ldots, n.$$

Case I: $\binom{n}{n} = 1$

Case II: $\binom{n}{0} = 1$ (Only subset having zero element → ϕ)

* $$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

$\binom{n}{0}$ (zero element subset), $\binom{n}{1}$ (1 element subset), $\binom{n}{n}$ (n element subset)

$= \#$ of all subsets $= 2^n$

Exercise: Counting Committees,

We start with a pool of n people. A chaired committee consists of $k \geq 1$ members, out of whom one member is designated as the chairperson. The expression $k\binom{n}{k}$ can be interpreted as the number of possible chaired committes with k members. This is because we have $\binom{n}{k}$ choices for the k members, and once the members are chosen, there are then k choices for the chairperson. Thus,

$$c = \underbrace{\sum_{k=1}^{n} k}_{\text{(2nd stage selection choice)}} \underbrace{\binom{n}{k}}_{\text{(1st stage/choice)}}$$ is the total number of possible chaired committees of any size.

Find the value of c (as a function of n) by thinking about a different way of forming a chaired committee: first choose the chairperson, then choose the other members of the committee. The answer is of the form: $c = (\alpha + n^{\beta})\, 2^{\gamma n + \delta}$

What are the values of α, β, γ and δ?

$\alpha =$ $\beta =$ $\gamma =$ $\delta =$

Soln → One chairperson → $\binom{n}{1} = n$ ways of choosing a chair person → 1st stage

2nd stage → choose $k \geq 1$ members out of $(n-1)$ people

$$\sum_{k=0}^{n-1} \binom{n-1}{k} = \underbrace{\binom{n-1}{0}}_{\substack{\text{zero element subset} \\ \text{(0 elements subset)}}} + \cdots + \underbrace{\binom{n-1}{k}}_{\text{(k element subset)}} = 2^{n-1}$$

→ each elements can belong to a subset or not →

$\Rightarrow$ Total possible choices $= n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$

Binomial Probabilities,

Binomial coefficient $\rightarrow \binom{n}{k} \rightarrow$ Binomial probabilities

- $n \geq 1$ independent coin tosses; $P(H) = p$ $\quad P(k \text{ heads}) = ?$

$$\text{eg. } \overset{n=6}{P(HTTHHH)} = P(H)P(T)P(T)P(H)P(H)P(H) = p^4(1-p)^2$$

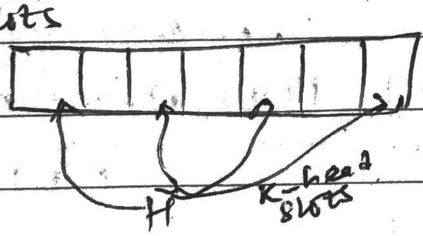
$$P(\text{particular sequence}) = p^{\# \text{heads}}(1-p)^{\# \text{tails}}$$

$$P(\text{particular } k\text{-head sequence}) = \binom{n}{k} p^k (1-p)^{n-k}$$

$\rightarrow$ (# of tosses)

(Can happen in many different ways) $\Rightarrow$ (# k-head sequence)

n slots



$\Rightarrow \binom{n}{k}$

Exercise: Binomial Probabilities

Recall that the probability of obtaining k Heads in n independent coin tosses is $\binom{n}{k}p^k(1-p)^{n-k}$, where p is the probability of Heads for any given coin toss. Find the value of $\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}$

Soln. "0 Heads", "1 Heads", ---, "n Heads" are disjoint, and their union is the entire sample space. Sum of probability of all these events = 1.

A Coin Tossing Example

- Given that there were 3 heads in 10 tosses, what is the probability that first two tosses were heads?
  - event A: the first two tosses were heads.
  - event B: 3 out of 10 tosses were heads.

$\binom{10}{3}$ $\qquad$ $\binom{2}{2} * \binom{8}{1}$

Assume: $\begin{cases} * \text{ Independence of coin tosses} \\ * \ P(H) = p \end{cases}$

$$P(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

(Independence) $\rightarrow k=1, n=8$

1st Method: $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(H_1 H_2 \cap \text{one more head in tosses } 3 \text{ to } 10)}{P(B)}$

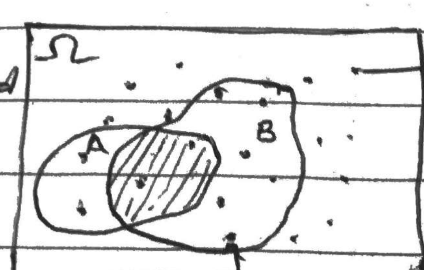
$$= \frac{p^2 \cdot \binom{8}{1} p^1 (1-p)^7}{\binom{10}{3} p^3 (1-p)^7} = \frac{8}{\binom{10}{3}}$$

2nd Method:

Conditional prob. $\rightarrow$ conditioned on B $\rightarrow$ new sample space is B

3 heads sequence $\rightarrow p^3(1-p)^7$

$$= \frac{\# \text{ in } A \cap B}{\# \text{ in } B} = \frac{8}{\binom{10}{3}}$$



→ length 10 sequence

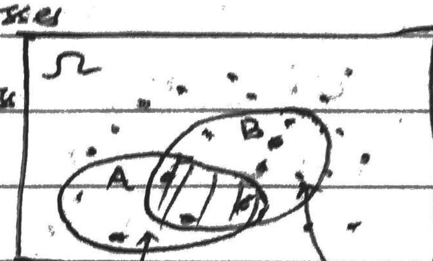
seq. of length 10 but has exactly 3 Heads.

Exercise:

Use the second method in the preceding segment to find the probability that 6 th toss out of a total of 10 tosses is Heads, given that there are exactly 2 Heads out of 10 tosses. As in the preceding segment, continue to assume that all coin tosses are independent and that each coin toss has the same fixed probability of Heads.

Soln.: event B: exactly 2 Heads out of 10 tosses

event A: 6th toss is Head out of 10 toss



(sequence of 10 having 6th toss as Head) (seq. of 10 having two Heads)

$$P(A \mid B) = \frac{\# \text{ in } A \cap B}{\# \text{ in } B}$$

$$= \frac{9}{\binom{10}{2}} = \frac{1}{5}$$

among 2 Heads → in how many ways one Head can occur on 6 th toss and remaining one Head in other 9 tosses? → $1 * \binom{9}{1} = 9$.

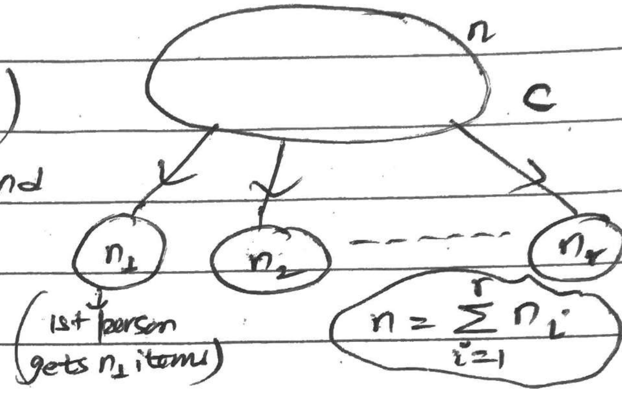
Partitions

- $n \geq 1$ distinct items ; $r \geq 1$ persons

given n_i items to person i.

- here $n_1, \ldots, n_r$ are given nonnegative integers
- with $n_1 + \cdots + n_r = n$.

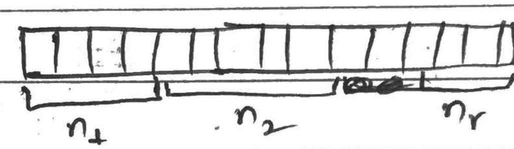
* Ordering n items: $n!$ $(\equiv {}^nP_n)$
  - Deal n_i to each person i, and then order



$c * n_1! \, n_2! \cdots n_r! = n!$

(# of partitions) $c = \dfrac{n!}{n_1! \, n_2! \cdots n_r!}$

(Multinomial coefficient)



Binomial: $r = 2,\ n_1 = k,\ n_2 = n-k \Rightarrow c = \dfrac{n!}{k!(n-k)!}$

Exercise: Counting partitions

We have 9 distinct items and three persons. Alice is to get 2 items, Bob is to get 3 items, and Charlie is to get 4 items.

1. As just discussed, this can be done in $\dfrac{a!}{b!\,3!\,4!}$ ways. Find a and b. $c = \dfrac{9!}{2!\,3!\,4!}$ $\Rightarrow a = 9$ and $b = 2$

2. A different way of generating the desired partition is as follow. We first choose 2 items to give to Alice. This can be done in $\binom{c}{d}$ different ways. Find c and d. (There are 2 possible values of d that are correct. Enter the small value.)

soln → We want the number of ways of choosing 2 items out of 9 items. This is the number of 2-element subsets of a 9-element set, so that $c=9$ and $d=2$.

3. We have 7 remaining items out of which we need to choose 3. Hence, $e=7$ and $f=3$.

$$\Rightarrow \quad C = \binom{9}{2} * \binom{7}{3} * \underbrace{\binom{4}{4}}_{\text{(Charlie)}} = \frac{9!}{2!\,3!\,4!}$$

Example: 52-card deck, dealt (fairly) to four players. Find P(each player gets an ace)

(all partitions are equally likely)

- Outcomes are: partition → (all partitions are equally likely)

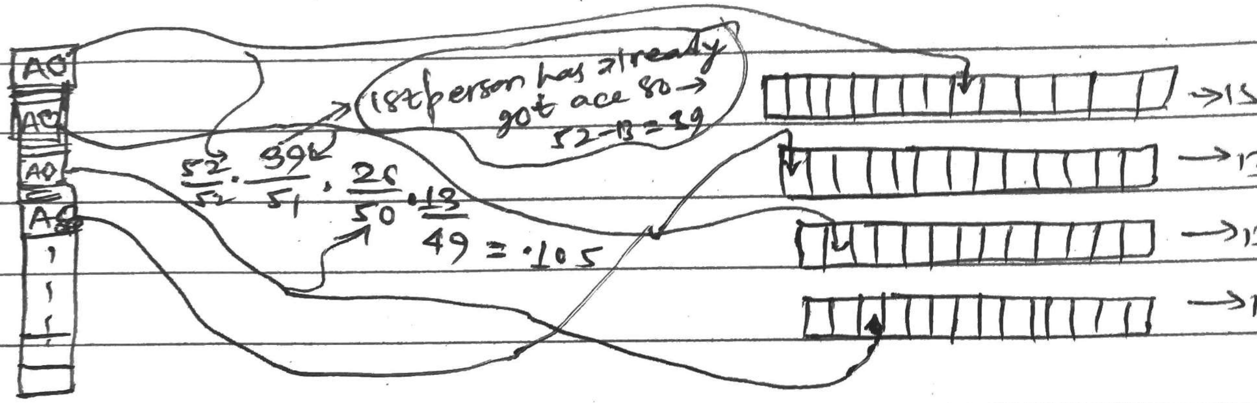
number of outcomes: $\frac{52!}{13!\,13!\,13!\,13!}$

- Constructing an outcome with one ace for each person:
  - distribute the aces → $4 \cdot 3 \cdot 2 \cdot 1$ (to four persons)
  - distribute the remaining 48 aces → $\frac{48!}{12!\,12!\,12!\,12!}$

$\#$ each player gets an ace $= 24 \times \dfrac{48!}{12!\,12!\,12!\,12!}$

Next Method:

Stack the deck, aces on top.



The birthday problem

Consider n people who are attending a party. We assume that every person has an equal probability of being born on any day during the year, independently of everyone else, and ignore the additional complication presented by leap years (i.e. nobody is born on February 29). What is the probability that each person has a distinct birthday.

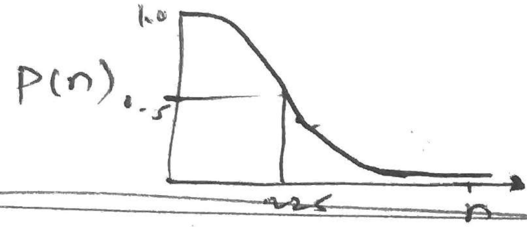
Soln. → $365 \cdot 365$ __ __ __ __ __ 365 → n person / can born on any day

$\Omega = (365)^n$ → possibilities

No two person has birthday on same day / distinct birthday: $365 \cdot 364$ __ __ __ $(365-n+1)$ → (n slots)

eg. $n=3$: $365 \cdot 364 \cdot 363$

(13)



~~# each player gets an ace~~ $= \cancel{24 * \frac{48!}{12!\,12!\,12!\,12!}}$

$$P(\text{no 2 birthday coincide}) = \frac{365 \cdot 364 \cdots (365-n+1)}{(365)^n}$$

Rocks on a Chessboard: Eight rocks are placed in distinct squares of an 8×8 chessboard, with all possible placements being equally likely. Find the probability that all the rocks are safe from one another, i.e. that there is no row or column with more than one rock.

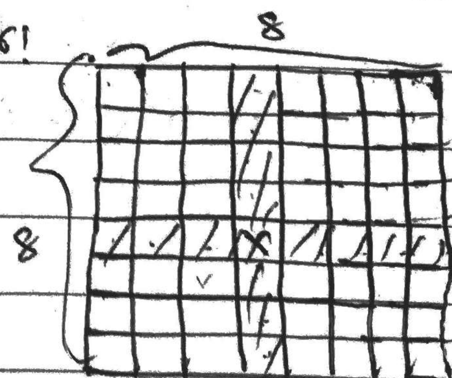
Discrete uniform law: $P(A) = \frac{|A|}{|\Omega|}$

$$P(\text{safe arrange}) = \frac{|\text{safe arranges}|}{|\text{total \# arrangements}|}$$

total # arrangements $= |\Omega| = \underbrace{64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57}_{8}$
$= 64!/56!$

|safe arrangements| = |A|

$$= \underbrace{64}_{\substack{\text{1st} \\ \text{(choices)} \\ \downarrow \\ 8\times8}} \cdot \underbrace{(64-15)}_{\substack{\text{(2nd} \\ \text{choice)} \\ \downarrow \\ 7\times7}} \cdot \underbrace{(64-15-13)}_{\substack{\text{(3rd choice)} \\ \downarrow \\ 6\times6}} \cdot \underbrace{25}_{\text{(4th)}} \cdot \underbrace{16}_{\text{(5th)}} \cdot \underbrace{9}_{\text{(6th)}} \cdot \underbrace{4}_{\text{(7th)}} \cdot \underbrace{1}_{\text{(8th)}}$$



(one placement of rock drops one row & one column)

Hypergeometric probabilities,

An urn ~~ball~~ contains n balls, out of which exactly m are red. We select k of the balls at random, w/o replacement. What is the probability that i of the selected balls are red?

Sol^n:

$$|\Omega| = \binom{n}{k}$$

red	non-red
(m)	$(n-m)$
i	$k-i$

e = # ways to get i red balls

$$= \binom{m}{i} * \binom{n-m}{k-i}$$

$$P(i \text{ red balls}) = \frac{\binom{m}{i}\binom{n-m}{k-i}}{\binom{n}{k}}$$

Ex: Deck of cards

$n = 52$

$M=4$	
3	4

$k = 7$; $i = 3$

$$P(3 \text{ Aces}) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}$$

Multinomial probabilities

An urn contains balls of r different colors. We draw n balls, with different draws being independent. For any given draw, there is a probability p_i, $i = 1, \ldots, r$, of getting a ball of color i. Here, the p_i's are nonnegative numbers that sum to 1. Let $n_1, n_2, \ldots, n_r$ be nonnegative integers that sum to n. What is the probability that we exactly obtain n_i balls of color i, for each $i = 1, \ldots, r$?

soln →
- balls of different colors: $i = 1, \ldots, r$
- probability of picking a ball of color i is p_i
- draw n balls, independently
- given nonnegative #s n_i w/ $n_1 + n_2 + \cdots + n_r = n$
- Find: P(n_1 balls of color 1, n_2 balls of color 2, ..., n_r balls of color r)

* Special case $r=2$: colors: "heads" "tails"

"heads" → p, "tails" → $1-p$

$n_1 \longleftrightarrow k$ heads

$n_2 \longleftrightarrow n-k$ tails $\qquad p_1 = p$; $p_2 = 1-p$

$$\binom{n}{k} p^k (1-p)^{n-k} \longrightarrow \frac{n!}{n_1!\, n_2!} p_1^{n_1} p_2^{n_2}$$

(Multinomial case for $n=2$)

* $r = 3$, $n = 7$ → $\{1\ 1\ 3\ 1\ 2\ 2\ 1\}$ → particular case

(1st color, 3rd color)

$\equiv (4, 2, 1)$ → type of outcome

Prob. $= p_1 p_2 p_3 p_1 p_2 p_2 p_1$ (using independence)

$= p_1^4 p_2^2 p_3$

$P(\underbrace{\text{particular sequence of "type" } (n_1, n_2, \ldots, n_r)}_{}) = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$

(How many sequences of this type?)

↓

partition of $\{1, \ldots, n\}$ into subsets of sizes $n_1, n_2, \ldots, n_r$

$$\boxed{P(\text{get type } (n_1, n_2, \ldots, n_r)) = \frac{n!}{n_1!\, n_2! \ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}}$$

Problem Set 3,

1. Customer Arriving at a Restaurant,

Six customers enter a three-floor restaurant. Each customer decides on which floor to have dinner. Assume that decisions of different customers are independent, and that for each customer, each floor is equally likely. Find the probability that exactly one customer dines on the first floor?

Soln: For the first floor, $p = \frac{1}{3}$, $(1-p) = \frac{2}{3}$

$$P(\text{exactly one on first floor}) = \binom{6}{1}\left(\frac{1}{3}\right)^1 (1-p)^{6-1} = \frac{64}{243}$$

2. A three-sided die

(a) The newest invention of the 6.431x staff is a three-sided die. On any roll of this die, the result is 1 with probability $\frac{1}{2}$, 2 with probability $\frac{1}{4}$, and 3 with probability $\frac{1}{4}$.

Consider a sequence of six independent rolls of this die. Find the probability that exactly two of the rolls result in a 3.

Soln $\rightarrow$ $p = \frac{1}{4} \rightarrow$ prob. of getting 3

$1-p = \frac{3}{4} \rightarrow$ prob. of getting other than 3

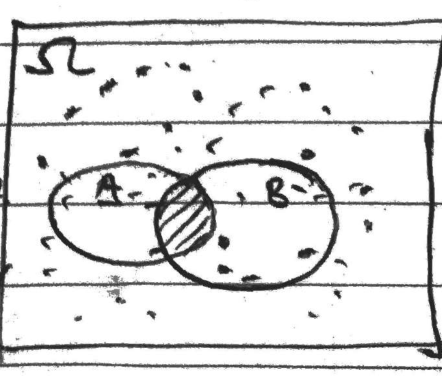
How many ways can we choose exact two 3 in die roll? $\longrightarrow \binom{6}{2}$ $(3\ 1\ 2\ 3\ 2\ 1) \rightarrow$ type

$$P(\text{exactly two } 3) = \binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{6-2}$$

(b) 1. Given that exactly two of the six rolls resulted in a 1, find the probability that the first roll resulted in a 1.

Soln $\rightarrow$ A: first roll resulted in 1

B: two of the six rolls resulted in 1



Here, all outcomes 3^6 are not equally likely $\not\Rightarrow$ (No) Discrete Uniform Law

Each roll can either be a 1 w/ prob. $\frac{1}{2}$ or not 1 with prob. $\frac{1}{2}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(\text{1st roll 1} \cap \text{1 in rest of 5 rolls})}{P(B)}$$

→ occuring 1 → not occurring 1

$$= \frac{1/2 * \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4}{\binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4} = \frac{1}{3}$$

2. We are told that exactly three of the rolls resulted in a 1 and exactly three rolls resulted in a 2. Given this information, find the probability that the six rolls resulted in the sequence (1, 2, 1, 2, 1, 2).

soln: $P\left(\{(121212)\} \,\middle|\, \text{exactly three 1's and three 2's}\right)$

$$= \frac{P(121212)}{P(\text{exactly three one's and three 2's})}$$

→ C → exactly three 1's & three 2's

Any particular sequence of three 1's and three 2's will have the same probability: $\left(\frac{1}{2}\right)^3\left(\frac{1}{4}\right)^3$

There are $\binom{6}{3}$ possible rolls with exactly three 1's and three 2's. (If we find out how to get exactly three 3's and we're left with — — — (three spots) in which we want all 2's → can be done 1 way)

$$= \frac{1}{\binom{6}{3}}$$

3. The conditional probability that exactly k rolls resulted in a 3, given that at least one roll resulted in a 3, is of the form: $\frac{1}{1-(c_1/c_2)^{c_3}} \binom{c_3}{k} \left(\frac{1}{c_2}\right)^k \left(\frac{c_1}{c_2}\right)^{c_3 k}$,

for $k = 1, 2, ---, 6$.

Find the values of c_1, c_2 and c_3.

Soln → $A \rightarrow$ exactly k rolls resulted in 3

$B \rightarrow$ at least one roll resulted in 3

$P(B) = 1 - P$ (zero roll resulted in 3)

$$= 1 - \binom{6}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^6 = 1 - \left(\frac{3}{4}\right)^6$$

$$P(A) = \binom{6}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{6-k}$$

$$P(A|B) = \frac{1}{1-\left(\frac{3}{4}\right)^6}\binom{6}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{6-k} \Rightarrow c_1 = 3$$

$c_2 = 4$

$c_3 = 6$

Problem 3: Forming a Committee,

Out of five men and five women, we form a committee consisting of four different people. Assuming that each committee of size 4 is equally likely, find the probabilities of the following events:

1. The committee consists of two men and two women.

Soln → $P(2M \cap 2W) = \frac{\binom{5}{2} * \binom{5}{2}}{\binom{10}{4}} = \frac{10}{21}$

$P(M=2 \cap W=2)$

$M \rightarrow$ men

W → women

2. The committee has more women than men.

Soln: $P(W=w > M=m) = P(\underbrace{\{(W=4, M=0), (W=3, M=1)\}}_{\text{(disjoint events)}})$

$$= P(W=4, M=0) + P(W=3, M=1)$$

$$= \frac{\binom{5}{4} * \binom{5}{0} + \binom{5}{3} * \binom{5}{1}}{\binom{10}{4}} = \frac{11}{42}$$

3. The committee has at least one man.

Soln: $P(M=\{m \geq 1\}) = 1 - P(M=0)$

$$= 1 - \frac{\binom{5}{0} * \binom{5}{4}}{\binom{10}{4}} = \frac{41}{42}$$

4. For the remainder of the problem, assume that Alice and Bob are among the ten people being considered.

Both Alice and Bob are members of the committee.

Soln: Since Bob and Alice are chosen, for the remaining two people the choices can be either both women or both men or one men ∩ one women.

$$= \underbrace{P(\text{Alice} \cap \text{Bob} \cap M=2)}_{\text{(we can only choose 2 men out of remaining 3)}} + P(\text{Alice} \cap \text{Bob} \cap W=2) + P(\text{Alice} \cap \text{Bob} \cap M=1, W=1)$$

$$= \frac{\binom{3}{2} + \binom{5}{2} * \binom{8}{0} + \binom{3}{1} * \binom{5}{1}}{\binom{10}{4}} = \frac{2}{15}$$

Problem 4, Proving Binomial Identities vs Counting

Binomial identities (i.e, identities involving binomial coefficients) can often be proved via a counting interpretation. For each of the binomial identities given below, select the counting problem that can be used to prove it.

can choose 1 chairperson out of 2n people → $\binom{2n}{1} = 2n$

1. $n \binom{2n}{n} = 2n \binom{2n-1}{n-1}$

(1 → select chair) 1st stage; chosen out of 2n (2nd stage); select rest $(n-1)$ from $(2n-1)$

⟺ Out of 2n people, we want to choose a committee of n people, one of whom will be its chair. In how many ways can this be done?

these are equivalent if you expand the sum in reverse order, n=n, n-1 & so on

2. $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$

⟺ In a group of 2n people, consisting of n boys and n girls, we want to select a committee of n people. In how many ways can this be done?

3. $2^{2n} = \sum_{i=0}^{2n} \binom{2n}{i}$

⟺ How many subsets does a set w/ 2n elements have?

⟶

22

can be n possible chairs

out of (n-1) remaining people, there're 2^{n-1} subsets

for each subset of size i → each of item can be a chair person

4. $n 2^{n-1} = \sum_{i=0}^{n} \binom{n}{i} i$

→ Out of n people, we want to form a committee consisting of a chair and other members. We allow the committee size to be any integer in the range 1, 2, ..., n. How many choices do we have in selecting a committee-chair combination?

Problem 5. Hats in a box

Each one of n persons, indexed by 1, 2, ..., n has a clean hat and throws it into a box. The persons then pick hats from the box, at random. Every assignment of the hats to the person is equally likely. In an equivalent model, each person picks a hat, one at a time, in the order of their index, with each one of the remaining hats being equally likely to be picked. Find the probability of the following events.

1. Every person gets his or her own hat back.

Soln → Total possible choices → $\underset{\text{1st pick}}{n}$ $\underset{\text{2nd pick}}{n-1}$ ------- $\underset{\text{last pick}}{1}$ $= n!$

P(getting their own hat back) $= \frac{1}{n!}$

(23)

2. Each one of persons $1, ---, m$ gets his or her own hat back, where $1 \le m \le n$.

soln → ~~P(exactly m person get their own hat)~~ =

P(first m person who picked hats get their own hats back) = $\dfrac{\text{how many ways } (n-m) \text{ hats can be distributed after the } m \text{ hats are picked up}}{n!}$

$$= \frac{(n-m)!}{m!}$$

3. Each one of persons $1, ---, m$ gets back a hat belonging to one of the last m persons $(n-m+1, ---, n)$ where $1 \le m \le n$.

soln There are $m!$ ways to distribute m hats among the first m persons, and $(n-m)!$ ways to distribute the remaining $n-m$ hats. The probability of an event with $\frac{m!}{(n-m)!}$ elements is $\frac{m!(n-m)!}{n!} = \frac{1}{\binom{m}{n}}$

Now assume that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). What is the probability that

a) the first m persons will pick up clean hats?

Soln → By independence assumption → $\underbrace{(1-p)}_{1} \cdots \underbrace{(1-p)}_{m} = (1-p)^m$

e) Exactly m persons will pick up clean hats?

Soln → $\binom{n}{m} (1-p)^m \, p^{n-m}$.