1. You have opened a timed exam with a **48 hours** time limit. Please use the timer to see the time remaining. If you had opened this exam too close to the exam **closing time, October 5 23:59UTC** , you will not have the full 48 hours, and the exam will close at the closing time.

2. This is an **open book exam** and you are allowed to refer back to all course material and use (online) calculators. However, **you must abide by the honor code and not ask for answers directly from any aide** .

3. As part of the honor code, you **must not share the exam content** with anyone in any way, i.e. **no posting of exam content anywhere on the internet** . Violators will be removed from the course. **If you find exam content anywhere outside of the this course site, you must still abide by the honor code and not refer to or use it in any way.**

4. You will be given **no feedback** during the exam. This means that unlike in the problem sets, you will not be shown whether any of your answers are correct or not. This is to test your understanding, to prevent cheating, and to encourage you to try your very best before submitting. Solutions will be available after the exam closes.

5. You will be given **3 attempts** for each (multipart) problem. Since you will be given no feedback, the extra attempts will be useful only in case you hit the "submit" button in a haste and wish to reconsider. **With no exception, your last submission will be the one that counts** . **DO NOT FORGET TO SUBMIT your answers to each question** . The “end your exam" button will **not** submit answers for you.

6. The exam will only be **graded 1 day after the due date** , and the **Progress Page will be disabled while the exam is open** .

7. **Error and bug reports:** While the exam is open, you are **not allowed to post on the discussion forum on anything related to the exam, except to report bugs/platform difficulties** . If you think you have found a bug, please **state on the forum only what needs to be checked on the forum** . You can still post questions relating to course material, but **the post must not comment on the exam** , and in particular **must not shed any light on the contents or concepts in the exam** . **Violators will receive a failing grade or a grade reduction in this exam** .

1.

Mid Term due Oct 6, 2020 19:59 EDT *Completed*

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True or False

4/4 points (graded)

Let A, B, and C be events associated with the same probabilistic model (i.e., subsets of a common sample space), and assume that $P(C) > 0$.

For each one of the following statements, decide whether the statement is True (always true), or False (not always true).

1. Suppose that $A \subset C$. Then, $P(A \mid C) \geq P(A)$.

- (●) True
- ( ) False

✓

2. Suppose that $A \subset B$. Then, $P(A \mid C) \leq P(B \mid C)$.

- (●) True
- ( ) False

3. Suppose that $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$.

- ○ True
- ◉ False

✓

4. Suppose that $A \subset C$, $B \subset C$, and $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$.

- ◉ True
- ○ False

✓

Solution:

1. Suppose that $A \subset C$. Then, $P(A \mid C) \geq P(A)$. This is **TRUE**:

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)}{P(C)} \geq P(A), \tag{7.1}$$

since $P(C) \leq 1$.

2. Suppose that $A \subset B$. Then, $P(A \mid C) \leq P(B \mid C)$. This is **TRUE**.

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} \leq \frac{P(B \cap C)}{P(C)} = P(B \mid C)$$

where the inequality follows from $A \cap C \subset B \cap C$.

3. Suppose that $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$.This is **FALSE**, with the following counter example: Suppose that A and B are disjoint events with positive probability and that $C = A$. Then, $P(A \mid C) = P(A) > 0$, whereas $P(B \mid C) = 0$.

4. Suppose that $A \subset C, B \subset C$, and $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$. This is **TRUE**: Since $A, B \subset C$, we have $P(A \mid C) = \dfrac{P(A)}{P(C)}$ and similarly $P(B \mid C) = \dfrac{P(B)}{P(C)}$. Then, $P(A) \leq P(B)$ implies $P(A \mid C) \leq P(B \mid C)$.

Submit You have used 3 of 3 attempts

Show Answer

Answers are displayed within the problem

2.

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A Drunk Person at the Theater

4.0/4.0 points (graded)

There are n people in line, indexed by $i = 1, \ldots, n$, to enter a theater with n seats one by one. However, the first person ($i = 1$) in the line is drunk. This person has lost her ticket and decides to take a random seat instead of her assigned seat. That is, the drunk person decides to take any one of the seats 1 to n with equal probability. Every other person $i = 2, \ldots, n$ that enters afterwards is sober and will take his assigned seat (seat i) unless his seat i is already taken, in which case he will take a random seat chosen uniformly from the remaining seats.

Suppose that $n = 3$. What is the probability that person 2 takes seat 2?

(Enter a fraction or a decimal accurate to at least 3 decimal places.)

2/3 ✔ **Answer:** 2/3

Suppose that $n = 5$. What is the probability that person 3 takes seat 3?

(Enter a fraction or a decimal accurate to at least 3 decimal places.)

3/4 ✔ **Answer:** 3/4

STANDARD NOTATION

Solution:

1. $\mathbf{P}\left(\text{Person 2 takes seat 2}\right) = \mathbf{P}\left(\text{Person 1 takes seat 1 or 3}\right) = \frac{2}{3}$.

2. $$\begin{aligned} & \mathbf{P}\left(\text{Person 3 takes seat 3}\right) \\ = & \mathbf{P}\left(\text{Person 1,2 does not take seat 3}\right) \\ = & \mathbf{P}\left(\text{Person 1 takes seat 1 or 4 or 5}\right) \cdot 1 + \mathbf{P}\left(\text{Person 1 takes seat 2}\right) \cdot \mathbf{P}\left(\text{Person 2 does not take seat 3}\right) \\ = & \frac{3}{5} \cdot 1 + \frac{1}{5} \cdot \frac{3}{4} = \frac{3}{4}. \end{aligned}$$

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Expectation 1

1/1 point (graded)

Compute $\mathbf{E}(X)$ for the following random variable X:

$$X = \text{Number of tosses until getting 4 (including the last toss) by tossing a fair 10-sided die.}$$

$\mathbf{E}(X) =$ 10 ✔ **Answer:** 10

Solution:

This is just the mean of a geometric random variable with parameter $1/10$. Hence, $\mathbf{E}(X) = 10$.

Submit You have used 1 of 3 attempts

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Expectation 2

1.3333333333333333/2.0 points (graded)

Compute $\mathbf{E}\left(X\right)$ for the following random variable X:

$$X = \text{Number of tosses until all 10 numbers are seen (including the last toss) by tossing a fair 10-sided die.}$$

To answer this, we will use induction and follow the steps below:

Let $\mathbf{E}\left(i\right)$ be the expected number of additional tosses until all 10 numbers are seen (including the last toss) **given i distinct numbers have already been seen**.

1. Find $\mathbf{E}\left(10\right)$.

   $\mathbf{E}\left(10\right) =$ 0 ✔ **Answer:** 0

2. Write down a relation between $\mathbf{E}\left(i\right)$ and $\mathbf{E}\left(i+1\right)$. Answer by finding the function $f\left(i\right)$ in the formula below.

   For $i = 0, 1, \ldots, 9$:

$$\mathbf{E}\left(i\right) = \mathbf{E}\left(i+1\right) + f\left(i\right)$$

   where $f\left(i\right) =$ 10/(10-i) **Answer:** 10/(10-i)

   $\frac{10}{10-i}$

3. Finally, using the results above, find $\mathbf{E}[X]$.

(Enter an answer accurate to at least 1 decimal place.)

$\mathbf{E}[X] =$ 29.289682539682538 ✔ **Answer:** 29.28968

Solution:

Recall $\mathbf{E}(i)$ is the expected number of additional tosses until all 10 numbers are seen (including the last toss) given i distinct numbers have already been seen.

1. $\mathbf{E}(10) = 0$

2. The induction step is as follows. For $i = 1, \ldots 9$:

$$\begin{aligned} \mathbf{E}(i) &= (\mathbf{E}(i) + 1) \times \frac{i}{10} + (\mathbf{E}(i+1) + 1) \times \left(1 - \frac{i}{10}\right) \\ \iff \mathbf{E}(i) &= \mathbf{E}(i+1) + \frac{10}{10-i}. \end{aligned}$$

Using $\mathbf{E}(10) = 0$ and the induction step, we have

$$\mathbf{E}(0) = \frac{10}{10} + \frac{10}{9} + \ldots + \frac{10}{2} + \frac{10}{1} + 0 \approx 29.28968.$$

4.

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Conditional Independence 1

4/4 points (graded)

Suppose that we have a box that contains two coins:

1. A fair coin: $\mathbf{P}\left(H\right) = \mathbf{P}\left(T\right) = 0.5$.

2. A two-headed coin: $\mathbf{P}\left(H\right) = 1$.

A coin is chosen at random from the box, i.e. either coin is chosen with probability $1/2$, and tossed twice. Conditioned on the identity of the coin, the two tosses are independent.

Define the following events:

- Event A: first coin toss is H.
- Event B: second coin toss is H.
- Event C: two coin tosses result in HH.
- Event D: the fair coin is chosen.

For the following statements, decide whether they are true or false.

1. A and B are independent.

- ( ) True
- (•) False

✓

2. A and C are independent.

- ( ) True
- (•) False

✓

3. A and B are independent given D.

- (•) True
- ( ) False

4. A and C are independent given D.

- [ ] True
- [x] False

Solution:

1. False. Since we do not know whether it is a fair coin or the two-headed one when the coin is being tossed, getting a Heads during one toss increases our belief the the coin is the two-headed one, so that also increases our belief that the other toss also results in a Heads. Or we can also verify by definition: $\mathbf{P}(A \cap B) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{5}{8} \neq \frac{9}{16} = \frac{3}{4} \cdot \frac{3}{4} = \mathbf{P}(A)\mathbf{P}(B)$.

2. False. $\mathbf{P}(A \cap C) = \mathbf{P}(C) \neq \mathbf{P}(A)\mathbf{P}(C)$.

3. True. Conditioned on D, A and B becomes the outcome of two independent fair coin tosses.

4. False. $\mathbf{P}(A \cap C|D) = \mathbf{P}(C|D) \neq \mathbf{P}(A|D)\mathbf{P}(C|D)$.

Conditional Independence 2

2/2 points (graded)

1. Suppose three random variables X, Y, Z have a joint distribution

$$\mathbf{P}_{X,Y,Z}(x,y,z) = \mathbf{P}_X(x)\,\mathbf{P}_{Z|X}(z \mid x)\,\mathbf{P}_{Y|Z}(y \mid z).$$

Then, X and Y are independent given Z.

- (●) True
- ( ) False

✔

2. Suppose random variables X and Y are independent given Z, then the joint distribution must be of the form

$$\mathbf{P}_{X,Y,Z}(x,y,z) = h(x,z)\,g(y,z),$$

where h, g are some functions.

- (●) True
- ( ) False

Solution:

1. True. Using $\mathbf{P}_{X,Y,Z}(x,y,z) = \mathbf{P}_X(x)\,\mathbf{P}_{Z|X}(z|x)\,\mathbf{P}_{Y|Z}(y|z)$, we have

$$\begin{aligned}
\mathbf{P}_{X,Y|Z}(x,y|z) &= \frac{\mathbf{P}_{X,Y,Z}(x,y,z)}{\mathbf{P}_Z(z)} \\
&= \frac{\mathbf{P}_X(x)\,\mathbf{P}_{Z|X}(z|x)\,\mathbf{P}_{Y|Z}(y|z)}{\mathbf{P}_Z(z)} \\
&= \frac{\mathbf{P}_X(x)\,\mathbf{P}_{Z|X}(z|x)}{\mathbf{P}_Z(z)}\mathbf{P}_{Y|Z}(y|z) \\
&= \mathbf{P}_{X|Z}(x|z)\,\mathbf{P}_{Y|Z}(y|z),
\end{aligned}$$

   which shows X and Y are conditionally independent given Z.

2. True. Since X and Y are conditionally independent given Z, we have

$$\begin{aligned}
\mathbf{P}_{X,Y,Z}(x,y,z) &= \mathbf{P}_Z(z)\,\mathbf{P}_{X,Y|Z}(x,y|z) \\
&= \mathbf{P}_Z(z)\,\mathbf{P}_{X|Z}(x|z)\,\mathbf{P}_{Y|Z}(y|z) \\
&= h(x,z)\,g(y,z),
\end{aligned}$$

   by letting $h(x,z) := \mathbf{P}_Z(z)\,\mathbf{P}_{X|Z}(x|z)$, $g(y,z) := \mathbf{P}_{Y|Z}(y|z)$. (In fact, by generalizing the argument for the part 1, we can show X and Y are conditionally independent given Z if and only if $\mathbf{P}_{X,Y,Z}(x,y,z) = h(x,z)\,g(y,z)$ for some h, g.)

Variance of Difference of Indicators

2.0/2.0 points (graded)

Let A be an event, and let I_A be the associated indicator random variable (I_A is 1 if A occurs, and zero if A does not occur). Similarly, let I_B be the indicator of another event, B. Suppose that $P(A) = p$, $P(B) = q$, and $P(A \cap B) = r$.

Find the variance of $I_A - I_B$, in terms of p, q, r.

$\text{Var}(I_A - I_B) =$ p+q-2*r-(p-q)^2 ✔ **Answer:** p-2*r+q-(p-q)^2

$p + q - 2 \cdot r - (p-q)^2$

STANDARD NOTATION

Solution:

$$\begin{aligned}
\text{Var}(I_A - I_B) &= \mathbf{E}\left[(I_A - I_B)^2\right] - \left(\mathbf{E}\left[(I_A - I_B)\right]\right)^2 \\
&= \mathbf{E}\left[I_A^2 - 2I_AI_B + I_B^2\right] - \left(\mathbf{E}\left[I_A\right] - \mathbf{E}\left[I_B\right]\right)^2 \\
&= \mathbf{E}\left[I_A^2\right] - 2\mathbf{E}\left[I_AI_B\right] + \mathbf{E}\left[I_B^2\right] - \left(\mathbf{E}\left[I_A\right]\right) - \left(\mathbf{E}\left[I_B\right]\right)^2 \\
&= \mathbf{E}\left[I_A\right] - 2\mathbf{E}\left[I_AI_B\right] + \mathbf{E}\left[I_B\right] - \left(\mathbf{E}\left[I_A\right]\right) - \left(\mathbf{E}\left[I_B\right]\right)^2 \\
&= P(A) - 2P(A \cap B) + P(B) - \left(P(A) - P(B)\right)^2 \\
&= p - 2r + q - (p-q)^2
\end{aligned}$$

6.

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For all problems on this page, use the following setup:

Let N be a positive integer random variable with PMF of the form

$$p_N(n) = \frac{1}{2} \cdot n \cdot 2^{-n}, \qquad n = 1, 2, \ldots.$$

Once we see the numerical value of N, we then draw a random variable K whose (conditional) PMF is uniform on the set $\{1, 2, \ldots, 2n\}$.

Joint PMF

1/1 point (graded)

Write down an expression for the joint PMF $p_{N,K}(n, k)$.

For $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2n$:

$p_{N,K}(n, k) =$ 2^(-(n+2)) ✔

STANDARD NOTATION

Solution:

We are given that:

$$p_{K|N}(k \mid n) = \frac{1}{2n}, \qquad k = 1, 2, \ldots, 2n. \tag{7.2}$$

By definition:

$$p_{N,K}(n, k) = p_{K|N}(k \mid n)\, p_N(n) = \frac{1}{2n}\frac{1}{2} \cdot n \cdot 2^{-n} = \left(\frac{1}{2}\right)^{n+2}, \qquad n = 1, 2, \ldots, \quad k = 1, 2, \ldots, 2n \tag{7.3}$$

Submit You have used 1 of 3 attempts

Show Answer

Answers are displayed within the problem

Marginal Distribution

1.5/1.5 points (graded)

Find the marginal PMF $p_K(k)$ as a function of k. For simplicity, provide the answer **only for the case when k is an even number**. (The formula for when k is odd would be slightly different, and you do not need to provide it).

Hint: You may find the following helpful: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $0 < r < 1$.

For $k = 2, 4, 6, \ldots$:

$p_K(k) =$ 1/2^((k/2) + 1)

✔ **Answer:** (1/2)^(k/2+1)

$$\frac{1}{2^{\left(\frac{k}{2}\right)+1}}$$

STANDARD NOTATION

Solution:

Solution

Observe that in the infinite sum $p_K(k) = \sum_{n=1}^{\infty} p_{N,K}(n,k)$ only the terms from $n = k/2$ and above have non-zero probability. Indeed, $K = k = 4$ has probability 0 if $n < k/2 = 4/2 = 2$.

Hence:

$$
\begin{aligned}
p_K(k) &= \sum_{n=k/2}^{\infty} p_{N,K}(n,k) = \sum_{n=k/2}^{\infty} \left(\frac{1}{2}\right)^{n+2} \\
&= \sum_{n=k/2}^{\infty} \left(\frac{1}{2}\right)^{n+2} = \frac{1}{4} \sum_{n=k/2}^{\infty} \left(\frac{1}{2}\right)^{n} \\
&= \frac{1}{4}\left[\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} - \sum_{0}^{k/2-1} \left(\frac{1}{2}\right)^{n}\right] \\
&= \frac{1}{4}\left[\frac{1}{1-\frac{1}{2}} - \frac{1-\left(\frac{1}{2}\right)^{k/2-1+1}}{1-\frac{1}{2}}\right] \\
&= \left(\frac{1}{2}\right)^{k/2+1} \qquad \text{for} k = 2, 4, \ldots
\end{aligned}
$$

Submit You have used 1 of 3 attempts

Show Answer

Answers are displayed within the problem

Discrete PMFs

2/2 points (graded)

Let A be the event that K is even. Find $P(A|N=n)$ and $P(A)$.

Let A be the event that K is even. Find $P(A|N=n)$ and $P(A)$.

$P(A \mid N=n) =$ 1/2 ✔ **Answer:** 1/2

$\frac{1}{2}$

$P(A) =$ 1/2 ✔ **Answer:** 1/2

$\frac{1}{2}$

STANDARD NOTATION

Solution:

Let A be then event that K is even. We need to check whether $P(A \mid N=n) = P(A)$ is true for the event A to be independent of N. Now because $p_{K|N}(k \mid n)$ is uniform over the $2n$-size set $\{1, 2, \ldots, 2n\}$ and there are exactly n even numbers in this set, we have that:

$$P(A \mid N=n) = \frac{n}{2n} = \frac{1}{2}, \qquad n \geq 1. \tag{7.4}$$

Intuitively, knowledge of n does not affect the beliefs about A, and we have independence. A full, formal argument goes as follows:

$$P(A) = \sum^{\infty} P(A \mid N=n) P(N=n)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} P(N = n) = \frac{1}{2},$$

where the last step follows because PMFs always sum to 1. So, $P(A \mid N = n) = P(A)$, for all n.
Equivalently, $P(A \text{ and } N = n) = P(A \mid N = n) \cdot P(N = n) = P(A) \cdot P(N = n)$, for all n, which is the defining property of independence.

Submit You have used 3 of 3 attempts

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Independence 2

0.5/0.5 points (graded)

Is the event A independent of N?

- (selected) yes
- no
- not enough information to determine