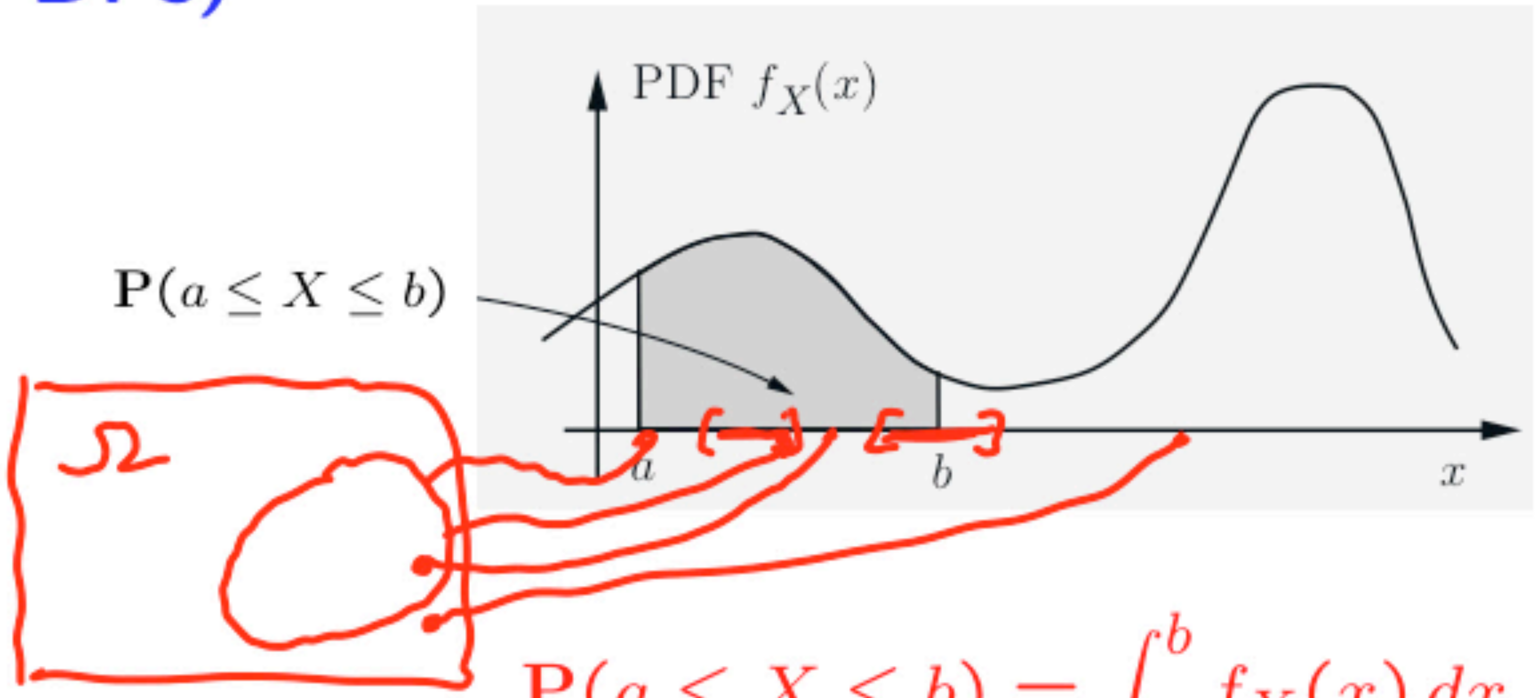
LECTURE 8: Continuous random variables and probability density functions

- Probability density functions
  - Properties
  - Examples
- Expectation and its properties
  - The expected value rule
  - Linearity
- Variance and its properties
- Uniform and exponential random variables
- Cumulative distribution functions
- Normal random variables
  - Expectation and variance
  - Linearity properties
  - Using tables to calculate probabilities

Probability density functions (PDFs)



$$\mathbf{P}(a \le X \le b) = \sum_{x:\, a \le x \le b} p_X(x)$$

$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\, dx$$

$p_X(x) \ge 0 \qquad \sum_x p_X(x) = 1$

- $f_X(x) \ge 0$
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

Definition: A random variable is **continuous if it can be described by a PDF**

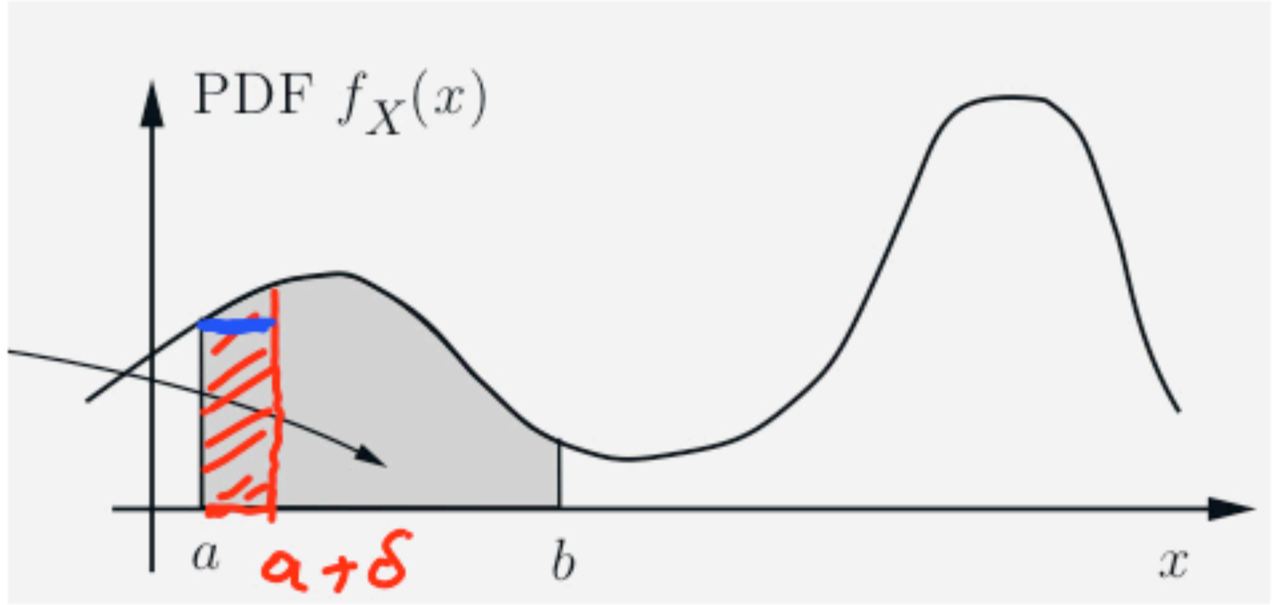
$\mathbf{P}(1 \le X \le 3 \text{ or } 4 \le X \le 5) = \mathbf{P}(1 \le X \le 3) + \mathbf{P}(4 \le X \le 5)$

Probability density functions (PDFs)

$\delta > 0$, small

$\mathbf{P}(a \le X \le a+\delta)$

$\approx f_X(a) \cdot \delta$



$$\boxed{\mathbf{P}(a \le X \le a+\delta) \approx f_X(a) \cdot \delta}$$

$$\boxed{\mathbf{P}(X = a) = 0}$$

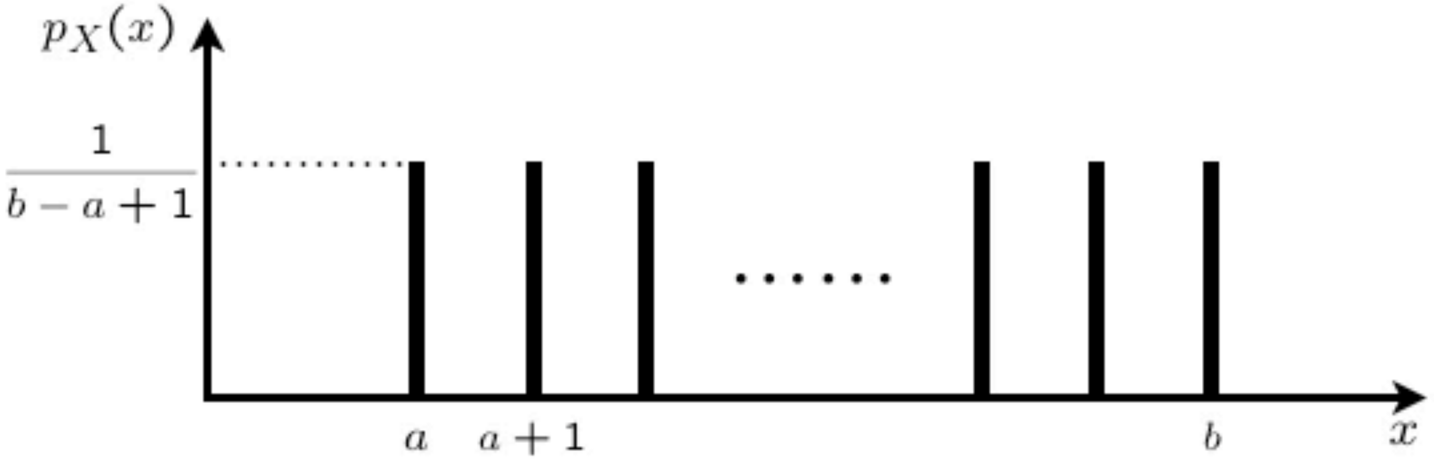
$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\, dx$$

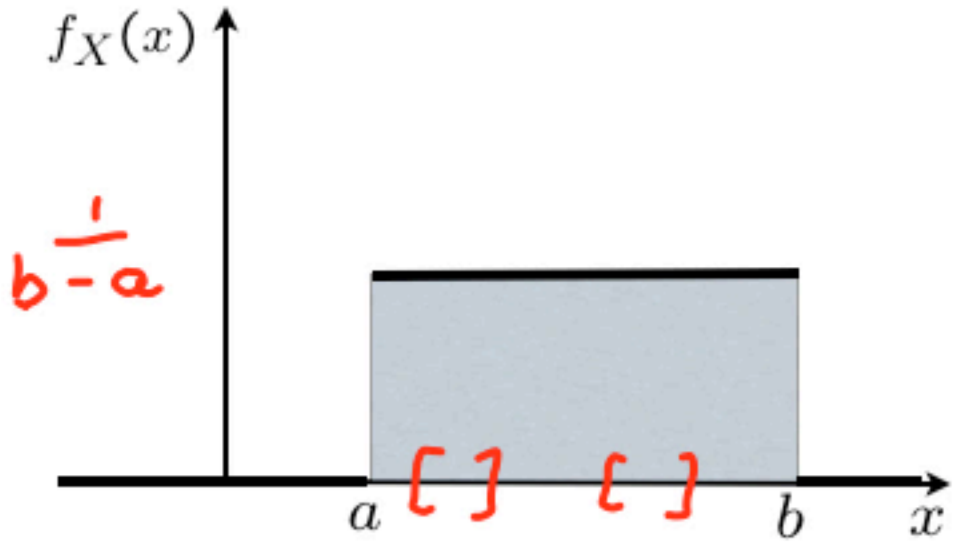
$$f_X(x) \ge 0$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

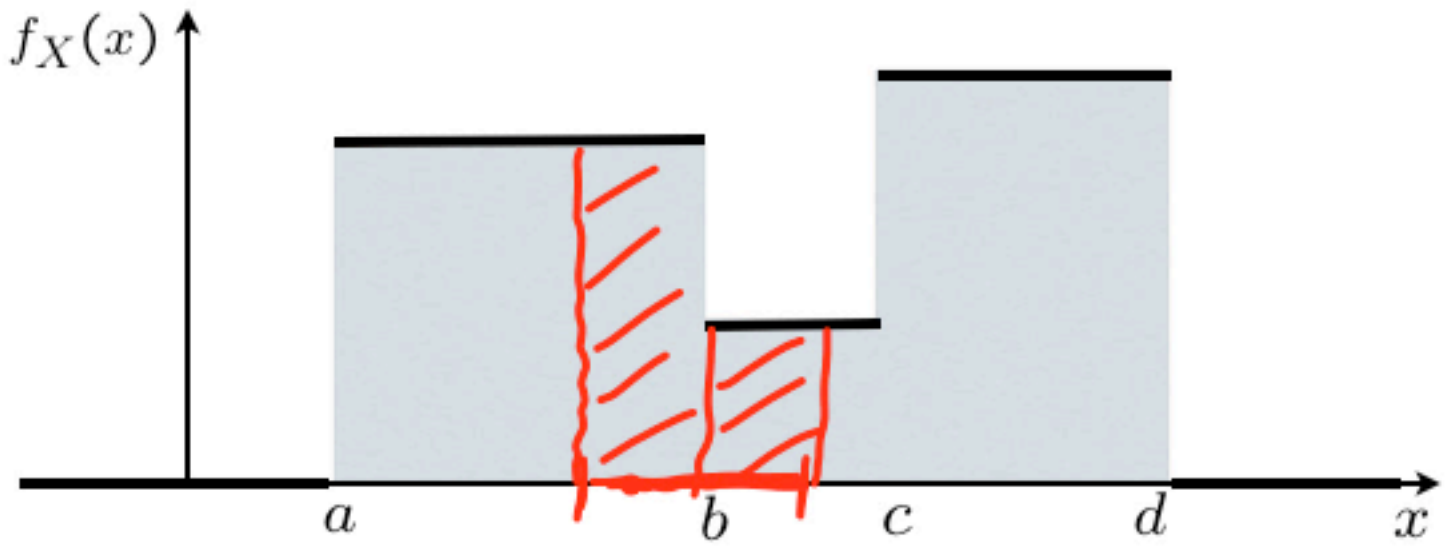
$\mathbf{P}(a \le X \le b) = \cancel{\mathbf{P}(X = a)} + \cancel{\mathbf{P}(X = b)} + \mathbf{P}(a < X < b)$

Example: continuous uniform PDF

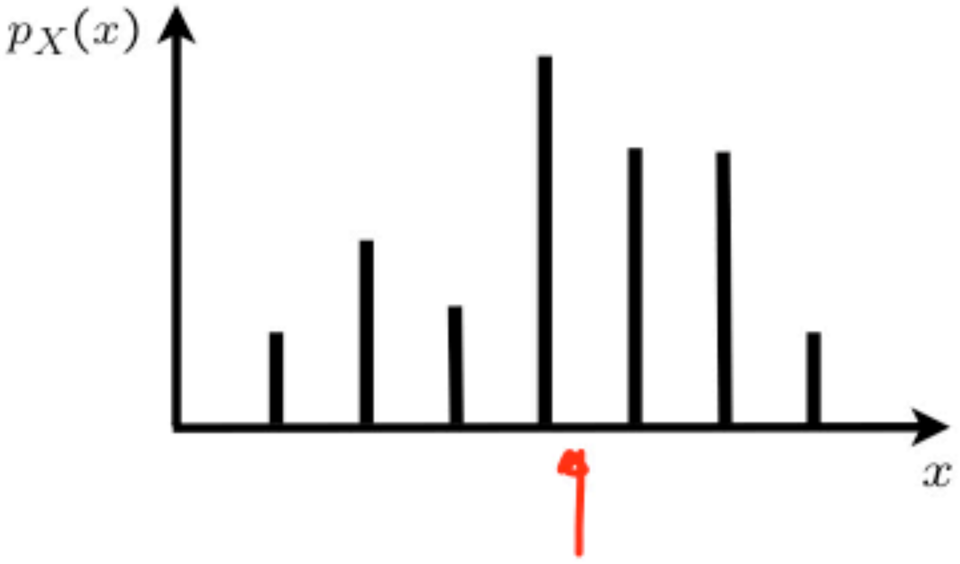




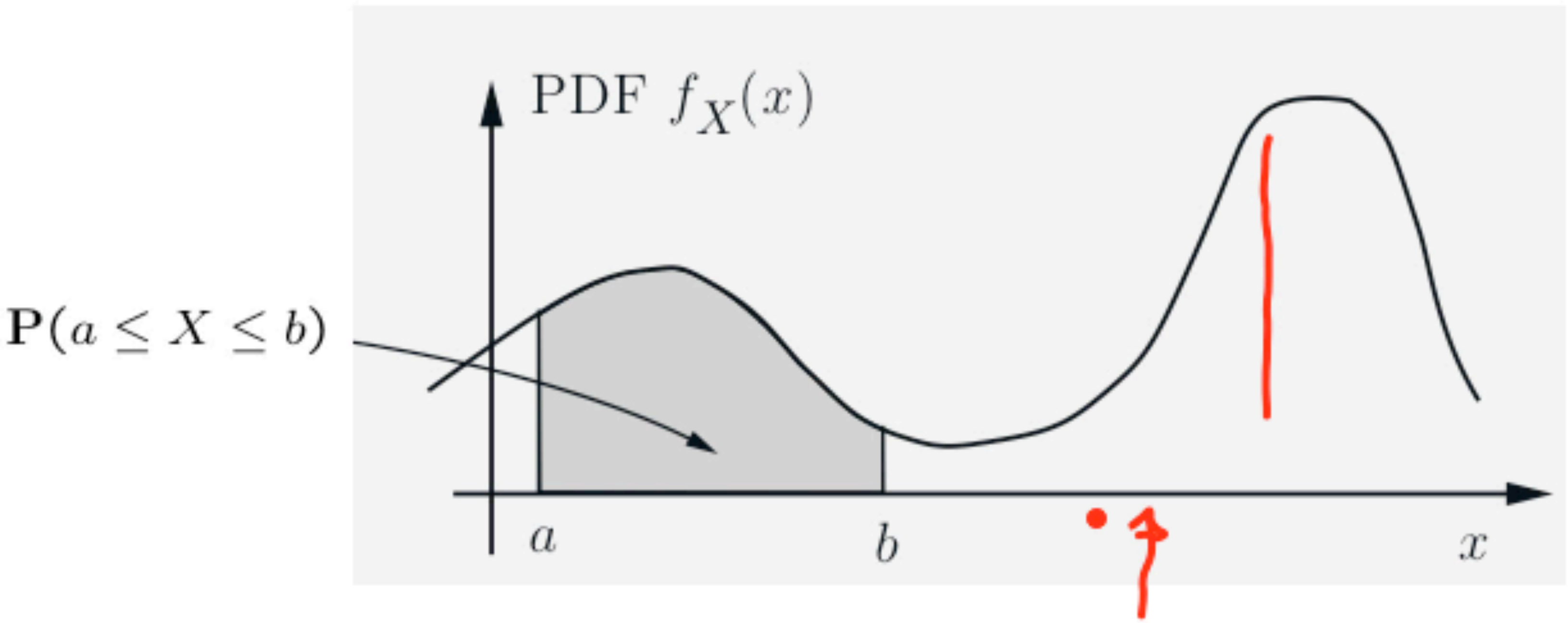
- Generalization: piecewise constant PDF



Expectation/mean of a continuous random variable



$$\mathbf{E}[X] = \sum_x x p_X(x)$$



$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- **Interpretation:** Average in large number of independent repetitions of the experiment

Fine print:
Assume $\int_{-\infty}^{\infty} |x| \, f_X(x) \, dx < \infty$

Properties of expectations

- If $X \geq 0$, then $\mathbf{E}[X] \geq 0$

- If $a \leq X \leq b$, then $a \leq \mathbf{E}[X] \leq b$

- Expected value rule:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

$$\boxed{\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

- Linearity

$$\boxed{\mathbf{E}[aX + b] = a\mathbf{E}[X] + b}$$

Variance and its properties

- **Definition of variance:** $\mathrm{var}(X) = \mathbf{E}[(X-\mu)^2]$

$\mu = E[X]$

- Calculation using the expected value rule, $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

$$\mathrm{var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$$

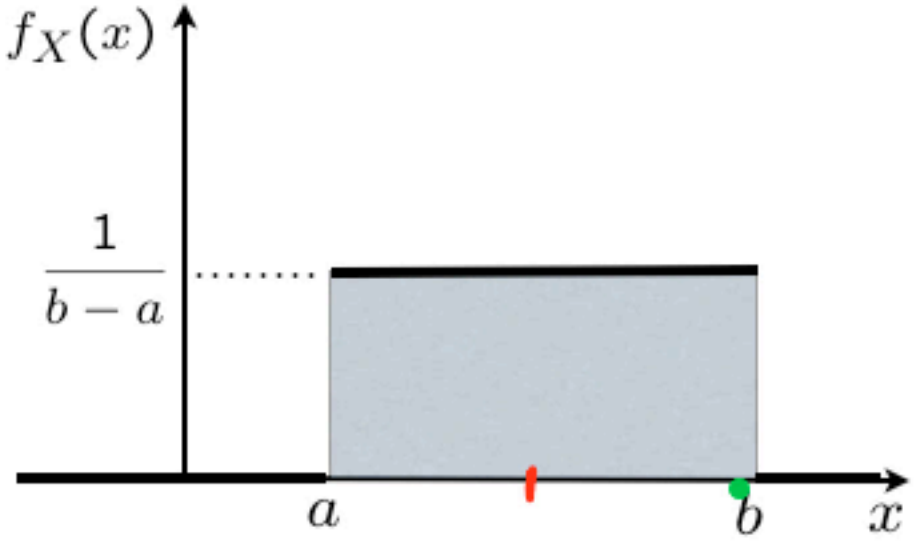
$g(x) = (x-\mu)^2$

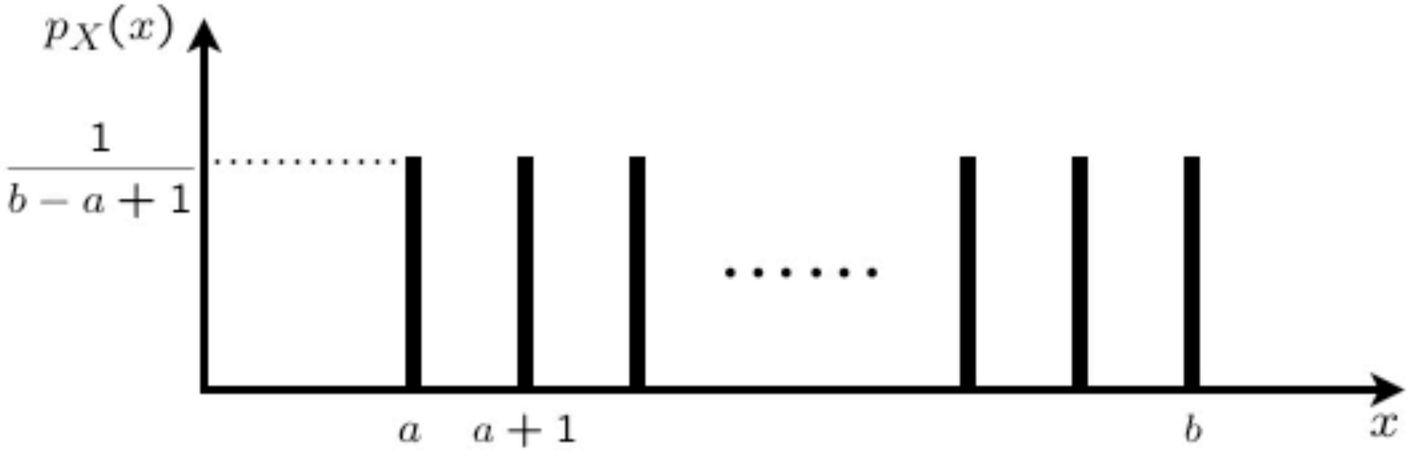
Standard deviation: $\sigma_X = \sqrt{\mathrm{var}(X)}$

✓ $\mathrm{var}(aX+b) = a^2 \mathrm{var}(X)$

✓ A useful formula: $\mathrm{var}(X) = \mathbf{E}[X^2] - \left(\mathbf{E}[X]\right)^2$

Continuous uniform random variable; parameters a, b





$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{a+b}{2}$$

$$\mathbf{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right)$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = \boxed{(b-a)^2/12}$$

$$\sigma = \frac{b-a}{\sqrt{12}}$$

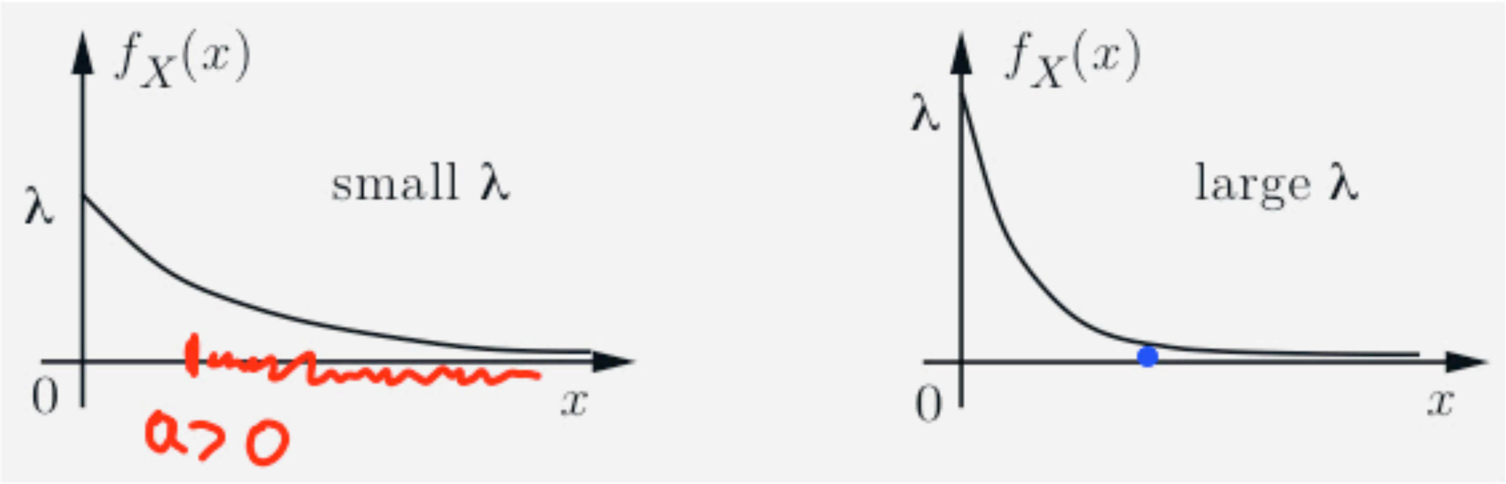
$$\mathbf{E}[X] = \frac{a+b}{2}$$

$$\mathrm{var}(X) = \frac{1}{12}(b-a)(b-a+\underline{2})$$

Exponential random variable; parameter $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

$\int f_X(x)\,dx = 1$

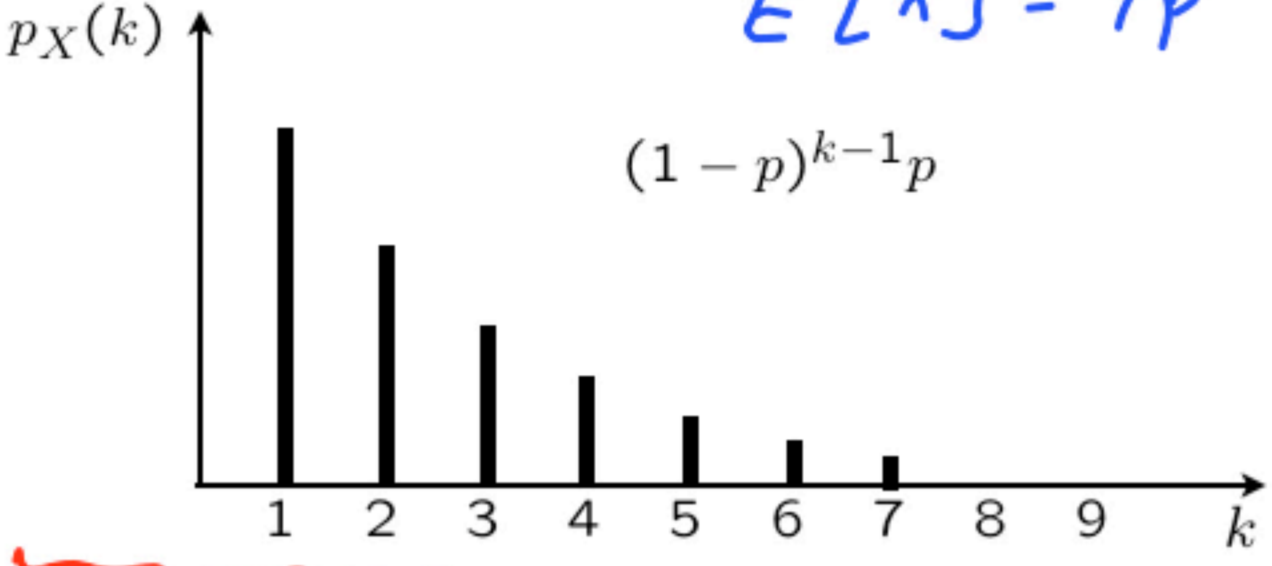


$$\mathbf{E}[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = 1/\lambda$$

$$\mathbf{E}[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx = 2/\lambda^2$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = 1/\lambda^2$$

$E[X] = 1/p$



$$\boxed{P(X \ge a)} = \int_a^\infty \lambda e^{-\lambda x}\,dx$$

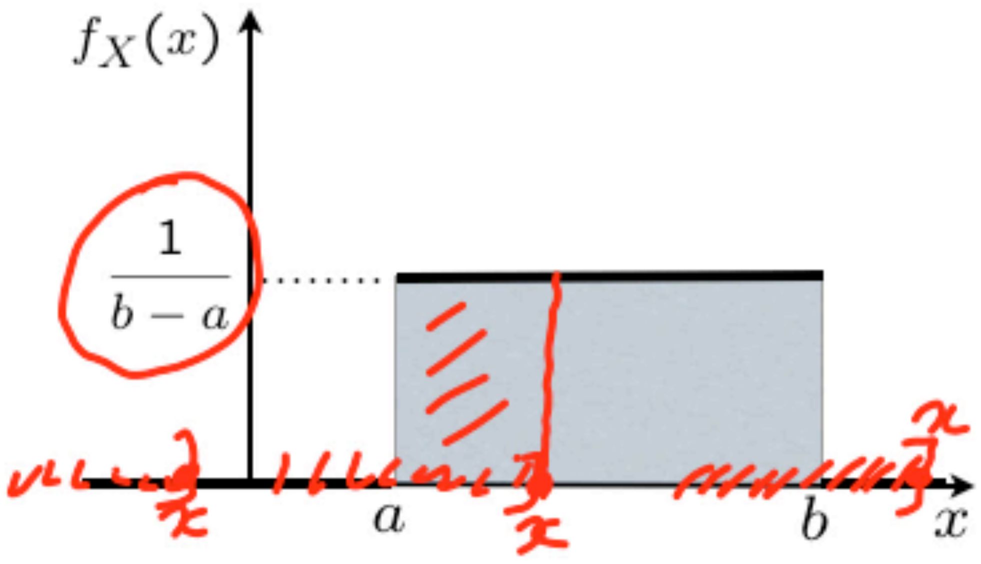
$$\left[\int e^{ax}\,dx = \frac{1}{a} e^{ax} \quad a \leftrightarrow -\lambda \right]$$

$$= \lambda \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x}\Big|_a^\infty$$

$$= -e^{-\lambda \cdot \infty} + e^{-\lambda a} = \boxed{e^{-\lambda a}}$$

Cumulative distribution function (CDF)

CDF definition: $F_X(x) = \mathrm{P}(X \leq x)$



- Continuous random variables:

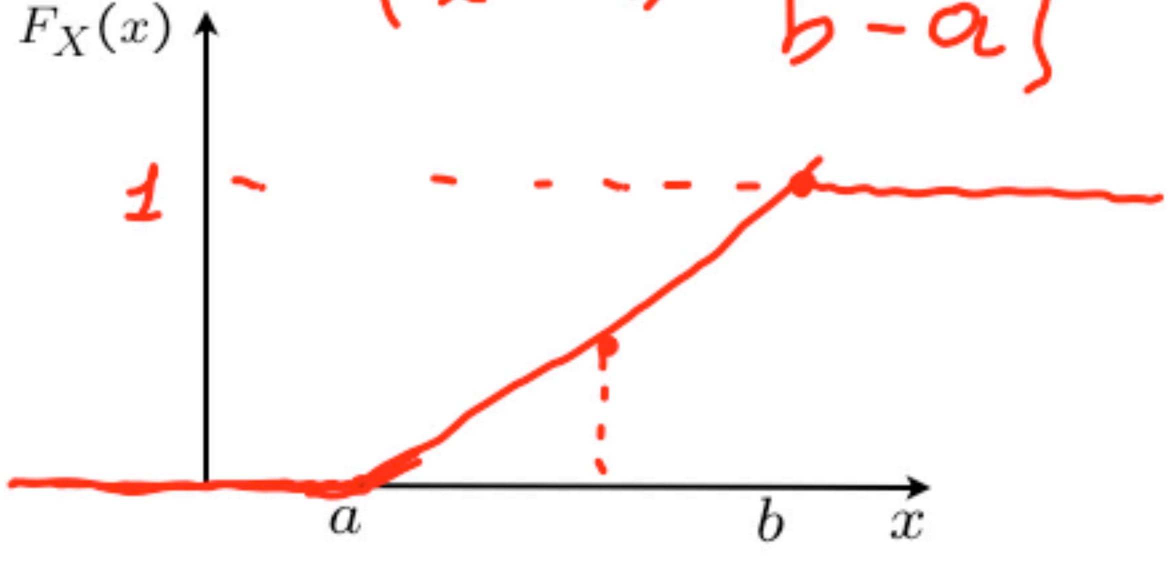
$$F_X(x) = \mathrm{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$$

3 4

$$\underbrace{P(X \leq 4)}_{} = \underbrace{P(X \leq 3)}_{} + P(3 < X \leq 4)$$

$$\boxed{\frac{dF_X}{dx}(x) = f_X(x)}$$

$(x-a) \cdot \frac{1}{b-a}$

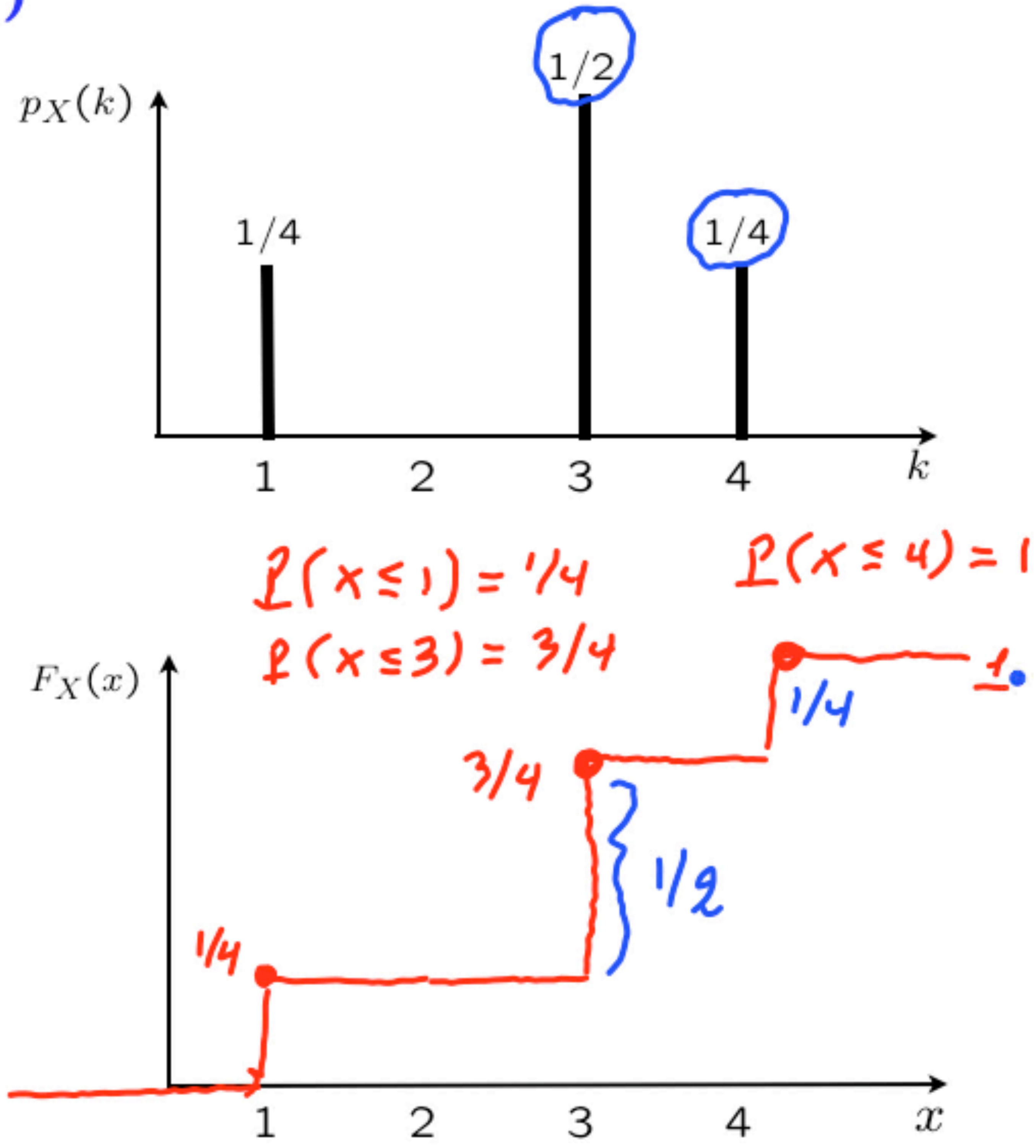


Cumulative distribution function (CDF)

CDF definition: $F_X(x) = \mathrm{P}(X \le x)$

- Discrete random variables:

$$F_X(x) = \mathrm{P}(X \le x) = \sum_{k \le x} p_X(k)$$



General CDF properties

$$F_X(x) = \mathbf{P}(X \leq x)$$

- Non-decreasing

If $y \geq x \Rightarrow F_X(y) \geq F_X(x)$

- $F_X(x)$ tends to 1, as $x \to \infty$

- $F_X(x)$ tends to 0, as $x \to -\infty$

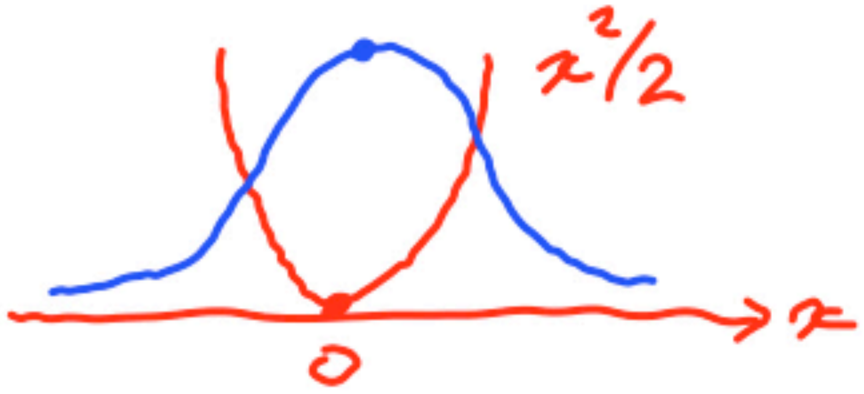
Normal (Gaussian) random variables

- Important in the theory of probability
  - Central limit theorem

- Prevalent in applications
  - Convenient analytical properties
  - Model of noise consisting of many, small independent noise terms

Standard normal (Gaussian) random variables

- Standard normal $N(0,1)$: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



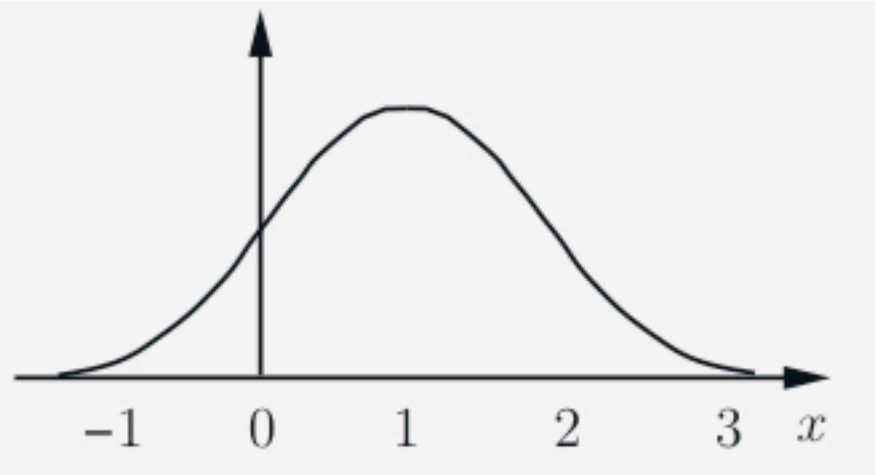
calculus:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

- $\mathbf{E}[X] = 0$

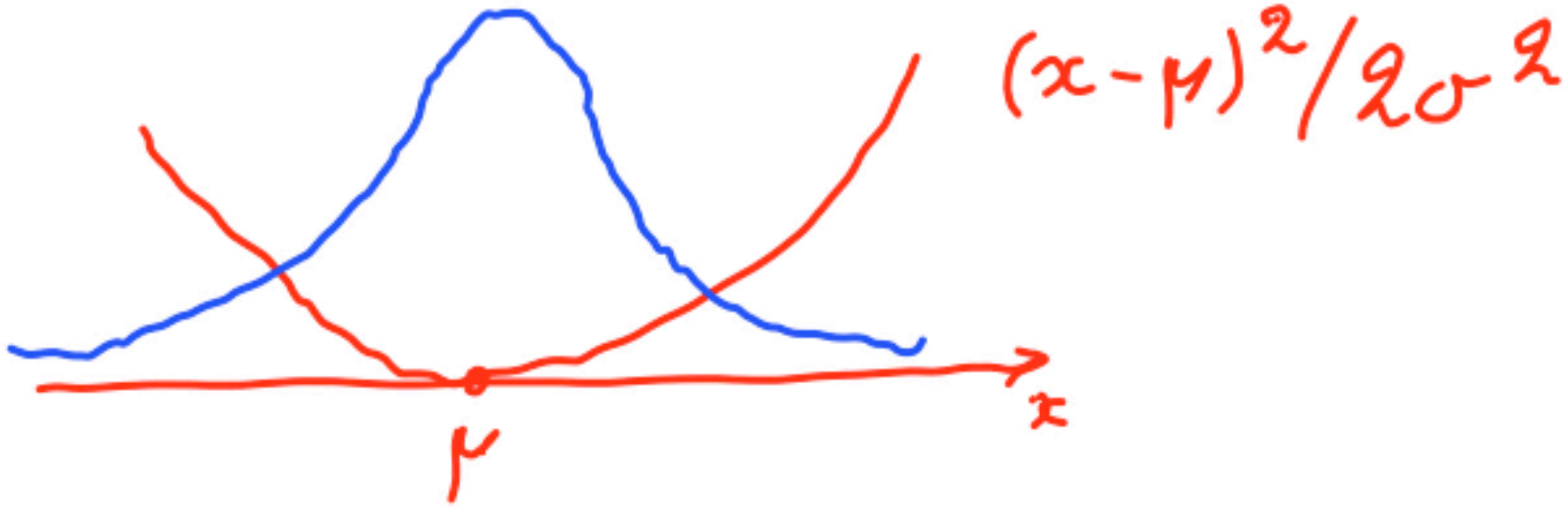
- $\mathrm{var}(X) = 1$ integrate by parts

General normal (Gaussian) random variables

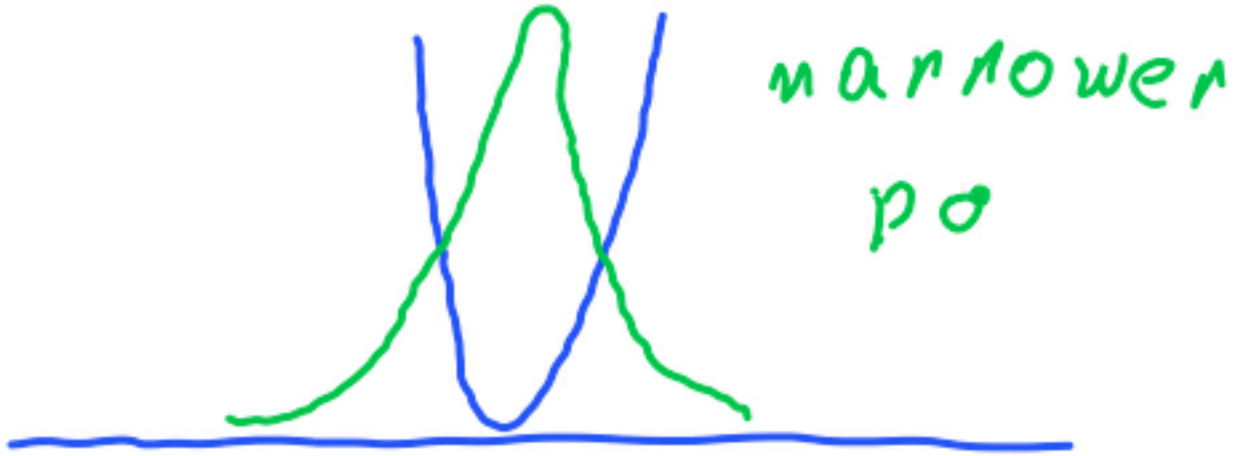


- General normal $N(\mu, \sigma^2)$: $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

$\sigma > 0$



σ small



- $\mathbf{E}[X] = \mu$

- $\mathrm{var}(X) = \sigma^2$

Linear functions of a normal random variable

- Let $Y = aX + b \quad X \sim N(\mu, \sigma^2)$

  $\mathbf{E}[Y] = a\mu + b$

  $\text{Var}(Y) = a^2 \sigma^2$

- Fact (will prove later in this course):

  $$\boxed{Y \sim N(a\mu + b, a^2\sigma^2)}$$

- Special case: $a = 0$?

  $Y = b$ discrete

  $N(b, 0)$

Standard normal tables

- No closed form available for CDF

but have tables, for the standard normal

$Y \sim N(0,1)$

$\Phi(y) = F_Y(y) = P(Y \le y)$

$\Phi(0) = P(Y \le 0) = 0.5$

$\Phi(1.16) = 0.8770$ $\quad \Phi(2.9) = 0.9981$

$\Phi(-2) = 1 - \Phi(2) = 1 - .9772$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Standardizing a random variable

- Let X have mean μ and variance $\sigma^2 > 0$

- Let $Y = \frac{X - \mu}{\sigma}$ $\quad E[Y] = 0 \quad Var(Y) = \frac{1}{\sigma^2} Var(X) = 1$

$$X = \mu + \sigma Y$$

- If also X is normal, then: $Y \sim N(0,1)$

Calculating normal probabilities

- Express an event of interest in terms of standard normal

$X \sim N(6,4) \qquad \sigma = 2$

$$\frac{2-6}{2} \le \frac{X-6}{2} \le \frac{8-6}{2}$$

st. normal

$$P(2 \le X \le 8) = P(-2 \le Y \le 1)$$

$$= P(Y \le 1) - P(Y \le -2)$$

$$= P(Y \le 1) - (1 - P(Y \le 2))$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
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0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
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1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
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1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

LECTURE 9: Conditioning on an event; Multiple continuous r.v.'s

- Conditioning a r.v. on an event
  - Conditional PDF
  - Conditional expectation and the expected value rule
  - Exponential PDF: memorylessness
  - Total probability and expectation theorems
  - Mixed distributions
- Jointly continuous r.v.'s and joint PDFs
  - From the joints to the marginals
  - Uniform joint PDF example
  - The expected value rule and linearity of expectations
  - The joint CDF

Conditional PDF, given an event

$\mathbf{P}(A) > 0$

$$p_X(x) = \mathbf{P}(X = x) \qquad f_X(x) \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta)$$

$$p_{X|A}(x) = \mathbf{P}(X = x \mid A) \qquad f_{X|A}(x) \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta \mid A)$$

$$\mathbf{P}(X \in B) = \sum_{x \in B} p_X(x) \qquad \mathbf{P}(X \in B) = \int_B f_X(x)\, dx$$

$$\mathbf{P}(X \in B \mid A) = \sum_{x \in B} p_{X|A}(x) \qquad \mathbf{P}(X \in B \mid A) = \int_B f_{X|A}(x)\, dx$$

Def

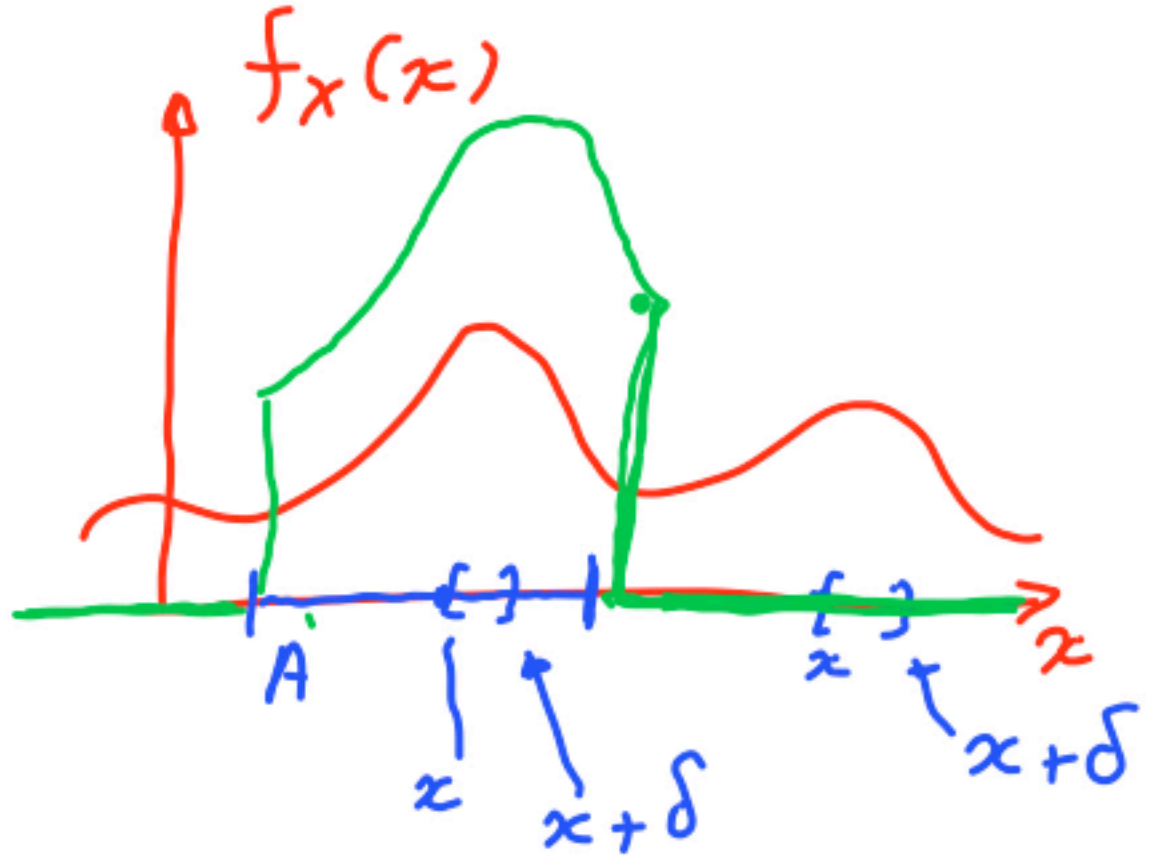
$$\sum_x p_{X|A}(x) = 1 \qquad \int f_{X|A}(x)\, dx = 1$$

Conditional PDF of X, given that $X \in A$

$$\mathbf{P}(x \le X \le x+\delta \mid X \in A) \approx f_{X|X\in A}(x)\cdot \cancel{\delta}$$

$$= \frac{\mathbf{P}(x \le X \le x+\delta,\ X \in A)}{\mathbf{P}(A)}$$

$$= \frac{\mathbf{P}(x \le X \le x+\delta)}{\mathbf{P}(A)} \approx \frac{f_X(x)\,\cancel{\delta}}{\mathbf{P}(A)}$$



$$f_{X|X\in A}(x) = \begin{cases} 0, & \text{if } x \notin A \\ \dfrac{f_X(x)}{\mathbf{P}(A)}, & \text{if } x \in A \end{cases}$$

Conditional expectation of X, given an event

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

$$\mathbf{E}[X] = \int x f_X(x)\, dx$$

$$\mathbf{E}[X \mid A] = \sum_x x p_{X|A}(x)$$

$$\mathbf{E}[X \mid A] = \int x f_{X|A}(x)\, dx$$

Def

Expected value rule:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

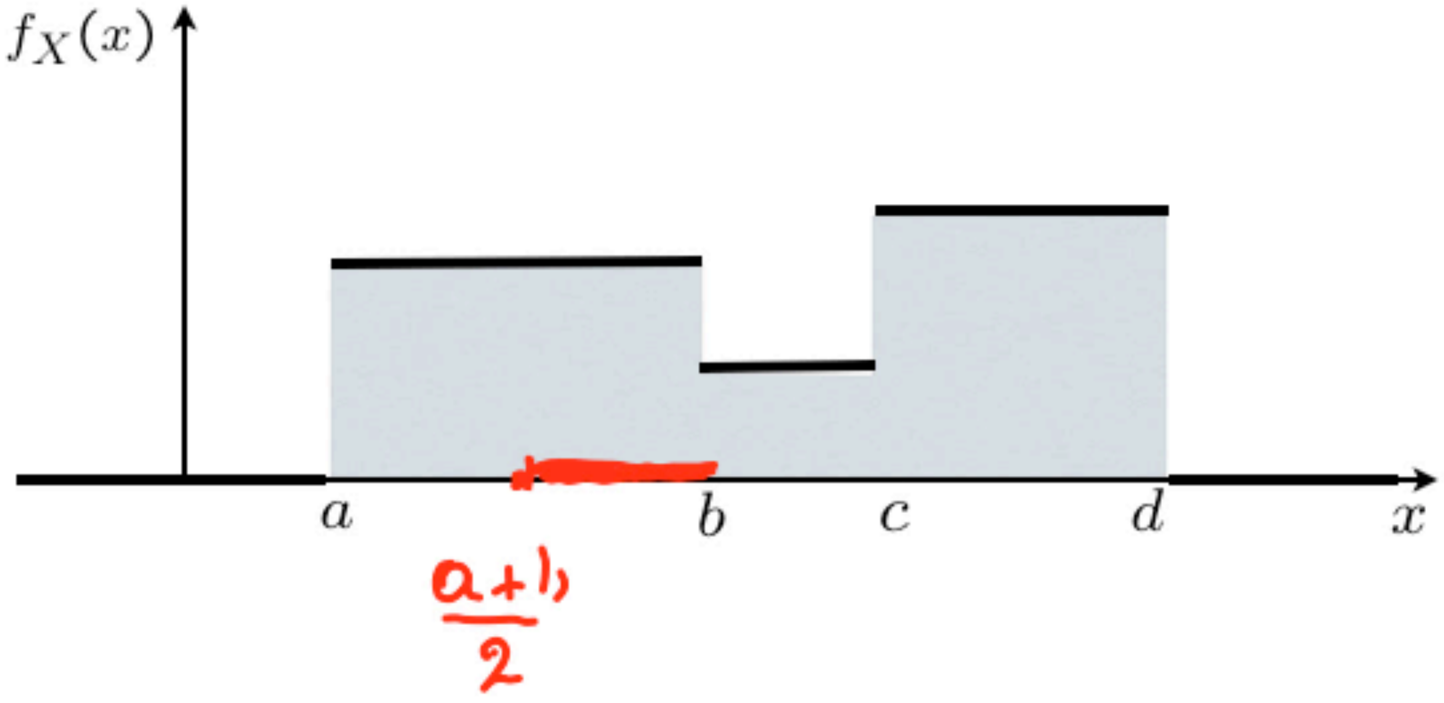
$$\mathbf{E}[g(X)] = \int g(x) f_X(x)\, dx$$

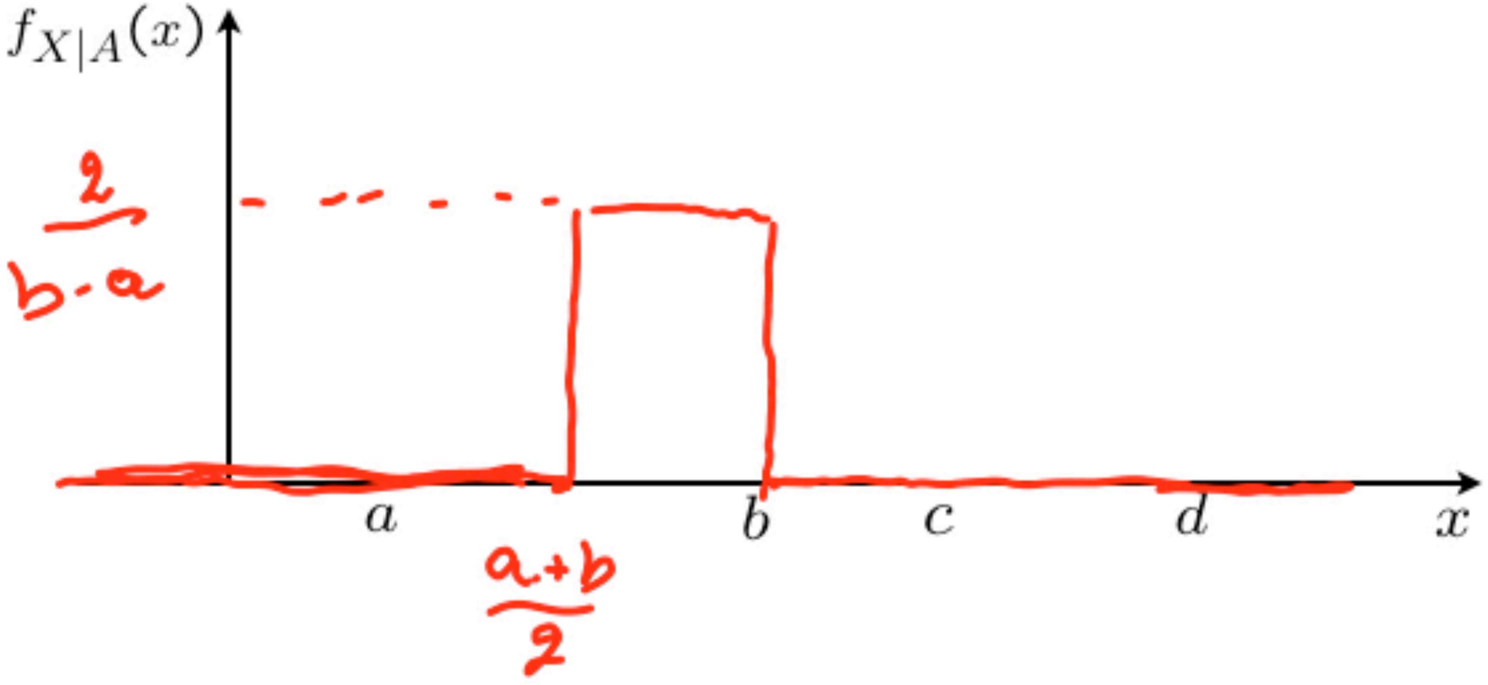
$$\mathbf{E}[g(X) \mid A] = \sum_x g(x) p_{X|A}(x)$$

$$\mathbf{E}[g(X) \mid A] = \int g(x) f_{X|A}(x)\, dx$$

Example

$$A: \quad \frac{a+b}{2} \le X \le b$$





$$\mathbf{E}[X \mid A] = \frac{1}{2} \cdot \frac{a+b}{2} + \frac{1}{2} b$$

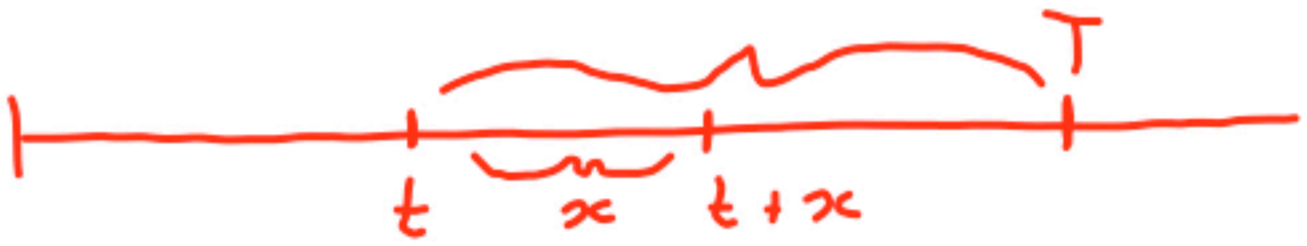
$$= \frac{1}{4} a + \frac{3}{4} b$$

$$\mathbf{E}[X^2 \mid A] =$$

$$\int_{\frac{a+b}{2}}^{b} \frac{2}{b-a} \cdot x^2 \, dx$$

Memorylessness of the exponential PDF

- Do you prefer a used or a new "exponential" light bulb? **Probabilistically identical!**



- Bulb lifetime T: exponential(λ)

  $\mathbf{P}(T > x) = e^{-\lambda x}$, for $x \geq 0$

  – we are told that $T > t$

  – r.v. X: remaining lifetime $= T - t$

  $\mathbf{P}(X > x \,|\, T > t) = e^{-\lambda x}$, for $x \geq 0$

$$= \frac{\mathbf{P}(T - t > x,\ T > t)}{\mathbf{P}(T > t)} = \frac{\mathbf{P}(T > t + x,\ T > t)}{\mathbf{P}(T > t)} = \frac{\mathbf{P}(T > t + x)}{\mathbf{P}(T > t)}$$

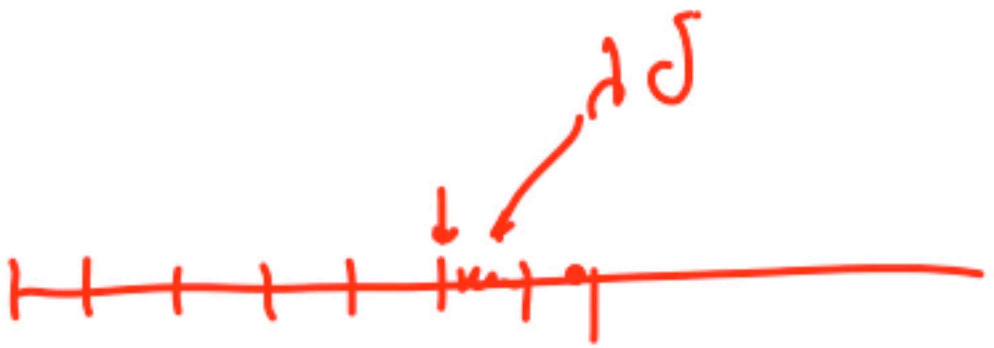
$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$

Memorylessness of the exponential PDF

$f_T(x) = \lambda e^{-\lambda x}$, for $x \geq 0$

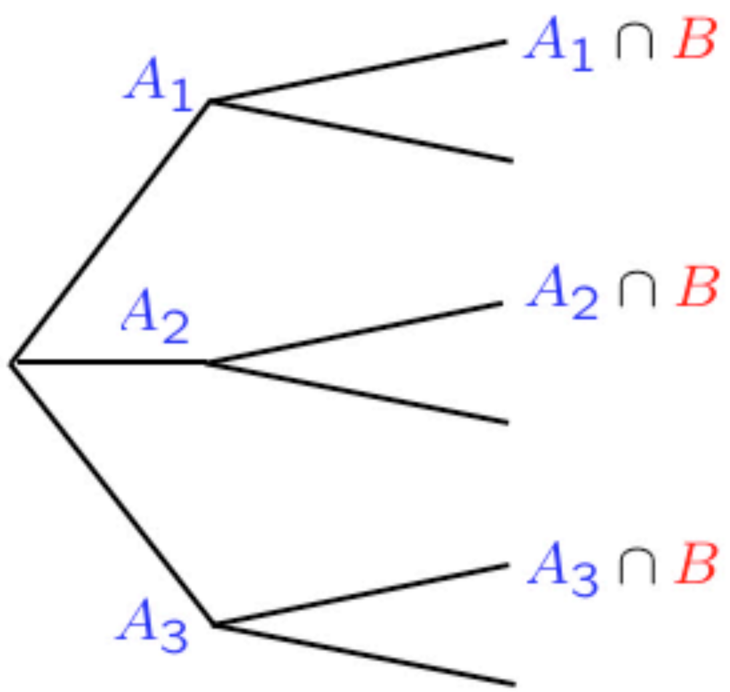
$\mathbf{P}(0 \leq T \leq \delta) \approx f_T(0) \cdot \delta = \lambda \delta$

$\mathbf{P}(t \leq T \leq t + \delta \mid T > t) = \approx \lambda \delta$



similar to an independent coin flip,
every δ time steps,
with $\mathbf{P}(\text{success}) \approx \lambda\delta$

Total probability and expectation theorems



$$\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)$$

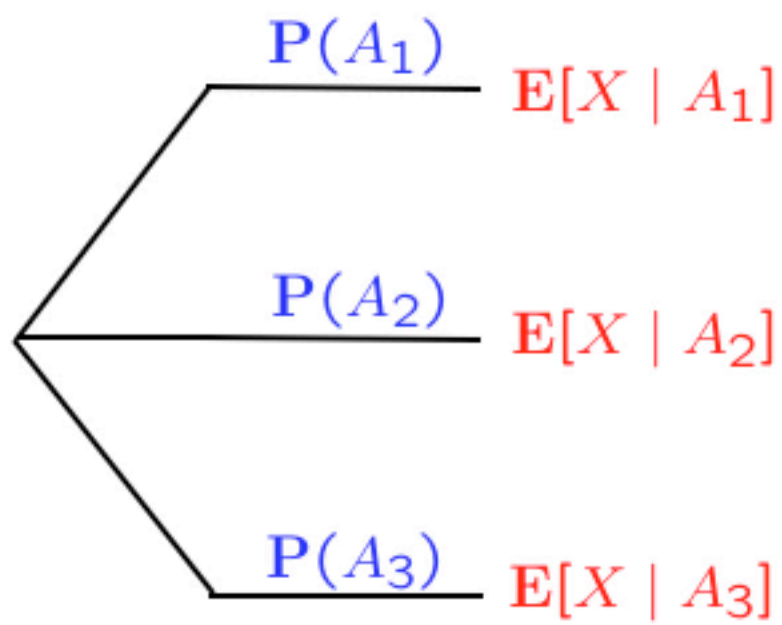
$$p_X(x) = \mathbf{P}(A_1)\, p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\, p_{X|A_n}(x)$$

$$F_X(x) = P(X \le x) = P(A_1)\, P(X \le x \mid A_1) + \cdots$$

$$= P(A_1)\, F_{X|A_1}(x) + \cdots$$

$$\boxed{f_X(x) = \mathbf{P}(A_1) f_{X|A_1}(x) + \cdots + \mathbf{P}(A_n) f_{X|A_n}(x)}$$

$$\int x f_X(x)\, dx = P(A_1) \int x f_{X|A_1}(x)\, dx + \cdots$$

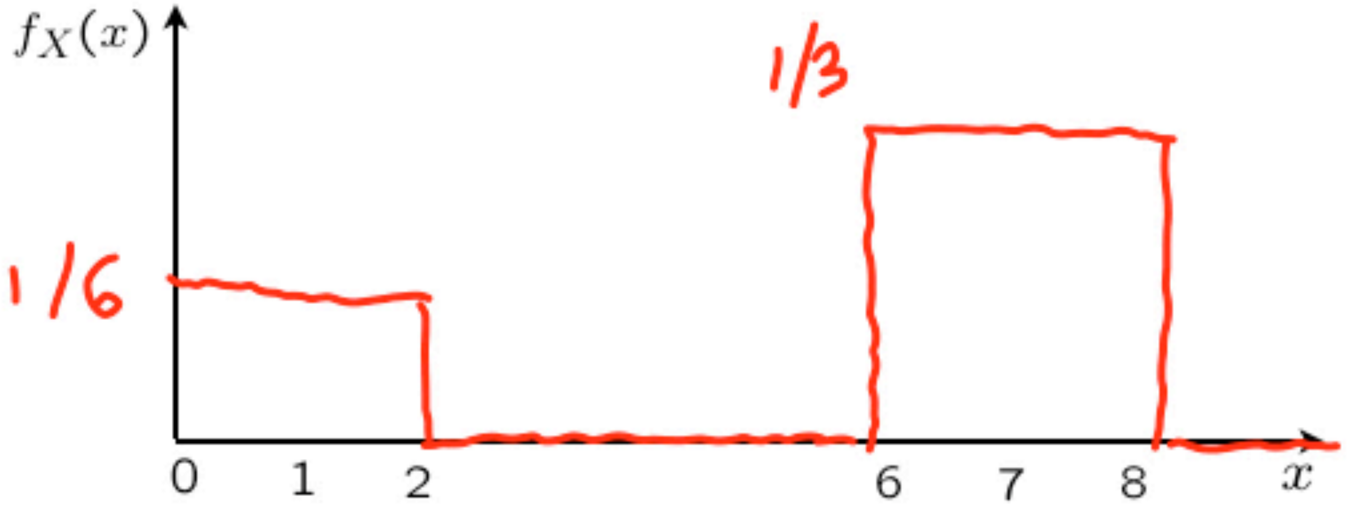


$$\boxed{\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]}$$

Example

- Bill goes to the supermarket shortly, with probability 1/3, at a time uniformly distributed between 0 and 2 hours from now; or with probability 2/3, later in the day at a time uniformly distributed between 6 and 8 hours from now

$P(A_1) = \frac{1}{3}$ $\quad f_{X|A_1} \sim \text{unif}[0,2]$ $\quad P(A_2) = \frac{2}{3}$ $\quad f_{X|A_2} \sim U[6,8]$



$$f_X(x) = \mathbf{P}(A_1) f_{X|A_1}(x) + \cdots + \mathbf{P}(A_n) f_{X|A_n}(x)$$

- $\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]$

$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 7$

Mixed distributions

$$X = \begin{cases} \text{uniform on } [0,2], & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}$$

Is X discrete? No

Is X continuous? No

$P(X=1) = 1/2$

Y discrete

Z continuous

$$X = \begin{cases} Y, & \text{with probability } p \\ Z, & \text{with probability } 1-p \end{cases}$$

X is mixed

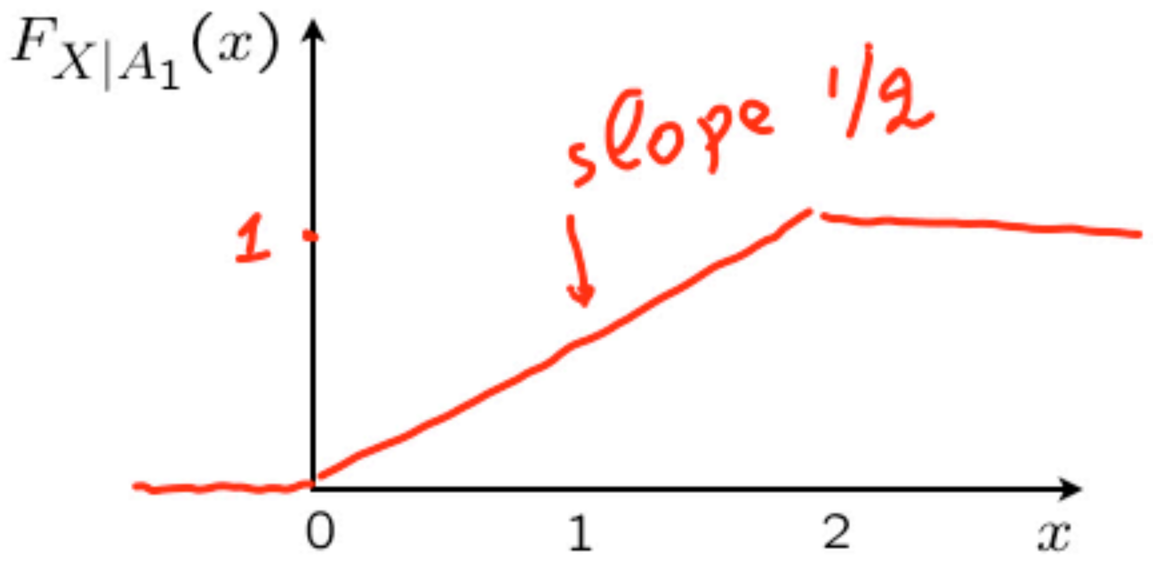
$$F_X(x) = p \cdot P(Y \le x) + (1-p)\, P(Z \le x)$$

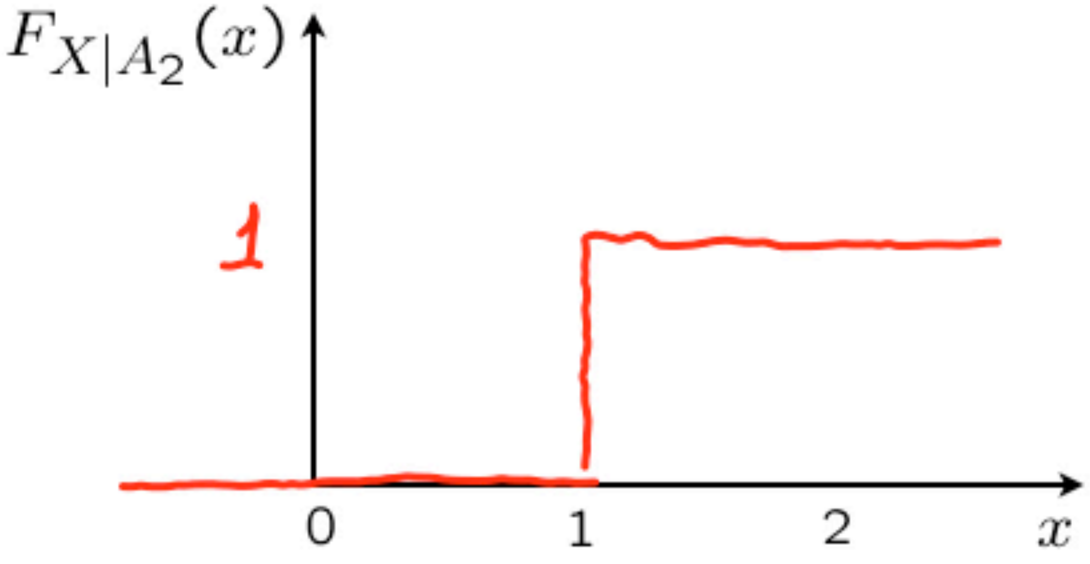
$$= p\, F_Y(x) + (1-p)\, F_Z(x)$$

$$E[X] = p\, E[Y] + (1-p)\, E[Z]$$

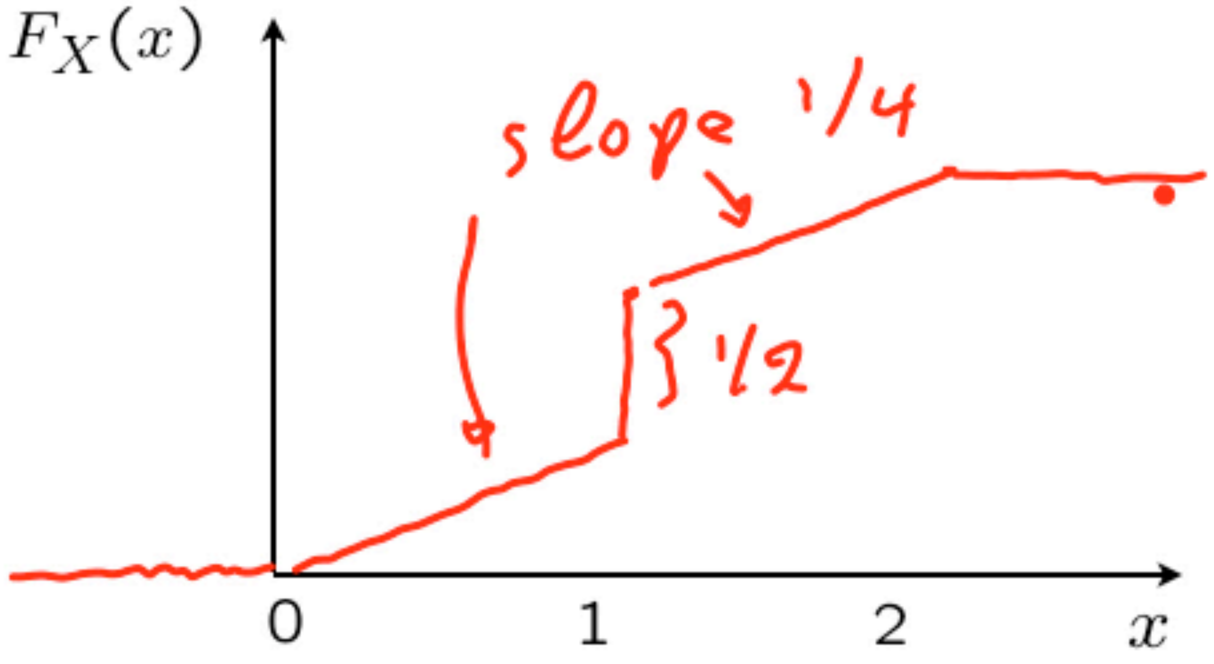
Mixed distributions

$$X = \begin{cases} \text{uniform on } [0,2], & \text{with probability } 1/2 \quad A_1 \\ 1, & \text{with probability } 1/2 \quad A_2 \end{cases}$$





$$F_X(x) = \mathbf{P}(A_1)F_{X|A_1}(x) + \mathbf{P}(A_2)F_{X|A_2}(x)$$



Jointly continuous r.v.'s and joint PDFs

$p_X(x)$	$f_X(x)$
$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$

$p_{X,Y}(x,y) = \mathbf{P}(X = x \text{ and } Y = y) \geq 0$

$f_{X,Y}(x,y) \geq 0$

$$\mathbf{P}\Big((X,Y) \in B\Big) = \sum_{(x,y)\in B}\sum p_{X,Y}(x,y)$$

$$\mathbf{P}\Big((X,Y) \in B\Big) = \int_{(x,y)\in B}\int f_{X,Y}(x,y)\,dx\,dy$$

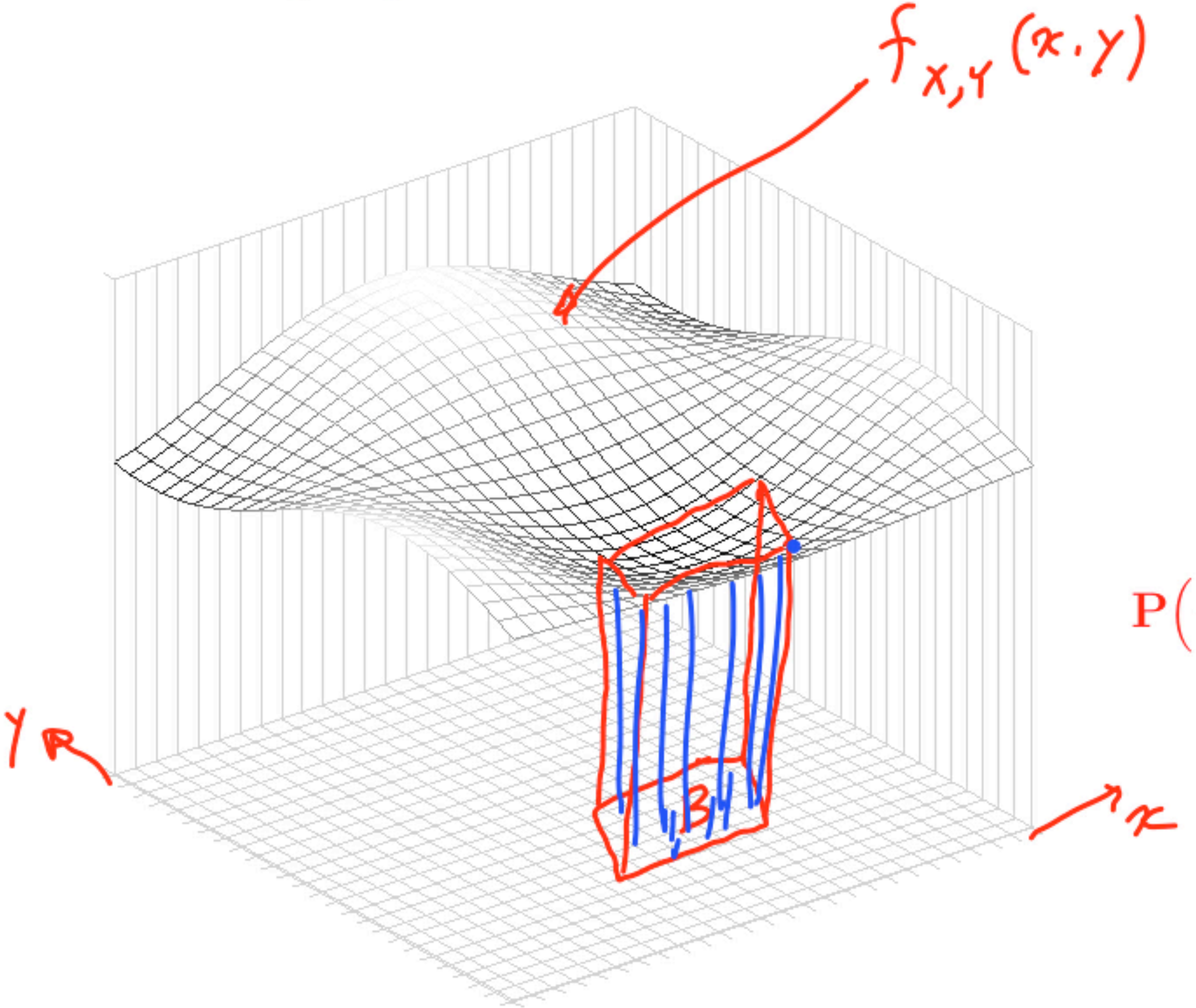
-

$$\sum_x \sum_y p_{X,Y}(x,y) = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Definition: Two random variables are **jointly continuous** if they can be described by a joint PDF

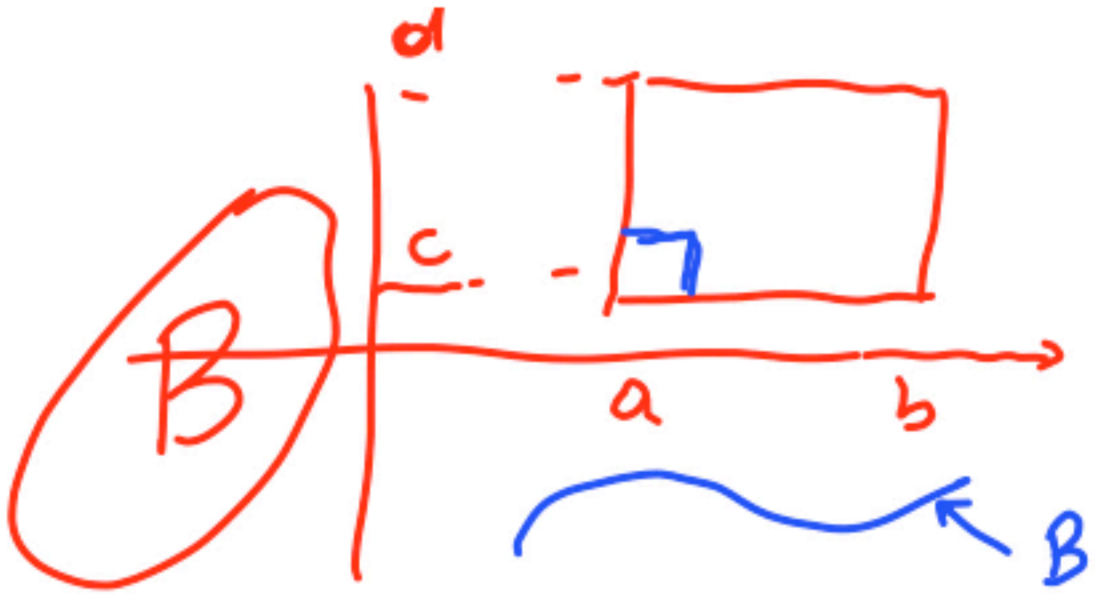
Visualizing a joint PDF



$$\mathbf{P}\big((X,Y) \in B\big) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

On joint PDFs

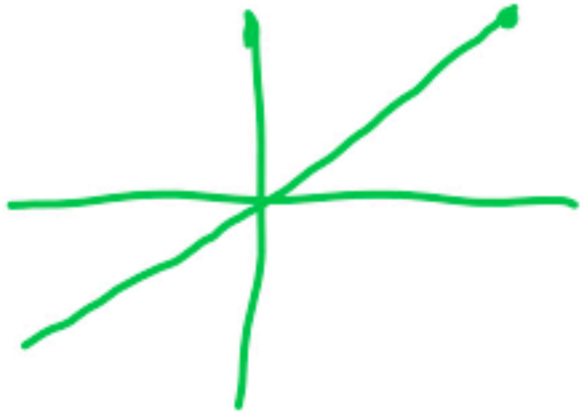
$$\mathbf{P}\Big((X,Y)\in B\Big) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$



$$\mathbf{P}(a\le X\le b,\ c\le Y\le d) = \int_c^d\int_a^b f_{X,Y}(x,y)\,dx\,dy$$

$$\boxed{\mathbf{P}(a\le X\le a+\delta,\ c\le Y\le c+\delta)\approx f_{X,Y}(a,c)\cdot\delta^2}$$

Y = X



$f_{X,Y}(x,y)$: probability per unit area

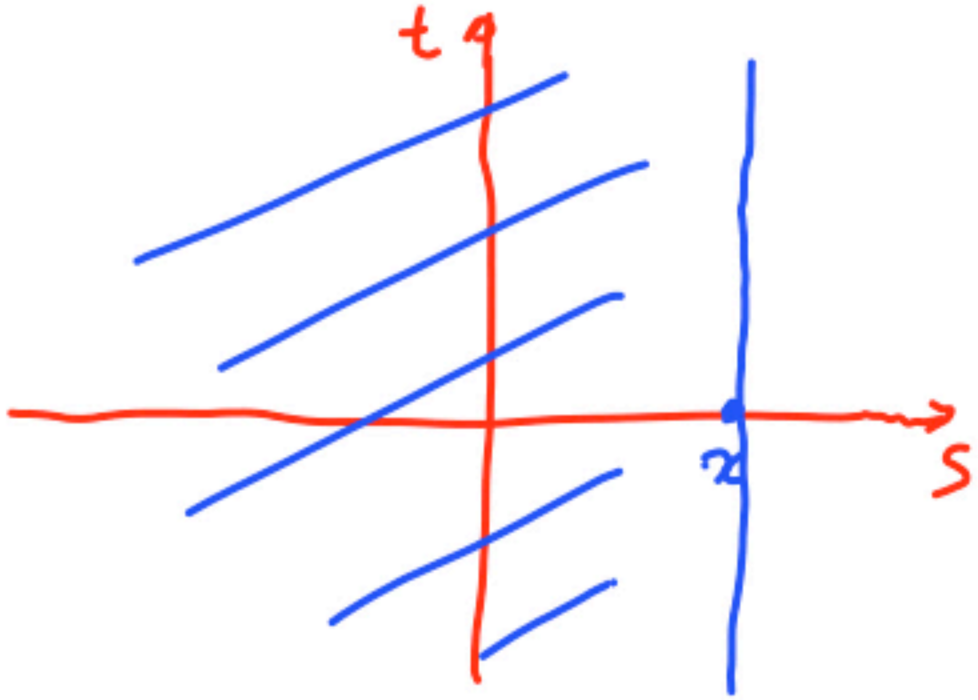
area$(B)=0 \quad\Rightarrow\quad \mathbf{P}\Big((X,Y)\in B\Big)=0$

From the joint to the marginals

$$p_X(x) = \sum_y p_{X,Y}(x,y) \qquad\qquad f_X(x) = \int f_{X,Y}(x,y)\,dy$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y) \qquad\qquad f_Y(y) = \int f_{X,Y}(x,y)\,dx$$

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,dt \right] ds$$

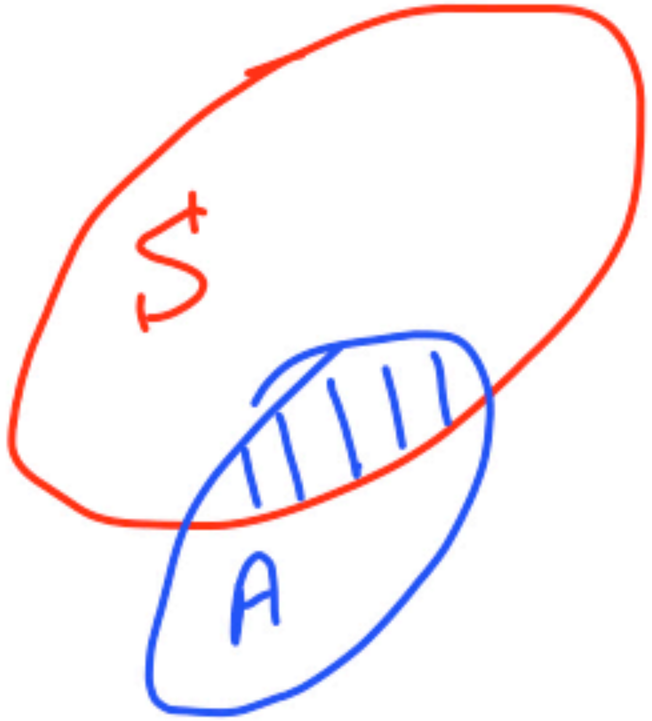


$$f_X(x) = \frac{dF_x}{dx}(x) = [\quad]$$

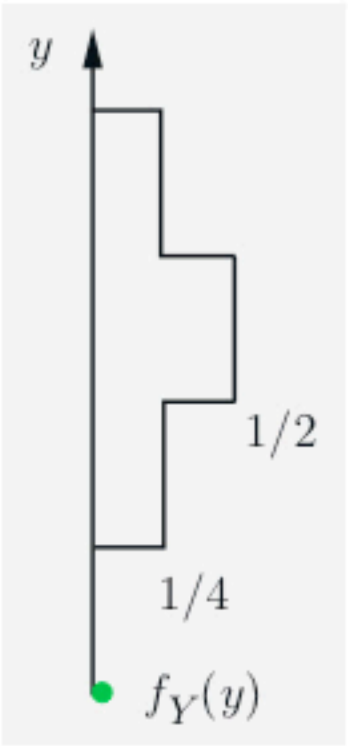
Uniform joint PDF on a set S

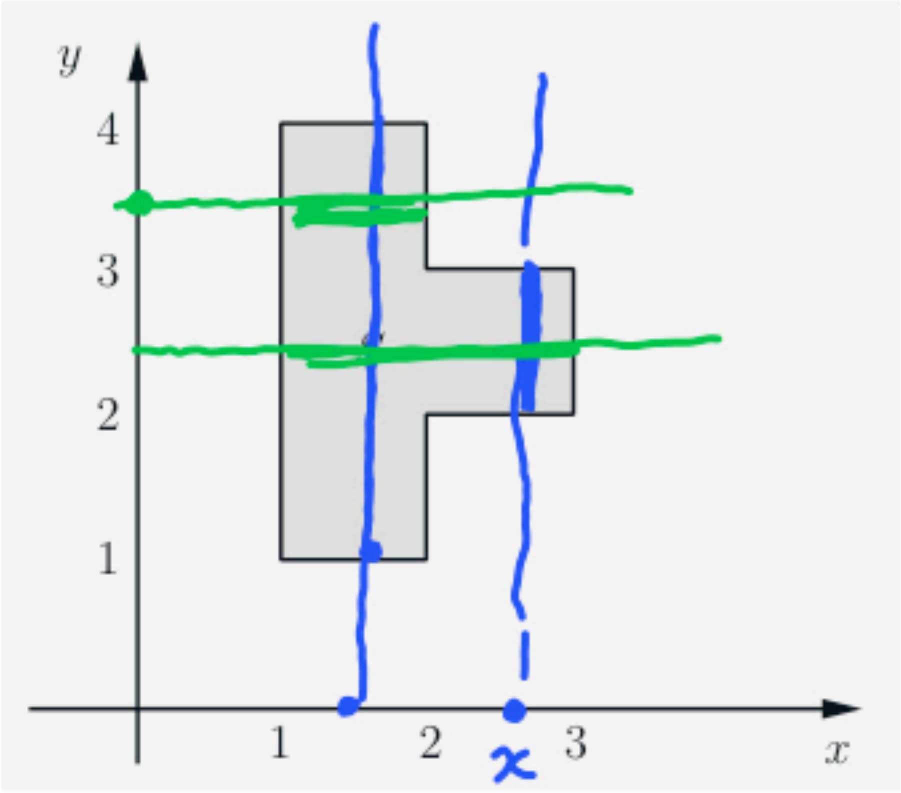
$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area of } S}, & \text{if } (x,y) \in S, \\ 0, & \text{otherwise.} \end{cases}$$

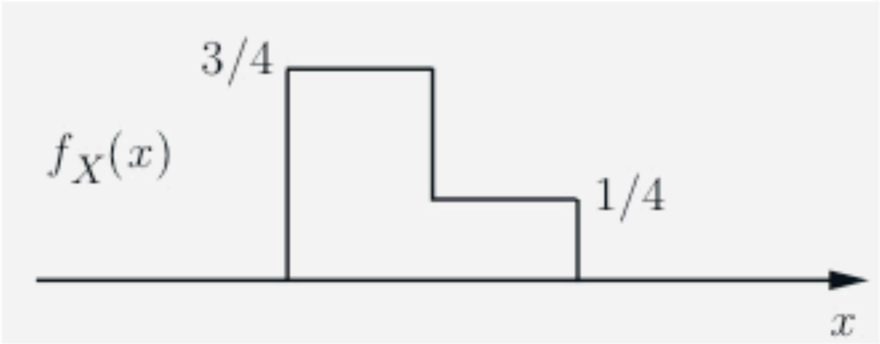
$f_{x,y} = \frac{1}{4}$



$\mathbf{P}(A) = \dfrac{\text{area}(A \cap S)}{\text{area}(S)}$







More than two random variables

$p_{X,Y,Z}(x, y, z)$

$f_{X,Y,Z}(x, y, z)$

$$\sum_x \sum_y \sum_z p_{X,Y,Z}(x, y, z) = 1$$

$$p_X(x) = \sum_y \sum_z p_{X,Y,Z}(x, y, z)$$

$$p_{X,Y}(x, y) = \sum_z p_{X,Y,Z}(x, y, z)$$

Functions of multiple random variables

$Z = g(X, Y)$

Expected value rule:

$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

$$\mathbf{E}[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

Linearity of expectations

$$\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$$

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

$$\boxed{\mathbf{E}[X_1 + \cdots + X_n] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n]}$$

The joint CDF

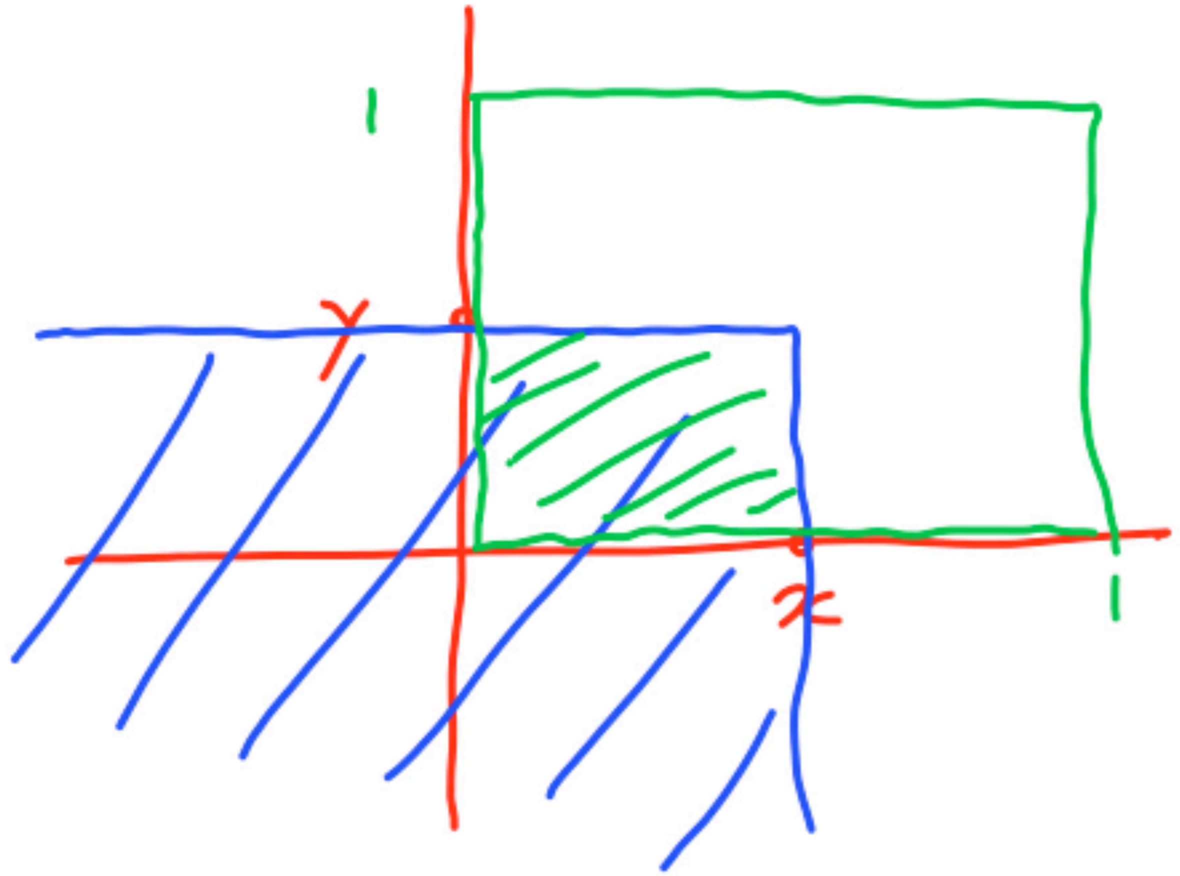
$$F_X(x) = \mathbf{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt \qquad f_X(x) = \frac{dF_X}{dx}(x)$$

$$F_{X,Y}(x,y) = \mathbf{P}(X \le x, Y \le y) = \int_{-\infty}^{y}\left[\int_{-\infty}^{x} \underbrace{f_{X,Y}(s,t)}_{}\,ds\right]dt$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x\,\partial y}(x,y)$$

$$F_{X,Y}(x,y) = xy$$

$$f_{X,Y}(x,y) = 1$$



LECTURE 10: Conditioning on a random variable; Independence; Bayes' rule

- Conditioning X on Y
  - Total probability theorem
  - Total expectation theorem
- Independence
  - independent normals
- A comprehensive example
- Four variants of the Bayes rule

Conditional PDFs, given another r.v.

$$p_{X|Y}(x \mid y) = \mathbf{P}(X = x \mid Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}, \quad \text{if } p_Y(y) > 0$$

$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$
$p_{X\|A}(x)$	$f_{X\|A}(x)$
$p_{X\|Y}(x \mid y)$	$f_{X\|Y}(x \mid y)$

Definition: $f_{X|Y}(x \mid y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ if $f_Y(y) > 0$

$\mathbf{P}(x \le X \le x + \delta \mid A) \approx f_{X|A}(x) \cdot \delta,$ where $\mathbf{P}(A) > 0$

$Y=y$ $\quad Y \approx y$

$$\mathbf{P}(x \le X \le x + \delta \mid y \le Y \le y + \epsilon) \approx \frac{f_{X,Y}(x,y)\,\delta\,\cancel{\epsilon}}{f_Y(y)\,\cancel{\epsilon}} = f_{X|Y}(x \mid y)\,\delta$$

Definition: $\mathbf{P}(X \in A \mid Y = y) = \int_A f_{X\,|\,Y}(x \mid y)\, dx$

Comments on conditional PDFs

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

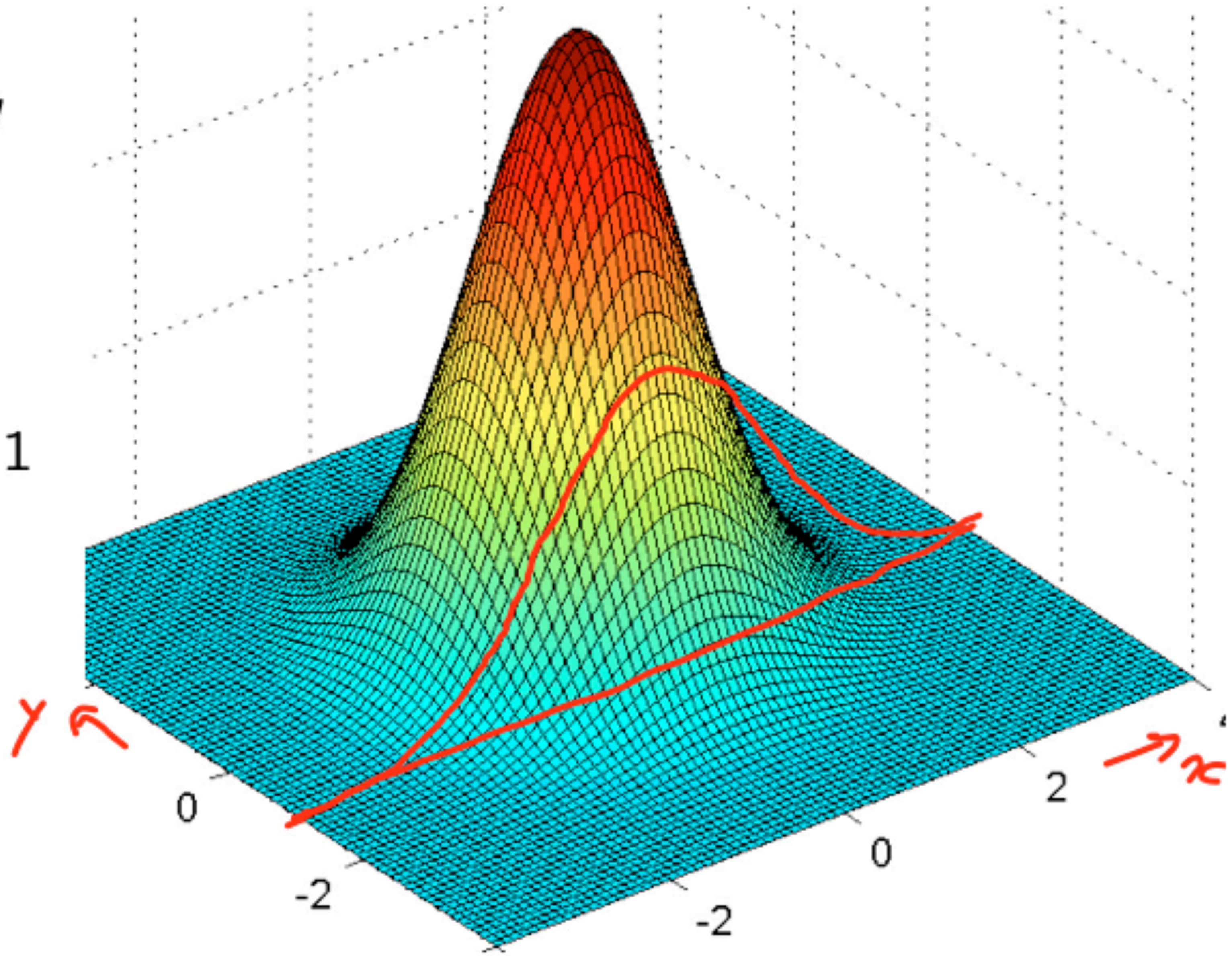
- $f_{X|Y}(x \mid y) \geq 0$

- Think of value of Y as fixed at some y
  shape of $f_{X|Y}(\cdot \mid y)$: slice of the joint

- $\displaystyle\int_{-\infty}^{\infty} f_{X|Y}(x \mid y)\, dx = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx}{f_Y(y)} = 1$

- Multiplication rule:

$$f_{X,Y}(x,y) = f_Y(y) \cdot f_{X|Y}(x \mid y)$$
$$= f_X(x) \cdot f_{Y|X}(y \mid x)$$



Total probability and expectation theorems

$$p_X(x) = \sum_y p_Y(y) p_{X|Y}(x \,|\, y)$$

$$f_X(x) = \int_{-\infty}^{\infty} \underbrace{f_Y(y) f_{X|Y}(x \,|\, y)}_{f_{X,Y}(x,y)} \, dy$$

Thm.

$$\mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \,|\, y)$$

$$\mathbf{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \,|\, y) \, dx$$

Def.

$$\mathbf{E}[X] = \sum_y p_Y(y) \mathbf{E}[X \mid Y = y]$$

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} f_Y(y) \mathbf{E}[X \mid Y = y] \, dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{\infty} x f_{X|Y}(x|y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y) \, dy}_{} \, dx$$

$$= \int_{-\infty}^{\infty} x f_X(x) \, dx = E[X]$$

- Expected value rule...

$$E[g(X) \,|\, Y = y]$$

$$= \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) \, dx$$

Independence

$p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$, for all x, y

$\boxed{f_{X,Y}(x,y) = \underline{f_X(x)}\,f_Y(y), \quad \text{for all } x \text{ and } y}$

$f_{Y|X} = f_Y$

$f_{X,Y}(x,y) = \underline{f_{X|Y}(x\,|\,y)}\,f_Y(y)$

- equivalent to: $f_{X|Y}(x\,|\,y) = f_X(x)$, for all y with $f_Y(y) > 0$ and all x

If X, Y are **independent:** $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$

$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$

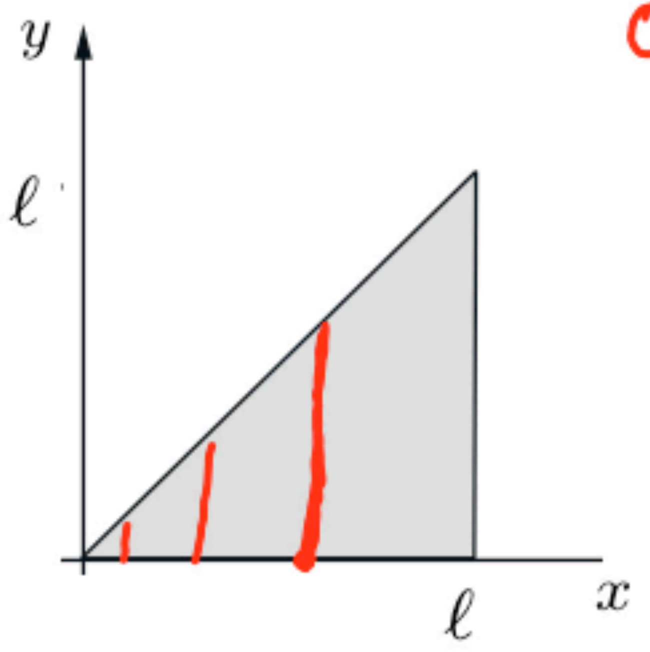
$g(X)$ and $h(Y)$ are also independent: $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$

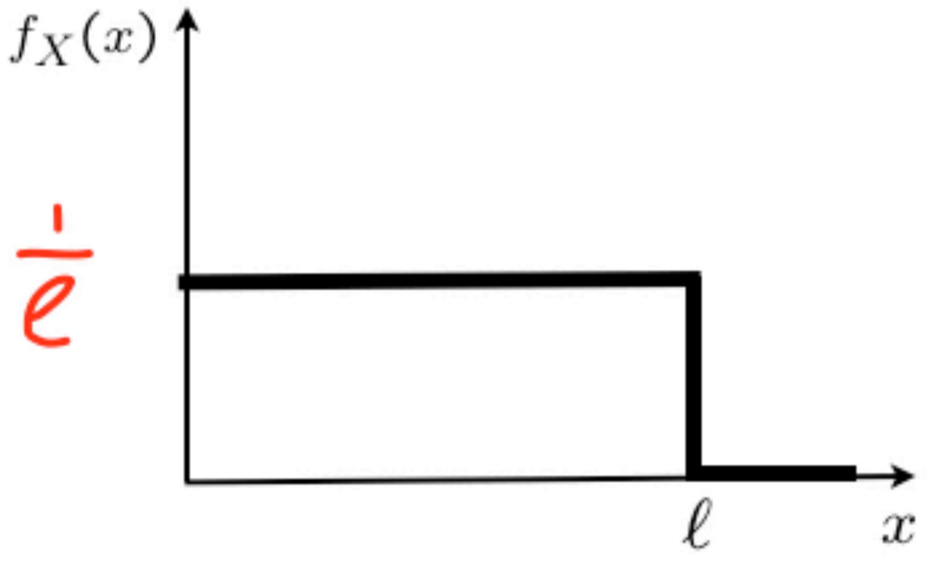
Stick-breaking example

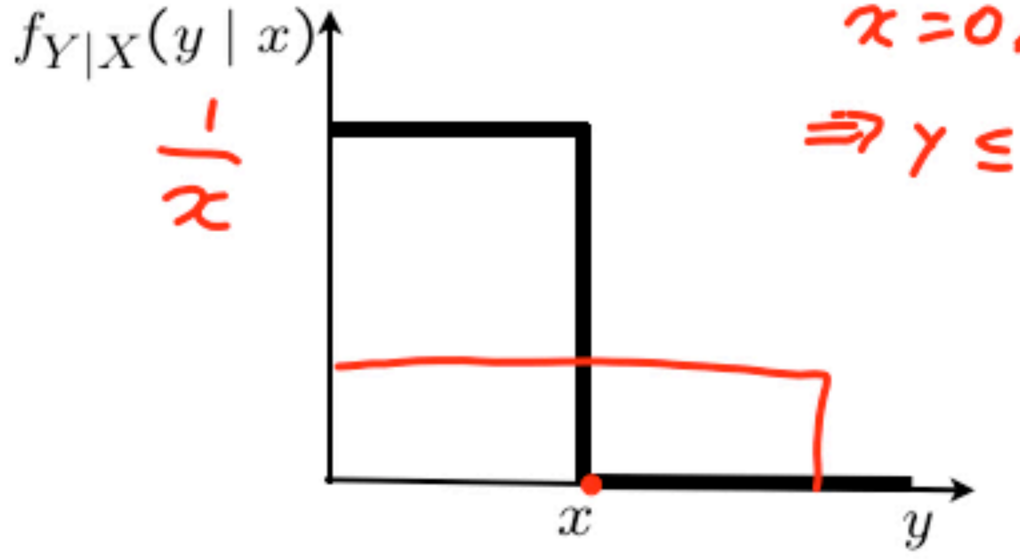
- Break a stick of length ℓ twice
  - first break at X: uniform in $[0, \ell]$
  - second break at Y: uniform in $[0, X]$

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) = \frac{1}{\ell x}$$

$0 \le y \le x \le \ell$



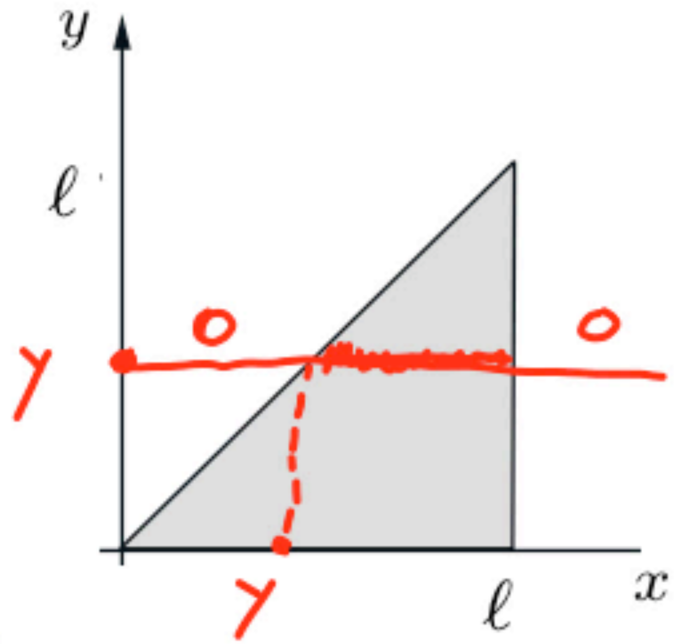




$x = 0.5$

$\Rightarrow y \le 0.5$

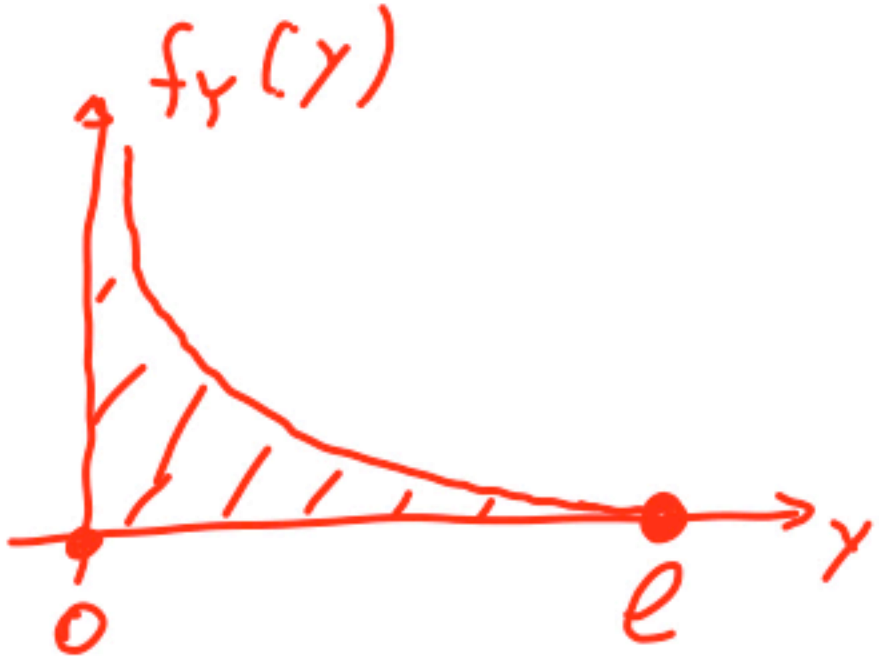
Stick-breaking example



$$f_{X,Y}(x,y) = \frac{1}{\ell x}, \qquad 0 \le y \le x \le \ell$$

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx = \int_y^{\ell} \frac{1}{\ell x}\,dx = \frac{1}{\ell}\log\left(\frac{\ell}{y}\right)$$

$$\mathbf{E}[Y] = \int_0^{\ell} y\,\frac{1}{\ell}\log\left(\frac{\ell}{y}\right)dy$$



- Using total expectation theorem:

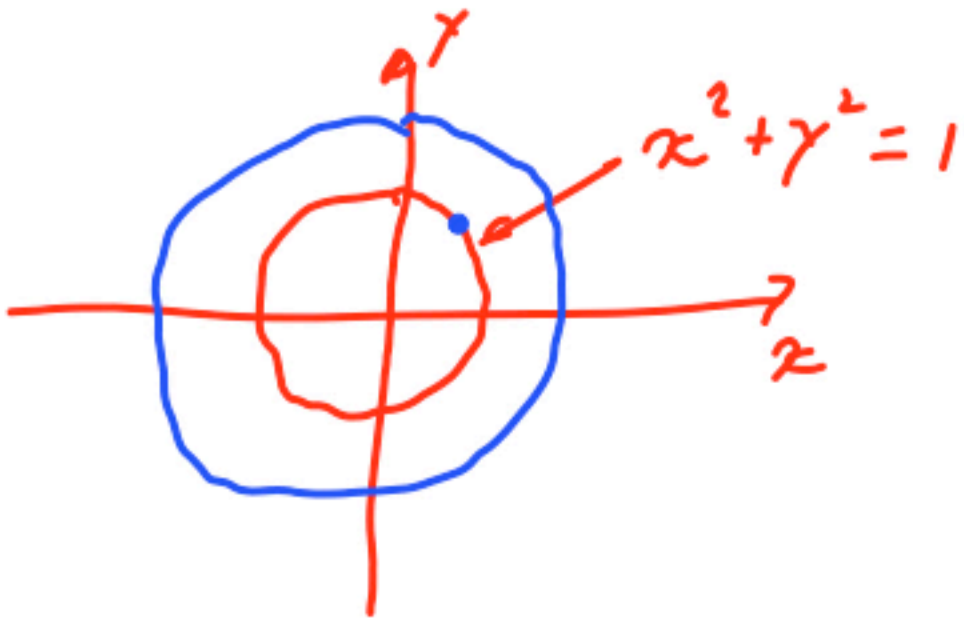
$$E[Y] = \int_0^{\ell} \frac{1}{\ell}\, E[Y \mid X = x]\,dx = \int_0^{\ell} \underbrace{\frac{1}{\ell}}_{f_X(x)} \frac{x}{2}\,dx = \frac{1}{2}E[X] = \frac{1}{2}\cdot\frac{\ell}{2} = \frac{\ell}{4}.$$

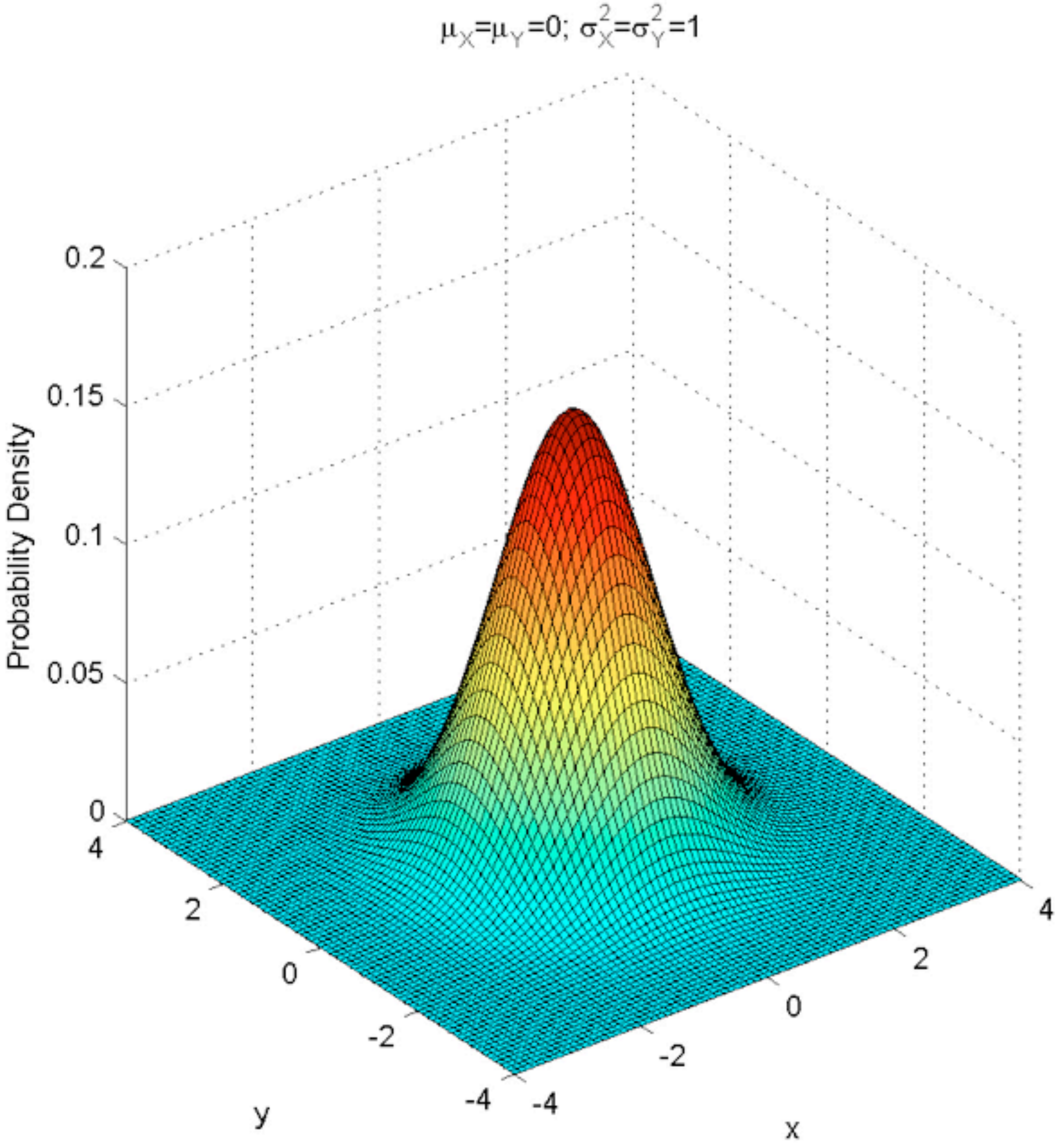
Independent standard normals

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x^2+y^2)\right\}$$

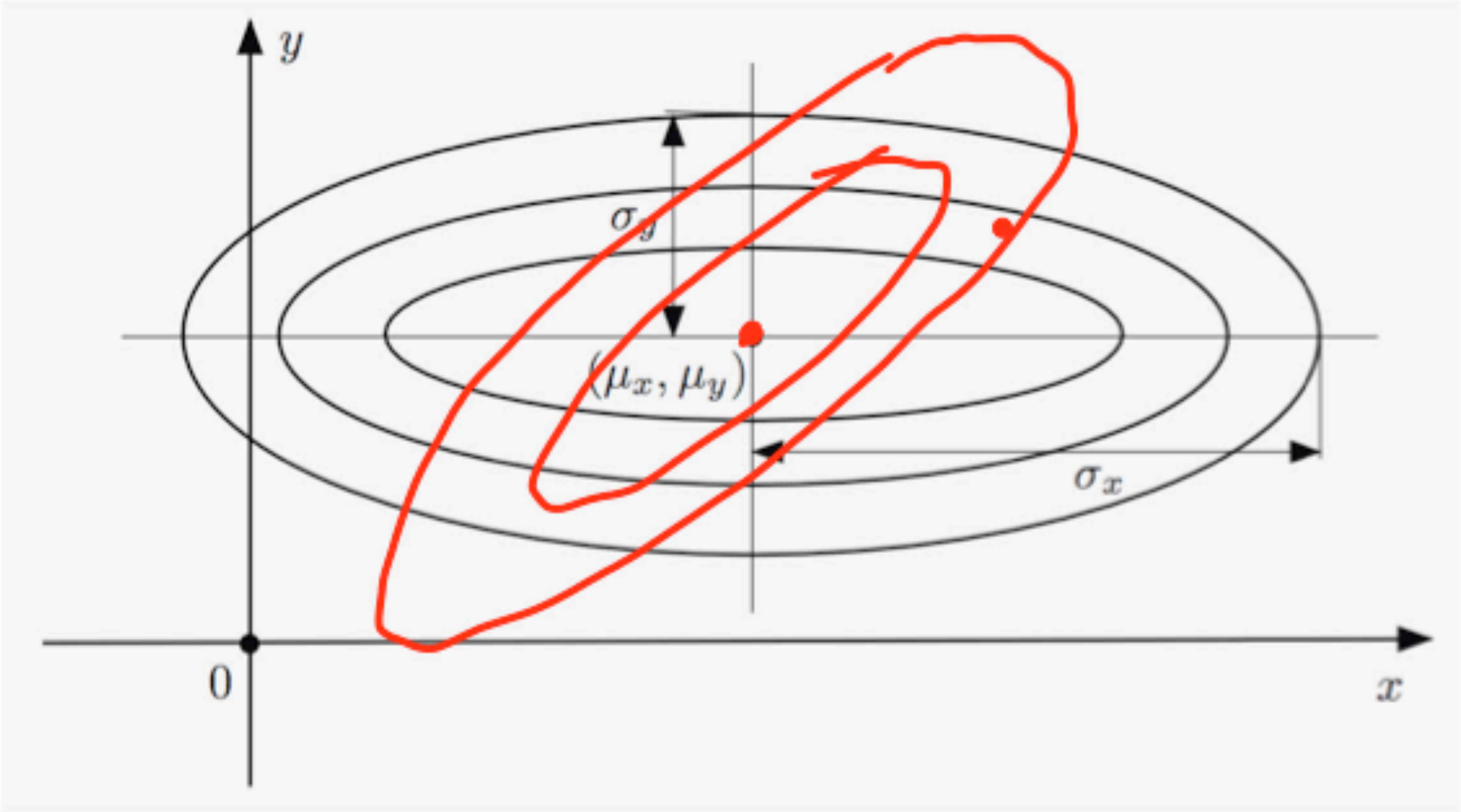


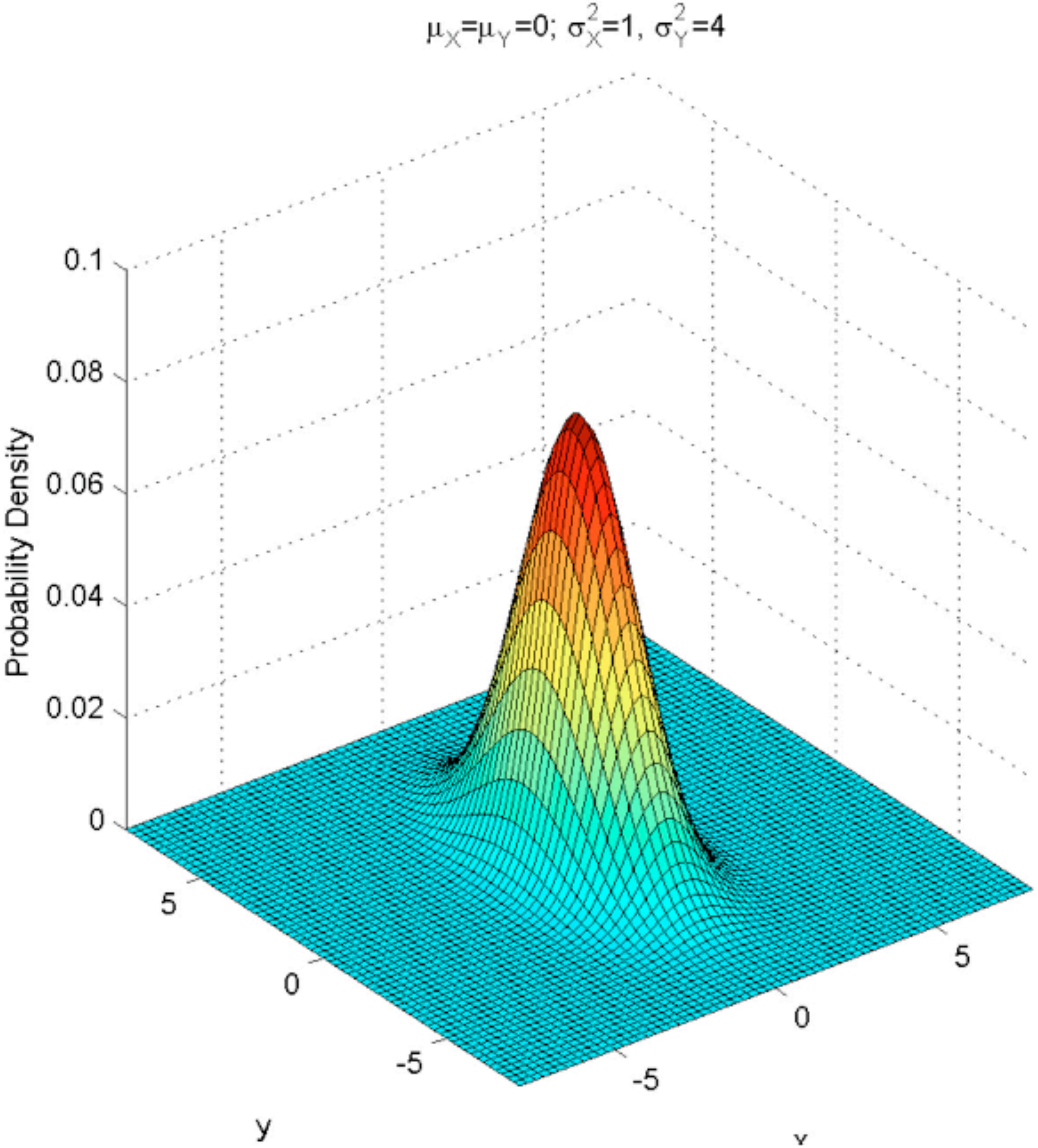


Independent normals

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \underbrace{\frac{1}{2\pi\sigma_x\sigma_y}} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} \right\}$$





The Bayes rule — a theme with variations

$p_{X|Y}(\cdot \mid y)$

$$\begin{tikzcd}[column sep=huge] X \arrow[r] & Y \arrow[l, bend right, "\text{Inference}"'] \end{tikzcd}$$

unobserved value x — prior $p_X(\cdot)$

observed value y — model $p_{Y|X}(\cdot \mid \cdot)$

$$\begin{aligned} p_{X,Y}(x,y) &= p_X(x)\, p_{Y|X}(y \mid x) \\ &= p_Y(y)\, p_{X|Y}(x \mid y) \end{aligned}$$

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)\, f_{Y|X}(y \mid x) \\ &= f_Y(y)\, f_{X|Y}(x \mid y) \end{aligned}$$

$$p_{X|Y}(x \mid y) = \frac{p_X(x)\, p_{Y|X}(y \mid x)}{p_Y(y)}$$

$$f_{X|Y}(x \mid y) = \frac{f_X(x)\, f_{Y|X}(y \mid x)}{f_Y(y)}$$

posterior

$$p_Y(y) = \sum_{x'} p_X(x')\, p_{Y|X}(y \mid x')$$

$$f_Y(y) = \int f_X(x')\, f_{Y|X}(y \mid x')\, dx'$$

The Bayes rule — one discrete and one continuous random variable

K: discrete $\qquad\qquad$ Y: continuous

$$P(K=k,\ y\le Y\le y+\delta) \qquad \delta>0,\ \delta\approx 0$$

$$= P(K=k)\,P(y\le Y\le y+\delta \mid K=k) \approx p_K(k)\,f_{Y|K}(y|k)\,\cancel{\delta}$$

$$= P(y\le Y\le y+\delta)\,P(K=k \mid y\le Y\le y+\delta) \approx f_Y(y)\,\cancel{\delta}\,p_{K|Y}(k|y)$$

$$\boxed{p_{K|Y}(k\,|\,y) = \frac{p_K(k)\,f_{Y|K}(y\,|\,k)}{f_Y(y)}} \qquad \boxed{f_{Y|K}(y\,|\,k) = \frac{f_Y(y)\,p_{K|Y}(k\,|\,y)}{p_K(k)}}$$

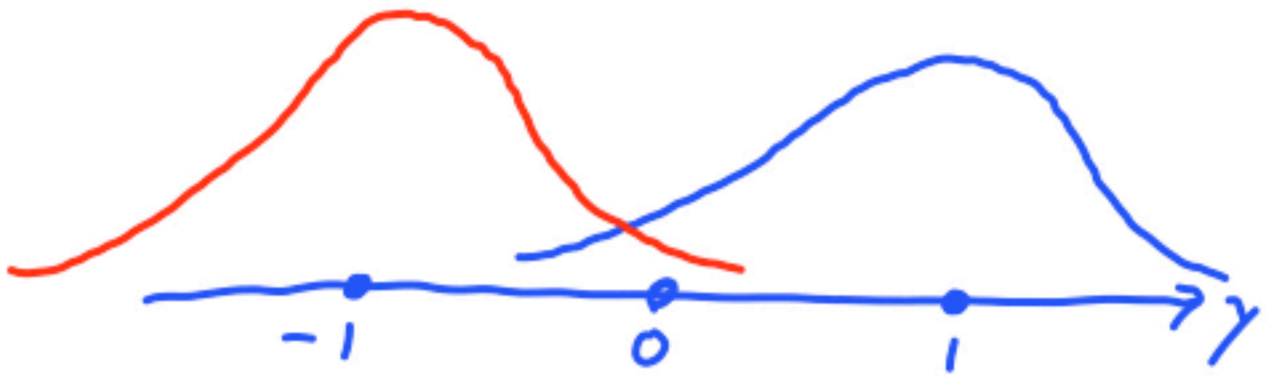
$$f_Y(y) = \sum_{k'} p_K(k')\,f_{Y|K}(y\,|\,k') \qquad p_K(k) = \int f_Y(y')\,p_{K|Y}(k\,|\,y')\,dy'$$

The Bayes rule — discrete unknown, continuous measurement

- unkown K: equally likely to be -1 or $+1$
- measurement Y: $\quad Y = K + W;\ \ W \sim \mathcal{N}(0,1)$

$Y \mid K=1 \sim N(1,1)$

$Y \mid K=-1 \sim N(-1,1)$



- Probability that $K = 1$, given that $Y = y$? $\quad p_{K|Y}(1 \mid y)$

$$p_{K|Y}(k \mid y) = \frac{p_K(k)\, f_{Y|K}(y \mid k)}{f_Y(y)}$$

$p_K(k) = 1/2 \qquad f_{Y|K}(y \mid k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-k)^2}$

$k = -1, +1$

$$f_Y(y) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2}$$

$$f_Y(y) = \sum_{k'} p_K(k')\, f_{Y|K}(y \mid k')$$

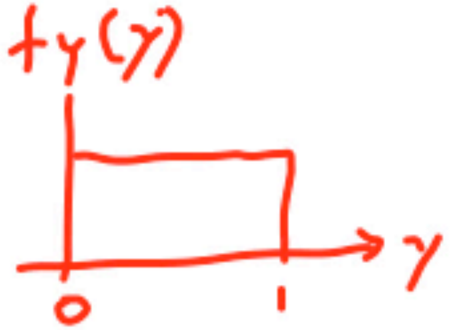
$$p_{K|Y}(1 \mid y) = \frac{1}{1+e^{-2y}}$$

algebra

The Bayes rule — continuous unknown, discrete measurement

- measurement K: Bernoulli with parameter Y

- unkown Y: uniform on $[0,1]$



- Distribution of Y given that $K = 1$? $f_{Y|K}(y|1)$

$$f_{Y|K}(y\,|\,\underset{1}{k}) = \frac{f_Y(y)\,p_{K|Y}(k\,|\,y)}{p_K(k)}$$

$$p_K(k) = \int f_Y(y')\,p_{K|Y}(k\,|\,y')\,dy'$$

$$f_Y(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$p_{K|Y}(1\,|\,y) = y$$

$$p_K(1) = \int_0^1 1 \cdot y\,dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$f_{Y|K}(y\,|\,1) = \frac{1 \cdot y}{1/2} = 2y\ ,\quad y \in [0,1]$$

UNIT 5: Continuous random variables — Summary

- r.v.'s and PDFs: $f_X(x)$, $f_{X,Y}(x,y)$, $f_{X|Y}(x\,|\,y)$, $f_{X|A}(x)$
- Expectation: $\mathbf{E}[X]$, $\mathbf{E}[X\,|\,A]$, $\mathbf{E}[X\,|\,Y=y]$

  Expected value rule: $\mathbf{E}[g(X,Y)]$, $\mathbf{E}[g(X,Y)\,|\,A]$, $\mathbf{E}[g(X,Y)\,|\,Z=z]$

  Linearity: $\mathbf{E}[aX+bY]=a\mathbf{E}[Y]+b\mathbf{E}[Y]$
- Variance: $\mathrm{var}(X)$, $\mathrm{var}(X\,|\,A)$, $\mathrm{var}(X\,|\,Y=y)$ $\qquad$ $\mathrm{var}(X)=\mathbf{E}[X^2]-\big(\mathbf{E}[X]\big)^2$
- Independence of r.v.'s: $f_{X,Y}=f_X\cdot f_Y$

  $\mathbf{E}[XY]=\mathbf{E}[X]\cdot\mathbf{E}[Y]$ $\qquad$ $\mathrm{var}(X+Y)=\mathrm{var}(X)+\mathrm{var}(Y)$
- Multiplication rule $\qquad$ $f_{X,Y,Z}(x,y,z)=f_Z(z)\,f_{Y|Z}(y\mid z)\,f_{X|Y,Z}(x\mid y,z)$
- Total probability theorem $\qquad$ $f_X(x)=\displaystyle\int f_Y(y)\,f_{X|Y}(x\,|\,y)\,dy$
- Total expectation theorem $\qquad$ $\mathbf{E}[X]=\displaystyle\int f_Y(y)\,\mathbf{E}[X\,|\,Y=y]\,dy$
- Examples: uniform, geometric, exponential, normal

What was new?

- Replace:
  - sums by integrals
  - PMFs by PDFs
- Densities are not probabilities: $\mathbf{P}(x \le X \le x+\delta) \approx f_X(x) \cdot \delta$
- Conditioning on events $\{Y = y\}$ that have zero probability
- CDF: $F_X(x) = \mathbf{P}(X \le x)$
- Bayes' rule variations and mixed (discrete/continuous) models

The Standard Normal Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

*For $z \geq 3.50$, the probability is greater than or equal to .9998.

The entries in this table provide the numerical values of $\Phi(z) = \mathbf{P}(Z \leq z)$, where Z is a standard normal random variable, for z between 0 and 3.49. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When z is negative, the value of $\Phi(z)$ can be found using the formula $\Phi(z) = 1 - \Phi(-z)$.

Define coordinates such that the stick extends from position 0 (the left end) to position 1 (the right end). Denote the position of the first break by X and the position of the second break by Y. We have $X < Y$. We assume that $X < Y$ and we later account for the case $Y < X$ by using symmetry.

Under the assumption $X < Y$, the three pieces have lengths X, $Y - X$, and $1 - Y$. In order that they form a triangle, the sum of the lengths of any two pieces must exceed the length of the third piece. Thus they form a triangle if

$$X < (Y-X)+(1-Y), \qquad (Y-X) < X+(1-Y), \qquad (1-Y) < X+(Y-X).$$

These conditions simplify to

$$X < 0.5, \qquad Y > 0.5, \qquad Y - X < 0.5.$$

For X and Y to satisfy these conditions, the pair (X, Y) must lie within the triangle with vertices $(0, 0.5)$, $(0.5, 0.5)$, and $(0.5, 1)$. This triangle has area $1/8$. Thus the probability of the event that the three pieces form a triangle *and* $X < Y$ is $1/8$. By symmetry, the probability of the event that the three pieces form a triangle *and* $X > Y$ is $1/8$. Since there two events are disjoint and form a partition of the event that the three pieces form a triangle, the desired probability is $1/8 + 1/8 = 1/4$.

The expected value in question is

$$\mathbf{E}[\text{Time}] = \big((5 + \mathbf{E}[\text{stay of 2nd student}])\big) \cdot \mathbf{P}(\text{1st stays no more than 5 minutes})$$

$$+ \big((\mathbf{E}[\text{stay of 1st} \mid \text{stay of 1st } \geq 5] + \mathbf{E}[\text{stay of 2nd}])\big)$$

$$\cdot\, \mathbf{P}(\text{1st stays more than 5 minutes}).$$

We have $\mathbf{E}[\text{stay of 2nd student}] = 30$, and, using the memorylessness property of the exponential distribution,

$$\mathbf{E}[\text{stay of 1st} \mid \text{stay of 1st } \geq 5] = 5 + \mathbf{E}[\text{stay of 1st}] = 35.$$

Also

$$\mathbf{P}(\text{1st student stays no more than 5 minutes}) = 1 - e^{-5/30},$$

$$\mathbf{P}(\text{1st student stays more than 5 minutes}) = e^{-5/30}.$$

By substitution we obtain

$$\mathbf{E}[\text{Time}] = (5+30)\cdot(1-e^{-5/30})+(35+30)\cdot e^{-5/30} = 35+30\cdot e^{-5/30} = 60.394.$$

(a) We have

$$\begin{aligned}
\mathbf{P}(Z \le z \mid X = x) &= \mathbf{P}(X + Y \le z \mid X = x) \\
&= \mathbf{P}(x + Y \le z \mid X = x) \\
&= \mathbf{P}(x + Y \le z) \\
&= \mathbf{P}(Y \le z - x),
\end{aligned}$$

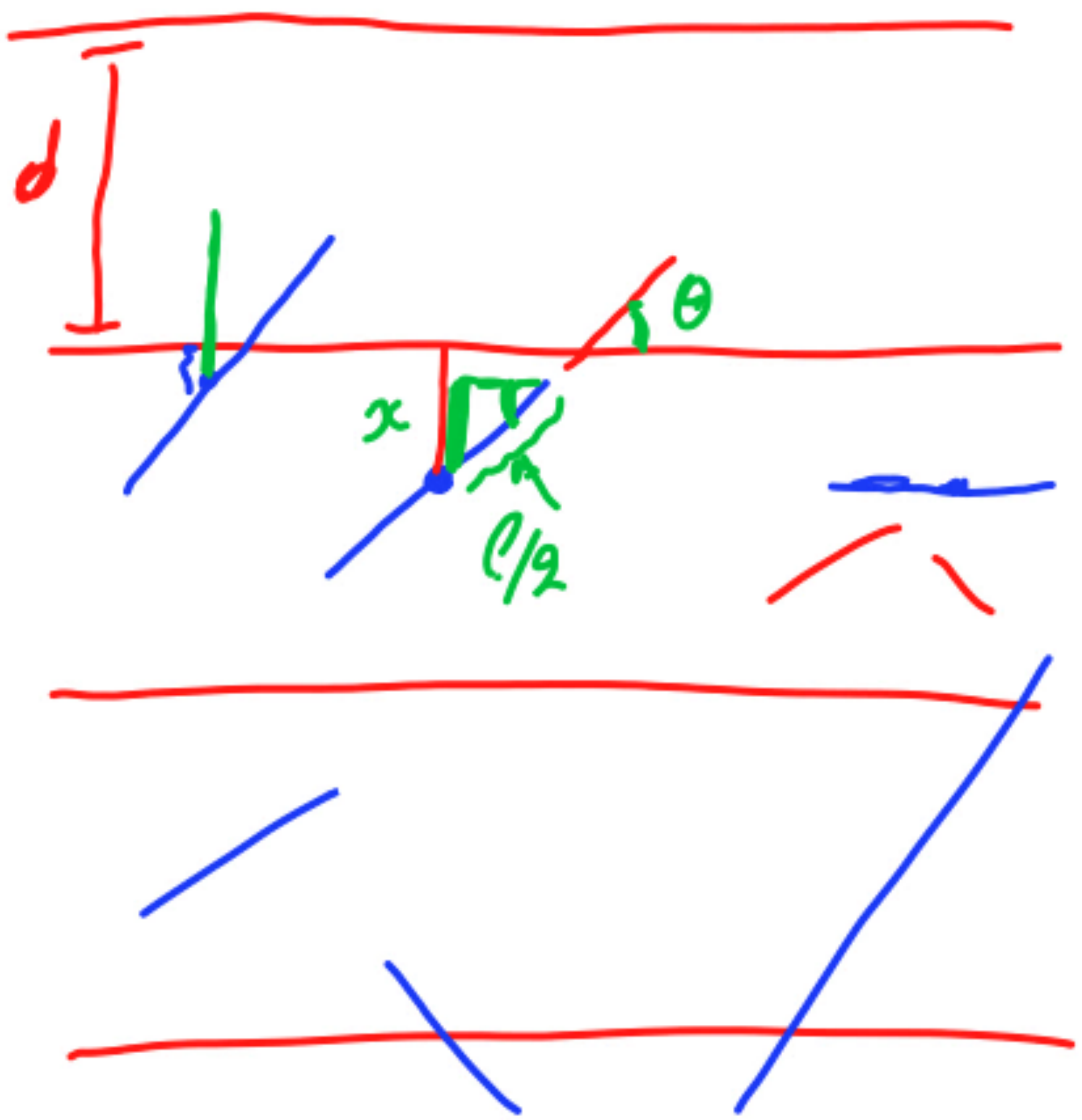
where the third equality follows from the independence of X and Y. By differentiating both sides with respect to z, the result follows.

(b) We have, for $0 \le x \le z$,

$$f_{X|Z}(x \mid z) = \frac{f_{Z|X}(z \mid x) f_X(x)}{f_Z(z)} = \frac{f_Y(z - x) f_X(x)}{f_Z(z)} = \frac{\lambda e^{-\lambda(z-x)} \lambda e^{-\lambda x}}{f_Z(z)} = \frac{\lambda^2 e^{-\lambda z}}{f_Z(z)}.$$

Since this is the same for all x, it follows that the conditional distribution of X is uniform on the interval $[0, z]$, with PDF $f_{X|Z}(x \mid z) = 1/z$.

Buffon's needle



- Parallel lines at distance d
  Needle of length ℓ (assume $\ell < d$)

- Find $\mathbf{P}$(needle intersects one of the lines)

- **Model:** (X, Θ) independent

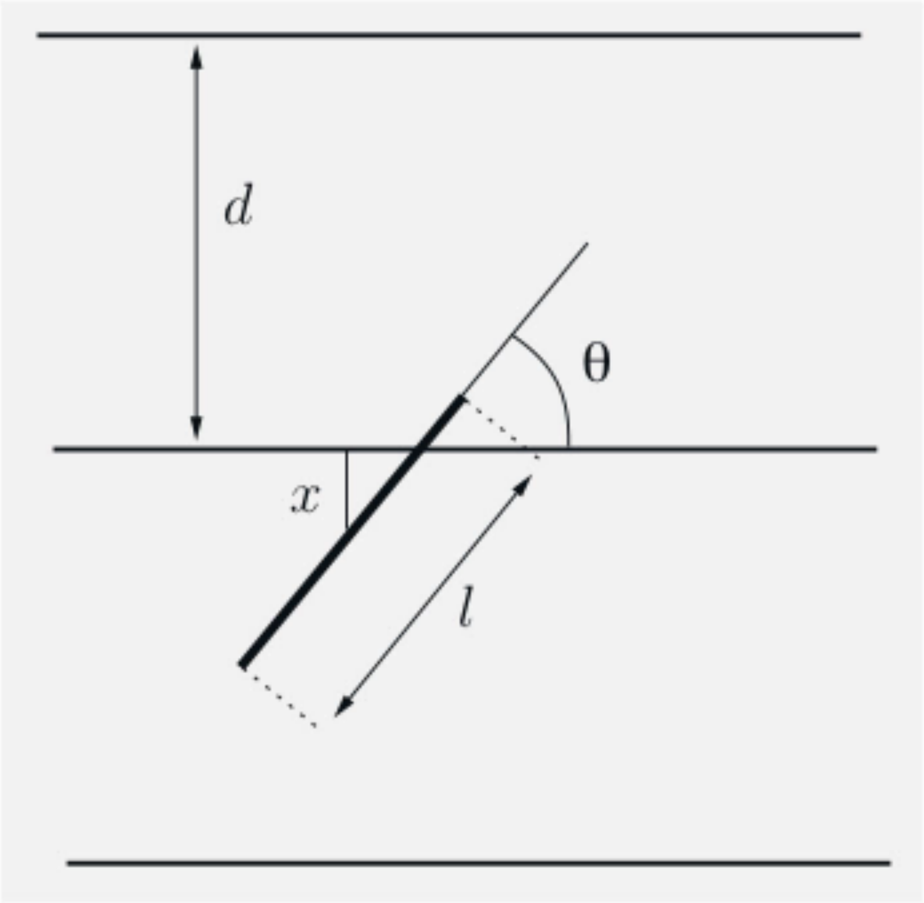
X: $f_X(x) = \frac{2}{d}$, $0 \le x \le \frac{d}{2}$

Θ: $f_\Theta(\theta) = \frac{2}{\pi}$, $0 \le \theta \le \frac{\pi}{2}$

- Intersect if
  - $X \le \frac{\ell}{2} \sin \Theta$

Buffon's needle

$$f_{X,\Theta}(x,\theta) = \frac{4}{\pi d}, \qquad 0 \le x \le d/2, \quad 0 \le \theta \le \pi/2$$



- Intersect if $X \le \frac{\ell}{2}\sin\Theta$

$$\boxed{\mathbf{P}\left(X \le \frac{\ell}{2}\sin\Theta\right) = \frac{2\ell}{\pi d}}$$

$$P\left(X \le \frac{\ell}{2}\sin\Theta\right)$$

$$= \int_0^{\pi/2}\int_0^{\frac{\ell}{2}\sin\theta} \frac{4}{\pi d}\,dx\,d\theta = \int_0^{\pi/2} \frac{4}{\pi d}\cdot\frac{\ell}{2}\sin\theta\cdot d\theta$$

$$= \frac{2\ell}{\pi d}\left(-\cos\theta\right)\Big|_0^{\pi/2} = \frac{2\ell}{\pi d}$$

- "Monte Carlo" method for experimental evaluation of π

1. We know that the PDF must integrate to 1. Therefore we have

$$\int_{-\infty}^{\infty} f_Z(z)\,dz \;=\; \int_{-2}^{1} \gamma(1+z^2) \;=\; \gamma\left(z+\frac{1}{3}z^3\right)\Big|_{-2}^{1} \;=\; 6\gamma.$$

   From this we conclude $\gamma = 1/6$.

2. To find the CDF, we integrate:

$$\begin{aligned} F_Z(z) \;&=\; \int_{-\infty}^{z} f_Z(t)\,dt \;=\; \begin{cases} 0, & \text{if } z < -2, \\ \frac{1}{6}\left(t+\frac{1}{3}t^3\right)\big|_{-2}^{z}, & \text{if } -2 \le z \le 1, \\ 1, & \text{if } z > 1 \end{cases} \\ &=\; \begin{cases} 0, & \text{if } z < -2, \\ \frac{1}{6}(z+\frac{1}{3}z^3+\frac{14}{3}), & \text{if } -2 \le z \le 1, \\ 1, & \text{if } z > 1. \end{cases} \end{aligned}$$

(a) We have

$$\mathbf{E}[X] = \int_1^3 \frac{x^2}{4}\,dx = \frac{x^3}{12}\Big|_1^3 = \frac{27}{12} - \frac{1}{12} = \frac{26}{12} = \frac{13}{6},$$

$$\mathbf{P}(A) = \int_2^3 \frac{x}{4}\,dx = \frac{x^2}{8}\Big|_2^3 = \frac{9}{8} - \frac{4}{8} = \frac{5}{8}.$$

We also have

$$\begin{aligned} f_{X|A}(x) &= \begin{cases} \dfrac{f_X(x)}{\mathbf{P}(A)}, & \text{if} \quad x \in A \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \dfrac{2x}{5}, & \text{if} \quad 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

from which we obtain

$$\mathbf{E}[X \mid A] = \int_2^3 x \cdot \frac{2x}{5}\,dx = \frac{2x^3}{15}\Big|_2^3 = \frac{54}{15} - \frac{16}{15} = \frac{38}{15}.$$

(b) We have

$$\mathbf{E}[Y] = \mathbf{E}[X^2] = \int_1^3 \frac{x^3}{4}\,dx = 5,$$

and

$$\mathbf{E}[Y^2] = \mathbf{E}[X^4] = \int_1^3 \frac{x^5}{4}\,dx = \frac{91}{3}.$$

Thus,

$$\operatorname{var}(Y) = \mathbf{E}[Y^2] - \big(\mathbf{E}[Y]\big)^2 = \frac{91}{3} - 5^2 = \frac{16}{3}.$$

(a) For $x \geq 0$,

$$F_X(x) \;=\; \int_{-\infty}^{x} f_X(t)\,dt \;=\; \int_0^x \lambda e^{-\lambda t}\,dt \;=\; \left[-e^{-\lambda t}\right]_0^x \;=\; 1 - e^{-\lambda x}.$$

For $x < 0$, we have $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = 0$. Thus we conclude

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$

(b) The key step in the following computation uses integration by parts, whereby

$$\int_0^\infty u\,dv \;=\; uv\Big|_0^\infty - \int_0^\infty v\,du$$

is applied with $u = x$ and $v = -e^{-\lambda x}$:

$$\mathbf{E}[X] \;=\; \int_{-\infty}^{\infty} x f_X(x)\,dx \;=\; \int_0^\infty x\lambda e^{-\lambda x}\,dx \;=\; \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,dx \;=\; \frac{1}{\lambda}.$$

(c) Integrating by parts with $u = x^2$ and $v = -e^{-\lambda x}$ in the second line below gives

$$\begin{aligned} \mathbf{E}[X^2] \;&=\; \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \;=\; \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx \\ &=\; \left[-x^2 e^{-\lambda x}\right]_0^\infty + 2\int_0^\infty x e^{-\lambda x}\,dx \;=\; \frac{2}{\lambda}\mathbf{E}[X] \;=\; \frac{2}{\lambda^2}. \end{aligned}$$

Combining with the previous computation, we obtain

$$\mathrm{var}(X) \;=\; \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \;=\; \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \;=\; \frac{1}{\lambda^2}.$$

(d) The maximum of a set is upper bounded by z when each element of the set is upper bounded by z. Thus for any positive z,

$$\begin{aligned} \mathbf{P}(Z \leq z) \;&=\; \mathbf{P}(\max\{X_1, X_2, X_3\} \leq z) \;=\; \mathbf{P}(X_1 \leq z,\, X_2 \leq z,\, X_3 \leq z) \\ &=\; \mathbf{P}(X_1 \leq z)\,\mathbf{P}(X_2 \leq z)\,\mathbf{P}(X_3 \leq z) \\ &=\; (1 - e^{-\lambda z})^3, \end{aligned}$$

where the third equality uses the independence of X_1, X_2, and X_3. Thus,

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ (1 - e^{-\lambda z})^3, & \text{if } z \geq 0. \end{cases}$$

Differentiating the CDF gives the desired PDF:

$$f_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 3\lambda e^{-\lambda z}(1 - e^{-\lambda z})^2, & \text{if } z \geq 0. \end{cases}$$

(e) The minimum of a set is lower bounded by w when each element of the set is lower bounded by w. Thus for any positive w,

$$\begin{aligned} \mathbf{P}(W \geq w) &= \mathbf{P}(\min\{X_1, X_2\} \geq w) = \mathbf{P}(X_1 \geq w,\, X_2 \geq w) \\ &= \mathbf{P}(X_1 \geq w)\,\mathbf{P}(X_2 \geq w) \\ &= (e^{-\lambda w})^2 = e^{-2\lambda w} \end{aligned}$$

where the third equality uses the independence of X_1 and X_2. Thus,

$$F_W(w) = \begin{cases} 0, & \text{if } w < 0, \\ 1 - e^{-2\lambda w}, & \text{if } w \geq 0. \end{cases}$$

We can recognize this as the CDF of an exponential random variable with parameter 2λ. The PDF is

$$f_W(w) = \begin{cases} 0, & \text{if } w < 0, \\ 2\lambda e^{-2\lambda w}, & \text{if } w \geq 0. \end{cases}$$

Let A be the event that Al will find a taxi waiting or will be picked up by the bus after 5 minutes. Note that the probability of boarding the next bus, given that Al has to wait, is

$$\mathbf{P}(\text{a taxi will take more than 5 minutes to arrive}) = \frac{1}{2}.$$

Al's waiting time, call it X, is a mixed random variable. With probability

$$\mathbf{P}(A) = \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{6},$$

it is equal to its discrete component Y (corresponding to either finding a taxi waiting, or boarding the bus), which has PMF

$$p_Y(y) = \begin{cases} \dfrac{2}{3\mathbf{P}(A)}, & \text{if } y = 0, \\ \\ \dfrac{1}{6\mathbf{P}(A)}, & \text{if } y = 5, \end{cases}$$

$$= \begin{cases} \dfrac{12}{15}, & \text{if } y = 0, \\ \\ \dfrac{3}{15}, & \text{if } y = 5. \end{cases}$$

[This equation follows from the calculation

$$p_Y(0) = \mathbf{P}(Y = 0 \mid A) = \frac{\mathbf{P}(Y = 0, A)}{\mathbf{P}(A)} = \frac{2}{3\mathbf{P}(A)}.$$

The calculation for $p_Y(5)$ is similar.] With the complementary probability $1 - \mathbf{P}(A)$, the waiting time is equal to its continuous component Z (corresponding to boarding a taxi after having to wait for some time less than 5 minutes), which has PDF

$$f_Z(z) = \begin{cases} 1/5, & \text{if } 0 \le z \le 5, \\ \\ 0, & \text{otherwise.} \end{cases}$$

The CDF is given by $F_X(x) = \mathbf{P}(A)F_Y(x) + \big(1 - \mathbf{P}(A)\big)F_Z(x)$, from which

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ \\ \dfrac{5}{6} \cdot \dfrac{12}{15} + \dfrac{1}{6} \cdot \dfrac{x}{5}, & \text{if } 0 \le x < 5, \\ \\ 1, & \text{if } 5 \le x. \end{cases}$$

The expected value of the waiting time is

$$\mathbf{E}[X] = \mathbf{P}(A)\mathbf{E}[Y] + \big(1 - \mathbf{P}(A)\big)\mathbf{E}[Z] = \frac{5}{6} \cdot \frac{3}{15} \cdot 5 + \frac{1}{6} \cdot \frac{5}{2} = \frac{15}{12}.$$

(a)

$$\begin{aligned}
\mathbf{P}(X \le 1.5) &= \Phi(1.5) \\
&\approx 0.9332.
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}(X \le -1) &= 1 - \mathbf{P}(X \le 1) \\
&= 1 - \Phi(1) \\
&\approx 1 - 0.8413 \\
&= 0.1587.
\end{aligned}$$

(b)

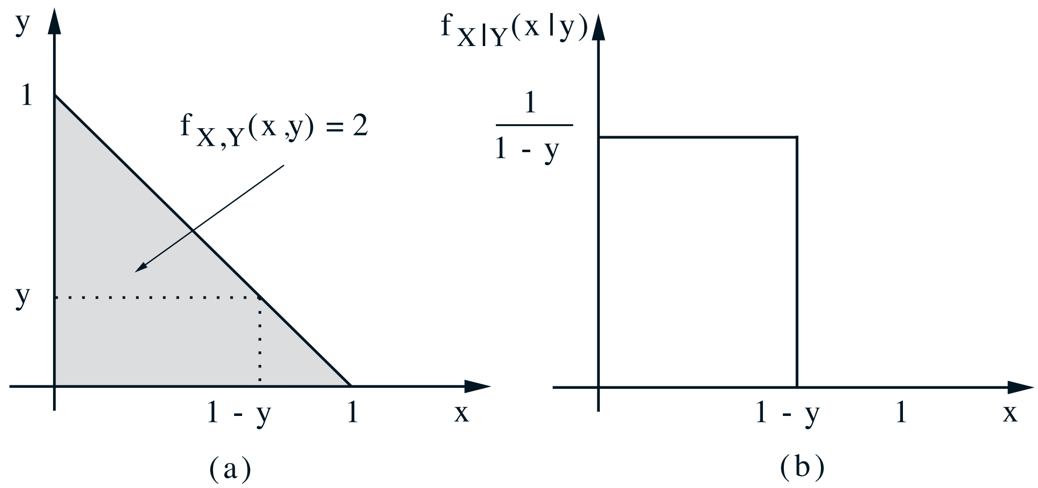
$$\begin{aligned}
\mathbf{E}\left[\frac{Y-1}{2}\right] &= \frac{1}{2}(\mathbf{E}[Y] - 1) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}\left(\frac{Y-1}{2}\right) &= \mathrm{var}\left(\frac{Y}{2}\right) \\
&= \frac{1}{4}\mathrm{var}Y \\
&= 1.
\end{aligned}$$

Thus, the distribution of $\frac{Y-1}{2}$ is $\mathcal{N}(0,1)$.

(c)

$$\begin{aligned}
\mathbf{P}(-1 \le Y \le 1) &= \mathbf{P}(\frac{-1-1}{2} \le \frac{Y-1}{2} \le \frac{1-1}{2}) \\
&= \Phi(0) - \Phi(-1) \\
&= \Phi(0) - (1 - \Phi(1)) \\
&\approx 0.3413.
\end{aligned}$$



a) The area of the triangle is 1/2, so that $f_{X,Y}(x, y) = 1/2$, on the triangle indicated in Fig. (a), and zero everywhere else.

(b) We have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_0^{1-y} 2\, dx = 2(1 - y), \qquad 0 \le y \le 1.$$

(c) We have

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{1}{1 - y}, \qquad 0 \le x \le 1 - y.$$

The conditional density is shown in the figure.

Intuitively, since the joint PDF is constant, the conditional PDF (which is a "slice" of the joint, at some fixed y) is also constant. Therefore, the conditional PDF must be a uniform distribution. Given that $Y = y$, X ranges from 0 to $1 - y$. Therefore, for the PDF to integrate to 1, its height must be equal to $1/(1 - y)$, in agreement with the figure.

(d) For $y > 1$ or $y < 0$, the conditional PDF is undefined, since these values of y are impossible. For $0 \le y < 1$, the conditional mean $\mathbf{E}[X \mid Y = y]$ is obtained using the uniform PDF in Fig. (b), and we have

$$\mathbf{E}[X \mid Y = y] = \frac{1 - y}{2}, \qquad 0 \le y < 1.$$

For $y = 1$, X must be equal to 0, with certainty, so $\mathbf{E}[X \mid Y = 1] = 0$. Thus, the above formula is also valid when $y = 1$. The conditional expectation is undefined when y is outside $[0, 1]$.

The total expectation theorem yields

$$\mathbf{E}[X] = \int_0^1 \frac{1-y}{2} f_Y(y)\, dy = \frac{1}{2} - \frac{1}{2}\int_0^1 y f_Y(y)\, dy = \frac{1 - \mathbf{E}[Y]}{2}.$$

(e) Because of symmetry, we must have $\mathbf{E}[X] = \mathbf{E}[Y]$. Therefore, $\mathbf{E}[X] = \big((1 - \mathbf{E}[X])\big)/2$, which yields $\mathbf{E}[X] = 1/3$.

The Bayes rule — two continuous random variables

- X, Y independent; $Z = X + Y$

  - Find $f_{X|Z}(x\,|\,z)$

$Z = x + Y$

$z = x + z - x$

- First find $\boxed{f_{Z|X}(z\,|\,x) = f_Y(z - x)}$

$$f_{X|Y}(x\,|\,y) = \frac{f_X(x)\,f_{Y|X}(y\,|\,x)}{f_Y(y)}$$

$$\underline{f_{X|Z}(x\,|\,z)} = \frac{f_X(x)\,f_{Z|X}(z\,|\,x)}{f_Z(z)}$$

$$P(Z \le z \,|\, X = x) = P(X + Y \le z \,|\, X = x)$$

$$= P(x + Y \le z \,|\, X = x)$$

$$= P(x + Y \le z) = P(Y \le z - x) = F_Y(z - x)$$

$$d/d_Y \quad = f_Y(z - x)$$

The Bayes rule — two continuous random variables

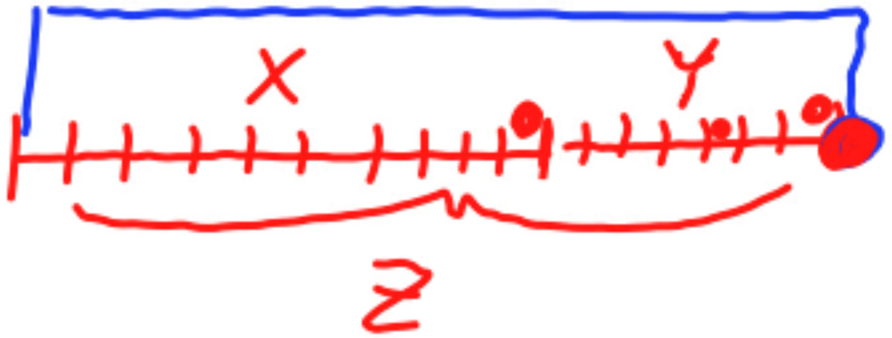
- X, Y independent; $Z = X + Y$
- Assume X and Y are exponential with parameter λ

$$f_{X|Z}(x\,|\,z) = \frac{f_X(x)\, f_{Z|X}(z\,|\,x)}{f_Z(z)}$$

$$f_{Z|X}(z\,|\,x) = f_Y(z-x)$$

$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

$f_Y(y) = \lambda e^{-\lambda y}, \quad y \geq 0$



Fix some $z \geq 0$:

uniform on $[0, z]$!

$$\underbrace{f_{X|Z}(x|z)} = \frac{1}{f_Z(z)} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)} = \boxed{\frac{1}{f_Z(z)} \lambda^2 e^{-\lambda z} \quad 0 \leq x \leq z}$$

$x \geq 0$

$z - x \geq 0$

$x \leq z$

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: PDFs

4/4 points (graded)

Let X be a continuous random variable with a PDF of the form

$$f_X(x) = \begin{cases} c(1-x), & \text{if } x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Find the following values.

1. $c =$ 2 ✔ Answer: 2

2. $\mathbf{P}(X = 1/2) =$ 0 ✔ Answer: 0

3. $\mathbf{P}\left(X \in \{1/k : k \text{ integer}, k \geq 2\}\right) =$ 0 ✔ Answer: 0

4. $\mathbf{P}(X \leq 1/2) =$ 3/4 ✔ Answer: 0.75

Solution:

1. We have $1 = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 c(1-x) = c(x - x^2/2)\big|_1^0 = c/2$, and therefore, $c = 2$.

2. Individual points have zero probability.

3. Using countable additivity and the fact that single points have zero probability, we have

$$\mathbf{P}(X \in \{1/2, 1/3, 1/4, 1/5, \ldots\}) = \sum_{n=2}^{\infty} \mathbf{P}(X = 1/n) = \sum_{n=2}^{\infty} 0 = 0.$$

4. $\mathbf{P}(X \le 1/2) = \int_{-\infty}^{1/2} f_X(x)\, dx = \int_0^{1/2} 2(1-x)\, dx = 2(x - x^2/2)\big|_{1/2}^{0} = \frac{3}{4}$.

Submit You have used 2 of 3 attempts

Show Answer

Answers are displayed within the problem

Exercise: Piecewise constant PDF

2/2 points (graded)

Consider a piecewise constant PDF of the form

$$f_X(x) = \begin{cases} 2c, & \text{if } 0 \le x \le 1, \\ c, & \text{if } 1 < x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the following values.

a) $c =$ 1/4 ✔ **Answer:** 0.25

b) $\mathbf{P}(1/2 \le X \le 3/2) =$ 3/8 ✔ **Answer:** 0.375

Solution:

a) The total area under the PDF is the sum of the areas of two rectangles and is equal to $(2c) \cdot 1 + c \cdot 2 = 4c$. Therefore, $c = 1/4$.

b) The total area under the PDF over the interval of interest is the sum of the areas of two smaller rectangles and is equal to $(2c) \cdot (1/2) + c \cdot (1/2) = c \cdot (3/2) = 3/8$.

Let X be uniform on the interval $[1,3]$. Suppose that $1 < a < b < 3$. Then,

(a) $\mathbf{P}(a \le X \le b) =$ (b-a)/2 ✔ **Answer:** (b-a)/2

$$\frac{b-a}{2}$$

(Your answer to part (a) should be an algebraic expression involving a and b.)

(b) $\mathbf{E}[X] =$ 2 ✔ **Answer:** 2

(c) $\mathbf{E}[X^3] =$ 10 ✔ **Answer:** 10

Solution:

(a) The value of the PDF on the interval $[1,3]$ must be equal to 1/2, so that it integrates to 1. Thus,

$$\mathbf{P}(a \le X \le b) = \int_a^b \frac{1}{2}\, dx = \frac{b-a}{2}.$$

(b) The expected value of a uniform is the midpoint of its range: $\mathbf{E}[X] = (1+3)/2 = 2$.

(c) Using the expected value rule, $\mathbf{E}[X^3] = \int_1^3 x^3 \cdot \frac{1}{2}\, dx = \frac{1}{2} \cdot \frac{1}{4} x^4 \Big|_1^3 = \frac{1}{2} \cdot \frac{1}{4} \cdot (81-1) = 10.$

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: Exponential PDF

1/2 points (graded)

Let X be an exponential random variable with parameter $\lambda = 2$. Find the values of the following. Use 'e' for the base of the natural logarithm (e.g., enter `e^(-3)` for e^{-3}).

a) $\mathbf{E}\left[(3X+1)^2\right] =$ 8.5 ✔ **Answer:** 8.5

b) $\mathbf{P}\left(1 \leq X \leq 2\right) =$ **Answer:** 0.11702

Solution:

a) By expanding the quadratic, using linearity of expectations, and the facts that $\mathbf{E}[X] = 1/\lambda$ and $\mathbf{E}[X^2] = 2/\lambda^2$, we have

$$\mathbf{E}\left[(3X+1)^2\right] = 9\mathbf{E}\left[X^2\right] + 6\mathbf{E}[X] + 1 = 9 \cdot \frac{2}{2^2} + 6 \cdot \frac{1}{2} + 1 = \frac{17}{2}.$$

b) We have seen that for $a > 0$, we have $\mathbf{P}(X \geq a) = e^{-\lambda a}$, so that $\mathbf{P}(X \leq a) = 1 - e^{-\lambda a}$. Therefore,

$$\mathbf{P}(1 \leq X \leq 2) = \mathbf{P}(X \leq 2) - \mathbf{P}(X \leq 1) = (1 - e^{-4}) - (1 - e^{-2}) = e^{-2} - e^{-4}.$$

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: Exponential CDF

2/2 points (graded)

Let X be an exponential random variable with parameter 2.

Find the CDF of X. Express your answer in terms of x using standard notation. Use 'e' for the base of the natural logarithm (e.g., enter `e^(-3*x)` for e^{-3x}).

a) For $x \leq 0$, $F_X(x) =$ 0 ✔ **Answer:** 0

0

b) For $x > 0$, $F_X(x) =$ 1-e^(-2*x) ✔ **Answer:** 1-e^(-2*x)

$1 - e^{-2 \cdot x}$

STANDARD NOTATION

Solution:

a) Since X is a nonnegative random variable, $F_X(x) = \mathbf{P}(X \leq x) = 0$ for $x \leq 0$.

b) We have seen that for an exponential random variable with parameter λ and for any $a > 0$, we have $\mathbf{P}(X \geq a) = e^{-\lambda a}$. Therefore, $F_X(x) = \mathbf{P}(X \leq x) = 1 - \mathbf{P}(X \geq x) = 1 - e^{-\lambda x} = 1 - e^{-2x}$.

14. Exercise: Normal random variables

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: Normal random variables

1/1 point (graded)

Choose the correct answer below.
According to our conventions, a normal random variable $X \sim N(\mu, \sigma^2)$ is a continuous random variable

- ( ) always.
- (•) if and only if $\sigma \neq 0$.
- ( ) if and only if $\mu \neq 0$ and $\sigma \neq 0$.

✔

Solution:

When $\sigma \neq 0$, the distribution of X is described by a PDF, and so X is a continuous random variable. But when $\sigma = 0$, then X has all of its probability assigned to a single point, and therefore it is not a continuous random variable. (For continuous random variables, any single point must have zero probability.)

3/3 points (graded)

Let X be a normal random variable with mean 4 and variance 9.

Use the normal table to find the following probabilities, to an accuracy of 4 decimal places.

Normal Table

Show

a) $\mathbf{P}(X \le 5.2) =$ .6554 ✔ **Answer:** 0.6554

b) $\mathbf{P}(X \ge 2.8) =$ .6554 ✔ **Answer:** 0.6554

c) $\mathbf{P}(X \le 2.2) =$ .2743 ✔ **Answer:** 0.2743

Solution:

a) Note that the standard deviation is 3. Subtracting the mean and dividing by the standard deviation, we obtain

$$\mathbf{P}(X \le 5.2) = \mathbf{P}\left(\frac{X-4}{3} \le \frac{5.2-4}{3}\right) = \Phi(0.4) = 0.6554.$$

b) Because of the symmetry around the mean, $\mathbf{P}(X \ge 2.8) = \mathbf{P}(X \le 5.2) = 0.6554.$

c) $\mathbf{P}(X \le 2.2) = \mathbf{P}\left(\frac{X-4}{3} \le \frac{2.2-4}{3}\right) = \Phi(-0.6) = 1 - \Phi(0.6) = 1 - 0.7257 = 0.2743.$

Exercise: A conditional PDF

1/1 point (graded)

Suppose that X has a PDF of the form

$$f_X(x) = \begin{cases} 1/x^2, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For any $x > 2$, the conditional PDF of X, given the event $X > 2$ is

2/(x^2) ✔ **Answer:** 2/(x^2)

$\frac{2}{x^2}$

(Your answer should be an algebraic function of x, in standard notation.)

STANDARD NOTATION

Solution:

The conditional PDF will be a scaled version of the unconditional, of the form $\frac{f_X(x)}{\mathbf{P}(X>2)}$. Now, $\mathbf{P}(X > 2) = \int_2^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|2_{\infty} = 1/2$, and so the answer is $2/x^2$.

6. Exercise: Memorylessness of the exponential

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: Memorylessness of the exponential

2/3 points (graded)

Let X be an exponential random variable with parameter λ.

a) The probability that $X > 5$ is

- ( ) $\lambda e^{-5\lambda}$
- (•) $e^{-5\lambda}$
- ( ) none of the above

✔

b) The probability that $X > 5$ given that $X > 2$ is

- ( ) $\lambda e^{-5\lambda}$

- ○ $e^{-5\lambda}$
- ○ $\lambda e^{-3\lambda}$
- ● $e^{-3\lambda}$
- ○ none of the above

c) Given that $X > 2$, and for a small $\delta > 0$, the probability that $4 \leq X \leq 4 + 2\delta$ is approximately

- ○ $\lambda\delta$
- ○ $2\lambda\delta$
- ○ $\delta e^{-4\lambda}$
- ○ $\lambda\delta e^{-4\lambda}$
- ○ $\lambda\delta e^{-2\lambda}$
- ● $2\lambda\delta e^{-2\lambda}$

none of the above

Solution:

a) We have seen in the past that for an exponential random variable with parameter λ, $\mathbf{P}(X > a) = e^{-\lambda a}$, and so $\mathbf{P}(X > 5) = e^{-5\lambda}$.

b) Because of the memorylessness property, given that $X > 2$, the remaining time $X - 2$ is again exponential with the same parameter. Thus, $\mathbf{P}(X > 5 \mid X > 2) = \mathbf{P}(X - 2 > 3 \mid X > 2) = \mathbf{P}(X > 3) = e^{-3\lambda}$.

c) By memorylessness, this is the same as the unconditional probability that an exponential takes values in the interval $[2, 2 + 2\delta]$, which is approximately the length, 2δ, of the small interval times the density evaluated at 2, yielding $2\lambda\delta e^{-2\lambda}$.

Submit You have used 2 of 2 attempts

Show Answer

8. Exercise: Total probability theorem II

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Exercises due Oct 15, 2020 19:59 EDT *Completed*

Exercise: Total probability theorem II

2/2 points (graded)

On any given day, mail gets delivered by either Alice or Bob. If Alice delivers it, which happens with probability $1/4$, she does so at a time that is uniformly distributed between 9 and 11. If Bob delivers it, which happens with probability $3/4$, he does so at a time that is uniformly distributed between 10 and 12. The PDF of the time X that mail gets delivered satisfies

a) $f_X(9.5) =$ 1/8 ✔ **Answer:** 0.125

b) $f_X(10.5) =$ 1/2 ✔ **Answer:** 0.5

Solution:

The PDF is $1/4$ times a uniform on $[9, 11]$ (of height $1/2$) plus $3/4$ times a uniform on $[10, 12]$ (again of height $1/2$).

a) At time 9.5, only the first uniform is nonzero, yielding $f_X(9.5) = (1/4) \cdot (1/2) = 1/8$.

b) At time 10.5 both uniforms are nonzero, yielding $f_X(10.5) = (1/4) \cdot (1/2) + (3/4) \cdot (1/2) = 1/2$.

10. Exercise: A mixed random variable

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT *Completed*

Exercise: A mixed random variable

1/1 point (graded)

A lightbulb is installed. With probability $1/3$, it burns out immediately when it is first installed. With probability $2/3$, it burns out after an amount of time that is uniformly distributed on $[0,3]$. The expected value of the time until the lightbulb burns out is

1

✔ **Answer:** 1

Solution:

The expected value of a uniform on $[0,3]$ is $3/2$. Using the definition of expectation of mixed random variables, the expected value is $\frac{1}{3}\cdot 0+\frac{2}{3}\cdot\frac{3}{2}=1.$

Submit You have used 1 of 3 attempts

Show Answer

12. Exercise: Jointly continuous r.v.'s

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT *Completed*

Exercise: Jointly continuous r.v.'s

2/2 points (graded)

The random variables X and Y are continuous. Is this enough information to determine the value of $\mathbf{P}\left(X^2 = e^{3Y}\right)$?

No ✔ **Answer:** No

The random variables X and Y are jointly continuous. Is this enough information to determine the value of $\mathbf{P}\left(X^2 = e^{3Y}\right)$?

Yes ✔ **Answer:** Yes

Solution:

a) There is no information on the relation between the two random variables. If, for example, $X = \sqrt{e^{3Y}}$, the probability is 1, whereas if $X = \sqrt{e^{3Y}} + 1$, then the probability is zero.

b) The set of points on the x-y plane that correspond to the event $X^2 = e^{3Y}$ is a one-dimensional curve, which has zero area, and therefore zero probability.

a) The probability of the event that $0 \leq Y \leq X \leq 1$ is of the form $\int_a^b \left(\int_c^d f_{X,Y}(x,y)\, dx \right) dy$.

Find the values of a, b, c, d. Each one of your answers should be one of the following: 0, x, y, or 1.

$a =$ 0 ✔ **Answer:** 0

0

$b =$ 1 ✔ **Answer:** 1

1

$c =$ y ✔ **Answer:** y

y

$d =$ 1 ✔ **Answer:** 1

1

b) The probability of the event that $0 \leq Y \leq X \leq 1$ is also of the form $\int_a^b \left(\int_c^d f_{X,Y}(x,y)\, dy \right) dx$. Note the different order of integration as compared to part (a).

Find the values of a, b, c, d. Each one of your answers should be one of the following: 0, x, y, or 1.

$a =$ 0 ✔ Answer: 0

0

$b =$ 1 ✔ Answer: 1

1

$c =$ 0 ✔ Answer: 0

0

$d =$ x ✔ Answer: x

x

Solution:

a) For any given $y \in [0,1]$, x ranges from y to 1, yielding $\int_0^1 \int_y^1 f_{X,Y}(x,y)\, dx\, dy$.

b) For any given $x \in [0,1]$, y ranges from 0 to x, yielding $\int_0^1 \int_0^x f_{X,Y}(x,y)\, dy\, dx$.

15. Exercise: Finding a marginal PDF

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT *Completed*

Exercise: Finding a marginal PDF

1/1 point (graded)

The random variables X and Y are described by a uniform joint PDF of the form $f_{X,Y}(x,y) = 3$ on the set $\{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ y \leq x^2\}$.

Then, $f_X(0.5) =$ .75 ✔ **Answer:** 0.75

Solution:

For any $x \in [0,1]$, and using also the fact that the PDF is zero outside the specified set of x-y pairs, we have $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{x^2} 3\,dy = 3x^2$. Therefore, $f_X(0.5) = 3/4$.

Submit You have used 1 of 3 attempts

Show Answer

Answers are displayed within the problem

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: From joint PDFs to the marginals

3/5 points (graded)

For each one of the following formulas, identify those that are always true. All integrals are meant to be from $-\infty$ to ∞.

$$f_{X,Z}(a,b) = \int f_{X,Y,Z}(a',b,c)\, da'$$

No ✓ **Answer:** No

$$f_{X,Z}(a,c) = \int f_{X,Y,Z}(a,b,c)\, db$$

Yes ✓ **Answer:** Yes

$$f_{X,Z}(a,b) = \int f_{X,Y,Z}(a,b,c)\, dc$$

No **Answer:** No

$$f_Y(a) = \int\int\int f_{U,V,X,Y}(a,b,c,s)\, db\, dc\, ds$$

No **Answer:** No

$$f_Y(a) = \int\int\int f_{U,V,X,Y}(s,c,b,a)\, db\, dc\, ds$$

Yes

✔ **Answer:** Yes

Solution:

In each case, we need to “integrate out" the arguments associated with random variables that do not appear on the left-hand side. Thus, the correct formulas are:

$$f_{X,Z}(a,c) = \int f_{X,Y,Z}(a,b,c)\, db$$

and

$$f_Y(a) = \int\int\int f_{U,V,X,Y}(s,c,b,a)\, db\, dc\, ds.$$

Submit You have used 1 of 1 attempt

Show Answer

Exercises due Oct 15, 2020 19:59 EDT *Completed*

Exercise: Joint CDFs

3/3 points (graded)

a) Is it always true that if $x < x'$, then $F_{X,Y}(x, y) \leq F_{X,Y}(x', y)$?

Yes

✔ **Answer:** Yes

b) Suppose that the random variables X and Y are jointly continuous and take values on the unit square, i.e., $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Is $F_{X,Y}(x, y) = (x + 2y)^2/9$ a legitimate joint CDF? *Hint:* Consider $F_{X,Y}(0, 1)$.

No

✔ **Answer:** No

c) As above, suppose that the random variables X and Y are jointly continuous and take values on the unit square, i.e., $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The joint CDF on that set is of the form $xy(x + y)/2$. Find an expression for the joint PDF which is valid for (x, y) in the unit square. Enter an algebraic function of x and y using standard notation.

x+y

✔ **Answer:** x+y

$x + y$

STANDARD NOTATION

STANDARD NOTATION

Solution:

a) Since $x < x'$, the event $\{X \leq x, Y \leq y\}$ is a subset of the event $\{X \leq x', Y \leq y\}$, and therefore $F_{X,Y}(x,y) = \mathbf{P}(X \leq x, Y \leq y) \leq \mathbf{P}(X \leq x', Y \leq y) = F_{X,Y}(x', y)$.

b) Since the random variables are nonnegative, we have $F_{X,Y}(0,1) = \mathbf{P}(X \leq 0 \text{ and } Y \leq 1) = \mathbf{P}(X = 0 \text{ and } Y \leq 1) \leq \mathbf{P}(X = 0) = 0$, where the last equality holds because X is a continuous random variable. But zero is different from $(0 + 2 \cdot 1)^2/9$. Therefore, we do not have a legitimate joint CDF.

c) The joint CDF is of the form $x^2y/2 + y^2x/2$. The partial derivative with respect to x is $xy + y^2/2$. Taking now the partial derivative with respect to y, we obtain $x + y$.

Submit You have used 1 of 3 attempts

Show Answer

Answers are displayed within the problem

Exercise: Conditional PDF

2/2 points (graded)

The random variables X and Y are jointly continuous, with a joint PDF of the form

$$f_{X,Y}(x,y) = \begin{cases} cxy, & \text{if } 0 \le x \le y \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

where c is a normalizing constant.

a) Is it true that $f_{X|Y}(2 \mid 0.5)$ is equal to zero?

Yes ✔ **Answer:** Yes

b) Is it true that $f_{X|Y}(0.5 \mid 2)$ is equal to zero?

No ✔ **Answer:** No

Solution:

a) Values of Y around 0.5 have positive probability, so that $f_Y(0.5) > 0$, and $f_{X|Y}(2 \mid 0.5)$ is therefore well-defined. But $x = 2$ is outside the range of values of X, and $f_{X,Y}(2, 0.5) = 0$, from which it follows that $f_{X|Y}(2 \mid 0.5) = 0$.

b) Since $y = 2$ is outside the range of values of Y, we have $f_Y(2) = 0$, and the conditional PDF $f_{X|Y}(0.5 \mid 2)$ is undefined.

$$f_{X,Y}(x,y) = \begin{cases} cxy, & \text{if } 0 \le x \le y \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

where c is a normalizing constant.

For $x \in [0, 0.5]$, the conditional PDF $f_{X|Y}(x \mid 0.5)$ is of the form ax^b. Find a and b. Your answers should be numbers.

$a =$ 8 ✔ **Answer:** 8

$b =$ 1 ✔ **Answer:** 1

Solution:

We have $f_{X|Y}(x \mid 0.5) = \dfrac{f_{X,Y}(x, 0.5)}{f_Y(0.5)}$.

Having fixed $y = 0.5$, the conditional PDF is to be viewed as a function of x. For those values of x that are possible (i.e., $x \in [0, 0.5]$), the conditional PDF will be proportional to the joint PDF, hence of the form ax, for some constant a. This implies that $b = 1$. To find the normalizing constant, we use the normalization equation

$$1 = \int_0^{0.5} f_{X|Y}(x \mid 0.5)\, dx = \int_0^{0.5} ax\, dx = a \cdot \frac{x^2}{2}\Big|0_{0.5} = \frac{a}{8},$$

7. Exercise: Expected value rule and total expectation theorem

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Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: Expected value rule and total expectation theorem

4/8 points (graded)

Let X, Y, and Z be jointly continuous random variables. Assume that all conditional PDFs and expectations are well defined. E.g., when conditioning on $X = x$, assume that x is such that $f_X(x) > 0$. For each one of the following formulas, state whether it is true for all choices of the function g or false (i.e., not true for all choices of g).

1. $\mathbf{E}[g(Y) \mid X = x] = \int g(y)\, f_{Y|X}(y \mid x)\, dy$

True ✔ **Answer:** True

2. $\mathbf{E}[g(y) \mid X = x] = \int g(y)\, f_{Y|X}(y \mid x)\, dy$

False ✔ **Answer:** False

3. $\mathbf{E}[g(Y)] = \int \mathbf{E}[g(Y) \mid Z = z]\, f_Z(z)\, dz$

True **Answer:** True

4. $\mathbf{E}\big[g(Y)\,|\,X=x, Z=z\big] = \int g(y)\, f_{Y|X,Z}(y\,|\,x,z)\,dy$

True ✔ **Answer:** True

5. $\mathbf{E}\big[g(Y)\,|\,X=x\big] = \int \mathbf{E}\big[g(Y)\,|\,X=x, Z=z\big]\, f_{Z|X}(z\,|\,x)\,dz$

True **Answer:** True

6. $\mathbf{E}\big[g(X,Y)\,|\,Y=y\big] = \mathbf{E}\big[g(X,y)\,|\,Y=y\big]$

True **Answer:** True

7. $\mathbf{E}\big[g(X,Y)\,|\,Y=y\big] = \mathbf{E}\big[g(X,y)\big]$

False ✔ **Answer:** False

8. $\mathbf{E}\big[g(X,Z)\,|\,Y=y\big] = \int g(x,z)\, f_{X,Z|Y}(x,z\,|\,y)\,dy$

False **Answer:** False

Solution:

1. True. This is the usual expected value rule, applied to a conditional model where we are given that $X = x$.

2. False. Here the quantity inside the expectation, $g(y)$, is a number (not a random variable). The left-hand side is a function of y, whereas on the right-hand side, y, is a dummy variable that gets integrated away. So, the formula is wrong on a purely syntactical basis (the left-hand side depends on y, while the right-hand side does not).

3. True. This is the total expectation theorem, where we condition on the events $Z = z$.

4. True. This is the usual expected value rule, applied to a conditional model where we are given that $X = x$ and $Z = z$.

5. True. This is the same total expectation theorem as in the third part, except that everything is calculated within a conditional model in which event $X = x$ is known to have occurred.

6. True. When we condition on $Y = y$, we know the value of Y, and we can replace $g(X, Y)$ by $g(X, y)$.

7. False. Given that $Y = y$, we need to somehow take into account the conditional distribution of X, whereas the right-hand side is determined by the unconditional PDF of X.

8. False. The left-hand side is a function of y, whereas the right-hand side (after y is integrated out) is a function of x and z. The correct form (expected value rule, in a conditional model) is:

$$\mathbf{E}\left[g(X, Z) \mid Y = y\right] = \int\int g(x, z)\, f_{X,Z|Y}(x, z \mid y)\, dx\, dz.$$

Submit

You have used 1 of 1 attempt

Show Answer

9. Exercise: Definition of independence

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: Definition of independence

1/1 point (graded)

Suppose that X and Y are independent, with a joint PDF that is uniform on a certain set S: $f_{X,Y}(x,y)$ is constant on S, and zero otherwise. The set S

- must be a square.
- must be a set of the form $\{(x,y) : x \in A,\ y \in B\}$ (known as the Cartesian product of two sets A and B).
- can be any set.

✔

Solution:

Let A be the set of all x on which $f_X(x)$ is positive and let B be the set of all y on which $f_Y(y)$ is positive. Then, the set S, on which $f_{X,Y}(x,y) = f_X(x)\, f_Y(y) > 0$, will be the Cartesian product of A with B; it is not necessarily a square, but it cannot be an arbitrary set.

10. Exercise: Independence and expectations II

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT *Completed*

Exercise: Independence and expectations II

3/3 points (graded)

Let X, Y, Z be independent jointly continuous random variables, and let g, h, r be some functions. For each one of the following formulas, state whether it is true for all choices of the functions g, h, and r, or false (i.e., not true for all choices of these functions). Do not attempt formal derivations; use an intuitive argument.

1. $\mathbf{E}\left[g(X,Y)\,h(Z)\right] = \mathbf{E}\left[g(X,Y)\right] \cdot \mathbf{E}\left[h(Z)\right]$

   True ✔ **Answer:** True

2. $\mathbf{E}\left[g(X,Y)\,h(Y,Z)\right] = \mathbf{E}\left[g(X,Y)\right] \cdot \mathbf{E}\left[h(Y,Z)\right]$

   False ✔ **Answer:** False

3. $\mathbf{E}\left[g(X)\,r(Y)\,h(Z)\right] = \mathbf{E}\left[g(X)\right] \cdot \mathbf{E}\left[r(Y)\right] \cdot \mathbf{E}\left[h(Z)\right]$

   True ✔ **Answer:** True

Solution:

1. True. Using our intuitive understanding of independence, the pair of random variables (X, Y) does not provide any information on Z. Therefore, (X, Y) and Z are independent. It follows that $g(X, Y)$ and $h(Z)$ are independent, from which the formula follows.

2. False. The random variable Y appears in both functions g and h, so that $g(X, Y)$ and $h(Y, Z)$ will be, in general, dependent. For an example, suppose that $g(X, Y) = h(Y, Z) = Y$, in which case the statement becomes $\mathbf{E}[Y^2] = (\mathbf{E}[Y])^2$, which we know to be false in general.

3. True. Using the first part, and then again the independence of X with Y, we have $\mathbf{E}\big[g(X)\, r(Y)\, h(Z)\big] = \mathbf{E}\big[g(X)\, r(Y)\big] \cdot \mathbf{E}\big[h(Z)\big] = \mathbf{E}\big[g(X)\big] \cdot \mathbf{E}\big[r(Y)\big] \cdot \mathbf{E}\big[h(Z)\big]$.

Submit You have used 1 of 1 attempt

Show Answer

Answers are displayed within the problem

11. Exercise: Independence and CDFs

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: Independence and CDFs

1/2 points (graded)

a) Suppose that X and Y are independent. Is it true that their joint CDF satisfies $F_{X,Y}(x,y) = F_X(x) F_Y(y)$, for all x and y?

Yes

Answer: Yes

b) Suppose that $F_{X,Y}(x,y) = F_X(x) F_Y(y)$, for all x and y. Is it true that X and Y are independent?
Hint: Recall the formula $f_{X,Y}(x,y) = (\partial^2 / \partial x \partial y) F_{X,Y}(x,y)$.

Yes

Answer: Yes

Solution:

a) Yes. We have

$$\begin{aligned}
F_{X,Y}(x,y) &= \mathbf{P}(X \le x, Y \le y) \\
&= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(x,y)\, dx\, dy \\
&= \int_{-\infty}^{x} f_X(x)\, dx \int_{-\infty}^{y} f_Y(y)\, dy \\
&= F_X(x)\, F_Y(y).
\end{aligned}$$

b) True. Using the formula in the hint, we find that

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) \\
&= \frac{\partial^2}{\partial x \partial y} F_X(x)\, F_Y(y) \\
&= \frac{\partial}{\partial x} F_X(x)\, \frac{\partial}{\partial y} F_Y(y) \\
&= f_X(x)\, f_Y(y),
\end{aligned}$$

and therefore we have independence.

Submit You have used 1 of 1 attempt

Show Answer

Exercise: Stick-breaking

3/3 points (graded)

Consider the same stick-breaking problem as in the previous clip, and let $\ell = 1$. Recall that $f_{X,Y}(x, y) = 1/x$ when $0 \le y \le x \le 1$.

a) Conditioned on $Y = 2/3$, the conditional PDF of X is nonzero when $a \le x \le b$. Find a and b.

$a =$ 2/3 ✔ **Answer:** 0.66667

$b =$ 1 ✔ **Answer:** 1

b) On the range found in part (a), the conditional PDF $f_{X|Y}(x \mid 2/3)$ is of the form cx^d for some constants c and d. Find d.

$d =$ -1 ✔ **Answer:** -1

Solution:

a) Since the joint PDF is nonzero only for $0 \le y \le x \le 1$, it follows that given that $Y = 2/3$, X ranges on the interval $[2/3, 1]$.

b) As a function of x, the conditional PDF has the same functional form (within a normalizing constant) as the joint PDF, and so it is of the form c/x, from which we conclude that $d = -1$.

The random variables X and Y have a joint PDF of the form $f_{X,Y}(x,y) = c \cdot \exp\left\{-\frac{1}{2}(4x^2 - 8x + y^2 - 6y + 13)\right\}$.

$\mathbf{E}[X] =$ 1 ✔ Answer: 1

$\mathsf{Var}(X) =$ 1/4 ✔ Answer: 0.25

$\mathbf{E}[Y] =$ 3 ✔ Answer: 3

$\mathsf{Var}(Y) =$ 1 ✔ Answer: 1

Solution:

We rewrite the joint PDF in the form

$$f_{X,Y}(x,y) = c \cdot \exp\left\{-\frac{1}{2}\left(\frac{(x-1)^2}{1/4} + (y-3)^2\right)\right\},$$

and we recognize that we are dealing with the joint PDF of two independent normals with $\mathbf{E}[X] = 1$, $\mathsf{Var}(X) = 1/4$,

17. Exercise: The discrete Bayes rule

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT Past Due

Exercise: The discrete Bayes rule

1 point possible (graded)

The bias of a coin (i.e., the probability of Heads) can take three possible values, $1/4$, $1/2$, or $3/4$, and is modeled as a discrete random variable Q with PMF

$$p_Q(q) = \begin{cases} 1/6, & \text{if } q = 1/4, \\ 2/6, & \text{if } q = 2/4, \\ 3/6, & \text{if } q = 3/4, \\ 0, & \text{otherwise.} \end{cases}$$

Let K be the total number of Heads in two independent tosses of the coin. Find $p_{Q|K}(3/4 \mid 2)$.

Answer: 0.75

Solution:

The Bayes rule for discrete random variables gives

$$p_{Q|K}(3/4\,|\,2) = \frac{p_Q(3/4)\,p_{K|Q}(2\,|\,3/4)}{p_K(2)} = \frac{(3/6)\cdot(3/4)^2}{p_K(2)} = \frac{(3/6)\cdot(3/4)^2}{3/8} = \frac{3}{4}.$$

To find $p_K(2)$, we used the total probability theorem:

$$p_K(2) = \sum_q p_Q(q)\,p_{K|Q}(2\,|\,q) = (1/6)\cdot(1/4)^2 + (2/6)\cdot(2/4)^2 + (3/6)\cdot(3/4)^2 = 3/8.$$

Submit You have used 0 of 3 attempts

Show Answer

Answers are displayed within the problem

20. Exercise: Discrete unknown, continuous measurement

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT Completed

Exercise: Discrete unknown, continuous measurement

1/1 point (graded)

Let K be a discrete random variable that can take the values $1, 2$, and 3, all with equal probability. Suppose that X takes values in $[0,1]$ and that for x in that interval we have

$$f_{X|K}(x \mid k) = \begin{cases} 1, & \text{if } k = 1, \\ 2x, & \text{if } k = 2, \\ 3x^2, & \text{if } k = 3. \end{cases}$$

Find the probability that $K = 1$, given that $X = 1/2$.

4/11

Answer: 0.36364

Solution:

Using the appropriate form of the Bayes rule, we have

$$p_{K|X}(1\,|\,1/2) = \frac{p_K(1)\,f_{X|K}(1/2\,|\,1)}{f_X(1/2)} = \frac{(1/3)\cdot 1}{f_X(1/2)} = \frac{1/3}{11/12} = 4/11.$$

To find $f_X(1/2)$, we used the total probability theorem:

$$\begin{aligned} f_X(1/2) &= \sum_k p_K(k)\,f_{X|K}(1/2\,|\,k) \\ &= (1/3)\cdot 1 + (1/3)\cdot(2\cdot(1/2)) + (1/3)\cdot(3\cdot(1/2)^2) \\ &= 11/12. \end{aligned}$$

Submit You have used 1 of 3 attempts

Show Answer

Answers are displayed within the problem

22. Exercise: Inference of the bias of a coin

Bookmark this page

Exercises due Oct 15, 2020 19:59 EDT *Completed*

Exercise: Inference of the bias of a coin

1/1 point (graded)

The random variable K is geometric with a parameter which is itself a uniform random variable Q on $[0, 1]$. Find the value $f_{Q|K}(0.5 \mid 1)$ of the conditional PDF of Q, given that $K = 1$. *Hint:* Use the result in the last segment.

1

✔ **Answer:** 1

Solution:

We identify Q with the variable Y in the last segment. The information that $K = 1$ is the information that the first coin flip resulted in Heads, which is the same as the information that $K = 1$ in the last segment. Therefore, the conditional PDF of Q is $2q$, which for $q = 0.5$ evaluates to 1.

Submit You have used 1 of 3 attempts

Show Answer

Problem 1. Normal random variables

5/5 points (graded)

Let X and Y be two normal random variables, with means 0 and 3, respectively, and variances 1 and 16, respectively. Find the following, using the standard normal table. Express your answers to an accuracy of **3 decimal places**.

Standard Normal Table

Show

1. $\mathbf{P}(X > -1) =$ 0.841 ✔ **Answer:** 0.841

2. $\mathbf{P}(X \leq -2)$ 0.023 ✔ **Answer:** 0.023

3. Let $V = (4 - Y)/3$. Find the mean and the variance of V.

   $\mathbf{E}[V] =$ 0.333 ✔ **Answer:** 1/3

   $\mathsf{Var}(V) =$ 1.778 ✔ **Answer:** 16/9

4. $\mathbf{P}(-2 < Y \leq 2) =$ 0.2957 ✔ **Answer:** 0.2957

1. Since the distribution of X is symmetric around 0, we have,

$$\mathbf{P}(X > -1) = \mathbf{P}(X < 1) = \Phi(1) = 0.841.$$

2. Using symmetry again,

$$\mathbf{P}(X \leq -2) = \mathbf{P}(X > 2) = 1 - \mathbf{P}(X < 2) = 1 - \Phi(2) = 0.023.$$

3. We have $\mathbf{E}[V] = 4/3 - \mathbf{E}[Y]/3 = 1/3$, and $\mathsf{Var}(V) = \frac{1}{3^2}\mathsf{Var}(16) = 16/9$.

4. By standardizing Y, and using the normal table, we have

$$\begin{aligned} \mathbf{P}(-2 < Y \leq 2) &= \mathbf{P}\left(\frac{-2-3}{4} \leq \frac{Y-3}{4} \leq \frac{2-3}{4}\right) \\ &= \mathbf{P}(-5/4 \leq Z \leq -1/4) \\ &= \mathbf{P}(1/4 \leq Z \leq 5/4) \\ &= \Phi(5/4) - \Phi(1/4) \\ &\approx 0.8944 - 0.5987 \\ &= 0.2957, \end{aligned}$$

where, Z is a standard normal random variable.

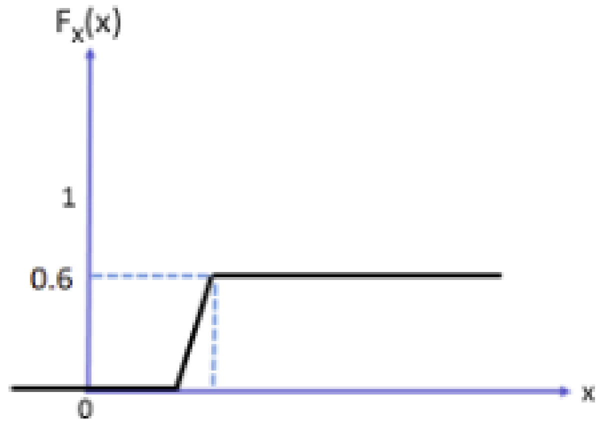
Problem Set due Oct 15, 2020 19:59 EDT Completed

Problem 2. CDF

4/4 points (graded)

For each one of the following figures, identify whether it is a valid CDF. The value of the CDF at points of discontinuity is indicated with a small solid circle.

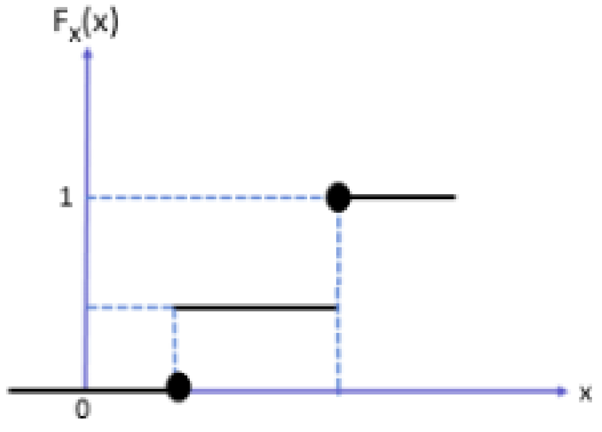
1.



Note: Assume that the above function never goes above 0.6.

It is not a valid CDF ✔ **Answer:** It is not a valid CDF

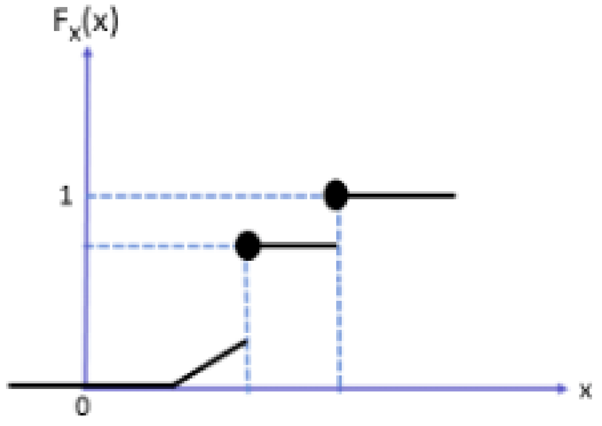
2.



It is not a valid CDF

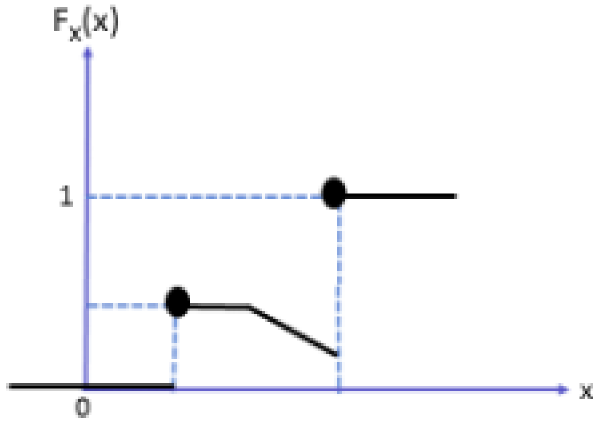
✔ **Answer:** It is not a valid CDF

3.



It is a valid CDF

✔ **Answer:** It is a valid CDF

4. 

It is not a valid CDF

✔ **Answer:** It is not a valid CDF

Solution:

1. No, because $\lim_{x\to\infty} F_X(x) \neq 1.$

2. No, because it is not right-continuous.

3. Yes, it is a valid CDF (limiting values are as required, and it is monotonic, and right-continuous).

4. No, because it is not monotonic.

Submit You have used 1 of 1 attempt

Problem 3. A joint PDF given by a simple formula

4/4 points (graded)

The random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax^2, & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the constant a.

   $a =$ 4/15 ✔ **Answer:** 4/15

2. Determine the marginal PDF $f_Y(y)$.
   (Your answer can be either numerical or an algebraic function of y).

   Useful fact: You may find the following fact useful: $\int_a^b x^2\, dx = \frac{1}{3}(b^3 - a^3)$.

   If $0 \le y \le 1$:

   $f_Y(y) =$ 28/45 ✔ **Answer:** 28/45

   $\frac{28}{45}$

If $1 < y \leq 2$:

$f_Y(y) =$ 4/45*(8-y^3) ✔ **Answer:** (32-4*y^3)/45

$\frac{4}{45} \cdot (8 - y^3)$

3. Determine the conditional expectation of $1/(X^2Y)$, given that $Y = 5/4$.

$\mathbf{E}\left[\frac{1}{X^2Y}\middle| Y = \frac{5}{4}\right] =$ 0.2976 ✔ **Answer:** 64/215

STANDARD NOTATION

Solution:

1. The joint PDF has to integrate to 1. From

$$\int_1^2 \int_0^x ax^2 \, dy \, dx = \int_1^2 ax^3 \, dx = \frac{15}{4}a = 1,$$

we get $a = 4/15$.

2. To find the marginal PDF of Y, we integrate the joint PDF over x:

$$= \begin{cases} \int_y^2 \frac{4}{15}x^2\,dx, & \text{if } 1 < y \leq 2, \\ 0, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} \frac{28}{45}, & \text{if } 0 \leq y \leq 1, \\ \frac{4}{45}(8 - y^3), & \text{if } 1 < y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

3. We first find the conditional PDF of X given $Y = 5/4$:

$$f_{X|Y}\left(x \,\middle|\, \frac{5}{4}\right) = \frac{f_{X,Y}\left(x, \frac{5}{4}\right)}{f_Y\left(\frac{5}{4}\right)} = \frac{\frac{4}{15}x^2}{\frac{4}{45}\left(8 - \left(\frac{5}{4}\right)^3\right)} = \frac{64}{129}x^2, \text{ for } \frac{5}{4} \leq x \leq 2.$$

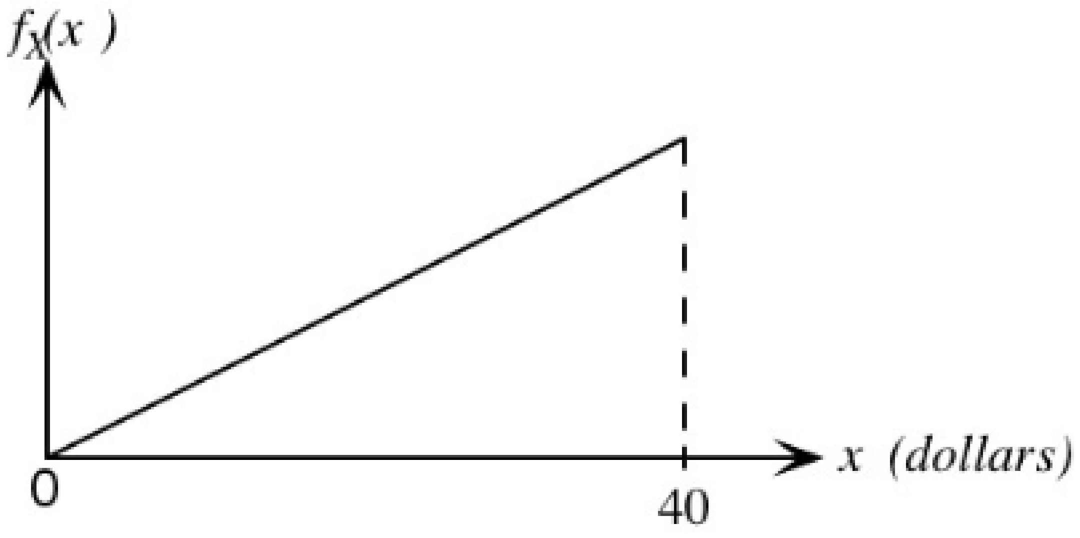
and equals 0 otherwise. Then,

$$\mathbf{E}\left[\frac{1}{X^2Y} \,\middle|\, Y = \frac{5}{4}\right] = \mathbf{E}\left[\frac{4}{5X^2} \,\middle|\, Y = \frac{5}{4}\right] = \int_{-\infty}^{\infty} \frac{4}{5x^2} \cdot f_{X|Y}\left(x \,\middle|\, \frac{5}{4}\right) dx,$$

which evaluates to

$$\int_{5/4}^{2} \frac{4}{5x^2} \cdot \frac{64}{129}x^2\,dx = \frac{64}{215}.$$

Sophia is vacationing in Monte Carlo. On any given night, she takes X dollars to the casino and returns with Y dollars. The random variable X has the PDF shown in the figure. Conditional on $X = x$, the continuous random variable Y is uniformly distributed between zero and $3x$.



1. Determine the joint PDF $f_{X,Y}(x, y)$.

   If $0 < x < 40$ and $0 < y < 3x$:

   $f_{X,Y}(x, y) =$ 1/2400 ✔ **Answer:** 1/2400

   If $y < 0$ or $y > 3x$:

   $f_{X,Y}(x, y) =$ 0 ✔ **Answer:** 0

2. On any particular night, Sophia makes a profit $Z = Y - X$ dollars. Find the probability that Sophia makes a positive profit, that is, find $\mathbf{P}(Z > 0)$.

$\mathbf{P}(Z > 0) =$ 2/3 ✔ **Answer:** 2/3

3. Find the PDF of Z. Express your answers in terms of z using standard notation.

Hint: Start by finding $f_{Z|X}(z \mid x)$.

If $-40 < z < 0$:

$f_Z(z) =$ **Answer:** (40+z)/2400

If $0 < z < 80$:

$f_Z(z) =$ **Answer:** (80-z)/4800

If $z < -40$ or $z > 80$:

$f_Z(z) =$ 0 ✔ **Answer:** 0

4. What is $\mathbf{E}[Z]$?

$\mathbf{E}[Z] =$ 40/3 ✔ **Answer:** 40/3

STANDARD NOTATION

Solution:

1. For this part, we will use the fact that $f_{X,Y}(x,y) = f_X(x)\, f_{Y|X}(y \mid x)$. Let us start by revealing $f_X(x)$. Clearly, $f_X(x) = ax$ for some ax, as shown in figure. Hence,

$$1 = \int_{-\infty}^{\infty} f_X(x)\ dx = \int_0^{40} ax\ dx = 800a.$$

Hence, $f_X(x) = \frac{x}{800}$. Using $f_{Y|X}(y \mid x) = \frac{1}{3x}$, for $0 < y < 3x$, we obtain the following expression for the joint density:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{2400}, & \text{if } 0 < x < 40 \text{ and } 0 < y < 3x \\ 0, & \text{otherwise.} \end{cases}$$

2. The first approach is to consider the region where Sophia makes positive profit. Notice that, this region consists of pairs

(x, y), where $y > x$. Intersecting this region with the region where the joint density is non-negative, we need to consider

$$\{(x, y) : 0 < x < 40, x < y < 3x\}.$$

Thus,

$$\mathbf{P}(Y > X) = \int_0^{40} \int_x^{3x} f_{X,Y}(x, y)\ dy\ dx = \int_0^{40} \int_x^{3x} \frac{1}{2400}\ dy\ dx = \int_0^{40} \frac{x}{1200} = \frac{2}{3}.$$

We could have also arrived at this answer by realizing that for each possible value of X, there is a $2/3$ probability that Y>X, and therefore by the total probability theorem,

$$\begin{aligned} \mathbf{P}(Y > X) &= \int_0^{40} \mathbf{P}(Y > X \mid X = x) f_X(x)\ dx \\ &= \int_0^{40} \frac{2}{3} f_X(x)\ dx \\ &= \frac{2}{3}, \end{aligned}$$

where the last equality follows because a PDF always integrates to 1, over the region where it is nonzero.

3. Given $X = x, Y$ is uniformly distributed on $[0, 3x]$, hence $Z = Y - x$ is uniform over $[-x, 2x]$. Thus,

$$f_{Z|X}(z \mid x) = \frac{1}{3x}, \quad \text{for } -x \le z \le 2x.$$

Therefore,

$$f_{X,Z}(x,z) = f_X(x)\, f_{Z|X}(z \mid x) = \frac{x}{800}\frac{1}{3x} = \frac{1}{2400}, \text{ for } 0 < x < 40 \text{ and } -x \le z \le 2x.$$

Now, we will integrate over x to compute the marginal density $f_Z(z)$. Note that, $x \ge -z$ and $x \ge \frac{z}{2}$ must be satisfied at the same time (in order for $f_{X,Z}$ to be non-zero).

If $-40 < z < 0$, the range of integration is $-z < x < 40$. Hence,

$$f_Z(z) = \int_{-z}^{40} \frac{1}{2400}\, dx = \frac{40+z}{2400}.$$

If $0 < z < 80$, the range of integration is $z/2 \le x \le 40$. Hence,

$$f_Z(z) = \int_{z/2}^{40} \frac{1}{2400}\, dx = \frac{80-z}{4800}.$$