[Ch 4] Sequences and Mathematical Induction

1. Sequences

1) Basics

- A sequence is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.
    - Example: 2,4,6,8,10,…
- A typical notation: $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$, where each element a_k is called a term and k is a subscript or index (and m is the subscript for the initial term, and n is the subscript for the last term).
    - Example: $a_1 = 2$, $a_2 = 4$, $a_3 = 6$, $a_4 = 8$, $a_5 = 10$, ..

- A sequence can be given/defined by an **explicit formula**.
    - Example: $\boldsymbol{a_k = 2 \cdot k}$ for all integers k ≥1. Then we can compute the first couple of terms and get $a_1 = 2 \cdot 1 = 2, a_2 = 2 \cdot 2 = 4, a_3 = 2 \cdot 3 = 6$
- Conversely, an explicit formula of a sequence can be inferred/derived by finding the pattern.
    - Example: A sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$ ➔ Formula: $\boldsymbol{a_k = \frac{1}{k^2}}$ for all integers k ≥1.

.2) Common sequences

1. Arithmetic series

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

2. Geometric series

$$\sum_{i=0}^{n} x^i = x^0 + x^1 + x^2 + \cdots + x^n = 1 + x^1 + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

3. Harmonic series

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

4. Factorial

n! = 1 * 2 * 3 * .. * (n-1) * n

2. Mathematical Induction

1) Proof by Mathematical Induction

- Another proof technique often used to prove theorems on values/objects that are composed of **sequence** (such as [1,2,3,4,..]).
- Suppose we want to show a theorem *P(n)* is true **<u>for all</u> positive integer *n* (>= n_0)**. Then we must show $P(n_0)$, $P(n_0+1)$, $P(n_0+2)$,... are all true. But instead of enumerating all possible (sometimes infinite number of) cases, we can show **2 cases**, and generalize inductively.
    1. **Show for the initial value of the sequence $P(n_0)$**
    2. **Show for a general case "if *P(n)* is true, then *P(n+1)* is true."**
- This scheme is analogous to **dominos.**
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P(1) P(2) P(3)

2) Weak form of Mathematical Induction

- Formal definition (worded slightly differently from the textbook)

> **<u>Principle of (Ordinary or 'Weak') Mathematical Induction:</u>**
>
> Let *P(n)* be a predicate that is defined for integer *n*, and let *a* be a fixed integer. Suppose the following two statements are true:
>
> 1. [**Basis Step**] *P(a)* is true.
> 2. [**Inductive Step**] For all integers *k* >= *a*,
>   - [Inductive Hypothesis] Assume *P(k)* is true.
>   - [Inductive statement] Then *P(k+1)* is true.
>
> Then, the statement "For all integers *n* >= *a*, *P(n)*" is true.

- *Example:*

Theorem: $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for *n* = 1, 2, ..

<u>*Proof:*</u> We show by induction on *n*.

<u>Basis Step:</u> When n = 1, $\sum_{i=1}^{1} i = 1$ and $\frac{n(n+1)}{2} = \frac{2}{2} = 1$ .... **(A)**

<u>Inductive Step:</u>
[Inductive Hypothesis] Assume $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ is true for all integer *n* >= 1.
[Inductive statement] We show that the equation is true for *n*+1, that is, $\sum_{i=1}^{n+1} i = 1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$.
Using the inductive hypothesis, we derive it as follows.

$1 + 2 + 3 + \cdots + n + (n+1)$
$= \frac{n(n+1)}{2} + (n+1)$ ... by inductive hypothesis

$= \dfrac{n(n+1) + 2(n+1)}{2}$

$= \dfrac{(n+1)(n+2)}{2}$

$= \sum_{i=1}^{n+1} i$

.... **(B)**

By **(A)** and **(B)**, the theorem is true (for all integers n >= 1).

- <u>Exercises</u>:
  1. $1 + 3 + 5 + .. + (2n-1) = n^2$
  2. $1 + 2 + 2^2 + .. + 2^n = 2^{n+1} - 1$
  3. $n! >= 2^{n-1}$
  4. $5^n - 1$ is divisible by 4, for n = 1, 2,..
  5. For all integers n >= 4, n cents can be obtained using 2-cent and 5-cent coins.

2) Strong form of Mathematical Induction

- Strong mathematical induction is similar to the ordinary version but more general.
- Difference is that the strong form assumes several cases for the basis step
  -- Rather than assume that P(k) is true to prove that P(k+1) is true, we assume that **P(i) is true for all i** where the (basis of induction) <= i <= k.
- In fact, the Weak version is a special case of the Strong version where the basis step assumes just one case
- Formal definition (worded slightly differently from the textbook)

<u>Principle of Strong Mathematical Induction:</u>

Let *P(n)* be a predicate that is defined for integer *n*, and let *a* and b be a fixed integers with a <= b. Suppose the following two statements are true:

1. [**Basis Step**] *P(a), P(a+1),…, and P(b) are all* true.
2. [**Inductive Step**] For all integers *k* >= b,
   - [Inductive Hypothesis] Assume *P(i)* is true for all integers *i*, a <= *i* <= k.
   - [Inductive statement] Then *P(k+1)* is true.

Then, the statement "For all integers *n* >= *a*, *P(n)*" is true.

- *<u>Example</u>:*

P(n) denotes the following: If b_0, b_1, b_2, b_3... is a sequence defined by the following:

$b_0 = 1, b_1 = 2, b_2 = 3$ and $b_k = b_{k-3} + b_{k-2} + b_{k-1}$ for all integers $k \geq 3$; then, b_n is $\leq 3^n$ for all integers $n \geq 0$.

<u>Proof:</u> We show by induction on *n*. We must show:

i) Basis step: prove that P(0), P(1) and P(2) are true. (Note: in sequence problems, we must prove all given base cases.)

$b_0 = 1$ which is $\leq 3^0$
$b_1 = 2$ which is $\leq 3^1$
$b_2 = 3$ which is $\leq 3^2$

ii) Induction hypothesis is to assume P(i) for $k > 2$: for all positive integers i where $1 \leq i < k$, $b_i \leq 3^i$, and show that $b_k \leq 3^k$.

By the basis cases, we have that $b_k = b_{k-3} + b_{k-2} + b_{k-1} \leq 3^k$ for all integers $k \geq 3$.
When we add these inequalities and apply the laws of basic algebra, we get

$b_{k-3} + b_{k-2} + b_{k-1} \leq 3^{k-1} + 3^{k-2} + 3^{k-3}$

We also know
$3^{k-1} + 3^{k-2} + 3^{k-3} \leq 3^{k-1} + 3^{k-1} + 3^{k-1}$
$3^{k-1} + 3^{k-1} + 3^{k-1} = 3 * 3^{k-1} = 3^k$

And thus by substitution, $b_k \leq 3^k$, which is what we set out to prove.

- Exercises:
  1. [Example 5.4.2, p. 270] Define a sequence s0, s1, s2,.. as follows

     $S_0 = 0$, $s_1 = 4$, $s_k = 6s_{k-1} – 5s_{k-2}$ for all integers k >= 2.

     a. Find the first four terms of this sequence.
     b. It is claimed that for each integer n >= 0, the nth term of the sequence has the same value as that given by the formula $5^n – 1$. In other words, the claim is that all the terms of the sequence satisfy the equation $s_n =5^n – 1$. Prove that this is true.

  2. Write the Strong Mathematical Induction version of the problem given earlier, “For all integer n >= 4, n cents can be obtained by using 2-cent and 5-cent coins.” Note the basis steps should prove P(4) and P(5).

  3. [Section 5.4, Exercise #5, p. 277] Suppose that $e_0, e_1, e_2,\ldots$ is a sequence defined as follows.

     $e_0 = 12$, $e_1 = 29$
     $e_k = 5e_{k-1} – 6e_{k-2}$ for all integers k >= 2.

     Prove that $e_n = 5*3^n + 7*2^n$ for all integer n >= 0. Use Strong Mathematical Induction.