09/08/2024 - 11h34

① Distribution of extreme values under definition 1 (largest data values)

Suppose $\boxed{X_1, \ldots, X_n}$ (data) are independently and identically distributed with CDF $F(\cdot)$ (cumulative distribution function).

By definition 1, the extreme value is: $M_n = \max(X_1, \ldots, X_n)$.

The CDF of M_n is $P(M_n \le x) = P(\max(X_1, \ldots, X_n) \le x)$

$$= P(X_1 \le x, \ldots, X_n \le x) \overset{\text{indep}}{=} \prod_{i=1}^{n} P(X_i \le x) = [F(x)]^n$$

how this behave when $n \to \infty$?

. People are usually interested in the behavior of M_n over large periods. That is, what is the distribution of M_n as $n \to \infty$?

$$\lim_{n \to \infty} P(M_n \le x) = \lim_{n \to \infty} [F(x)]^n$$

$$= \begin{cases} 0 & \text{if } 0 \le F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

⚠ This is not a useful result for practical application!

→ Classical Deviation problem:

Suppose $X_1, \ldots, X_n$ are iid with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$

Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ denote the sample mean.

1) $\lim_{n \to \infty} \bar{X} = \mu \Rightarrow$ "Strong Law of large Numbers"

"Not a very useful results" because limit is fix (constant)!

2) $\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \xrightarrow[n \to \infty]{D} \mathcal{N}(0,1)$ "Central limit theorem" (CLT) "Very useful result"

② we look at the limit of $P\left(\frac{M_n - b_n}{a_n} \le x\right)$.

$$\Rightarrow \lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \lim_{n\to\infty} P(M_n \le b_n + a_n x)$$

$$= \lim_{n\to\infty} P(\max(X_1, \dots, X_n) \le b_n + a_n x)$$

$$= \lim_{n\to\infty} P(X_i \le b_n + a_n x \ \forall i \in [\![1; n]\!])$$

$$\overset{\text{indep}}{=} \lim_{n\to\infty} \prod_{i=1}^{n} P(X_i \le b_n + a_n x) = \lim_{n\to\infty} [F(b_n + a_n x)]^n \quad (*)$$

what is the limit of $(*)$?

$\boxed{\text{Extremal types theorem}}$

suppose $X_1, \dots, X_n$ are independent and identical with CDF $F(\cdot)$.
if there exists $a_n > 0$ and $b_n \in [-\infty, \infty]$ such that
$\lim_{n\to\infty} [F(b_n + a_n x)]^n = G(x)$ for some non-degenerate
CDF $G(x)$ then it must be of the $\boxed{\text{same type}}$ as:

(Gumbel type).

$$\text{I} : \Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

$$\text{II} : \Phi_\alpha(x) = \begin{cases} 0, & \text{if } x < 0 \\ e^{-x^{-\alpha}}, & \text{if } x \ge 0 \end{cases} \quad \text{(Fréchet type)}$$

$$\text{III} : \Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha}, & \text{if } x < 0 \\ 1, & \text{if } x \ge 0 \end{cases}$$

(Weibull type)

non-degenerate : Something not equal to 0 or 1, $\boxed{\text{useful}}$

③ Definition of "Same type"

Two CDFs G_1 and G_2 are of the same type if

$G_2(x) = G_1(ax + b)$ for all x where $a > 0$ and $b \in \mathbb{R}$

Ex 1: $G_1(x) = e^{-e^{-x}}$, $G_2(x) = e^{-e^{-2x-4}}$

$\Rightarrow a = 2, b = 4$; $G_2(x) = G_1(2x+4)$

$\Rightarrow G_1$ and G_2 are of the same type.

Ex 2: $G_1(x) = e^{-\frac{1}{x}}$, $G_2(x) = e^{-\frac{1}{3-2x}}$

$\Rightarrow a = -2$, $b = 3$, $a < 0$

$\Rightarrow G_1$ and G_2 are not of the same type.

~~How do you we know which of the functions are we going to attend?~~

$\boxed{\text{How to know which of the 3 limits is attained (if there is a limit)}}$

Conditions for extremal type theorem

- type I (Gumbel type) will be attained if there exists $\gamma(t) > 0$ such that:
$$\lim_{t \to \omega(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$$

- type II (Fréchet type) will be attained if $\omega(F) = \infty$ and
$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad \alpha > 0$$

- type III (Weibull type) will be obtained if $\omega(F) < \infty$ and $\lim_{t \to 0} \frac{1 - F(\omega(F) - tx)}{1 - F(\omega(F) - t)} = x^{\alpha}, \alpha > 0$

Only one of ~~these~~ these three conditions will ever be satisfied.

• Definiti of $\omega(F)$

$\omega(F)$ = " upper end point " of the suppot of F.

Formally, $\omega(F) = \sup \{x : F(x) < 1\}$

An easy way to find $\omega(F)$ is to set $F(x) = 1$ and then solve for x.

Example 1 : $F(x) = 1 - e^{-x} = 1$

$\Rightarrow e^{-x} = 0$

$\Rightarrow -x = -\infty$

$\Rightarrow x = +\infty$

$\Rightarrow \omega(F) = +\infty$

Example 2 : $F(x) = 1 - \frac{1}{x} = 1$ (Pareto CDF, $x \geq 1$)

$\Rightarrow \frac{1}{x} = 0$

$\Rightarrow x = +\infty$

$\Rightarrow \omega(F) = +\infty$

Example 3 : $F(x) = x, \quad 0 < x < 1$

$x = 1$

$\Rightarrow \omega(F) = 1$

Examples on Checking Conditions I - III

Example 1 Suppose $F(x) = 1 - e^{-x}$

we know by previous example 1 that $\omega(F) = +\infty$

$$(I): \lim_{t \to \omega(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t\to\infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]}{1 - [1 - e^{-t}]}$$

$$= \lim_{t\to\infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t\to\infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if } \gamma(t) = 1 \ \forall t.$$

Hence, the Gumbel type is attained. That is, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that $[F(a_n x + b_n)]^n \xrightarrow[n\to\infty]{} e^{-e^{-x}}$

Example 2: Suppose $F(x) = 1 - \frac{1}{x}$

we saw earlier that $\cancel{\omega(F) = 1}$ $\omega(F) = +\infty$

$$I: \lim_{t\to\infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t\to\infty} \frac{1 - \left(1 - \frac{1}{t + x\gamma(t)}\right)}{1 - \left(1 - \frac{1}{t}\right)} = \lim_{t\to\infty} \frac{t}{t + x\gamma(t)}$$

⑥

For condition (I) to be satisfied this limit must be equal to e^{-x} and there is no chance for that.

$\Rightarrow$ condition (I) fails to hold.

Let's ~~check~~ see condition (II).

$$\text{II} : \lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)}$$

$$= \lim_{t\to\infty} \frac{1-\left(1-\frac{1}{tx}\right)}{1-\left(1-\frac{1}{t}\right)} = \lim_{t\to\infty} \frac{t}{tx} = \frac{1}{x} = x^{-\alpha}$$

when $\alpha = 1$.

Hence, condition II holds with $\alpha = 1$. That is, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that: $\left[F_n(b_n + a_n x)\right]^n \to \begin{cases} 0 & \text{if } x < 0 \\ e^{-x^{-1}} & \text{if } x \geq 0 \end{cases}$ as $n \to \infty$

Example 3: Suppose $F(x) = x$ the CDF of Uni $[0,1]$

we saw previously that $\omega(F) = 1$

$$\text{I}: \lim_{t\to 1} \frac{1-F(t+x\gamma(t))}{1-F(t)}$$

$$= \lim_{t\to 1} \frac{1-(t+x\gamma(t))}{1-t}$$

$$= \lim_{t\to 1} 1 - \frac{x\gamma(t)}{1-t}.$$

For condition I to be satisfied, this must be equal to e^{-x} and there is no chance for that, so condition I fails to hold.

(7) II : we do not need to calculate the limit because

$\omega(F) = 1 < \infty$

$\Rightarrow$ Condition II fails to hold

III : $\lim\limits_{t \to 0} \dfrac{1 - F(1 - tx)}{1 - F(1 - t)}$

$= \lim\limits_{t \to 0} \dfrac{1 - (1 - tx)}{1 - (1 - t)}$

$= \lim\limits_{t \to 0} \dfrac{tx}{t} = x = x^{\alpha}$ when $\alpha = 1$

Hence, condition III holds with $\alpha = 1$.

According to the extremal types theorem, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that: $[F(a_n x + b_n)]^n \longrightarrow \begin{cases} e^{x} & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ as $n \to \infty$.

$\boxed{\text{Alternative Conditions for extremal types theorem}}$: Alternative Conditions to check which of the 3 limits is attained.

Let $f(t) = \dfrac{dF(t)}{dt}$, $f(t)$ = density function

$a(t) = F^{-1}\left(1 - \frac{1}{t}\right)$,

$b(t) = t\, f(a(t))$.

⑧

I: $\lim\limits_{t\to\infty} \dfrac{b(tx)}{b(t)} = 1$

II: $\omega(F)=\infty$ and $\lim\limits_{t\to\infty} a(t)\, b(t) = \alpha > 0$

III: $\omega(F)<\infty$ and $\lim\limits_{t\to\infty} \{\omega(F) - a(t)\}\, b(t) = \alpha > 0$

These conditions are equivalent to the previous conditions.
Once again, only one of the conditions will ever be satisfied.

Example 1: $F(x) = 1 - e^{-x}$

$f(x) = \dfrac{dF(x)}{dx} = e^{-x}$

~~Set $F(x) = \frac{1}{\quad}$~~ Set $F(x) = 1 - \dfrac{1}{t}$

$\Rightarrow 1 - e^{-x} = 1 - \dfrac{1}{t}$

$\Rightarrow e^{-x} = \dfrac{1}{t} \Rightarrow -x = -\log(t)$

$\Rightarrow x = \log(t)$

$\Rightarrow a(t) = \log(t)$

. $b(t) = t\, f(a(t))$

$= t\, e^{-a(t)}$

$= t\, e^{-\log(t)}$

$= 1$

I: $\lim_{t\to\infty} \frac{b(tx)}{b(t)} = \lim_{t\to\infty} \frac{1}{1} = 1$

Hence, condition I holds.

<u>Example 2</u>: $F(x) = 1 - \frac{1}{x}$; $\omega(F) = \infty$

$f(x) = \frac{dF(x)}{dx} = \frac{1}{x^2}$.

$F(x) = 1 - \frac{1}{t} \Rightarrow 1 - \frac{1}{x} = 1 - \frac{1}{t}$

$\Rightarrow x = t. \Rightarrow a(t) = t$

• $b(t) = t.f(a(t)) \cancel{= t\,a(t)}$

$= t. \frac{1}{(a(t))^2}$

$= t. \frac{1}{t^2} = \frac{1}{t}$

I: $\lim_{t\to\infty} \frac{b(tx)}{b(t)} = \lim_{t\to\infty} \frac{\frac{1}{tx}}{\frac{1}{t}} = \frac{1}{x} \neq 1$.

$\Rightarrow$ Condition I fails to hold

II: $\omega(F) = \infty$ ✓

$\lim_{t\to\infty} a(t)\, b(t) = \lim_{t\to\infty} t . \frac{1}{t} = 1 > 0$

Hence, condition II hold with $\alpha = 1$

(10) Example 3 : $F(x) = x, \quad 0 < x < 1$

$w(F) = 1 \; ; \quad f(x) = \frac{dF(x)}{dx} = 1$

~~$F(x) = \frac{1}{x}$~~ $\quad F(x) = 1 - \frac{1}{t}$

$\Rightarrow x = 1 - \frac{1}{t} \Rightarrow a(t) = 1 - \frac{1}{t}$

$\cdot \; b(t) = t \cdot f(a(t)) = t \cdot 1 = t$

I: $\lim_{t \to \infty} \frac{b(tx)}{b(t)} = \lim_{t \to \infty} \frac{tx}{t} = x \neq 1$

$\Rightarrow$ Conditions I fails to hold

II: $w(F) = 1 \neq \infty \Rightarrow$ Condition II fails to hold.

III: $w(F) = 1 < \infty$ ✓

$$\lim_{t \to \infty} \{1 - a(t)\} \, b(t) = \lim_{t \to \infty} \left\{1 - \left(1 - \frac{1}{t}\right)\right\} t$$

$$= 1$$

Hence, Condition III holds with $\alpha = 1$

⚠ There are examples where none of the three conditions are satisfied.

$\boxed{\text{Formulas for Norming constants}}$

How to choose a_n and b_n

I: $\begin{cases} a_n = \gamma\left(F^{-1}\left(1 - \frac{1}{n}\right)\right) \\ b_n = F^{-1}\left(1 - \frac{1}{n}\right) \end{cases}$

II : $\begin{cases} a_n = F^{-1}(1-\frac{1}{n}) \\ b_n = 0 \end{cases}$

III : $\begin{cases} a_n = w(F) - F^{-1}(1-\frac{1}{n}) \\ b_n = w(F) \end{cases}$

$F^{-1}(\cdot)$ denotes the inverse function of $F(\cdot)$

Example 1 : $F(x) = 1 - e^{-x}$, $x > 0$

Previously, condition I holds with $\gamma(t) \equiv 1$

$\Rightarrow a_n = \gamma(F^{-1}(1-\frac{1}{n}))$

$= 1$

$\Rightarrow F(x) = 1 - \frac{1}{n} \Rightarrow 1 - e^{-x} = 1 - \frac{1}{n}$

$\Rightarrow x = \log n$

$\Rightarrow b_n = \log(n)$

By The ETT (extremal types theorems) :

$$[F(x + \log(n))]^n \xrightarrow[n\to\infty]{} e^{-e^{-x}}$$

Example 2 : $F(x) = 1 - \frac{1}{x}$

We know that, condition (II) holds,

$F(x) = 1 - \frac{1}{n} \Rightarrow 1 - \frac{1}{x} = 1 - \frac{1}{n}$

$\Rightarrow x = n \Rightarrow a_n = n$ and $b_n = 0$

(12) By the ETT, $[F(nx)]^n \xrightarrow[\text{as } n\to\infty]{} \begin{cases} 0, & x<0 \\ e^{-x^{-1}} & x \geq 0 \end{cases}$

Example 3 : $F(x) = x$, $0 < x < 1$

we know that, condition (III) holds with $\alpha = 1$ and $\omega(F) = 1$

$F(x) = 1 - \frac{1}{n} \Rightarrow x = 1 - \frac{1}{n}$

$\Rightarrow F^{-1}(1-\frac{1}{n}) = 1 - \frac{1}{n}$

$\Rightarrow a_n = 1 - (1 - \frac{1}{n}) = \frac{1}{n}$

$b_n = 1$

By the ETT, $[F(\frac{x}{n} + 1)]^n \xrightarrow[\text{as } n\to\infty]{} \begin{cases} e^x, & x<0 \\ 1, & x \geq 0 \end{cases}$

$\boxed{\text{Example where Extremal types theorem (ETT) fails}}$

Here is an example where none of conditions I - III will be satisfied:

Suppose $F(x) = 1 - \frac{1}{\log(x)}$, $x > e$

→ find $\omega(F)$?

$F(x) = 1 \Rightarrow 1 - \frac{1}{\log x} = 1$

$\Rightarrow \frac{1}{\log x} = 0 \Rightarrow \log x = +\infty$

$\Rightarrow x = +\infty$

$\Rightarrow \omega(F) = +\infty$

I: $\lim_{t\to\infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t\to\infty} \frac{1 - \left[1 - \frac{1}{\log(t + x\gamma(t))}\right]}{1 - \left[1 - \frac{1}{\log t}\right]}$

$= \lim_{t\to\infty} \frac{\log(t)}{\log(t + x\gamma(t))} \neq e^{-x} \Rightarrow$ Condition I fails.

13 II: $\omega(F) = +\infty$ ✓

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} \overset{\lim_{t\to\infty}}{=} \frac{1-\left[1-\frac{1}{\log(tx)}\right]}{1-\left[1-\frac{1}{\log t}\right]}$$

$$= \lim_{t\to\infty} \frac{\log(t)}{\log(tx)} = \lim_{t\to\infty} \frac{\log(t)}{\log t + \log x}$$

$$= \lim_{t\to\infty} \frac{1}{1+\underbrace{\frac{\log x}{\log(t)}}_{\downarrow 0}} = 1 \neq x^{-a}, \alpha > 0$$

$\Rightarrow$ Condition (II) fails.

III) $\omega(F) < \infty$, which is not the case here, because $\omega(F) = \infty$

$\Rightarrow$ Condition III fails.

All three conditions I-III fail. Hence, the ETT does not hold.

A Condition to check if ETT holds for discrete RV

Checking of Conditions I-III can be time consuming. It is convenient to have a single condition to find out whether the ETT will hold or not.

Suppose X is a "discrete" RV with CDF $F(\cdot)$. Let $\omega(F)$ denote it upper end point. Then the ETT will hold if and only if:

$$\boxed{\lim_{k\to\omega(F)} \frac{P(X=k)}{1-F(k-1)} = 0}$$

(14) **Example 1:** Suppose $X_1, \dots, X_n$ are IID Geom(p).

Does the ETT hold?

$P(X=k) = p(1-p)^{k-1}$, $k = 1, 2, \dots$

$\omega(F) = +\infty$, largest point of the support $k \in \mathbb{N}$.

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = \lim_{k\to\infty} \frac{p(1-p)^{k-1}}{1-[1-(1-p)^{k-1}]}$$

$$= \lim_{k\to\infty} \frac{p(1-p)^{k-1}}{(1-p)^{k-1}}$$

$$= p \neq 0$$

Hence, the ETT fails to hold.

Example 2: Suppose $X_1, \dots, X_n$ are IID Bin(m, p). Does the ETT hold?

$$P(X=k) = \binom{m}{k} p^k (1-p)^{m-k}$$

support $= \{0, \dots, m\} \Rightarrow \omega(F) = m$

$$\Rightarrow \lim_{k\to m} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=m)}{1-F(m-1)}$$

$$= \frac{P(X=m)}{1-P(X\leq m-1)} = \frac{P(X=m)}{P(X>m-1)} = \frac{P(X=m)}{P(X=m)} = 1 \neq 0$$

Hence, the ETT fails to hold.

15 **Example 3**: Suppose $X_1, \dots, X_n$ are IID with

$$P(X=k) = \frac{1}{N} \; ; \; k = 1, 2, \dots, N$$

This is the discrete uniform distribution.

Does the ETT hold?

$\omega(F) = N$.

$$\Rightarrow \lim_{k \to N} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=N)}{1-F(N-1)} = \frac{P(X=N)}{1-P(X \le N-1)}$$

$$= \frac{P(X=N)}{P(X>N-1)} = \frac{P(X=N)}{P(X=N)}$$

$$= 1 \neq 0$$

Hence, the ETT fails to hold.

Can the 3 limits be combined into 1 ?

The ETT says that there can be 3 possible limits for the sample maximum. A practitioner may not know to check the conditions I–III giving rise to the 3 limits. So it will be convenient if the 3 limits can be combined into one mathematical form.

The combined form is known as the Generalised extreme value (GEV) distribution.

GEV distribution

A random variable X follows the GEV distribution if its CDF is $G(x) = \exp\left[-\left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right]$

(16) where $1 + \xi \frac{x-\mu}{\sigma} > 0$

$-\infty < \xi < \infty$ " Shape parameter "

$\sigma > 0$ " Scale parameter "

$-\infty < \mu < \infty$ " location parameter "

Notation : $X \sim GEV(\mu, \sigma, \xi)$

Domain (support) of X

- $\xi < 0$

$$1 + \xi \frac{x-\mu}{\sigma} > 0 \iff \xi \frac{x-\mu}{\sigma} > -1$$
$$\iff \frac{x-\mu}{\sigma} < -\frac{1}{\xi}$$
$$\iff x < \mu - \frac{\sigma}{\xi}$$

domain is $(-\infty, \mu - \frac{\sigma}{\xi})$

- $\xi = 0$, $1 + \xi \frac{x-\mu}{\sigma} = 1 > 0$

$\Rightarrow$ domain is $(-\infty, \infty)$

- $\xi > 0$

$$1 + \xi \frac{x-\mu}{\sigma} > 0 \iff \xi \frac{x-\mu}{\sigma} > -1$$
$$\iff \frac{x-\mu}{\sigma} > -\frac{1}{\xi}$$
$$\iff x > \mu - \frac{\sigma}{\xi}$$

$\Rightarrow$ domain is $(\mu - \frac{\sigma}{\xi}, \infty)$

$$\text{Domain of } X = \begin{cases} \left(\mu - \frac{\sigma}{\xi}, \infty\right) & \text{if } \xi > 0 \\ (-\infty, \infty) & \text{if } \xi = 0 \\ \left(-\infty, \mu - \frac{\sigma}{\xi}\right) & \text{if } \xi < 0 \end{cases}$$

How come the GEV combines the 3 limiting distributions

$\xi < 0$

$$G(x) = \exp\left\{-\left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right\}$$

$$= \exp\left\{-\left(1 - \frac{\xi\mu}{\sigma} + \frac{\xi x}{\sigma}\right)^{-\frac{1}{\xi}}\right\}$$

$$= \exp\left\{-\left(-\left(\frac{\xi\mu}{\sigma} - 1 + \left(-\frac{\xi}{\sigma}\right)x\right)\right)^{-\frac{1}{\xi}}\right\}$$

Set $a = -\frac{\xi}{\sigma} > 0$ and $b = \frac{\xi\mu}{\sigma} - 1 \in \mathbb{R}$

$$G(x) = \exp\left\{-\left(-(ax+b)\right)^{-\frac{1}{\xi}}\right\}$$; set $\alpha = -\frac{1}{\xi} > 0$ since $\xi < 0$

$$= \exp\left\{-\left(-(ax+b)\right)^{\alpha}\right\}$$

$=$ "Same type" as $\exp\left\{-(-x)^{\alpha}\right\}$

$\Rightarrow$ Weibull limit is a particular case of the GEV for $\xi < 0$.

(18)

$\xi > 0$

$$G(x) = \exp\left\{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right\}$$

$$= \exp\left\{-\left(1-\frac{\xi\mu}{\sigma}+\frac{\xi}{\sigma}x\right)^{-\frac{1}{\xi}}\right\}$$

Set $b = 1-\frac{\xi\mu}{\sigma} \in \mathbb{R}$, $a = \frac{\xi}{\sigma} > 0$ and $\alpha = \frac{1}{\xi} > 0$

$$G(x) = \exp\left\{-(ax+b)^{-\alpha}\right\}$$

$=$ Same type as $\exp(-x^{-\alpha})$

$\Rightarrow$ Fréchet limit is a particular case of the GEV for $\xi > 0$

$\xi = 0$

$$G(x) = \lim_{\xi\to 0}\exp\left\{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right\}$$

$$= \lim_{\xi\to 0}\exp\left\{-\left(1+\frac{\frac{x-\mu}{\sigma}}{\frac{1}{\xi}}\right)^{-\frac{1}{\xi}}\right\}$$

Set $m = \frac{1}{\xi}$, $\xi \to 0 \Rightarrow m \to +\infty$

$$G(x) = \lim_{m\to\infty}\exp\left\{-\left(1+\frac{\frac{x-\mu}{\sigma}}{m}\right)^{-m}\right\}$$

$$= \lim_{m\to\infty}\exp\left\{-\left[\left(1+\frac{\frac{x-\mu}{\sigma}}{m}\right)^{m}\right]^{-1}\right\}$$

Reminder: $\lim_{m \to \infty} \left(1 + \frac{z}{m}\right) = e^z$

So $G(x) = \exp\left\{ - \left[\exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-1} \right\}$

$= \exp\left\{ - \exp\left(-\frac{x-\mu}{\sigma}\right)\right\}$; Set $a = \frac{1}{\sigma} > 0$, $b = -\frac{\mu}{\sigma} \in \mathbb{R}$

$= \exp\left\{ - \exp\left\{-(ax+b)\right\}\right\}$

$=$ same type as $\exp\{-\exp(-x)\}$

$\Rightarrow$ Gumbel limit is the particular case of the GEV for $\xi = 0$.

In a nutshell:
$\begin{cases} \text{Gumbel limit is the particular case of GEV for } \xi = 0 \\ \text{Weibull} \quad " \quad " \quad " \quad " \quad " \quad " \quad " \quad " \quad \xi < 0 \\ \text{Fréchet} \quad " \quad " \quad " \quad " \quad " \quad " \quad " \quad " \quad \xi > 0 \end{cases}$

PDF of GEV distribution

$$g(x) = \frac{d}{dx} G(x)$$

$$= \frac{d}{dx} \exp\left[- \left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \right]$$

$$= \frac{1}{\sigma} \left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot \exp\left[- \left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \right]$$

Quantile of GEV distribution

A quantile x_p of a RV X is define by

$$\boxed{P(X \leq x_p) = p} \quad (+)$$

For the GEV distribution,

$$(+) \Leftrightarrow G(x_p) = p$$

$$\Leftrightarrow \exp\left[-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right] = p$$

$$\Leftrightarrow 1+\xi\frac{x-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\Leftrightarrow x_p = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right] = p^{th} \text{ quantile of the GEV distribution.}$$

In particular, $x_{\frac{1}{2}}$ = median of GEV

$$= \mu + \frac{\sigma}{\xi}\left[(\log 2)^{-\xi} - 1\right]$$

Return level of GEV distribution

A return level with period T (in years) is the level expected to be exceed on average once in every T years. Let x_T denote the return level with period T.

Then : $\boxed{P(X > x_T) = \frac{1}{T}} \Leftrightarrow P(X \leq x_T) = 1 - \frac{1}{T}$

$$\Leftrightarrow G(x_T) = 1 - \frac{1}{T} \Leftrightarrow \exp\left\{-\left(1+\xi\frac{x_T-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right\} = 1 - \frac{1}{T}$$

$$\Leftrightarrow x_T = \mu + \frac{\sigma}{\xi}\left[\left(-\log\left(1-\frac{1}{T}\right)\right)^{-\xi} - 1\right]$$

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Estimation of GEV distribution

Suppose $x_1, x_2, \dots, x_n$ are independent and identical data on $X \sim GEV(\mu, \sigma, \xi)$.

The most popular method for estimation is the method of maximum likelihood.

The likelihood function of (μ, σ, ξ) is defines as:

$$L(\mu,\sigma,\xi) = \prod_{i=1}^{n} \left\{ \frac{1}{\sigma}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot \exp\left\{-\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right\}\right\}$$

$$= \frac{1}{\sigma^n}\left[\prod_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} \cdot \exp\left[-\sum_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right]$$

The log-likelihood function is:

$$\log L(\mu,\sigma,\xi) = -n\log\sigma - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i-\mu}{\sigma}\right) - \sum_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

The MLEs of μ, σ and ξ are the simultaneous solutions

$$\frac{\partial \log L}{\partial \mu} = 0 \; ; \quad \frac{\partial \log L}{\partial \sigma} = 0 \; ; \quad \frac{\partial \log L}{\partial \xi} = 0$$

But these solutions ~~do~~ not have analytical forms so we must do it with some computer program packages.

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Definition 2 of extreme values

Extreme values = all data exceeding a threshold u.

Suppose X = variables of interest

u = threshold

if $X > u$ then it is an extreme value.

What is the distribution of X when $X > u$?

There is a result due to Pickands (1975) which says:

$$P(\underbrace{\boxed{X-u}}_{\text{"excess amount"}} > x \mid \underbrace{\boxed{X > u}}_{X \text{ is an extreme value.}}) \longrightarrow \left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

as. $u \to \omega(F)$, where $F(\cdot)$ denotes the CDF of X.

Suppose u is large enough such that $P(X-u>x \mid X>u) \simeq \left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$

$$\Rightarrow \frac{P(X-u>x,\ X>u)}{P(X>u)} \simeq \left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}} \Rightarrow \frac{P(X>u+x,\ X>u)}{P(X>u)} \simeq \left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\Rightarrow \frac{P(X>u+x)}{P(X>u)} \simeq \left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}} \Rightarrow P(X>u+x) \simeq P(X>u)\left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\Rightarrow 1-P(X\leq u+x) \simeq P(X>u)\left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\Rightarrow 1-F(u+x) \simeq P(X>u)\left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

Set $y = u+x \Rightarrow 1-F(y) \simeq P(X>u)\left(1+\xi\frac{y-u}{\sigma}\right)^{-\frac{1}{\xi}}$

$$\Rightarrow \boxed{F(y) \simeq 1-P(X>u)\left(1+\xi\frac{y-u}{\sigma}\right)^{-\frac{1}{\xi}}}$$

Generalised Pareto (GP) distribution.

The GP distribution

A random variable X has the GP distribution if its CDF is:

$$F(y) = 1 - P(X > u)\left(1 + \xi \frac{y-u}{\sigma}\right)^{-\frac{1}{\xi}}$$

where $1 + \xi \frac{y-u}{\sigma} > 0$, $y > u$

$\sigma > 0$ " scale parameter "

$-\infty < \xi < \infty$ " shape parameter "

Notation : $X \sim GP(\sigma, \xi)$

Domain of $X \sim GP(\sigma, \xi)$

- $\xi > 0$

$1 + \xi \frac{y-u}{\sigma} > 0$ & $y > u$

$\Leftrightarrow y > u - \overset{>0}{\frac{\sigma}{\xi}}$ & $y > u \Leftrightarrow y > u$

$\Rightarrow$ the domain is $(u, +\infty)$

- $\xi < 0$

$1 + \xi \frac{y-\mu}{\sigma} > 0$ & $y > 0$

$\Leftrightarrow \cancel{y < u +}\ y < u - \frac{\sigma}{\xi}$ & $y > u$

$\Leftrightarrow u < y < u - \frac{\sigma}{\xi}$

$\Rightarrow$ the domain is $\left(u, u - \frac{\sigma}{\xi}\right)$

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• $\xi = 0$

$1 > 0$ & $y > u \Leftrightarrow y > u$

$\Rightarrow$ the domain is $(u, +\infty)$

Hence,

$$\text{Domain of } GP(\sigma, \xi) = \begin{cases} (u, \infty) & \text{if } \xi \geq 0 \\ (u, u - \frac{\sigma}{\xi}) & \text{if } \xi < 0 \end{cases}$$

Special case: The $GP(\sigma, 0)$ distribution

$$F(y) = \lim_{\xi \to 0} \left[1 - P(X > u) \left(1 + \xi \frac{y-u}{\sigma}\right)^{-\frac{1}{\xi}} \right]$$

$$= 1 - P(X > u) \lim_{\xi \to 0} \left(1 + \xi \frac{y-u}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$= 1 - P(X > u) \lim_{\xi \to 0} \left(1 + \frac{(y-u)/\sigma}{1/\xi}\right)^{-\frac{1}{\xi}}$$

Set $m = \frac{1}{\xi}$; $\xi \to 0 \Rightarrow m \to +\infty$

$$\Rightarrow F(y) = 1 - P(X > u) \left[\lim_{m \to \infty} \left(1 + \frac{\frac{y-u}{\sigma}}{m}\right)^{m} \right]^{-1}$$

we know that: $\lim_{m \to +\infty} \left(1 + \frac{z}{m}\right) = e^{z}$

$$\Rightarrow F(y) = 1 - P(X > u) \left(e^{-\frac{y-u}{\sigma}}\right)$$

$$= 1 - P(X > u) e^{-\frac{y-u}{\sigma}}$$ = CDF of a shifted exponential distribution.

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PDF of $GP(\sigma, \xi)$ distribution

$$f(y) = \frac{d}{dy} F(y)$$

$$= \frac{d}{dy}\left[1 - P(X>u)\left(1+\xi\,\frac{y-u}{\sigma}\right)^{-\frac{1}{\xi}}\right]$$

$$= P(X>u)\cdot\frac{1}{\sigma}\cdot\left(1+\xi\,\frac{y-u}{\sigma}\right)^{-\frac{1}{\xi}-1}$$

Quantile of $GP(\sigma, \xi)$ distribution

if X is a RV its p^{th} quantile is defined by

$$P(X \leq x_p) = p$$

if $X \sim GP(\sigma, \xi)$ then :

$$1 - P(X>u)\left(1+\xi\,\frac{x_p-u}{\sigma}\right)^{-\frac{1}{\xi}} = p$$

$$\Leftrightarrow 1+\xi\,\frac{x_p-u}{\sigma} = \left[\frac{1-p}{P(X>u)}\right]^{-\xi}$$

$$\Leftrightarrow x_p = u + \frac{\sigma}{\xi}\left\{\left[\frac{1-p}{P(X>u)}\right]^{-\xi} - 1\right\}$$

In particular, $x_{\frac{1}{2}} = \text{median}(X)$

$$= u + \frac{\sigma}{\xi}\left\{\left[\frac{1}{2P(X>u)}\right]^{-\xi} - 1\right\}$$

Return level of $GP(\sigma, \xi)$ distribution

x_T is the level that will be exceeded on average once in every T years.

(26)

By definition: $P(X > x_T) = \frac{1}{mT}$ $(++)$

Where m denotes the average ~~of m~~ number of extremes values per year.

$(++) \Leftrightarrow P(X \leq x_T) = 1 - \frac{1}{mT}$

$\Leftrightarrow 1 - P(X > u)\left(1 + \xi \frac{x_T - u}{\sigma}\right)^{-\frac{1}{\xi}} = 1 - \frac{1}{mT}$

$\Leftrightarrow x_T = u + \frac{\sigma}{\xi}\left\{\left[\frac{1}{mT\,P(X>u)}\right]^{-\xi} - 1\right\}$

Estimation of $GP(\sigma, \xi)$ distribution

Suppose $(x_1, \dots, x_n)$ are IID data from $GP(\sigma, \xi)$.

The likelihood function is:

$$L(\sigma, \xi) = \prod_{i=1}^{n}\left[\frac{P(X>u)}{\sigma}\left(1 + \xi \frac{x_i - u}{\sigma}\right)^{-\frac{1}{\xi} - 1}\right]$$

$$= \frac{[P(X>u)]^n}{\sigma^n}\prod_{i=1}^{n}\left(1 + \xi \frac{x_i - u}{\sigma}\right)^{-\frac{1}{\xi} - 1}$$

The log likelihood function is:

$$\log L(\sigma, \xi) = n \log P(X>u) - n \log \sigma - \left(\frac{1}{\xi} + 1\right)\sum_{i=1}^{n} \log\left(1 + \xi \frac{x_i - u}{\sigma}\right)$$

The MLEs of σ and ξ are the simultaneous solutions of:

$$\frac{\partial \log L}{\partial \sigma} = 0, \quad \frac{\partial \log L}{\partial \xi} = 0.$$

(27)

Definition 3 of extreme values

Extreme values = first few largest data values

Let $M_n^{(1)}$ = 1^{st} Largest value

$M_n^{(2)}$ = 2^{nd} largest value

$M_n^{(3)}$ = 3^{rd} largest value

$\vdots$

$M_n^{(r)}$ = r^{th} largest value.

what is the distribution of $(M_n^{(1)}, M_n^{(2)}, \dots, M_n^{(r)})$?

A result due to Weissman (1978) is:

$$P\left[\frac{M_n^{(i)} - b_n}{a_n} < x_i \mid i \in [\![1, r]\!] \right] \xrightarrow{\text{as } n \to \infty}$$

$$\sum_{s_1=0}^{1} \sum_{s_2=0}^{2-s_1} \cdots \sum_{s_{r-1}=0}^{r-1-s_1-\cdots-s_{r-2}} \frac{(\gamma_2-\gamma_1)^{s_1}}{s_1!} \cdots \frac{(\gamma_r-\gamma_{r-1})^{s_{r-1}}}{s_{r-1}!} e^{-\gamma_r} \quad (*)$$

$\gamma_i = (1+\xi x_i)^{-\frac{1}{\xi}}$; a_n = same as in ETT,

b_n = same as in ETT.

It can be shown from $(*)$ that the joint PDF of

$(M_n^{(1)}, \dots, M_n^{(r)})$ is $\sigma^{-r} \exp\left\{-\left(1+\xi\frac{x_r-u}{\sigma}\right)^{-\frac{1}{\xi}} - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{r}\log\left(1+\xi\frac{x_i-u}{\sigma}\right)\right\}$

for $x_1 \geq x_2 \geq \dots \geq x_r$ and

$1+\xi\frac{x_i-u}{\sigma} > 0, \quad i = 1, 2, \dots, r$

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Estimation under definition 3

Suppose $(x_{1,1}, x_{1,2}, \dots, x_{1,r})$ for 1^{st} year

$(x_{2,1}, x_{2,2}, \dots x_{2,r})$ for 2^{nd} "

$\vdots$

$(x_{T,1}; x_{T,2}, \dots x_{T,r})$ for T^{th} "

The data are the r largest observations for T years.

The likelihood function is:

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{T} \left[\sigma^{-r} \exp\left\{ -\left(1+\xi \frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}} - \left(\frac{1}{\xi}+1\right) \sum_{j=1}^{r} \log\left(1+\xi \frac{x_{i,j}-\mu}{\sigma}\right) \right\} \right]$$

The log-likelihood function is:

$$\log L(\mu, \sigma, \xi) = -Tr\log\sigma - \sum_{i=1}^{T}\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}} - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{T}\sum_{j=1}^{r}\log\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)$$

The MLEs of μ, σ, ξ are the simultaneous solutions of:

$$\frac{\partial \log L}{\partial \mu} = 0 \ ; \quad \frac{\partial \log L}{\partial \sigma} = 0 \ ; \quad \frac{\partial \log L}{\partial \xi} = 0.$$

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Copula

Suppose (X, Y) is a random vector, There are many situations where x and Y wile be dependent on each other.

EX1 : (X, Y) = (oil price, car price)

EX2: (X, Y) : (oil price, food price)

A Copula can be used to model the dependence among 2 or more variables.

For 2 variables, a Copula is a function :

$$C : \underbrace{\boxed{[0,1]\times[0,1]}}_{\leftarrow \text{ unit square}} \longrightarrow [0,1] \leftarrow \text{unit intervals}$$

with $C(u, 1) = u$, $C(1, v) = v$ } marginal CDF of C are uniformly distributed over $[0, 1]$.

Suppose (X, Y) has joint CDF $F_{X,Y}(x, y)$ and marginal CDFs $F_X(x)$, $F_Y(y)$.

Statement 1: For every $F_{X,Y}$ there is a corresponding copula.

Proof: ~~the~~ we have

$$F_{X,Y}(x, y) = P(X \le x, Y \le y)$$

$$= P(F_X(X) \le F_X(x),\ F_Y(Y) \le F_Y(y))$$

Probability integral transform

$$= P(\mathrm{Uni}[0,1] \le F_X(x),\ \mathrm{Uni}[0,1] \le F_Y(y))$$

$= C(F_X(x), F_Y(y))$

The proof is complete.

Statement 2: For every C there is a corresponding $F_{X,Y}$.

Proof: We have

$$C(u,v) = P(\mathrm{Uni}[0,1] \leq u, \ \mathrm{Uni}[0,1] \leq v)$$

Probability integral transform

$$= P(F_X^{-1}(\mathrm{Uni}[0,1]) \leq F_X^{-1}(u), F_Y^{-1}(\mathrm{Uni}[0,1]) \leq F_Y^{-1}(v))$$

$$= P(X \leq F_X^{-1}(u), \ Y \leq F_Y^{-1}(v))$$

$$= F_{X,Y}(F_X^{-1}(u), \ F_Y^{-1}(v))$$

The proof is complete.

Two simple Copulas

1) Independence Copula is $C(u,v) = uv$

$\Leftrightarrow P(\mathrm{Uni}[0,1] \leq u, \ \mathrm{Uni}[0,1] \leq v) = uv$

$\Leftrightarrow P(F_X^{-1}(\mathrm{Uni}[0,1]) \leq F_X^{-1}(u), \ F_Y^{-1}(\mathrm{Uni}[0,1]) \leq F_Y^{-1}(v)) = uv$

$\Leftrightarrow P(X \leq \underbrace{F_X^{-1}(u)}_{=x}, \ Y \leq \underbrace{F_X^{-1}(v)}_{=y}) = uv$

$\Leftrightarrow$ Set $x = F_X^{-1}(u) \Rightarrow u = F_X(x)$

" $y = F_Y^{-1}(v) \Rightarrow v = F_Y(y)$

$\Leftrightarrow P(X \leq x, \ Y \leq x) = F_X(x) \, F_Y(y)$

$\Leftrightarrow \boxed{F_{X,Y} = F_X(x) \, F_Y(y)}$ usual definition of independence between X and Y

2) Completely dependent copula is

$$C(u,v) = \min(u,v)$$

$\Leftrightarrow P(\text{Uni}[0,1] \leq u, \text{Uni}[0,1] \leq v) = \min(u,v)$

$\Leftrightarrow P(F_X^{-1}(\text{Uni}[0,1]) \leq F_X^{-1}(u), F_Y^{-1}(\text{Uni}[0,1]) \leq F_Y^{-1}(v)) = \min(u,v)$

$\Leftrightarrow P(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v)) = \min(u,v)$

Set $x = F_X^{-1}(u) \Rightarrow u = F_X(x)$

Set $y = F_Y^{-1}(v) \Rightarrow v = F_Y(y)$

$\Leftrightarrow P(X \leq x, Y \leq y) = \min(F_X(x), F_Y(y))$

$\Leftrightarrow \boxed{F_{X,Y}(x,y) = \min(F_X(x), F_Y(y))}$

$\Leftrightarrow \boxed{F_{X,Y}(x,y) = C(F_X(x), F_Y(y))}$

Usual definition of Complete dependence between X and Y.

Formal definition of a Copula

A function $C : [0,1] \times [0,1] \longrightarrow [0,1]$ is a Copula if

(i) $C(u,0) = 0 \quad \forall u$

(ii) $C(0,v) = 0 \quad \forall v$

(iii) $C(u,1) = u \quad \forall u$

(iv) $C(1,v) = v \quad \forall v$

(v) $\dfrac{\partial}{\partial u} C(u,v) \geq 0 \quad \forall u,v$

(vi) $\dfrac{\partial}{\partial v} C(u,v) \geq 0 \quad \forall u,v$

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Ex1: Check $C(u,v) = uv$ is a copula

i) $C(u,0) = u \cdot 0 = 0$ ✓

ii) $C(0,v) = 0 \cdot v = 0$ ✓

iii) $C(u,1) = u \cdot 1 = u$ ✓

iv) $C(1,v) = 1 \cdot v = v$ ✓

v) $\frac{\partial}{\partial u} C(u,v) = v \geq 0$; vi) $\frac{\partial}{\partial v} C(u,v) = u \geq 0$

Hence, $C(u,v)$ is a copula.

Ex2: Check $C(u,v) = \min(u,v)$ is a copula.

i) $C(u,0) = \min(u,0) = 0$ ✓

ii) $C(0,v) = \min(0,v) = 0$ ✓

iii) $C(u,1) = \min(u,1) = u$ ✓

iv) $C(1,v) = \min(1,v) = v$ ✓

v) $\frac{\partial}{\partial u} C(u,v) = \frac{\partial}{\partial u} \min(u,v) = \frac{\partial}{\partial u} \begin{cases} u & \text{if } u \leq v \\ v & \text{if } u > v \end{cases} = \begin{cases} 1 & \text{if } u \leq v \\ 0 & u > v \end{cases} \geq 0$

vi) $\frac{\partial}{\partial v} C(u,v) = \frac{\partial}{\partial v} \min(u,v) = \frac{\partial}{\partial v} \begin{cases} u & \text{if } u \leq v \\ v & \text{if } u > v \end{cases} = \begin{cases} 0 & u \leq v \\ 1 & u > 0 \end{cases} \geq 0$

Hence, C is a copula.

Ex3: Check $C(u,v) = uv[1 + \theta(1-u)(1-v)]$ is a copula

$-1 \leq \theta \leq 1$

i) $C(u,0) = u \cdot 0 = 0$ ✓; ii) $C(0,v) = 0 \cdot v[1+\theta(1-u)(1-v)] = 0$ ✓

iii) $C(u,1) = u[1 + \theta \times 0] = u$ ✓; iv) $C(1,v) = v[1 + \theta . 0 . (1-v)]$
$= v$ ✓

v) $\frac{\partial}{\partial u} C(u,v) = v[1+\theta(1-v)(1-2u)]$

$= v[1+\underbrace{\theta}_{-1\to 1}\underbrace{(1-2u)}_{-1\to 1}\underbrace{(1-v)}_{0\to 1}]$

$\underbrace{\qquad\qquad}_{-1\to 1}$

≥ 0

(vi) $\frac{\partial}{\partial v} C(u,v) \geq 0$ (Same argument like Condition (v))

Hence, C is a Copula.

$\boxed{\text{The Two most popular Copulas}}$

1) Normal (Gaussian) Copula is

$$C(u_1,u_2) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$$

Where $\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-\frac{x^2}{2}}\,du$

= CDF of standard distribution.

and $\Phi_2(u_1,u_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{u_1}\int_{-\infty}^{u_2} e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}}\,dy\,dx$

= Joint CDF of standard bivariate normal distribution.

The ~~most~~ normal / Gaussian copula was the most popular copular up until 2008. ("Read the formula that killed wall street")

Financial data do not follow the normal distribution well. Hence, Gaussian Copula is not a good model for financial data.

2) t Copula is

$$C(u_1, u_2) = T_2\left(t_a^{-1}(u_1),\ t_a^{-1}(u_2)\right)$$

where $$t_a(u) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\,\Gamma\left(\frac{a}{2}\right)} \int_{-\infty}^{u} \left(1+\frac{x^2}{a}\right)^{\frac{a+1}{2}} dx$$

= CDF of student's t distribution with degree of freedom equal to a.

and $$T_2(u_1, u_2) = \frac{\Gamma\left(\frac{a+2}{2}\right)}{a\pi\,\Gamma\left(\frac{a}{2}\right)\sqrt{1-\rho^2}} \int_{-\infty}^{u_1}\int_{-\infty}^{u_2} \left[1+\frac{x^2+y^2-2\rho xy}{a(1-\rho^2)}\right]^{-\frac{a+2}{2}} dy\,dx$$

= Joint CDF of bivariate t distribution with degree of freedom a.

"This copula is being increasingly used especially in the financial sector to model the dependence between different stocks."