WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2021

MTMACOR10T-MATHEMATICS (CC10)

RING THEORY AND LINEAR ALGEBRA-I

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times 5 = 10$

(a) Show that $\mathbb{Z}_n$ is not a field, when n is not a prime.

(b) Show that the set $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z}\right\}$ of diagonal matrices is a subring of the ring of all 2×2 matrices over $\mathbb{Z}$.

(c) Give an example of a ring having exactly 25 points.

(d) Are the rings $\mathbb{Z}$ and $2\mathbb{Z}$ isomorphic? — Justify your answer.

(e) Show that any field is a simple ring i.e., it has no non-trivial proper ideal.

(f) Extend the set $S = \{(1, 2, 1), (2, 1, 1)\}$ to obtain a basis of the vector space $\mathbb{R}^3$.

(g) Show that the intersection of any family of subspaces of a vector space V over a field F is a subspace of V.

(h) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by $T(a, b) = (a + 3b, 0, 2a - 4b)$. Let β and γ be the standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Find $[T]_\beta^\gamma$.

(i) Determine all possible linear transformations from the vector space of all real numbers to itself.

2. (a) If R is an integral domain of prime characteristic p, then prove that $(a+b)^p = a^p + b^p$. — 4

(b) Prove that the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{Q}\right\}$ is a field, where $\mathbb{Q}$ is the set of all rational numbers. — 4

3. (a) Let R be a ring with identity $1 \neq 0$, such that R has no non-trivial left ideal. Show that R is a division ring. — 4

(b) Let $n \in \mathbb{Z}$ be a fixed positive integer. If n is a prime, show that $\mathbb{Z}/\langle n\rangle$ is a field, where $\langle n\rangle = \{qn : q \in \mathbb{Z}\}$ and $\mathbb{Z}/\langle n\rangle = \{a + \langle n\rangle : a \in \mathbb{Z}\}$. — 4

4. (a) Give an example to show that the homomorphic image of an integral domain need not be an integral domain. — 4

(b) Let f be a homomorphism of a ring R into a ring R'. Then show that $f(R)$ is an ideal of R' and $R/\ker f \simeq f(R)$. — 4

5. (a) Suppose F is a field and there is a ring homomorphism from $\mathbb{Z}$ onto F. Show that $F \simeq \mathbb{Z}_p$, for some prime p. 4

   (b) Let f be a homomorphism of a ring R into a ring R'. Then show that 2+2

   (i) if R is commutative, then $f(R)$ is commutative and

   (ii) if R has an identity and $f(R) = R'$, then R' has an identity.

6. (a) Let S be a non-empty subset of a vector space V over a field F. Then show that $L(S)$, the linear span of S is the smallest subspace of V containing S. 4

   (b) Show that $S = \{(x, y, z) \in \mathbb{R}^3 :\ 2x + y - z = 0\}$ is a subspace of the vector space $\mathbb{R}^3$. Find a basis and the dimension of S. 3+1

7. (a) Let W_1, W_2 be two subspaces of a finite dimensional vector space V over a field F. Show that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. 4

   (b) Determine all possible subspaces of the vector space $\mathbb{R}^3$ over $\mathbb{R}$. 4

8. (a) Let V be a vector space over a field F, with a basis consisting of n elements. Then show that any $n+1$ elements of V are linearly dependent. 4

   (b) Show that $S = \{(x, y) \in R^2 : x^2 = y^2\}$ is not a subspace of the vectors space R^2 over R. Find the smallest subspace of R^2 containing S. 3+1

9. (a) Let V and W be vector spaces over the same field F and let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. If $\beta_1, \ldots, \beta_n$ be any n vectors in W, then prove that there is precisely one linear transformation $T : V \to W$ such that $T(\alpha_i) = \beta_i$, $i = 1, \ldots, n$. 4

   (b) Show that $T :\ R^3 \to R^3$ defined by $T(x, y, z) = (y+z,\ z+x,\ x+y)$, $\forall (x, y, z) \in R^3$ is a linear transformation. 2+1+1

   Is it one-one? Justify your answer.

   Is it onto? Justify your answer.

10. (a) Let V and W be vector spaces over a field F of equal (finite) dimension and let $T : V \to W$ be linear. If $\operatorname{rank}(T) = \dim(V)$, then show that T is one-to-one and onto. 4

   (b) A linear transformation $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is defined by 4

   $T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$, where $P_2(\mathbb{R})$ is the collection of all polynomials over $\mathbb{R}$ of degree atmost 2 and $M_{2\times 2}(\mathbb{R})$ is the collection of all 2 × 2 matrices over $\mathbb{R}$. Find $\operatorname{rank}(T)$.

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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