Desarrollo de péndulo esférico

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Formulación de Euler-Lagrange

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad , \quad \mathcal{L} = T - V$$

Coordenadas generalizadas

$$q_1 = \theta$$
$$q_2 = \varphi$$

$$x = l\cos\theta\sin\varphi$$
$$y = l\sin\theta\sin\varphi$$
$$z = -l\cos\varphi$$

$$\dot{x} = -l\dot{\theta}\sin\theta\sin\varphi + l\dot{\varphi}\cos\theta\cos\varphi$$
$$\dot{y} = l\dot{\theta}\cos\theta\sin\varphi + l\dot{\varphi}\sin\theta\cos\varphi$$
$$\dot{z} = l\dot{\varphi}\sin\varphi$$

$$\dot{x}^2 = l^2\dot{\theta}^2\sin^2\theta\sin^2\varphi + l^2\dot{\varphi}^2\cos^2\theta\cos^2\varphi - 2l^2\dot{\theta}\dot{\varphi}\cos\theta\sin\theta\cos\varphi\sin\varphi$$
$$\dot{y}^2 = l^2\dot{\theta}^2\cos^2\theta\sin^2\varphi + l^2\dot{\varphi}^2\sin^2\theta\cos^2\varphi + 2l^2\dot{\theta}\dot{\varphi}\cos\theta\sin\theta\cos\varphi\sin\varphi$$
$$\dot{z}^2 = l^2\dot{\varphi}^2\sin^2\varphi$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2\dot{\theta}^2\sin^2\varphi + l^2\dot{\varphi}^2$$

Energía potencial

$$V = mgz \to V = -mgl\cos\varphi$$

Energía cinética

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$T = \frac{1}{2}m(l^2\dot{\theta}^2\sin^2\varphi + l^2\dot{\varphi}^2)$$

Lagrangiano

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2}m(l^2\dot{\theta}^2\sin^2\varphi + l^2\dot{\varphi}^2) + mgl\cos\varphi$$

Derivadas parciales del Lagrangiano

$$\frac{\partial\mathcal{L}}{\partial\theta} = 0$$
$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = ml^2\dot{\theta}\sin^2\varphi$$
$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] = ml^2\ddot{\theta}\sin^2\varphi + ml^2\dot{\theta}\dot{\varphi}\sin 2\varphi$$

$$\frac{\partial\mathcal{L}}{\partial\varphi} = ml^2\dot{\theta}^2\sin\varphi\cos\varphi - mgl\sin\varphi$$
$$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\dot{\varphi}$$
$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right] = ml^2\ddot{\varphi}$$

Ecuación de movimiento

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}\left[ml^2\dot{\theta}\sin^2\varphi\right] = 0 \quad \rightarrow \quad \frac{d}{dt}\left[\dot{\theta}\sin^2\varphi\right] = 0$$

$$\boxed{h = \dot{\theta}\sin^2\varphi \quad \rightarrow \quad h \equiv \text{cte.}}$$

$$ml^2\ddot{\theta}\sin^2\varphi + ml^2\dot{\theta}\dot{\varphi}\sin 2\varphi = 0 \quad \rightarrow \quad \ddot{\theta}\sin^2\varphi + 2\dot{\theta}\dot{\varphi}\cos\varphi\sin\varphi = 0$$

$$\boxed{\ddot{\theta}\sin\varphi + 2\dot{\theta}\dot{\varphi}\cos\varphi = 0}$$

$$\ddot{\theta}\sin^3\varphi + 2\dot{\theta}\dot{\varphi}\cos\varphi\sin^2\varphi = 0 \quad \rightarrow \quad \ddot{\theta}\sin^3\varphi + 2h\dot{\varphi}\cos\varphi = 0$$

$$\boxed{\ddot{\theta}(\varphi, \dot{\varphi}) = -2h\dot{\varphi}\frac{\cos\varphi}{\sin^3\varphi}}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right] - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$ml^2\ddot{\varphi} - ml^2\dot{\theta}^2 \sin\varphi\cos\varphi + mgl\sin\varphi = 0$$

$$\ddot{\varphi} - \dot{\theta}^2 \sin\varphi\cos\varphi + \frac{g}{l}\sin\varphi = 0 \quad \rightarrow \quad \ddot{\varphi} - h^2\frac{\cos\varphi}{\sin^3\varphi} + \frac{g}{l}\sin\varphi = 0$$

$$\omega_0^2 = \frac{g}{l} \quad \rightarrow \quad \ddot{\varphi} - h^2\frac{\cos\varphi}{\sin^3\varphi} + \omega_0^2\sin\varphi = 0 \quad \rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}\dot{\varphi}^2 + \frac{h^2}{2\sin^2\varphi} - \omega_0^2\cos\varphi\right] = 0$$

$$\boxed{\varepsilon = \frac{1}{2}\dot{\varphi}^2 + \frac{h^2}{2\sin^2\varphi} - \omega_0^2\cos\varphi \quad \rightarrow \quad \varepsilon \equiv \text{cte.}}$$

$$\boxed{\ddot{\varphi}(\varphi) = h^2\frac{\cos\varphi}{\sin^3\varphi} - \omega_0^2\sin\varphi}$$

Solución númerica por método de Euler

$$f_{i+1} = f_i + h\dot{f}(f_i, t_i)$$

$$h = \dot{\theta}\sin^2\varphi \quad \rightarrow \quad \dot{\theta}(\varphi) = \frac{h}{\sin^2\varphi}$$

$$\varepsilon = \frac{1}{2}\dot{\varphi}^2 + \frac{h^2}{2\sin^2\varphi} - \omega_0^2\cos\varphi \quad \rightarrow \quad \dot{\varphi}(\varphi) = \sqrt{2\varepsilon - \frac{h^2}{\sin^2\varphi} + 2\omega_0^2\cos\varphi}$$

$$\boxed{\begin{aligned} t_{i+1} &= t_i + h \\ \theta_{i+1} &= \theta_i + h\dot{\theta}(\varphi_i) \\ \varphi_{i+1} &= \varphi_i + h\dot{\varphi}(\varphi_i) \end{aligned}}$$

Solución númerica por método de Euler

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$$\ddot{\varphi}(\varphi) = h^2\frac{\cos\varphi}{\sin^3\varphi} - \omega_0^2\sin\varphi$$

$$\boxed{\begin{aligned} t_{i+1} &= t_i + h \\ \\ \theta_{i+1} &= \theta_i + h\dot{\theta}_i \\ \dot{\theta}_{i+1} &= \dot{\theta}_i + h\ddot{\theta}(\varphi_i, \dot{\varphi}_i) \\ \\ \varphi_{i+1} &= \varphi_i + h\dot{\varphi}_i \\ \dot{\varphi}_{i+1} &= \dot{\varphi}_i + h\ddot{\varphi}(\varphi_i) \end{aligned}}$$