Desarrollo y simulación de péndulo de masa doble

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Formulación de Euler-Lagrange

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]-\frac{\partial \mathcal{L}}{\partial q}=0 \quad , \quad \mathcal{L}=T-V$$

Coordenadas generalizadas

$$q_1=\theta_1$$
$$q_2=\theta_2$$

$$S_1=2l_1 \quad , \quad S_2=l_2$$
$$x_1=\frac{S_1}{2}\sin\theta_1 \quad , \quad \dot{x}_1=\frac{S_1}{2}\dot{\theta}_1\cos\theta_1$$
$$y_1=-\frac{S_1}{2}\cos\theta_1 \quad , \quad \dot{y}_1=\frac{S_1}{2}\dot{\theta}_1\sin\theta_1$$

$$x_2=2x_1+l_2\sin\theta_2 \rightarrow x_2=S_1\sin\theta_1+l_2\sin\theta_2 \quad , \quad \dot{x}_2=S_1\dot{\theta}_1\cos\theta_1+l_2\dot{\theta}_2\cos\theta_2$$
$$y_2=2y_1-l_2\cos\theta_2 \rightarrow y_2=-S_1\cos\theta_1-l_2\cos\theta_2 \quad , \quad \dot{y}_2=S_1\dot{\theta}_1\sin\theta_1+l_2\dot{\theta}_2\sin\theta_2$$

Energía potencial

$$V=m_1gy_1+m_2gy_2 \rightarrow V=-m_1g\frac{S_1}{2}\cos\theta_1+m_2g(-S_1\cos\theta_1-l_2\cos\theta_2)$$

$$V=-m_1g\frac{S_1}{2}\cos\theta_1-m_2gS_1\cos\theta_1-m_2gl_2\cos\theta_2$$

$$\mu=\frac{1}{2}m_1+m_2$$

$$V=-\mu gS_1\cos\theta_1-m_2gl_2\cos\theta_2$$

Energía cinética

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 (\dot{x}_2 + \dot{y}_2)$$

$$\dot{x}_2^2 = (S_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2)^2 \rightarrow \dot{x}_2^2 = S_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + l_2^2 \dot{\theta}_2^2 \cos^2\theta_2 + 2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2$$

$$\dot{y}_2^2 = (S_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2)^2 \rightarrow \dot{y}_2^2 = S_1^2 \dot{\theta}_1^2 \sin^2\theta_1 + l_2^2 \dot{\theta}_2^2 \sin^2\theta_2 + 2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2$$

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 +$$
$$\frac{1}{2} m_2 \Big(S_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + l_2^2 \dot{\theta}_2^2 \cos^2\theta_2 + 2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 +$$
$$S_1^2 \dot{\theta}_1^2 \sin^2\theta_1 + l_2^2 \dot{\theta}_2^2 \sin^2\theta_2 + 2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2 \Big)$$

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 +$$
$$\frac{1}{2} m_2 \Big(S_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + 2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2 \Big)$$

Energía cinética
Producto de cosenos

$$\cos a \cos b = \frac{1}{2}\left[cos(a+b) + cos(a-b)\right]$$

Producto de senos

$$\sin a \sin b = \frac{1}{2}\left[cos(a-b) - cos(a+b)\right]$$

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}m_2\left(S_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2S_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)$$

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}m_2S_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2S_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$

Lagrangiano

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2}I_1\dot{\theta}_1^2+\frac{1}{2}I_2\dot{\theta}_2^2+\frac{1}{2}m_2S_1^2\dot{\theta}_1^2+\frac{1}{2}m_2l_2^2\dot{\theta}_2^2+m_2S_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)+\mu gS_1\cos\theta_1+m_2gl_2\cos\theta_2$$

Derivadas del Lagrangiano

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -m_2S_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - \mu gS_1\sin\theta_1$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = I_1\dot{\theta}_1 + m_2S_1^2\dot{\theta}_1 + m_2S_1l_2\dot{\theta}_2\cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right] = I_1\ddot{\theta}_1+m_2S_1^2\ddot{\theta}_1+m_2S_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)+m_2S_1l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)-m_2S_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = m_2S_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2gl_2\sin\theta_2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = I_2\dot{\theta}_2 + m_2l_2^2\dot{\theta}_2 + m_2S_1l_2\dot{\theta}_1\cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right] = I_2\ddot{\theta}_2+m_2l_2^2\ddot{\theta}_2+m_2S_1l_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-m_2S_1l_2\dot{\theta}_1^2\sin(\theta_1-\theta_2)+m_2S_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)$$

Ecuaciones de Lagrange

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right] - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0$$

$$I_1\ddot{\theta}_1 + m_2 S_1^2 \ddot{\theta}_1 + m_2 S_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 S_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - m_2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \mu g S_1 \sin\theta_1 = 0$$

$$I_1\ddot{\theta}_1 + m_2 S_1^2 \ddot{\theta}_1 + m_2 S_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 S_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + \mu g S_1 \sin\theta_1 = 0$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right] - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0$$

$$I_2\ddot{\theta}_2 + m_2 l_2^2 \ddot{\theta}_2 + m_2 S_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 S_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 S_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

$$I_2\ddot{\theta}_2 + m_2 l_2^2 \ddot{\theta}_2 + m_2 S_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 S_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

Ecuaciones de movimiento

$$\boxed{\begin{aligned}
&\ddot{\theta}_1 + (I_1 + m_2 S_1^2)^{-1}\left[m_2 S_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 S_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + \mu g S_1 \sin\theta_1\right] = 0 \\
&\ddot{\theta}_2 + (I_2 + m_2 l_2^2)^{-1}\left[m_2 S_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 S_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2\right] = 0
\end{aligned}}$$

Solución numerica por
Metodo de Runge Kutta de orden 4

$\frac{d\theta_1}{dt} = \omega_1$	$\frac{d\omega_1}{dt} = f(\theta_1, \theta_2, \omega_1, \omega_2, t)$
$k_1 = h\omega_1$ $k_2 = h(\omega_1 + \frac{1}{2}c_1)$ $k_3 = h(\omega_1 + \frac{1}{2}c_2)$ $k_4 = h(\omega_1 + c_3)$	$c_1 = hf(\theta_1, \theta_2, \omega_1, \omega_2, t)$ $c_2 = hf(\theta_1 + \frac{1}{2}k_1, \theta_2 + \frac{1}{2}n_1, \omega_1 + \frac{1}{2}c_1, \omega_2 + \frac{1}{2}m_1, t + \frac{1}{2}h)$ $c_3 = hf(\theta_1 + \frac{1}{2}k_2, \theta_2 + \frac{1}{2}n_2, \omega_1 + \frac{1}{2}c_2, \omega_2 + \frac{1}{2}m_2, t + \frac{1}{2}h)$ $c_4 = hf(\theta_1 + k_3, \theta_2 + n_3, \omega_1 + c_3, \omega_2 + m_3, t + h)$
$\theta_1(t+h) = \theta_1(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	$\omega_1(t+h) = \omega_1(t) + \frac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4)$

$\frac{d\theta_2}{dt} = \omega_2$	$\frac{d\omega_2}{dt} = u(\theta_1, \theta_2, \omega_1, \omega_2, t)$
$n_1 = h\omega_2$ $n_2 = h(\omega_2 + \frac{1}{2}m_1)$ $n_3 = h(\omega_2 + \frac{1}{2}m_2)$ $n_4 = h(\omega_2 + m_3)$	$m_1 = hu(\theta_1, \theta_2, \omega_1, \omega_2, t)$ $m_2 = hu(\theta_1 + \frac{1}{2}k_1, \theta_2 + \frac{1}{2}n_1, \omega_1 + \frac{1}{2}c_1, \omega_2 + \frac{1}{2}m_1, t + \frac{1}{2}h)$ $m_3 = hu(\theta_1 + \frac{1}{2}k_2, \theta_2 + \frac{1}{2}n_2, \omega_1 + \frac{1}{2}c_2, \omega_2 + \frac{1}{2}m_2, t + \frac{1}{2}h)$ $m_4 = hu(\theta_1 + k_3, \theta_2 + n_3, \omega_1 + c_3, \omega_2 + m_3, t + h)$
$\theta_2(t+h) = \theta_2(t) + \frac{1}{6}(n_1 + 2n_2 + 2n_3 + n_4)$	$\omega_2(t+h) = \omega_2(t) + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

Cambio de variable

$$\frac{d\theta_1}{dt} = \omega_1 \rightarrow \frac{d^2\theta_1}{dt^2} = \dot{\omega}_1 \quad , \quad \frac{d\theta_2}{dt} = \omega_2 \rightarrow \frac{d^2\theta_2}{dt^2} = \dot{\omega}_2$$

$$\dot{\omega}_1 = -(I_1{+}m_2S_1^2)^{-1}\left[m_2S_1l_2\dot{\omega}_2\cos(\theta_1-\theta_2) + m_2S_1l_2\omega_2^2\sin(\theta_1-\theta_2) + \mu gS_1\sin\theta_1\right]$$
$$\dot{\omega}_2 = -(I_2{+}m_2l_2^2)^{-1}\left[m_2S_1l_2\dot{\omega}_1\cos(\theta_1-\theta_2) - m_2S_1l_2\omega_1^2\sin(\theta_1-\theta_2) + m_2gl_2\sin\theta_2\right]$$

Simplificación y despeje

$$ms = m_2S_2^2$$
$$ml = m_2l_2^2$$
$$msl = m_2S_2l_2$$
$$i_1 = (I_1 + m_2S_1^2)^{-1}$$
$$i_2 = (I_2 + m_2l_2^2)^{-1}$$
$$\varepsilon_1 = m_2S_1l_2\omega_2^2\sin(\theta_1-\theta_2) + \mu gS_1\sin\theta_1$$
$$\varepsilon_2 = -msl\omega_1^2\sin(\theta_1-\theta_2) + m_2gl_2\sin\theta_2$$

$$\dot{\omega}_1 = -i_1\left[msl\dot{\omega}_2\cos(\theta_1-\theta_2) + \varepsilon_1\right]$$
$$\dot{\omega}_2 = -i_2\left[msl\dot{\omega}_1\cos(\theta_1-\theta_2) + \varepsilon_2\right]$$

$$\dot{\omega}_1 = -i_1msl\dot{\omega}_2\cos(\theta_1-\theta_2) - i_1\varepsilon_1$$
$$\dot{\omega}_1 = i_1i_2msl\cos(\theta_1-\theta_2)\left[msl\dot{\omega}_1\cos(\theta_1-\theta_2) + \varepsilon_2\right] - i_1\varepsilon_1$$
$$\dot{\omega}_1 = i_1i_2(msl)^2\dot{\omega}_1\cos^2(\theta_1-\theta_2) + i_1i_2msl\cos(\theta_1-\theta_2)\varepsilon_2 - i_1\varepsilon_1$$
$$\dot{\omega}_1[1 - i_1i_2(msl)^2\cos^2(\theta_1-\theta_2)] = i_1i_2msl\cos(\theta_1-\theta_2)\varepsilon_2 - i_1\varepsilon_1$$

$$\dot{\omega}_1 = \frac{[i_1i_2msl\cos(\theta_1-\theta_2)\varepsilon_2 - i_1\varepsilon_1]}{[1 - i_1i_2(msl)^2\cos^2(\theta_1-\theta_2)]}$$

$$\dot{\omega}_1 = \frac{(I_1 + m_2 S_1^2)^{-1}(I_2 + m_2 l_2^2)^{-1} m_2 S_2 l_2 \cos(\theta_1 - \theta_2)[-msl\omega_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2]}{[1 - (I_1 + m_2 S_1^2)^{-1}(I_2 + m_2 l_2^2)^{-1} m_2^2 S_2^2 l_2^2 \cos^2(\theta_1 - \theta_2)]}$$
$$- \frac{(I_1 + m_2 S_1^2)^{-1}[m_2 S_1 l_2 \omega_2^2 \sin(\theta_1 - \theta_2) + \mu g S_1 \sin\theta_1]}{[1 - (I_1 + m_2 S_1^2)^{-1}(I_2 + m_2 l_2^2)^{-1} m_2^2 S_2^2 l_2^2 \cos^2(\theta_1 - \theta_2)]}$$

Funciones de iteración

$$f(\theta_1, \theta_2, \omega_1, \omega_2, t) = \frac{d\omega_1}{dt}$$
$$f(\theta_1, \theta_2, \omega_1, \omega_2, t) =$$
$$\frac{(I_1 + m_2 S_1^2)^{-1}(I_2 + m_2 l_2^2)^{-1} m_2 S_2 l_2 \cos(\theta_1 - \theta_2)[-msl\omega_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2]}{[1 - (I_1 + m_2 S_1^2)^{-1}(I_2 + m_2 l_2^2)^{-1} m_2^2 S_2^2 l_2^2 \cos^2(\theta_1 - \theta_2)]}$$
$$- \frac{(I_1 + m_2 S_1^2)^{-1}[m_2 S_1 l_2 \omega_2^2 \sin(\theta_1 - \theta_2) + \mu g S_1 \sin\theta_1]}{[1 - (I_1 + m_2 S_1^2)^{-1}(I_2 + m_2 l_2^2)^{-1} m_2^2 S_2^2 l_2^2 \cos^2(\theta_1 - \theta_2)]}$$

$$u(\theta_1, \theta_2, \omega_1, \omega_2, t) = \frac{d\omega_2}{dt}$$
$$u(\theta_1, \theta_2, \omega_1, \omega_2, t) = -(I_2 + m_2 l_2^2)^{-1} \big[m_2 S_1 l_2 \cos(\theta_1 - \theta_2) f(\theta_1, \theta_2, \omega_1, \omega_2, t)$$
$$-m_2 S_1 l_2 \omega_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2\big]$$