Desarrollo y simulación de péndulo simple doble

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Formulación de Euler-Lagrange

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad , \quad \mathcal{L} = T - V$$

Coordenadas generalizadas

$$q_1 = \theta_1$$

$$q_2 = \theta_2$$

$$x_1 = l_1 \sin\theta_1 \quad , \quad \dot{x}_1 = l_1\dot{\theta}_1 \cos\theta_1$$

$$y_1 = -l_1 \cos\theta_1 \quad , \quad \dot{y}_1 = l_1\dot{\theta}_1 \sin\theta_1$$

$$x_2 = x_1 + l_2\sin\theta_2 \rightarrow x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \quad , \quad \dot{x}_2 = l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2$$

$$y_2 = y_1 - l_2\cos\theta_2 \rightarrow y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2 \quad , \quad \dot{y}_2 = l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2$$

Energía potencial

$$V = m_1 g y_1 + m_2 g y_2 \rightarrow V = -m_1 g l_1 \cos\theta_1 + m_2 g(-l_1\cos\theta_1 - l_2\cos\theta_2)$$

$$\mu = m_1 + m_2 \quad , \quad L = \frac{l_2}{l_1}$$

$$V = -g\mu l_1\cos\theta_1 - g m_2 l_2 \cos\theta_2$$

Energía cinética

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1^2 = l_1^2\dot{\theta}_1^2\cos^2\theta_1$$

$$y_1^2 = l_1^2\dot{\theta}_1^2\sin^2\theta_1$$

$$\dot{x}_2^2 = (l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2)^2 \rightarrow \dot{x}_2^2 = l_1^2\dot{\theta}_1^2\cos^2\theta_1 + l_2^2\dot{\theta}_2^2\cos^2\theta_2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2$$

$$\dot{y}_2^2 = (l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2)^2 \rightarrow \dot{y}_2^2 = l_1^2\dot{\theta}_1^2\sin^2\theta_1 + l_2^2\dot{\theta}_2^2\sin^2\theta_2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2$$

$$T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2$$

Producto de cosenos

$$\cos a\cos b = \frac{1}{2}\left[cos(a+b) + cos(a-b)\right]$$

Producto de senos

$$\sin a\sin b = \frac{1}{2}\left[cos(a-b) - cos(a+b)\right]$$

$$T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

Lagrangiano

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + g\mu l_1 \cos\theta_1 + g m_2 l_2 \cos\theta_2$$

Derivadas del Lagrangiano

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g\mu l_1 \sin\theta_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \mu l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right] = \mu l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g m_2 l_2 \sin\theta_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right] = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$

Ecuaciones de Lagrange

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right] - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0$$

$$\mu l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g\mu l_1 \sin\theta_1 = 0$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right] - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0$$

$$l_2^2 \ddot{\theta}_2 + l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g l_2 \sin\theta_2 = 0$$

$$\mu = m_1 + m2 \quad , \quad L = \frac{l_2}{l_1}$$

Ecuaciones de movimiento

$$\boxed{\begin{aligned} \ddot{\theta}_1 + \tfrac{m_2}{\mu} L \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + \tfrac{m_2}{\mu} L \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + \tfrac{g}{l_1} \sin\theta_1 = 0 \\ \ddot{\theta}_2 + L^{-1} \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - L^{-1} \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + \tfrac{g}{l_2} \sin\theta_2 = 0 \end{aligned}}$$

Solución numerica por
Metodo de Runge Kutta de orden 4

$\frac{d\theta_1}{dt} = \omega_1$	$\frac{d\omega_1}{dt} = f(\theta_1, \theta_2, \omega_1, \omega_2, t)$
$k_1 = h\omega_1$ $k_2 = h(\omega_1 + \frac{1}{2}c_1)$ $k_3 = h(\omega_1 + \frac{1}{2}c_2)$ $k_4 = h(\omega_1 + c_3)$	$c_1 = hf(\theta_1, \theta_2, \omega_1, \omega_2, t)$ $c_2 = hf(\theta_1 + \frac{1}{2}k_1, \theta_2 + \frac{1}{2}n_1, \omega_1 + \frac{1}{2}c_1, \omega_2 + \frac{1}{2}m_1, t + \frac{1}{2}h)$ $c_3 = hf(\theta_1 + \frac{1}{2}k_2, \theta_2 + \frac{1}{2}n_2, \omega_1 + \frac{1}{2}c_2, \omega_2 + \frac{1}{2}m_2, t + \frac{1}{2}h)$ $c_4 = hf(\theta_1 + k_3, \theta_2 + n_3, \omega_1 + c_3, \omega_2 + m_3, t + h)$
$\theta_1(t+h) = \theta_1(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	$\omega_1(t+h) = \omega_1(t) + \frac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4)$

$\frac{d\theta_2}{dt} = \omega_2$	$\frac{d\omega_2}{dt} = u(\theta_1, \theta_2, \omega_1, \omega_2, t)$
$n_1 = h\omega_2$ $n_2 = h(\omega_2 + \frac{1}{2}m_1)$ $n_3 = h(\omega_2 + \frac{1}{2}m_2)$ $n_4 = h(\omega_2 + m_3)$	$m_1 = hu(\theta_1, \theta_2, \omega_1, \omega_2, t)$ $m_2 = hu(\theta_1 + \frac{1}{2}k_1, \theta_2 + \frac{1}{2}n_1, \omega_1 + \frac{1}{2}c_1, \omega_2 + \frac{1}{2}m_1, t + \frac{1}{2}h)$ $m_3 = hu(\theta_1 + \frac{1}{2}k_2, \theta_2 + \frac{1}{2}n_2, \omega_1 + \frac{1}{2}c_2, \omega_2 + \frac{1}{2}m_2, t + \frac{1}{2}h)$ $m_4 = hu(\theta_1 + k_3, \theta_2 + n_3, \omega_1 + c_3, \omega_2 + m_3, t + h)$
$\theta_2(t+h) = \theta_2(t) + \frac{1}{6}(n_1 + 2n_2 + 2n_3 + n_4)$	$\omega_2(t+h) = \omega_2(t) + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

Cambio de variable

$$\frac{d\theta_1}{dt} = \omega_1 \rightarrow \frac{d^2\theta_1}{dt^2} = \dot{\omega}_1 \quad , \quad \frac{d\theta_2}{dt} = \omega_2 \rightarrow \frac{d^2\theta_2}{dt^2} = \dot{\omega}_2$$

$$\dot{\omega}_1 = -\frac{m_2}{\mu} L \cos(\theta_1 - \theta_2)\dot{\omega}_2 - \frac{m_2}{\mu} L \sin(\theta_1 - \theta_2)\omega_2^2 - \frac{g}{l_1}\sin\theta_1$$

$$\dot{\omega}_2 = -L^{-1}\cos(\theta_1 - \theta_2)\dot{\omega}_1 + L^{-1}\sin(\theta_1 - \theta_2)\omega_1^2 - \frac{g}{l_2}\sin\theta_2$$

$$\dot{\omega}_1 = \frac{\frac{g}{l_1}\left[\frac{m_2}{\mu}\cos(\theta_1 - \theta_2)\sin\theta_2 - \sin\theta_1\right] - \frac{m_2}{\mu}\sin(\theta_1 - \theta_2)\left[\cos(\theta_1 - \theta_2)\omega_1^2 + L\omega_2^2\right]}{\left[1 - \frac{m_2}{\mu}\cos^2(\theta_1 - \theta_2)\right]}$$

Funciones de iteración

$$f(\theta_1, \theta_2, \omega_1, \omega_2, t) = \frac{d\omega_1}{dt}$$
$$f(\theta_1, \theta_2, \omega_1, \omega_2, t) = \frac{\frac{g}{l_1}\left[\frac{m_2}{\mu}\cos(\theta_1-\theta_2)\sin\theta_2 - \sin\theta_1\right] - \frac{m_2}{\mu}\sin(\theta_1-\theta_2)\left[\cos(\theta_1-\theta_2)\omega_1^2 + L\omega_2^2\right]}{\left[1-\frac{m_2}{\mu}\cos^2(\theta_1-\theta_2)\right]}$$

$$u(\theta_1, \theta_2, \omega_1, \omega_2, t) = \frac{d\omega_2}{dt}$$
$$u(\theta_1, \theta_2, \omega_1, \omega_2, t) = -L^{-1}\cos(\theta_1-\theta_2) f(\theta_1, \theta_2, \omega_1, \omega_2, t) + L^{-1}\sin(\theta_1-\theta_2)\omega_1^2 - \frac{g}{l_2}\sin\theta_2$$