Desarrollo y simulación de péndulo simple

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Formulación de Newton

$$\sum \vec{F} = m\vec{a}$$

Sumatoria de Fuerzas

$$\vec{W} \equiv \text{peso}$$
$$\vec{T} \equiv \text{tensión}$$

$$\sum \vec{F} = \vec{T} + \vec{W}$$

Descomposición de fuerzas(eje normal y eje tangente)

$$\sum \vec{F} = \sum \vec{F}_n + \sum \vec{F}_t \quad , \quad \sum F = \left|\sum \vec{F}\right|$$
$$\vec{W} = \vec{W}_n + \vec{W}_t \quad , \quad W = |\vec{W}| \rightarrow W = mg$$
$$\vec{T} = |\vec{T}|\hat{n} \quad , \quad T = |\vec{T}| \rightarrow \vec{T} = T\hat{n}$$

Fuerzas en el Eje normal

$$\vec{W}_n = |\vec{W}|\vec{n} \rightarrow \vec{W}_n = -W\cos\theta\hat{n}$$
$$\vec{W}_n = -mg\cos\theta\hat{n}$$

$$\sum \vec{F}_n = 0 \rightarrow \vec{T} + \vec{W}_n = 0$$
$$\vec{T} = mg\cos\theta\hat{n} \rightarrow T = mg\cos\theta$$

Fuerzas en el Eje tangente

$$\vec{W}_t = |\vec{W}|\vec{t} \rightarrow \vec{W}_t = -W\sin\theta\hat{t}$$
$$\sum \vec{F}_t = \vec{W}_t \rightarrow \sum \vec{F}_t = -mg\sin\theta\hat{t} \rightarrow \sum F_t = -mg\sin\theta$$

Ecuación de movimiento

$$\sum \vec{F} = \sum \cancelto{0}{\vec{F}_n} + \sum \vec{F}_t \rightarrow \sum \vec{F} = \sum \vec{F}_t$$

$$\sum \vec{F}_t = m\vec{a}_t$$

$$m\vec{a}_t = -mg\sin\theta\hat{t} \rightarrow a_t = -g\sin\theta$$

$$\vec{a}_t = \vec{r} \times \vec{\omega} \quad , \quad |\vec{r}| = l \equiv \text{cte.} \rightarrow a_t = l\omega$$

$$a_t = l\frac{d^2\theta}{dt^2} \rightarrow l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$\boxed{\therefore \quad \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0}$$

Formulación de Euler-Lagrange

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad , \quad \mathcal{L} = T - V$$

Coordenadas generalizadas

$$q_1 = \theta$$

$$x = l\sin\theta \quad , \quad \dot{x} = l\dot{\theta}\cos\theta$$

$$y = -l\cos\theta \quad , \quad \dot{y} = l\dot{\theta}\sin\theta$$

Energía potencial

$$V = mgy \rightarrow V = -mgl\cos\theta$$

Energía cinética

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$T = \frac{1}{2}m[(l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2] \rightarrow T = \frac{1}{2}m[l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta]$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

Lagrangiano

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

Derivadas del Lagrangiano

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] = ml^2\ddot{\theta}$$

Ecuación de movimiento

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \rightarrow l^2\ddot{\theta} + gl\sin\theta = 0$$

$$\boxed{\therefore \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

Solución analítica aproximada para ángulos pequeños

$$\theta \ll \frac{\pi}{2} \rightarrow \sin\theta \approx \theta$$

$$\therefore \quad \frac{g}{l}\sin\theta \approx \frac{g}{l}\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \rightarrow \quad \ddot{\theta} + \frac{g}{l}\theta = 0$$

Ecuación no lineal de segundo orden

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\theta = e^{\beta t}$$
$$\dot{\theta} = \beta e^{\beta t}$$
$$\ddot{\theta} = \beta^2 e^{\beta t}$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \rightarrow \beta^2 e^{\beta t} + \omega^2 e^{\beta t} = 0$$
$$\beta^2 = -\omega^2 \rightarrow \beta = \sqrt{-\omega^2} \rightarrow \beta = \omega\sqrt{-1}$$
$$\beta = \pm i\omega$$

$$\theta = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$
$$\theta = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$
$$e^{\pm ik} = \cos k \pm i \sin k$$
$$\theta = C_1 \cos(\omega t) + iC_1 \sin(\omega t) + C_2 \cos(\omega t) - iC_2 \sin(\omega t)$$
$$\theta = (C_1 + C_2)\cos(\omega t) + (C_1 - C_2) i \sin(\omega t)$$
$$A = C_1 + C_2 \quad , \quad B = i(C_1 - C_2)$$

$$\theta = A\cos(\omega t) + B\sin(\omega t)$$
$$\dot{\theta} = -A\omega \sin(\omega t) + B\omega \cos(\omega t)$$

$$\theta(t=0) = \theta_0 \quad , \quad \theta(t=0) = A\cancelto{1}{\cos(\omega(0))} + B\cancelto{0}{\sin(\omega(0))} \rightarrow A = \theta_0$$
$$\dot{\theta}(t=0) = 0 \quad , \quad \dot{\theta}(t=0) = -A\omega\cancelto{0}{\sin(\omega(0))} + B\omega\cancelto{1}{\cos(\omega(0))} \rightarrow B = 0$$

$$\therefore \quad \theta = Acos(\phi + \omega t)$$

$$\boxed{\theta(t) = Acos\left(\phi + \sqrt{\frac{g}{l}}t\right)}$$

Solución númerica por método de Euler

$$x_{i+1} = x_i + h\dot{x}(x_i, t_i)$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$
$$\omega = \dot{\theta} \rightarrow \dot{\omega} = \ddot{\theta}$$
$$\dot{\omega} = -\frac{g}{l}\sin\theta$$

$$\boxed{\begin{aligned} t_{i+1} &= t_i + h \\ \theta_{i+1} &= \theta_i + h\omega_{i+1} \\ \omega_{i+1} &= \omega_i - h\tfrac{g}{l}\sin\theta_i \end{aligned}}$$