Potencial doble Delta de Dirac

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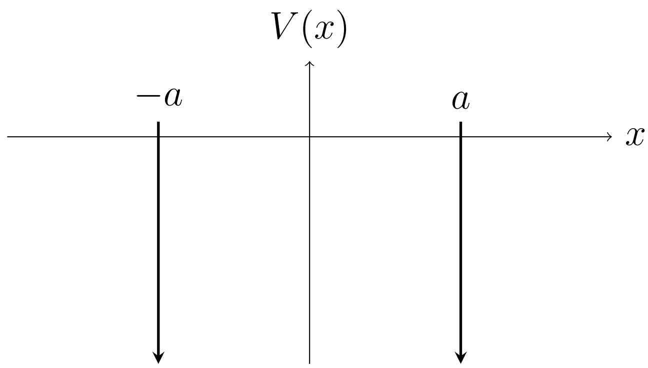
Potencial

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)]$$



Ecuación de Schrödinger Independiente del tiempo

$$\hat{H}\psi = E\psi \quad \rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha[\delta(x+a) + \delta(x-a)]\psi(x) = E\psi(x)$$

Regiones sin potencial

$$V(x) = 0 \quad : \quad |x| \neq a$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x)$$

Energía menor al máximo $E < 0$

$$k \equiv \frac{\sqrt{-2mE}}{\hbar} \quad \rightarrow \quad k \in \mathbb{R} \quad : \quad E < 0$$

$$\boxed{\frac{d^2\psi}{dx^2} - k^2\psi(x) = 0}$$

Región I: $x < -a$

$$\psi_I(x) = A' e^{-kx} + A e^{kx}$$

$$\lim_{x \to -\infty} A' e^{-kx} \to \infty \quad \to \quad A' = 0$$

$$\therefore \quad \psi_I(x) = A e^{kx}$$

Región II: $-a < x < a$

$$\psi_{II}(x) = C e^{-kx} + D e^{kx}$$

Región III: $x > a$

$$\psi_{III}(x) = B e^{-kx} + B' e^{kx}$$

$$\lim_{x \to \infty} B' e^{kx} \to \infty \quad \to \quad B' = 0$$

$$\therefore \quad \psi_{III}(x) = B e^{-kx}$$

$$\boxed{\psi(x) = \begin{cases} A e^{kx} & : \quad x < -a \\ C e^{-kx} + D e^{kx} & : \quad -a < x < a \\ B e^{-kx} & : \quad x > a \end{cases}}$$

Continuidad de la función en $-a$

$$\psi_I(-a) = \psi_{II}(-a)$$

$$Ae^{-ka} = Ce^{ka} + De^{-ka} \quad \rightarrow \quad \boxed{A = D + Ce^{2ka}}$$

Continuidad de la función en a

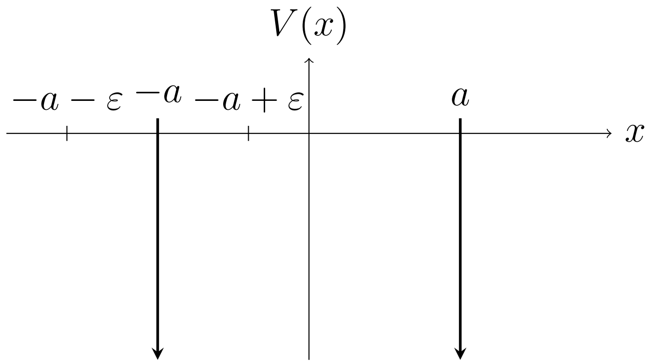
$$\psi_{II}(a) = \psi_{III}(a)$$

$$Ce^{-ka} + De^{ka} = Be^{-ka} \quad \rightarrow \quad \boxed{B = C + De^{2ka}}$$

Derivada de $\psi(x)$

$$\boxed{\frac{d\psi}{dx} = \begin{cases} kAe^{kx} & : \quad x < -a \\ -kCe^{-kx} + kDe^{kx} & : \quad -a < x < a \\ -kBe^{-kx} & : \quad x > a \end{cases}}$$

Integrando alrededor de $-a$



$$-\frac{\hbar^2}{2m}\int_{-a-\varepsilon}^{-a+\varepsilon}\frac{d^2\psi}{dx^2}dx-\alpha\int_{-a-\varepsilon}^{-a+\varepsilon}[\delta(x+a)+\delta(x-a)]\psi(x)dx=E\int_{-a-\varepsilon}^{-a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\int_{-a-\varepsilon}^{-a+\varepsilon}\frac{d^2\psi}{dx^2}dx-\alpha\int_{-a-\varepsilon}^{-a+\varepsilon}\delta(x+a)\psi(x)dx-\alpha\cancelto{0}{\int_{-a-\varepsilon}^{-a+\varepsilon}\delta(x-a)\psi(x)dx}=E\int_{-a-\varepsilon}^{-a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\int_{-a-\varepsilon}^{-a+\varepsilon}\frac{d^2\psi}{dx^2}dx-\alpha\int_{-a-\varepsilon}^{-a+\varepsilon}\delta(x+a)\psi(x)dx=E\int_{-a-\varepsilon}^{-a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\left|\frac{d\psi}{dx}\right|_{-a-\varepsilon}^{-a+\varepsilon}-\alpha\psi(-a)=E\int_{-a-\varepsilon}^{-a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi_{II}}{dx}(-a+\varepsilon)-\frac{d\psi_I}{dx}(-a-\varepsilon)\right]-\alpha\psi(-a)=E\int_{-a-\varepsilon}^{-a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\lim_{\varepsilon\to 0}\left[\frac{d\psi_{II}}{dx}(-a+\varepsilon)-\frac{d\psi_I}{dx}(-a-\varepsilon)\right]-\alpha\psi(-a)=0$$

$$-\frac{\hbar^2}{2m}\lim_{\varepsilon\to 0}\left[-kCe^{-k(-a+\varepsilon)}+kDe^{k(-a+\varepsilon)}-kAe^{k(-a-\varepsilon)}\right]-\alpha Ae^{-ka}=0$$

$$-\frac{\hbar^2}{2m}\left(-kCe^{ka}+kDe^{-ka}-kAe^{-ka}\right)-\alpha Ae^{-ka}=0$$

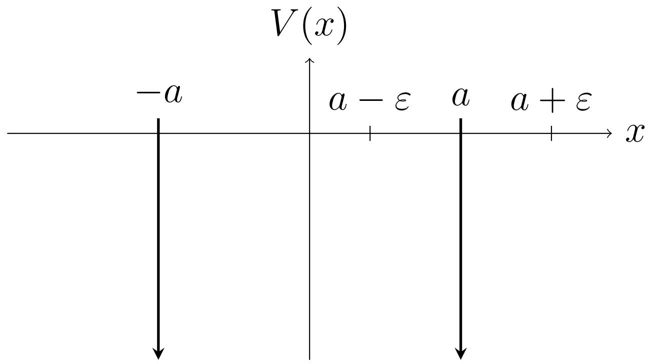
$$-Ce^{ka}+De^{-ka}-Ae^{-ka}+\frac{2m\alpha}{k\hbar^2}Ae^{-ka}=0$$

$$-Ce^{ka}+\left(D-A+\frac{2m\alpha}{k\hbar^2}A\right)e^{-ka}=0$$

$$\left[D+\left(\frac{2m\alpha}{k\hbar^2}-1\right)A\right]e^{-ka}=Ce^{ka}$$

$$\boxed{D+\left(\frac{2m\alpha}{k\hbar^2}-1\right)A=Ce^{2ka}}$$

Integrando alrededor de a



$$-\frac{\hbar^2}{2m}\int_{a-\varepsilon}^{a+\varepsilon}\frac{d^2\psi}{dx^2}dx-\alpha\int_{a-\varepsilon}^{a+\varepsilon}[\delta(x+a)+\delta(x-a)]\psi(x)dx=E\int_{a-\varepsilon}^{a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\int_{a-\varepsilon}^{a+\varepsilon}\frac{d^2\psi}{dx^2}dx-\alpha\cancelto{0}{\int_{a-\varepsilon}^{a+\varepsilon}\delta(x+a)\psi(x)dx}-\alpha\int_{a-\varepsilon}^{a+\varepsilon}\delta(x-a)\psi(x)dx=E\int_{a-\varepsilon}^{a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\int_{a-\varepsilon}^{a+\varepsilon}\frac{d^2\psi}{dx^2}dx-\alpha\int_{a-\varepsilon}^{a+\varepsilon}\delta(x-a)\psi(x)dx=E\int_{a-\varepsilon}^{a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\left|\frac{d\psi}{dx}\right|_{a-\varepsilon}^{a+\varepsilon}-\alpha\psi(a)=E\int_{a-\varepsilon}^{a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi_{III}}{dx}(a+\varepsilon)-\frac{d\psi_{II}}{dx}(a-\varepsilon)\right]-\alpha\psi(a)=E\int_{a-\varepsilon}^{a+\varepsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\lim_{\varepsilon\to 0}\left[\frac{d\psi_{III}}{dx}(a+\varepsilon)-\frac{d\psi_{II}}{dx}(a-\varepsilon)\right]-\alpha\psi(a)=0$$

$$-\frac{\hbar^2}{2m}\lim_{\varepsilon\to 0}\left[-kBe^{-k(a+\varepsilon)}-\left(-kCe^{-k(a-\varepsilon)}+kDe^{k(a-\varepsilon)}\right)\right]-\alpha Be^{-ka}=0$$

$$-\frac{\hbar^2}{2m}\left(-kBe^{-ka}+kCe^{-ka}-kDe^{ka}\right)-\alpha Be^{-ka}=0$$

$$-Be^{-ka}+Ce^{-ka}-De^{ka}+\frac{2m\alpha}{k\hbar^2}Be^{-ka}=0$$

$$-De^{ka}+\left(C-B+\frac{2m\alpha}{k\hbar^2}B\right)e^{-ka}=0$$

$$\left[C+\left(\frac{2m\alpha}{k\hbar^2}-1\right)B\right]e^{-ka}=De^{ka}$$

$$\boxed{C+\left(\frac{2m\alpha}{k\hbar^2}-1\right)B=De^{2ka}}$$

Sistema de 4 ecuaciónes

$$A = D + Ce^{2ka}$$

$$B = C + De^{2ka}$$

$$D + \left(\frac{2m\alpha}{k\hbar^2} - 1\right) A = Ce^{2ka}$$

$$C + \left(\frac{2m\alpha}{k\hbar^2} - 1\right) B = De^{2ka}$$

$$D + \left(\frac{2m\alpha}{k\hbar^2} - 1\right)\left(D + Ce^{2ka}\right) = Ce^{2ka} \quad \rightarrow \quad \frac{2m\alpha}{k\hbar^2}\left(D + Ce^{2ka}\right) = 2Ce^{2ka}$$

$$\boxed{\frac{m\alpha}{k\hbar^2}\left(D + Ce^{2ka}\right) = Ce^{2ka}}$$

$$C + \left(\frac{2m\alpha}{k\hbar^2} - 1\right)\left(C + De^{2ka}\right) = De^{2ka} \quad \rightarrow \quad \frac{2m\alpha}{k\hbar^2}\left(C + De^{2ka}\right) = 2De^{2ka}$$

$$\boxed{\frac{m\alpha}{k\hbar^2}\left(C + De^{2ka}\right) = De^{2ka}}$$

$$\frac{m\alpha}{k\hbar^2}\left(D + Ce^{2ka}\right) = Ce^{2ka} \quad \rightarrow \quad \frac{m\alpha}{k\hbar^2} D = \left(1 - \frac{m\alpha}{k\hbar^2}\right) Ce^{2ka}$$

$$D = \frac{k\hbar^2}{m\alpha}\left(1 - \frac{m\alpha}{k\hbar^2}\right) Ce^{2ka} = \left(\frac{k\hbar^2}{m\alpha} - 1\right) Ce^{2ka}$$

$$\frac{m\alpha}{k\hbar^2}\left(C + De^{2ka}\right) = De^{2ka} \quad \rightarrow \quad \left(\frac{m\alpha}{k\hbar^2} - 1\right) De^{2ka} + \frac{m\alpha}{k\hbar^2}C = 0$$

$$\left(\frac{m\alpha}{k\hbar^2} - 1\right)\left(\frac{k\hbar^2}{m\alpha} - 1\right) Ce^{4ka} + \frac{m\alpha}{k\hbar^2}C = 0 \quad \rightarrow \quad -\left(\frac{m\alpha}{k\hbar^2} + \frac{k\hbar^2}{m\alpha} - 2\right) e^{4ka} + \frac{m\alpha}{k\hbar^2} = 0$$

$$\frac{m\alpha}{k\hbar^2}\left(\frac{m\alpha}{k\hbar^2} + \frac{k\hbar^2}{m\alpha} - 2\right) e^{4ka} - \left(\frac{m\alpha}{k\hbar^2}\right)^2 = 0 \quad \rightarrow \quad \left[\left(\frac{m\alpha}{k\hbar^2}\right)^2 + 1 - \frac{2m\alpha}{k\hbar^2}\right] e^{4ka} - \left(\frac{m\alpha}{k\hbar^2}\right)^2 = 0$$

$$\left(\frac{m\alpha}{k\hbar^2} - 1\right)^2 e^{4ka} - \left(\frac{m\alpha}{k\hbar^2}\right)^2 = 0 \quad \rightarrow \quad \left(\frac{m\alpha}{k\hbar^2} - 1\right)^2 e^{4ka} = \left(\frac{m\alpha}{k\hbar^2}\right)^2$$

$$\left(\frac{m\alpha}{k\hbar^2} - 1\right) e^{2ka} = \pm\frac{m\alpha}{k\hbar^2} \quad \rightarrow \quad \left(\frac{m\alpha}{\hbar^2} - k\right) e^{2ka} = \pm\frac{m\alpha}{\hbar^2} \quad \rightarrow \quad k = \frac{m\alpha}{\hbar^2} \mp \frac{m\alpha}{\hbar^2} e^{-2ka}$$

$$\boxed{k_\pm = \frac{m\alpha}{\hbar^2}\left(1 \pm e^{-2k_\pm a}\right)}$$

Normalización

$$\psi_{\pm}(x) = \begin{cases} Ae^{k_{\pm}x} & : \quad x < -a \\ Ce^{-k_{\pm}x} + De^{k_{\pm}x} & : \quad -a < x < a \\ Be^{-k_{\pm}x} & : \quad x > a \end{cases}$$

$$\int_{-\infty}^{\infty} |\psi_{\pm}(x)|^2 dx = 1$$

$$\int_{-\infty}^{-a} |\psi_{\pm I}|^2 dx + \int_{-a}^{a} |\psi_{\pm II}|^2 dx + \int_{a}^{\infty} |\psi_{\pm III}|^2 dx = 1$$

$$A^2 \int_{-\infty}^{-a} e^{2k_{\pm}x} dx + \int_{-a}^{a} \left(Ce^{-k_{\pm}x} + De^{k_{\pm}x}\right)^2 dx + B^2 \int_{a}^{\infty} e^{-2k_{\pm}x} dx = 1$$

$$A^2 \int_{-\infty}^{-a} e^{2k_{\pm}x} dx + C^2 \int_{-a}^{a} e^{-2k_{\pm}x} dx + D^2 \int_{-a}^{a} e^{2k_{\pm}x} dx + 2CD \int_{-a}^{a} dx + B^2 \int_{a}^{\infty} e^{-2k_{\pm}x} dx = 1$$

$$\frac{A^2}{2k_{\pm}} \left|e^{2k_{\pm}x}\right|_{-\infty}^{-a} - \frac{C^2}{2k_{\pm}} \left|e^{-2k_{\pm}x}\right|_{-a}^{a} + \frac{D^2}{2k_{\pm}} \left|e^{2k_{\pm}x}\right|_{-a}^{a} + 2CD|x|_{-a}^{a} - \frac{B^2}{2k_{\pm}} \left|e^{-2k_{\pm}x}\right|_{a}^{\infty} = 1$$

$$A^2 \left|e^{2k_{\pm}x}\right|_{-\infty}^{-a} - C^2 \left|e^{-2k_{\pm}x}\right|_{-a}^{a} + D^2 \left|e^{2k_{\pm}x}\right|_{-a}^{a} + 4k_{\pm}CD|x|_{-a}^{a} - B^2 \left|e^{-2k_{\pm}x}\right|_{a}^{\infty} = 2k_{\pm}$$

$$A^2 e^{-2k_{\pm}a} - C^2 \left(e^{-2k_{\pm}a} - e^{2k_{\pm}a}\right) + D^2 \left(e^{2k_{\pm}a} - e^{-2k_{\pm}a}\right) + 8k_{\pm}aCD + B^2 e^{-2k_{\pm}a} = 2k_{\pm}$$

$$\left(A^2 + B^2\right) e^{-2k_{\pm}a} + \left(C^2 + D^2\right) \left(e^{2k_{\pm}a} - e^{-2k_{\pm}a}\right) + 8k_{\pm}aCD = 2k_{\pm}$$

Solucón para k_+

$$k_+ = \frac{m\alpha}{\hbar^2}\left(1 + e^{-2k_+a}\right)$$

$$A = B \quad , \quad C = D$$

$$C = \frac{A}{1 + e^{2k_+a}}$$

$$2A^2e^{-2k_+a} + 2C^2\left(e^{2k_+a} - e^{-2k_+a}\right) + 8k_+aC^2 = 2k_+$$

$$A^2e^{-2k_+a} + C^2\left(e^{2k_+a} - e^{-2k_+a}\right) + 4k_+aC^2 = k_+$$

$$A^2e^{-2k_+a} + \frac{A^2}{(1 + e^{2k_+a})^2}\left(e^{2k_+a} - e^{-2k_+a}\right) + \frac{4k_+aA^2}{(1 + e^{2k_+a})^2} = k_+$$

$$A^2e^{-2k_+a}(1 + e^{2k_+a})^2 + A^2\left(e^{2k_+a} - e^{-2k_+a}\right) + 4k_+aA^2 = k_+(1 + e^{2k_+a})^2$$

$$A^2(e^{-k_+a} + e^{k_+a})^2 + A^2\left(e^{2k_+a} - e^{-2k_+a}\right) + 4k_+aA^2 = k_+(1 + e^{2k_+a})^2$$

$$A^2(e^{-2k_+a} + e^{2k_+a} + 2) + A^2\left(e^{2k_+a} - e^{-2k_+a}\right) + 4k_+aA^2 = k_+(1 + e^{2k_+a})^2$$

$$A^2\left[(e^{-2k_+a} + e^{2k_+a} + 2) + \left(e^{2k_+a} - e^{-2k_+a}\right) + 4k_+a\right] = k_+(1 + e^{2k_+a})^2$$

$$A^2\left(2e^{2k_+a} + 2 + 4k_+a\right) = k_+(1 + e^{2k_+a})^2 \quad \rightarrow \quad A^2 = \frac{k_+(1 + e^{2k_+a})^2}{2\left(e^{2k_+a} + 2k_+a + 1\right)}$$

$$A \in \mathbb{R} \quad , \quad A > 0$$

$$\boxed{A_+ = \sqrt{\frac{k_+(1 + e^{2k_+a})^2}{2\left(e^{2k_+a} + 2k_+a + 1\right)}}}$$

Solucón para k_-

$$k_- = \frac{m\alpha}{\hbar^2}\left(1 + e^{-2k_-a}\right)$$

$$A = -B \quad , \quad C = -D$$

$$C = -\frac{A}{1 - e^{2k_-a}}$$

$$2A^2 e^{-2k_-a} + 2C^2\left(e^{2k_-a} - e^{-2k_-a}\right) - 8k_-aC^2 = 2k_-$$

$$A^2 e^{-2k_-a} + C^2\left(e^{2k_-a} - e^{-2k_-a}\right) - 4k_-aC^2 = k_-$$

$$A^2 e^{-2k_-a} + \frac{A^2}{(1 - e^{2k_-a})^2}\left(e^{2k_-a} - e^{-2k_-a}\right) - \frac{4k_-aA^2}{(1 - e^{2k_-a})^2} = k_-$$

$$A^2 e^{-2k_-a}(1 - e^{2k_-a})^2 + A^2\left(e^{2k_-a} - e^{-2k_-a}\right) - 4k_-aA^2 = k_-(1 - e^{2k_-a})^2$$

$$A^2(e^{-k_-a} - e^{k_-a})^2 + A^2\left(e^{2k_-a} - e^{-2k_-a}\right) - 4k_-aA^2 = k_-(1 - e^{2k_-a})^2$$

$$A^2(e^{-2k_-a} + e^{2k_-a} - 2) + A^2\left(e^{2k_-a} - e^{-2k_-a}\right) - 4k_-aA^2 = k_-(1 - e^{2k_-a})^2$$

$$A^2\left[(e^{-2k_-a} + e^{2k_-a} - 2) + \left(e^{2k_-a} - e^{-2k_-a}\right) - 4k_-a\right] = k_-(1 - e^{2k_-a})^2$$

$$A^2\left(2e^{2k_-a} - 2 - 4k_-a\right) = k_-(1 - e^{2k_-a})^2 \quad \rightarrow \quad A^2 = \frac{k_-(1 - e^{2k_-a})^2}{2\left(e^{2k_-a} - 2k_-a - 1\right)}$$

$$A \in \mathbb{R} \quad , \quad A > 0$$

$$\boxed{A_- = \sqrt{\frac{k_-(1 - e^{2k_-a})^2}{2\left(e^{2k_-a} - 2k_-a - 1\right)}}}$$

Función de onda

$$\psi_\pm(x) = \begin{cases} Ae^{k_\pm x} & : \quad x < -a \\ \frac{\pm A}{1\pm e^{2k_\pm a}}\left(e^{-k_\pm x} \pm e^{k_\pm x}\right) & : \quad -a < x < a \\ \pm Ae^{-k_\pm x} & : \quad x > a \end{cases} \quad , \quad E_\pm = -\frac{k_\pm^2 \hbar^2}{2m}$$

$$k_\pm = \frac{m\alpha}{\hbar^2}\left(1 \pm e^{-2k_\pm a}\right) \quad , \quad A_\pm = \sqrt{\frac{k_\pm(1\pm e^{2k_\pm a})^2}{2\left(e^{2k_\pm a} \pm 2k_\pm a \pm 1\right)}}$$

Solución numérica

$$\beta = \frac{m}{\hbar^2}$$

$$k_{\pm} = \alpha\beta\left(1 \pm e^{-2k_{\pm}a}\right) \quad \rightarrow \quad k_{\pm} - \alpha\beta\left(1 \pm e^{-2k_{\pm}a}\right) = 0$$

$$E_{\pm} = -\frac{k_{\pm}^2}{2\beta}$$

Método de Newton Raphson

$$f(k_{\pm}) = k_{\pm} - \alpha\beta\left(1 \pm e^{-2k_{\pm}a}\right) \quad \rightarrow \quad f'(k_{\pm}) = 1 \pm 2a\alpha\beta e^{-2k_{\pm}a}$$

$$k_{\pm(n+1)} = k_{\pm(n)} - \frac{f\left(k_{\pm(n)}\right)}{f'\left(k_{\pm(n)}\right)}$$

Solución numérica