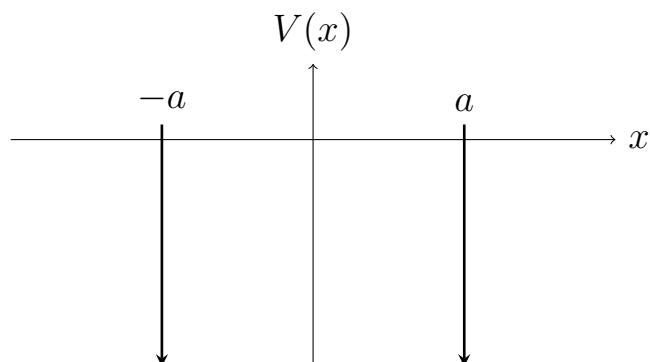


Potencial doble Delta de Dirac

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Potencial

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)]$$



Ecuación de Schrödinger Independiente del tiempo

$$\hat{H}\psi = E\psi \quad \rightarrow \quad -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - \alpha[\delta(x+a) + \delta(x-a)]\psi(x) = E\psi(x)$$

Regiones sin potencial

$$V(x) = 0 \quad : \quad |x| \neq a$$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi(x)$$

Energía menor al máximo  $E < 0$

$$k \equiv \frac{\sqrt{-2mE}}{\hbar} \quad \rightarrow \quad k \in \mathbb{R} \quad : \quad E < 0$$

$$\boxed{\frac{d^2\psi}{dx^2} - k^2\psi(x) = 0}$$

Región I:  $x < -a$

$$\psi_I(x) = A'e^{-kx} + Ae^{kx}$$

$$\lim_{x \rightarrow -\infty} A'e^{-kx} \rightarrow \infty \quad \rightarrow \quad A' = 0$$

$$\therefore \quad \psi_I(x) = Ae^{kx}$$

Región II:  $-a < x < a$

$$\psi_{II}(x) = Ce^{-kx} + De^{kx}$$

Región III:  $x > a$

$$\psi_{III}(x) = Be^{-kx} + B'e^{kx}$$

$$\lim_{x \rightarrow \infty} B'e^{kx} \rightarrow \infty \quad \rightarrow \quad B' = 0$$

$$\therefore \quad \psi_{III}(x) = Be^{-kx}$$

$$\psi(x) = \begin{cases} Ae^{kx} & : \quad x < -a \\ Ce^{-kx} + De^{kx} & : \quad -a < x < a \\ Be^{-kx} & : \quad x > a \end{cases}$$

Continuidad de la función en  $-a$

$$\psi_I(-a) = \psi_{II}(-a)$$

$$Ae^{-ka} = Ce^{ka} + De^{-ka} \quad \rightarrow \quad \boxed{A = D + Ce^{2ka}}$$

Continuidad de la función en  $a$

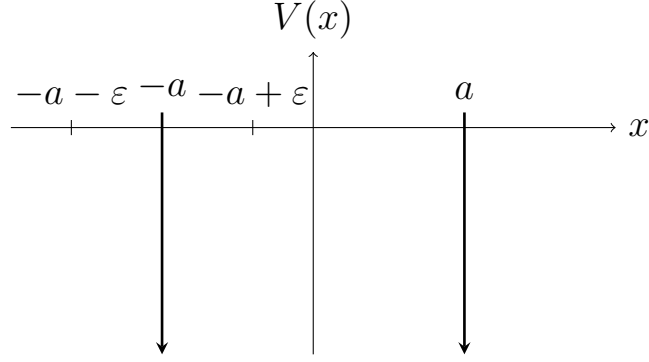
$$\psi_{II}(a) = \psi_{III}(a)$$

$$Ce^{-ka} + De^{ka} = Be^{-ka} \quad \rightarrow \quad \boxed{B = C + De^{2ka}}$$

Derivada de  $\psi(x)$

$$\frac{d\psi}{dx} = \begin{cases} kAe^{kx} & : \quad x < -a \\ -kCe^{-kx} + kDe^{kx} & : \quad -a < x < a \\ -kB e^{-kx} & : \quad x > a \end{cases}$$

Integrando alrededor de  $-a$



$$-\frac{\hbar^2}{2m} \int_{-a-\varepsilon}^{-a+\varepsilon} \frac{d^2\psi}{dx^2} dx - \alpha \int_{-a-\varepsilon}^{-a+\varepsilon} [\delta(x+a) + \delta(x-a)] \psi(x) dx = E \int_{-a-\varepsilon}^{-a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \int_{-a-\varepsilon}^{-a+\varepsilon} \frac{d^2\psi}{dx^2} dx - \alpha \int_{-a-\varepsilon}^{-a+\varepsilon} \delta(x+a) \psi(x) dx - \alpha \int_{-a-\varepsilon}^{-a+\varepsilon} \delta(x-a) \psi(x) dx = E \int_{-a-\varepsilon}^{-a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \int_{-a-\varepsilon}^{-a+\varepsilon} \frac{d^2\psi}{dx^2} dx - \alpha \int_{-a-\varepsilon}^{-a+\varepsilon} \delta(x+a) \psi(x) dx = E \int_{-a-\varepsilon}^{-a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \left[ \frac{d\psi}{dx} \right]_{-a-\varepsilon}^{-a+\varepsilon} - \alpha \psi(-a) = E \int_{-a-\varepsilon}^{-a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \left[ \frac{d\psi_{II}}{dx}(-a+\varepsilon) - \frac{d\psi_I}{dx}(-a-\varepsilon) \right] - \alpha \psi(-a) = E \int_{-a-\varepsilon}^{-a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \lim_{\varepsilon \rightarrow 0} \left[ \frac{d\psi_{II}}{dx}(-a + \varepsilon) - \frac{d\psi_I}{dx}(-a - \varepsilon) \right] - \alpha\psi(-a) = 0$$

$$-\frac{\hbar^2}{2m} \lim_{\varepsilon \rightarrow 0} \left[ -kCe^{-k(-a+\varepsilon)} + kDe^{k(-a+\varepsilon)} - kAe^{k(-a-\varepsilon)} \right] - \alpha Ae^{-ka} = 0$$

$$-\frac{\hbar^2}{2m} (-kCe^{ka} + kDe^{-ka} - kAe^{-ka}) - \alpha Ae^{-ka} = 0$$

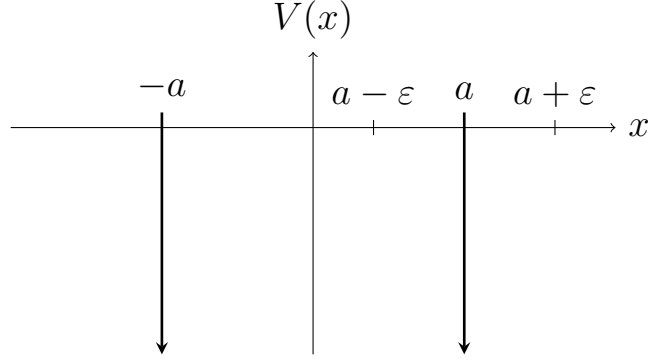
$$-Ce^{ka} + De^{-ka} - Ae^{-ka} + \frac{2m\alpha}{k\hbar^2} Ae^{-ka} = 0$$

$$-Ce^{ka} + \left( D - A + \frac{2m\alpha}{k\hbar^2} A \right) e^{-ka} = 0$$

$$\left[ D + \left( \frac{2m\alpha}{k\hbar^2} - 1 \right) A \right] e^{-ka} = Ce^{ka}$$

$$\boxed{D + \left( \frac{2m\alpha}{k\hbar^2} - 1 \right) A = Ce^{2ka}}$$

Integrando alrededor de  $a$



$$-\frac{\hbar^2}{2m} \int_{a-\varepsilon}^{a+\varepsilon} \frac{d^2\psi}{dx^2} dx - \alpha \int_{a-\varepsilon}^{a+\varepsilon} [\delta(x+a) + \delta(x-a)] \psi(x) dx = E \int_{a-\varepsilon}^{a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \int_{a-\varepsilon}^{a+\varepsilon} \frac{d^2\psi}{dx^2} dx - \alpha \int_{a-\varepsilon}^{a+\varepsilon} \delta(x+a) \psi(x) dx - \alpha \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) \psi(x) dx = E \int_{a-\varepsilon}^{a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \int_{a-\varepsilon}^{a+\varepsilon} \frac{d^2\psi}{dx^2} dx - \alpha \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) \psi(x) dx = E \int_{a-\varepsilon}^{a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \left[ \frac{d\psi}{dx} \right]_{a-\varepsilon}^{a+\varepsilon} - \alpha \psi(a) = E \int_{a-\varepsilon}^{a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \left[ \frac{d\psi_{III}}{dx}(a+\varepsilon) - \frac{d\psi_{II}}{dx}(a-\varepsilon) \right] - \alpha \psi(a) = E \int_{a-\varepsilon}^{a+\varepsilon} \psi(x) dx$$

$$-\frac{\hbar^2}{2m} \lim_{\varepsilon \rightarrow 0} \left[ \frac{d\psi_{III}}{dx}(a + \varepsilon) - \frac{d\psi_{II}}{dx}(a - \varepsilon) \right] - \alpha\psi(a) = 0$$

$$-\frac{\hbar^2}{2m} \lim_{\varepsilon \rightarrow 0} \left[ -kBe^{-k(a+\varepsilon)} - \left( -kCe^{-k(a-\varepsilon)} + kDe^{k(a-\varepsilon)} \right) \right] - \alpha Be^{-ka} = 0$$

$$-\frac{\hbar^2}{2m} \left( -kBe^{-ka} + kCe^{-ka} - kDe^{ka} \right) - \alpha Be^{-ka} = 0$$

$$-Be^{-ka} + Ce^{-ka} - De^{ka} + \frac{2m\alpha}{k\hbar^2} Be^{-ka} = 0$$

$$-De^{ka} + \left( C - B + \frac{2m\alpha}{k\hbar^2} B \right) e^{-ka} = 0$$

$$\left[ C + \left( \frac{2m\alpha}{k\hbar^2} - 1 \right) B \right] e^{-ka} = De^{ka}$$

$$\boxed{C + \left( \frac{2m\alpha}{k\hbar^2} - 1 \right) B = De^{2ka}}$$



Sistema de 4 ecuaciones

$$A = D + Ce^{2ka}$$

$$B = C + De^{2ka}$$

$$D + \left( \frac{2m\alpha}{k\hbar^2} - 1 \right) A = Ce^{2ka}$$

$$C + \left( \frac{2m\alpha}{k\hbar^2} - 1 \right) B = De^{2ka}$$

$$D + \left( \frac{2m\alpha}{k\hbar^2} - 1 \right) (D + Ce^{2ka}) = Ce^{2ka} \quad \rightarrow \quad \frac{2m\alpha}{k\hbar^2} (D + Ce^{2ka}) = 2Ce^{2ka}$$

$$\boxed{\frac{m\alpha}{k\hbar^2} (D + Ce^{2ka}) = Ce^{2ka}}$$

$$C + \left( \frac{2m\alpha}{k\hbar^2} - 1 \right) (C + De^{2ka}) = De^{2ka} \quad \rightarrow \quad \frac{2m\alpha}{k\hbar^2} (C + De^{2ka}) = 2De^{2ka}$$

$$\boxed{\frac{m\alpha}{k\hbar^2} (C + De^{2ka}) = De^{2ka}}$$

$$\frac{m\alpha}{k\hbar^2} (D + Ce^{2ka}) = Ce^{2ka} \quad \rightarrow \quad \frac{m\alpha}{k\hbar^2} D = \left( 1 - \frac{m\alpha}{k\hbar^2} \right) Ce^{2ka}$$

$$D = \frac{k\hbar^2}{m\alpha} \left( 1 - \frac{m\alpha}{k\hbar^2} \right) Ce^{2ka} = \left( \frac{k\hbar^2}{m\alpha} - 1 \right) Ce^{2ka}$$

$$\frac{m\alpha}{k\hbar^2} (C + De^{2ka}) = De^{2ka} \quad \rightarrow \quad \left( \frac{m\alpha}{k\hbar^2} - 1 \right) De^{2ka} + \frac{m\alpha}{k\hbar^2} C = 0$$

$$\left( \frac{m\alpha}{k\hbar^2} - 1 \right) \left( \frac{k\hbar^2}{m\alpha} - 1 \right) Ce^{4ka} + \frac{m\alpha}{k\hbar^2} C = 0 \quad \rightarrow \quad - \left( \frac{m\alpha}{k\hbar^2} + \frac{k\hbar^2}{m\alpha} - 2 \right) e^{4ka} + \frac{m\alpha}{k\hbar^2} = 0$$

$$\frac{m\alpha}{k\hbar^2} \left( \frac{m\alpha}{k\hbar^2} + \frac{k\hbar^2}{m\alpha} - 2 \right) e^{4ka} - \left( \frac{m\alpha}{k\hbar^2} \right)^2 = 0 \quad \rightarrow \quad \left[ \left( \frac{m\alpha}{k\hbar^2} \right)^2 + 1 - \frac{2m\alpha}{k\hbar^2} \right] e^{4ka} - \left( \frac{m\alpha}{k\hbar^2} \right)^2 = 0$$

$$\left( \frac{m\alpha}{k\hbar^2} - 1 \right)^2 e^{4ka} - \left( \frac{m\alpha}{k\hbar^2} \right)^2 = 0 \quad \rightarrow \quad \left( \frac{m\alpha}{k\hbar^2} - 1 \right)^2 e^{4ka} = \left( \frac{m\alpha}{k\hbar^2} \right)^2$$

$$\left( \frac{m\alpha}{k\hbar^2} - 1 \right) e^{2ka} = \pm \frac{m\alpha}{k\hbar^2} \quad \rightarrow \quad \left( \frac{m\alpha}{\hbar^2} - k \right) e^{2ka} = \pm \frac{m\alpha}{\hbar^2} \quad \rightarrow \quad k = \frac{m\alpha}{\hbar^2} \mp \frac{m\alpha}{\hbar^2} e^{-2ka}$$

$$\boxed{k_{\pm} = \frac{m\alpha}{\hbar^2} (1 \pm e^{-2k_{\pm}a})}$$

## Normalización

$$\psi_{\pm}(x) = \begin{cases} Ae^{k_{\pm}x} & : x < -a \\ Ce^{-k_{\pm}x} + De^{k_{\pm}x} & : -a < x < a \\ Be^{-k_{\pm}x} & : x > a \end{cases}$$

$$\int_{-\infty}^{\infty} |\psi_{\pm}(x)|^2 dx = 1$$

$$\int_{-\infty}^{-a} |\psi_{\pm I}|^2 dx + \int_{-a}^a |\psi_{\pm II}|^2 dx + \int_a^{\infty} |\psi_{\pm III}|^2 dx = 1$$

$$A^2 \int_{-\infty}^{-a} e^{2k_{\pm}x} dx + \int_{-a}^a (Ce^{-k_{\pm}x} + De^{k_{\pm}x})^2 dx + B^2 \int_a^{\infty} e^{-2k_{\pm}x} dx = 1$$

$$A^2 \int_{-\infty}^{-a} e^{2k_{\pm}x} dx + C^2 \int_{-a}^a e^{-2k_{\pm}x} dx + D^2 \int_{-a}^a e^{2k_{\pm}x} dx + 2CD \int_{-a}^a dx + B^2 \int_a^{\infty} e^{-2k_{\pm}x} dx = 1$$

$$\frac{A^2}{2k_{\pm}} \left| e^{2k_{\pm}x} \right|_{-\infty}^{-a} - \frac{C^2}{2k_{\pm}} \left| e^{-2k_{\pm}x} \right|_{-a}^a + \frac{D^2}{2k_{\pm}} \left| e^{2k_{\pm}x} \right|_{-a}^a + 2CD \left| x \right|_{-a}^a - \frac{B^2}{2k_{\pm}} \left| e^{-2k_{\pm}x} \right|_a^{\infty} = 1$$

$$A^2 \left| e^{2k_{\pm}x} \right|_{-\infty}^{-a} - C^2 \left| e^{-2k_{\pm}x} \right|_{-a}^a + D^2 \left| e^{2k_{\pm}x} \right|_{-a}^a + 4k_{\pm}CD \left| x \right|_{-a}^a - B^2 \left| e^{-2k_{\pm}x} \right|_a^{\infty} = 2k_{\pm}$$

$$A^2 e^{-2k_{\pm}a} - C^2 (e^{-2k_{\pm}a} - e^{2k_{\pm}a}) + D^2 (e^{2k_{\pm}a} - e^{-2k_{\pm}a}) + 8k_{\pm}aCD + B^2 e^{-2k_{\pm}a} = 2k_{\pm}$$

$$(A^2 + B^2) e^{-2k_{\pm}a} + (C^2 + D^2) (e^{2k_{\pm}a} - e^{-2k_{\pm}a}) + 8k_{\pm}aCD = 2k_{\pm}$$

Solución para  $k_+$

$$k_+ = \frac{m\alpha}{\hbar^2} (1 + e^{-2k_+a})$$

$$A = B \quad , \quad C = D$$

$$C = \frac{A}{1 + e^{2k_+a}}$$

$$2A^2e^{-2k_+a} + 2C^2 (e^{2k_+a} - e^{-2k_+a}) + 8k_+aC^2 = 2k_+$$

$$A^2e^{-2k_+a} + C^2 (e^{2k_+a} - e^{-2k_+a}) + 4k_+aC^2 = k_+$$

$$A^2e^{-2k_+a} + \frac{A^2}{(1 + e^{2k_+a})^2} (e^{2k_+a} - e^{-2k_+a}) + \frac{4k_+aA^2}{(1 + e^{2k_+a})^2} = k_+$$

$$A^2e^{-2k_+a}(1 + e^{2k_+a})^2 + A^2 (e^{2k_+a} - e^{-2k_+a}) + 4k_+aA^2 = k_+(1 + e^{2k_+a})^2$$

$$A^2(e^{-k_+a} + e^{k_+a})^2 + A^2 (e^{2k_+a} - e^{-2k_+a}) + 4k_+aA^2 = k_+(1 + e^{2k_+a})^2$$

$$A^2(e^{-2k_+a} + e^{2k_+a} + 2) + A^2 (e^{2k_+a} - e^{-2k_+a}) + 4k_+aA^2 = k_+(1 + e^{2k_+a})^2$$

$$A^2 [(e^{-2k_+a} + e^{2k_+a} + 2) + (e^{2k_+a} - e^{-2k_+a}) + 4k_+a] = k_+(1 + e^{2k_+a})^2$$

$$A^2 (2e^{2k_+a} + 2 + 4k_+a) = k_+(1 + e^{2k_+a})^2 \quad \rightarrow \quad A^2 = \frac{k_+(1 + e^{2k_+a})^2}{2(e^{2k_+a} + 2k_+a + 1)}$$

$$A \in \mathbb{R} \quad , \quad A > 0$$

$$A_+ = \sqrt{\frac{k_+(1 + e^{2k_+a})^2}{2(e^{2k_+a} + 2k_+a + 1)}}$$

Solución para  $k_-$

$$k_- = \frac{m\alpha}{\hbar^2} (1 + e^{-2k_-a})$$

$$A = -B \quad , \quad C = -D$$

$$C = -\frac{A}{1 - e^{2k_-a}}$$

$$2A^2e^{-2k_-a} + 2C^2(e^{2k_-a} - e^{-2k_-a}) - 8k_-aC^2 = 2k_-$$

$$A^2e^{-2k_-a} + C^2(e^{2k_-a} - e^{-2k_-a}) - 4k_-aC^2 = k_-$$

$$A^2e^{-2k_-a} + \frac{A^2}{(1 - e^{2k_-a})^2}(e^{2k_-a} - e^{-2k_-a}) - \frac{4k_-aA^2}{(1 - e^{2k_-a})^2} = k_-$$

$$A^2e^{-2k_-a}(1 - e^{2k_-a})^2 + A^2(e^{2k_-a} - e^{-2k_-a}) - 4k_-aA^2 = k_-(1 - e^{2k_-a})^2$$

$$A^2(e^{-k_-a} - e^{k_-a})^2 + A^2(e^{2k_-a} - e^{-2k_-a}) - 4k_-aA^2 = k_-(1 - e^{2k_-a})^2$$

$$A^2(e^{-2k_-a} + e^{2k_-a} - 2) + A^2(e^{2k_-a} - e^{-2k_-a}) - 4k_-aA^2 = k_-(1 - e^{2k_-a})^2$$

$$A^2[(e^{-2k_-a} + e^{2k_-a} - 2) + (e^{2k_-a} - e^{-2k_-a}) - 4k_-a] = k_-(1 - e^{2k_-a})^2$$

$$A^2(2e^{2k_-a} - 2 - 4k_-a) = k_-(1 - e^{2k_-a})^2 \quad \rightarrow \quad A^2 = \frac{k_-(1 - e^{2k_-a})^2}{2(e^{2k_-a} - 2k_-a - 1)}$$

$$A \in \mathbb{R} \quad , \quad A > 0$$

$$A_- = \sqrt{\frac{k_-(1 - e^{2k_-a})^2}{2(e^{2k_-a} - 2k_-a - 1)}}$$

## Función de onda

$$\psi_{\pm}(x) = \begin{cases} Ae^{k_{\pm}x} & : x < -a \\ \frac{\pm A}{1 \pm e^{2k_{\pm}a}} (e^{-k_{\pm}x} \pm e^{k_{\pm}x}) & : -a < x < a \\ \pm Ae^{-k_{\pm}x} & : x > a \end{cases}, \quad E_{\pm} = -\frac{k_{\pm}^2 \hbar^2}{2m}$$

$$k_{\pm} = \frac{m\alpha}{\hbar^2} (1 \pm e^{-2k_{\pm}a}) \quad , \quad A_{\pm} = \sqrt{\frac{k_{\pm}(1 \pm e^{2k_{\pm}a})^2}{2(e^{2k_{\pm}a} \pm 2k_{\pm}a \pm 1)}}$$

## Solución numérica

$$\beta = \frac{m}{\hbar^2}$$

$$k_{\pm} = \alpha\beta (1 \pm e^{-2k_{\pm}a}) \quad \rightarrow \quad k_{\pm} - \alpha\beta (1 \pm e^{-2k_{\pm}a}) = 0$$

$$E_{\pm} = -\frac{k_{\pm}^2}{2\beta}$$

## Método de Newton Raphson

$$f(k_{\pm}) = k_{\pm} - \alpha\beta (1 \pm e^{-2k_{\pm}a}) \quad \rightarrow \quad f'(k_{\pm}) = 1 \pm 2a\alpha\beta e^{-2k_{\pm}a}$$

$$k_{\pm(n+1)} = k_{\pm(n)} - \frac{f(k_{\pm(n)})}{f'(k_{\pm(n)})}$$

## Solución numérica

