Desarrollo y simulación de masas en mesa sin fricción

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$$\vec{r}_1 = r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath} + 0\hat{k}$$

$$\dot{\vec{r}}_1 = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{\imath} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{\jmath} + 0\hat{k}$$

$$\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1 = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 \quad \rightarrow \quad \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\vec{r}_2 = 0\hat{\imath} + 0\hat{\jmath} + (r - L)\hat{k}$$

$$\dot{\vec{r}}_2 = 0\hat{\imath} + 0\hat{\jmath} + \dot{r}\hat{k}$$

$$\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2 = \dot{r}^2$$

$$T = \frac{1}{2}m(\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1 + \dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2) \quad \rightarrow \quad T = \frac{1}{2}m(2\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = mgy_2 \quad \rightarrow \quad V = mg(r - L)$$

$$\mathcal{L} = T - V \quad \rightarrow \quad \mathcal{L} = \frac{1}{2}m(2\dot{r}^2 + r^2\dot{\theta}^2) - mg(r - L)$$

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - mg$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = 2m\dot{r} \quad \rightarrow \quad \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = 2m\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \rightarrow \quad \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] - \frac{\partial \mathcal{L}}{\partial r} = 0$$

$$2m\ddot{r} - (mr\dot{\theta}^2 - mg) = 0$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \rightarrow \quad \frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \rightarrow \quad M = r^2\dot{\theta}$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0 \quad \rightarrow \quad 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$$

Ecuaciones de movimiento

$$\ddot{r} - \frac{1}{2}r\dot{\theta}^2 + \frac{1}{2}g = 0$$

$$\ddot{\theta} + 2\dot{\theta}\frac{\dot{r}}{r} = 0$$

Casos particulares

Masa 2 sin aceleración

$$\ddot{r} = 0 \quad \rightarrow \quad r\dot{\theta}^2 = g \quad \rightarrow \quad \dot{\theta} = \pm\sqrt{\frac{g}{r}}$$

Masa 2 sin movimiento

$$\ddot{r} = 0 \quad \rightarrow \quad r\dot{\theta}^2 = g \quad \rightarrow \quad \dot{\theta} = \pm\sqrt{\frac{g}{r}}$$

$$\ddot{\theta} = 0 \quad \rightarrow \quad 2\dot{\theta}\frac{\dot{r}}{r} = 0 \quad \rightarrow \quad \dot{r} = 0$$

Masa 1 sin rotación

$$\dot{\theta} = 0 \quad \rightarrow \quad \ddot{r} = -\frac{1}{2}g \quad \rightarrow \quad \dot{r} = -\frac{1}{2}gt \quad \rightarrow \quad r = -\frac{1}{4}gt^2$$

$$\ddot{\theta} + 2\dot{\theta}\frac{\dot{r}}{r} = 0 \quad \rightarrow \quad \frac{\ddot{\theta}}{\dot{\theta}} = -2\frac{\dot{r}}{r}$$

$$\omega = \dot{\theta} \quad \rightarrow \quad \frac{\dot{\omega}}{\omega} = -2\frac{\dot{r}}{r} \quad \rightarrow$$

$$\frac{\dot{f}}{f} = \frac{d}{dt}\left(\frac{1}{f}\right) \quad ?$$

$$\frac{d}{dt}\left(\frac{1}{f}\right) = \frac{d}{dt}f^{-1} = -f^{-2}\frac{d}{dt}f = -f^{-2}\dot{f}$$

$$\therefore \quad \dot{f} = -f^2\frac{d}{dt}\left(\frac{1}{f}\right)$$

$$\ddot{\theta} + 2\dot{\theta}\frac{\dot{r}}{r} = 0 \quad \rightarrow \quad r^3\ddot{\theta} + 2r^2\dot{\theta}\dot{r} = 0 \quad \rightarrow \quad r^3\ddot{\theta} + 2M\dot{r} = 0$$

$$\ddot{\theta} = -2M\frac{\dot{r}}{r^3}$$

$$\ddot{r} - \frac{1}{2} r\dot{\theta}^2 + \frac{1}{2} g = 0$$

$$r = e^{\omega t} \quad \rightarrow \quad \dot{r} = \omega e^{\omega t} \quad \rightarrow \quad \ddot{r} = \omega^2 e^{\omega t}$$

$$\omega^2 e^{\omega t} - \frac{1}{2} e^{\omega t}\dot{\theta}^2 + \frac{1}{2} g = 0$$

$$2\omega^2 e^{\omega t} - e^{\omega t}\dot{\theta}^2 + g = 0 \quad \rightarrow \quad e^{\omega t}(\omega^2 - \dot{\theta}^2) + g = 0$$

Solución númerica

$$\ddot{r} - \frac{1}{2} r\dot{\theta}^2 + \frac{1}{2} g = 0$$

$$\ddot{\theta} + 2\dot{\theta}\frac{\dot{r}}{r} = 0$$

$$\omega = \frac{d\theta}{dt} \quad \rightarrow \quad \dot{\omega} = \frac{d^2\theta}{dt^2}$$

$$q = \frac{dr}{dt} \quad \rightarrow \quad \dot{q} = \frac{d^2 r}{dt^2}$$

$$\dot{q} - \frac{1}{2} r\omega^2 + \frac{1}{2} g = 0$$

$$\dot{\omega} + 2\omega\frac{q}{r} = 0$$

$$\dot{q} = \frac{1}{2} r\omega^2 - \frac{1}{2} g$$

$$\dot{\omega} = -2\omega\frac{q}{r}$$

Metodo de Runge Kutta de orden 4

$\frac{d\theta}{dt} = \omega$	$\frac{d\omega}{dt} = f(\theta, r, \omega, q, t)$
$k_1 = h\omega$ $k_2 = h(\omega + \frac{1}{2}c_1)$ $k_3 = h(\omega + \frac{1}{2}c_2)$ $k_4 = h(\omega + c_3)$	$c_1 = h(\theta, r, \omega, q, t)$ $c_2 = hf(\theta + \frac{1}{2}k_1, r + \frac{1}{2}n_1, \omega + \frac{1}{2}c_1, q + \frac{1}{2}m_1, t + \frac{1}{2}h)$ $c_3 = hf(\theta + \frac{1}{2}k_2, r + \frac{1}{2}n_2, \omega + \frac{1}{2}c_2, q + \frac{1}{2}m_2, t + \frac{1}{2}h)$ $c_4 = hf(\theta + k_3, r + n_3, \omega + c_3, q + m_3, t + h)$
$\theta(t+h) = \theta(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	$\omega(t+h) = \omega(t) + \frac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4)$

$\frac{dr}{dt} = q$	$\frac{dq}{dt} = u(\theta, r, \omega, q, t)$
$n_1 = hq$ $n_2 = h(q + \frac{1}{2}m_1)$ $n_3 = h(q + \frac{1}{2}m_2)$ $n_4 = h(q + m_3)$	$m_1 = hu(\theta, r, \omega, q, t)$ $m_2 = hu(\theta + \frac{1}{2}k_1, r + \frac{1}{2}n_1, \omega + \frac{1}{2}c_1, q + \frac{1}{2}m_1, t + \frac{1}{2}h)$ $m_3 = hu(\theta + \frac{1}{2}k_2, r + \frac{1}{2}n_2, \omega + \frac{1}{2}c_2, q + \frac{1}{2}m_2, t + \frac{1}{2}h)$ $m_4 = hu(\theta + k_3, r + n_3, \omega + c_3, q + m_3, t + h)$
$r(t+h) = r(t) + \frac{1}{6}(n_1 + 2n_2 + 2n_3 + n_4)$	$q(t+h) = q(t) + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$$f(\theta, r, \omega, q, t) = -2\omega\frac{q}{r} \quad , \quad u(\theta, r, \omega, q, t) = \frac{1}{2}r\omega^2 - \frac{1}{2}g$$