Desarrollo y simulación de péndulo de masa

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Formulación de Newton

$$\sum \vec{\tau} = I\vec{\alpha}$$
$$\sum \vec{\tau} = \sum \vec{r} \times \vec{F}$$

Sumatoria de torcas

$$\vec{r} \equiv \text{vector pivote - centro de masa}$$
$$\vec{W} \equiv \text{peso}$$
$$\sum \vec{\tau} = \vec{r} \times \vec{W}$$

$$\therefore \quad I\vec{\alpha} = \vec{r} \times \vec{W}$$

Forma escalar

$$|\vec{r}| = l \quad , \quad l \equiv \text{cte}$$
$$I\alpha = Wl\sin\theta$$

Ecuación de movimiento

$$W = -mg$$
$$\alpha = \frac{d^2\theta}{dt^2}$$

$$\boxed{\therefore \quad \frac{d^2\theta}{dt^2} + \frac{m}{I}gl\sin\theta = 0}$$

Formulación de Euler-Lagrange

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad , \quad \mathcal{L} = T - V$$

Coordenadas generalizadas

$$q_1 = \theta_1$$

$$l = \frac{L}{2}$$

$$x = l \sin \theta_1 \quad , \quad \dot{x} = l\dot{\theta}_1 \cos \theta_1$$

$$y = -l \cos \theta_1 \quad , \quad \dot{y} = l\dot{\theta}_1 \sin \theta_1$$

Energía potencial

$$V = mgy \to V = -mgl \cos \theta_1$$

Energía cinética

$$T = \frac{1}{2}\dot{\vec{\theta}} \cdot \vec{L} \to T = \frac{1}{2}\dot{\vec{\theta}} \cdot I\dot{\vec{\theta}} \to T = \frac{1}{2}I\dot{\theta}^2$$

$$T = \frac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 + I_3\dot{\theta}_3^2) \to T = \frac{1}{2}(I_1\dot{\theta}_1^2 + I_2\cancelto{0}{\dot{\theta}_2^2} + I_3\cancelto{0}{\dot{\theta}_3^2})$$

$$T = \frac{1}{2}I_1\dot{\theta}_1^2$$

$$\therefore \quad \theta_1 = \theta \quad , \quad I_1 = I \to V = -mgl \cos \theta \quad , \quad T = \frac{1}{2}I\dot{\theta}^2$$

Lagrangiano

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2} I \dot{\theta}^2 + mgl \cos\theta$$

Derivadas del Lagrangiano

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgl \sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] = I\ddot{\theta}$$

Ecuación de movimiento

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$I\ddot{\theta} + mgl \sin\theta = 0$$

$$\boxed{\therefore \quad \ddot{\theta} + \frac{m}{I} gl \sin\theta = 0}$$

Solución analítica aproximada para ángulos pequeños

$$\theta \ll \frac{\pi}{2} \rightarrow \sin\theta \approx \theta$$

$$\therefore \quad \frac{m}{I} gl \sin\theta \approx \frac{m}{I} gl\theta$$

$$\ddot{\theta} + \frac{m}{I} gl \sin\theta = 0 \rightarrow \quad \ddot{\theta} + \frac{m}{I} gl\theta = 0$$

Ecuación no lineal de segundo orden

$$\ddot{\theta} + \frac{m}{I} gl\theta = 0$$

$$\omega = \sqrt{\frac{m}{I} gl}$$

$$\theta = e^{\beta t}$$

$$\dot{\theta} = \beta e^{\beta t}$$

$$\ddot{\theta} = \beta^2 e^{\beta t}$$

$$\ddot{\theta} + \frac{m}{I} gl\theta = 0 \rightarrow \beta^2 e^{\beta t} + \omega^2 e^{\beta t} = 0$$

$$\beta^2 = -\omega^2 \rightarrow \beta = \sqrt{-\omega^2} \rightarrow \beta = \omega\sqrt{-1}$$

$$\beta = \pm i\omega$$

$$\theta = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$
$$\theta = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$
$$e^{\pm ik} = \cos k \pm i \sin k$$
$$\theta = C_1 \cos(\omega t) + iC_1 \sin(\omega t) + C_2 \cos(\omega t) - iC_2 \sin(\omega t)$$
$$\theta = (C_1 + C_2)\cos(\omega t) + (C_1 - C_2) i \sin(\omega t)$$
$$A = C_1 + C_2 \quad , \quad B = i(C_1 - C_2)$$

$$\theta = A\cos(\omega t) + B\sin(\omega t)$$
$$\dot{\theta} = -A\omega \sin(\omega t) + B\omega \cos(\omega t)$$

$$\theta(t=0) = \theta_0 \quad , \quad \theta(t=0) = A\cancelto{1}{\cos(\omega(0))} + B\cancelto{0}{\sin(\omega(0))} \rightarrow A = \theta_0$$
$$\dot{\theta}(t=0) = 0 \quad , \quad \dot{\theta}(t=0) = -A\omega \cancelto{0}{\sin(\omega(0))} + B\omega \cancelto{1}{\cos(\omega(0))} \rightarrow B = 0$$

$$\therefore \quad \theta = Acos(\phi + \omega t)$$

$$\boxed{\theta(t) = Acos\left(\phi + t\sqrt{\frac{m}{I}gl}\right)}$$

Solución númerica por método de Euler

$$x_{i+1} = x_i + h\dot{x}(x_i, t_i)$$

$$\ddot{\theta} + \frac{m}{I} gl \sin\theta = 0$$
$$\omega = \dot{\theta} \to \dot{\omega} = \ddot{\theta}$$
$$\dot{\omega} = -\frac{m}{I} gl \sin\theta$$

$$\boxed{\begin{aligned} t_{i+1} &= t_i + h \\ \theta_{i+1} &= \theta_i + h\omega_{i+1} \\ \omega_{i+1} &= \omega_i - h\tfrac{m}{I} gl \sin\theta_i \end{aligned}}$$

Tensor de inercia

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

$$I_{ij} = I_{ji} = \int_M \left[\delta_{ij} r^2 - x_i x_j\right] dm = \int_V \rho \left[\delta_{ij} r^2 - x_i x_j\right] dV$$

Momentos principales

$$\begin{aligned} I_i &= I_{ii} \\ I_1 &= \int_V \rho(x_2^2 + x_3^2) dV \\ I_2 &= \int_V \rho(x_1^2 + x_3^2) dV \\ I_3 &= \int_V \rho(x_1^2 + x_2^2) dV \end{aligned}$$

Barra de densidad constante

Momento de inercia en barra unidimensional

$$I = \lambda \int_0^L r^2 dr \rightarrow I = \frac{1}{3}\lambda |r^3|_0^L$$

$$I = \frac{1}{3}\lambda L^3$$

$$m = \lambda L$$

$$\boxed{\therefore \quad I = \frac{1}{3} m L^2}$$

Cilindro de densidad constante

Parametrización con coordenadas polares

$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta$$

$$x_3 = x_3$$

$$I_1 = \rho\int_V (x_2(r,\theta)^2 + x_3^2)|J(r,\theta,x_3)|dV$$

$$I_2 = \rho\int_V (x_1(r,\theta)^2 + x_3^2)|J(r,\theta,x_3)|dV$$

$$I_3 = \rho\int_V (x_1(r,\theta)^2 + x_2(r,\theta)^2)|J(r,\theta,x_3)|dV$$

Jacobiano

$$|J(r,\theta,x_3)| = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial x_3} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_2}{\partial x_3} \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \theta} & \frac{\partial x_3}{\partial x_3} \end{vmatrix} \to |J(r,\theta,x_3)| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|J(r,\theta,x_3)| = r$$

Momento de inercia eje 1

$$I_1 = \rho \int_V (x_2(r,\theta)^2 + x_3^2)|J(r,\theta,x_3)|dV \to I_1 = \rho \int_V (r^2 \sin^2\theta + x_3^2) r dV$$

$$I_1 = \rho \left[\int_V r^3 \sin^2\theta dV + \int_V x_3^2 r dV \right]$$

$$I_1 = \rho \left[\int_0^R r^3 \int_0^{2\pi} \sin^2\theta \int_0^L dx_3 d\theta dr + \int_0^L x_3^2 \int_0^R r \int_0^{2\pi} d\theta dr dx_3 \right]$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \to I_1 = \rho \left[\int_0^R r^3 \left(\frac{1}{2}\int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} \cos 2\theta d\theta \right) \int_0^L dx_3 d\theta dr + \int_0^L x_3^2 \int_0^R r \int_0^{2\pi} d\theta dr dx_3 \right]$$

$$I_1 = \rho \left[\frac{1}{4} \left| r^4 \right|_0^R \left(\frac{1}{2} |\theta|_0^{2\pi} - \frac{1}{4} \cancelto{0}{|\sin 2\theta|_0^{2\pi}} \right) |x_3|_0^L + \frac{1}{3} \left| x_3^3 \right|_0^L \frac{1}{2} \left| r^2 \right|_0^R |\theta|_0^{2\pi} \right] \to I_1 = \frac{1}{4}\rho\pi L R^4 + \frac{1}{3}\rho\pi L^3 R^2$$

Momento de inercia eje 2

$$I_2 = \rho \int_V (x_1(r,\theta)^2 + x_3^2)|J(r,\theta,x_3)|dV \to I_2 = \rho \int_V (r^2 \cos^2\theta + x_3^2) r dV$$

$$I_2 = \rho \left[\int_V r^3 \cos^2\theta dV + \int_V x_3^2 r dV \right]$$

$$I_2 = \rho \left[\int_0^R r^3 \int_0^{2\pi} \cos^2\theta \int_0^L dx_3 d\theta dr + \int_0^L x_3^2 \int_0^R r \int_0^{2\pi} d\theta dr dx_3 \right]$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \to I_2 = \rho \left[\int_0^R r^3 \left(\frac{1}{2}\int_0^{2\pi} d\theta + \frac{1}{2}\int_0^{2\pi} \cos 2\theta d\theta \right) \int_0^L dx_3 d\theta dr + \int_0^L x_3^2 \int_0^R r \int_0^{2\pi} d\theta dr dx_3 \right]$$

$$I_2 = \rho \left[\frac{1}{4} \left| r^4 \right|_0^R \left(\frac{1}{2} |\theta|_0^{2\pi} + \frac{1}{4} \cancelto{0}{|\sin 2\theta|_0^{2\pi}} \right) |x_3|_0^L + \frac{1}{3} \left| x_3^3 \right|_0^L \frac{1}{2} \left| r^2 \right|_0^R |\theta|_0^{2\pi} \right] \to I_2 = \frac{1}{4}\rho\pi L R^4 + \frac{1}{3}\rho\pi L^3 R^2$$

Momento de inercia eje 3

$$I_3 = \rho \int_V (x_1(r,\theta)^2 + x_2(r,\theta)^2)|J(r,\theta,x_3)|dV \rightarrow I_3 = \rho \int_V r^3 dV$$

$$I_3 = \rho \int_0^R r^3 \int_0^{2\pi} \int_0^L dx_3 d\theta dr \rightarrow I_3 = \rho \frac{1}{4} \left| r^4 \right|_0^R |\theta|_0^{2\pi} |x_3|_0^L \rightarrow I_3 = \frac{1}{2} \rho \pi L R^4$$

Momentos principales

$$V = \pi L R^2 \quad , \quad m = \rho V$$

$$\boxed{\begin{aligned} I_1 &= \tfrac{1}{4} m R^2 + \tfrac{1}{3} m L^2 \\ I_2 &= \tfrac{1}{4} m R^2 + \tfrac{1}{3} m L^2 \\ I_3 &= \tfrac{1}{2} m R^2 \end{aligned}}$$