Introduction to Electrodynamics

Fourth Edition

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Solucionario

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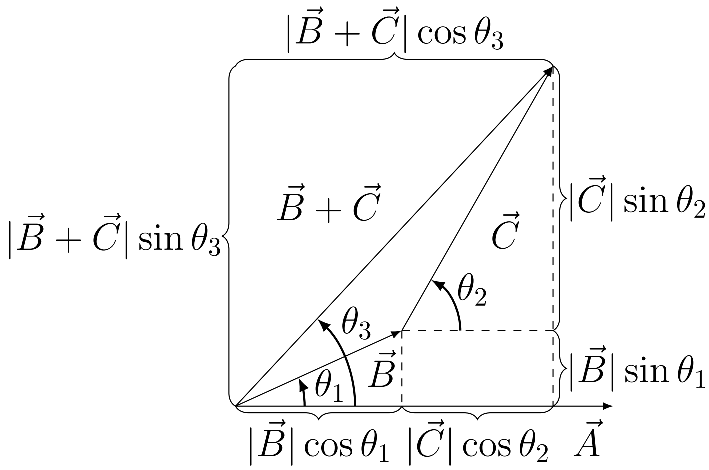
Problem 1.1 Using the definitions in Eqs. 1.1 and 1.4, and appropriate diagrams, show that the dot product and cross product are distributive,

a) when the three vectors are coplanar;

b) in the general case.

Vectores coplanares



Como podemos ver en el diagrama, cuando los vectores $\vec{A}$, $\vec{B}$ y $\vec{C}$ son coplanares se mantienen las siguientes relaciones:

$$|\vec{B}|\cos\theta_1 + |\vec{C}|\cos\theta_2 = |\vec{B}+\vec{C}|\cos\theta_3$$
$$|\vec{B}|\sin\theta_1 + |\vec{C}|\sin\theta_2 = |\vec{B}+\vec{C}|\sin\theta_3$$

Ecuación 1.1

$$\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta$$

Ecuación 1.4

$$\mathbf{A}\times\mathbf{B} \equiv AB\sin\theta\hat{\mathbf{n}}$$

Propiedad distributiva en el producto escalar

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C} \quad ?$$

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C} \quad \rightarrow \quad |\vec{A}||\vec{B}+\vec{C}|\cos\theta_3 = |\vec{A}||\vec{B}|\cos\theta_1+|\vec{A}||\vec{C}|\cos\theta_2$$

$$A = |\vec{A}|$$

$$A|\vec{B}+\vec{C}|\cos\theta_3 = A|\vec{B}|\cos\theta_1 + A|\vec{C}|\cos\theta_2 = A\left(|\vec{B}|\cos\theta_1 + |\vec{C}|\cos\theta_2\right)$$

$$A|\vec{B}+\vec{C}|\cos\theta_3 = A|\vec{B}+\vec{C}|\cos\theta_3$$

$$\boxed{1.1\text{ a)} \quad \vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C} \quad \blacksquare}$$

Propiedad distributiva en el producto vectorial

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C} \quad ?$$

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C} \quad \rightarrow \quad |\vec{A}||\vec{B}+\vec{C}|\cos\theta_3\hat{n} = |\vec{A}||\vec{B}|\cos\theta_1\hat{n}+|\vec{A}||\vec{C}|\sin\theta_2\hat{n}$$

$$A = |\vec{A}|$$

$$A|\vec{B}+\vec{C}|\sin\theta_3\hat{n} = A|\vec{B}|\sin\theta_1\hat{n} + A|\vec{C}|\sin\theta_2\hat{n} = A\left(|\vec{B}|\sin\theta_1 + |\vec{C}|\sin\theta_2\right)\hat{n}$$

$$A|\vec{B}+\vec{C}|\sin\theta_3\hat{n} = A|\vec{B}+\vec{C}|\sin\theta_3\hat{n}$$

$$\boxed{\text{1.1 a)}\quad \vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C} \quad \blacksquare}$$

Caso general

$$\vec{A} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} \quad , \quad \vec{B} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k} \quad , \quad \vec{C} = c_x\hat{\imath} + c_y\hat{\jmath} + c_z\hat{k}$$

$$\vec{B} + \vec{C} = (b_x + c_x)\hat{\imath} + (b_y + c_y)\hat{\jmath} + (b_z + c_z)\hat{k}$$

Propiedad distributiva en el producto escalar

$$\boxed{\vec{A} \cdot (\vec{B} + \vec{C}) = a_x(b_x + c_x) + a_y(b_y + c_y) + a_z(b_z + c_z)}$$

$$\vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z \quad , \quad \vec{A} \cdot \vec{C} = a_x c_x + a_y c_y + a_z c_z$$

$$\boxed{\vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} = a_x(b_x + c_x) + a_y(b_y + c_y) + a_z(b_z + c_z)}$$

$$\boxed{\text{1.1 b)} \quad \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \quad \blacksquare}$$

Propiedad distributiva en el producto vectorial

$$\vec{A} \times (\vec{B} + \vec{C}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ (b_x + c_x) & (b_y + c_y) & (b_z + c_z) \end{vmatrix}$$

$$\boxed{\begin{gathered} \vec{A} \times (\vec{B} + \vec{C}) = \\ [a_y(b_z + c_z) - a_z(b_y + c_y)]\hat{\imath} + [a_z(b_x + c_x) - a_x(b_z + c_z)]\hat{\jmath} + [a_x(b_y + c_y) - a_y(b_x + c_x)]\hat{k} \end{gathered}}$$

.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\imath}(a_y b_z - a_z b_y) - \hat{\jmath}(a_x b_z - a_z b_x) + \hat{k}(a_x b_y - a_y b_x)$$

$$\boxed{\vec{A} \times \vec{B} = (a_y b_z - a_z b_y)\hat{\imath} + (a_z b_x - a_x b_z)\hat{\jmath} + (a_x b_y - a_y b_x)\hat{k}}$$

$$\vec{A} \times \vec{C} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ c_x & c_y & c_z \end{vmatrix} = \hat{\imath}(a_y c_z - a_z c_y) - \hat{\jmath}(a_x c_z - a_z c_x) + \hat{k}(a_x c_y - a_y c_x)$$

$$\boxed{\vec{A} \times \vec{C} = (a_y c_z - a_z c_y)\hat{\imath} + (a_z c_x - a_x c_z)\hat{\jmath} + (a_x c_y - a_y c_x)\hat{k}}$$

$$\boxed{\begin{gathered} \vec{A} \times \vec{B} + \vec{A} \times \vec{C} = \\ [a_y(b_z + c_z) - a_z(b_y + c_y)]\hat{\imath} + [a_z(b_x + c_x) - a_x(b_z + c_z)]\hat{\jmath} + [a_x(b_y + c_y) - a_y(b_x + c_x)]\hat{k} \end{gathered}}$$

.

$$\boxed{\text{1.1 b)} \quad \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad \blacksquare}$$

Problem 1.2 Is the cross product associative?

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \stackrel{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

If so, *prove* it; if not, provide a counterexample (the simpler the better).

$$(\vec{A}\times\vec{B})\times\vec{C}=\vec{A}\times(\vec{B}\times\vec{C}) \quad ?$$

$$\vec{A}=a_x\hat{\imath}+a_y\hat{\jmath}+a_z\hat{k} \quad , \quad \vec{B}=b_x\hat{\imath}+b_y\hat{\jmath}+b_z\hat{k} \quad , \quad \vec{C}=c_x\hat{\imath}+c_y\hat{\jmath}+c_z\hat{k}$$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\hat{\imath}(a_yb_z-a_zb_y)-\hat{\jmath}(a_xb_z-a_zb_x)+\hat{k}(a_xb_y-a_yb_x)$$

$$\boxed{\vec{A}\times\vec{B}=(a_yb_z-a_zb_y)\hat{\imath}+(a_zb_x-a_xb_z)\hat{\jmath}+(a_xb_y-a_yb_x)\hat{k}}$$

$$(\vec{A}\times\vec{B})\times\vec{C}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ (a_yb_z-a_zb_y) & (a_zb_x-a_xb_z) & (a_xb_y-a_yb_x)\\ c_x & c_y & c_z\end{vmatrix}$$

$$\begin{gathered}(\vec{A}\times\vec{B})\times\vec{C}=\\ \hat{\imath}[(a_zb_x-a_xb_z)c_z-(a_xb_y-a_yb_x)c_y]\\ -\hat{\jmath}[(a_yb_z-a_zb_y)c_z-(a_xb_y-a_yb_x)c_x]\\ +\hat{k}[(a_yb_z-a_zb_y)c_y-(a_zb_x-a_xb_z)c_x]\end{gathered}$$

$$\boxed{\begin{gathered}(\vec{A}\times\vec{B})\times\vec{C}=\\ [(a_zb_x-a_xb_z)c_z-(a_xb_y-a_yb_x)c_y]\hat{\imath}\\ +[(a_xb_y-a_yb_x)c_x-(a_yb_z-a_zb_y)c_z]\hat{\jmath}\\ +[(a_yb_z-a_zb_y)c_y-(a_zb_x-a_xb_z)c_x]\hat{k}\end{gathered}}$$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \hat{\imath}(b_y c_z - b_z c_y) - \hat{\jmath}(b_x c_z - b_z c_x) + \hat{k}(b_x c_y - b_y c_x)$$

$$\boxed{\vec{B} \times \vec{C} = (b_y c_z - b_z c_y)\hat{\imath} + (b_z c_x - b_x c_z)\hat{\jmath} + (b_x c_y - b_y c_x)\hat{k}}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ (b_y c_z - b_z c_y) & (b_z c_x - b_x c_z) & (b_x c_y - b_y c_x) \end{vmatrix}$$

$$\begin{gathered} \vec{A} \times (\vec{B} \times \vec{C}) = \\ \hat{\imath}[(b_x c_y - b_y c_x)a_y - (b_z c_x - b_x c_z)a_z] \\ -\hat{\jmath}[(b_x c_y - b_y c_x)a_x - (b_y c_z - b_z c_y)a_z] \\ +\hat{k}[(b_z c_x - b_x c_z)a_x - (b_y c_z - b_z c_y)a_y] \end{gathered}$$

$$\boxed{\begin{gathered} \vec{A} \times (\vec{B} \times \vec{C}) = \\ [(b_x c_y - b_y c_x)a_y - (b_z c_x - b_x c_z)a_z]\hat{\imath} \\ +[(b_y c_z - b_z c_y)a_z - (b_x c_y - b_y c_x)a_x]\hat{\jmath} \\ +[(b_z c_x - b_x c_z)a_x - (b_y c_z - b_z c_y)a_y]\hat{k} \end{gathered}}$$

$$(a_z b_x - a_x b_z)c_z - (a_x b_y - a_y b_x)c_y \neq (b_x c_y - b_y c_x)a_y - (b_z c_x - b_x c_z)a_z$$
$$(a_x b_y - a_y b_x)c_x - (a_y b_z - a_z b_y)c_z \neq (b_y c_z - b_z c_y)a_z - (b_x c_y - b_y c_x)a_x$$
$$(a_y b_z - a_z b_y)c_y - (a_z b_x - a_x b_z)c_x \neq (b_z c_x - b_x c_z)a_x - (b_y c_z - b_z c_y)a_y$$

$$\therefore \quad (\vec{A} \times \vec{B}) \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})$$

1.2

$$(\vec{A} \times \vec{B}) \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C}) \quad \blacksquare$$

El producto vectorial es no asociativo.

Problem 1.3 Find the angle between the body diagonals of a cube.

Supondremos un cubo con arista de longitud a, con vertices en $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, $(0,0,a)$, $(a,a,0)$, $(0,a,a)$, $(a,0,a)$, y (a,a,a), siendo las diagonales aquellos vectores $\vec{A}$ de $(0,0,0)$ a (a,a,a), y $\vec{B}$ de $(0,0,a)$ a $(a,a,0)$.

Vectores posicionados en el origen

$$\vec{A} = a\hat{\imath} + a\hat{\jmath} + a\hat{k} \quad , \quad \vec{B} = a\hat{\imath} + a\hat{\jmath} - a\hat{k}$$

Magnitudes

$$A = |\vec{A}| = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}a$$

$$B = |\vec{B}| = \sqrt{a^2 + a^2 + (-a)^2} = \sqrt{3}a$$

Producto escalar

$$\vec{A} \cdot \vec{B} = a^2 + a^2 - a^2 = a^2$$

$$\vec{A} \cdot \vec{B} = AB\cos\theta \quad \rightarrow \quad \theta = \mathrm{acos}\left(\frac{\vec{A} \cdot \vec{B}}{AB}\right)$$

$$\theta = \mathrm{acos}\left(\frac{a^2}{(\sqrt{3}a)(\sqrt{3}a)}\right) = \mathrm{acos}\left(\frac{1}{2}\right)$$

$$\boxed{1.3 \quad \theta \approx 1.231\mathrm{rad}}$$

Problem 1.4 Use the cross product to find the components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the shaded plane in Fig 1.11.

En el plano de la figura 1.11 están ubicados los vertices $(1, 0, 0)$, $(0, 2, 0)$, y $(0, 0, 3)$, por lo que tomaremos los vectores $\vec{A}$ de $(1, 0, 0)$ a $(0, 2, 0)$, y $\vec{B}$ de $(1, 0, 0)$ a $(0, 0, 3)$.

Vectores posicionados en el origen

$$\vec{A} = -1\hat{\imath} + 2\hat{\jmath} + 0\hat{k} \quad , \quad \vec{B} = -1\hat{\imath} + 0\hat{\jmath} + 3\hat{k}$$

Producto vectorial - vector normal

$$\vec{n} = \vec{A} \times \vec{B}$$

$$\vec{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = [2(3) - 0]\hat{\imath} - [-1(3) - 0]\hat{\jmath} + [0 - 2(-1)]\hat{k}$$

$$\vec{n} = 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$$

$$n = |\vec{n}| = \sqrt{6^2 + 3^2 + 2^2} = 7$$

Vector normal unitario

$$\hat{n} = \frac{\vec{n}}{n} = \frac{1}{7}(6\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$$

$$\boxed{1.4 \quad \hat{n} = \frac{6}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{2}{7}\hat{k}}$$

Problem 1.5 Prove the **BAC-CAB** rule by writing out both sides in component form.

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) \quad ?$$

$$\begin{gathered}
\vec{A}\times(\vec{B}\times\vec{C})= \\
[(b_xc_y-b_yc_x)a_y-(b_zc_x-b_xc_z)a_z]\hat{\imath} \\
+[(b_yc_z-b_zc_y)a_z-(b_xc_y-b_yc_x)a_x]\hat{\jmath} \\
+[(b_zc_x-b_xc_z)a_x-(b_yc_z-b_zc_y)a_y]\hat{k}
\end{gathered}$$

$$\vec{B}(\vec{A}\cdot\vec{C})=(a_xc_x+a_yc_y+a_zc_z)(b_x\hat{\imath}+b_y\hat{\jmath}+b_z\hat{k})$$

$$\vec{C}(\vec{A}\cdot\vec{B})=(a_xb_x+a_yb_y+a_zb_z)(c_x\hat{\imath}+c_y\hat{\jmath}+c_z\hat{k})$$

$$\begin{gathered}
\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})= \\
[(a_yc_y+a_zc_z)b_x-(a_yb_y+a_zb_z)c_x]\hat{\imath} \\
+[(a_xc_x+a_zc_z)b_y-(a_xb_x+a_zb_z)c_y]\hat{\jmath} \\
+[(a_xc_x+a_yc_y)b_z-(a_xb_x+a_yb_y)c_z]\hat{k}=
\end{gathered}$$

$$\begin{gathered}
(a_yc_yb_x+a_zc_zb_x-a_yb_yc_x-a_zb_zc_x)\hat{\imath} \\
+(a_xc_xb_y+a_zc_zb_y-a_xb_xc_y-a_zb_zc_y)\hat{\jmath} \\
+(a_xc_xb_z+a_yc_yb_z-a_xb_xc_z-a_yb_yc_z)\hat{k}
\end{gathered}$$

$$\begin{gathered}
\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})= \\
[(b_xc_y-b_yc_x)a_y-(b_zc_x-b_xc_z)a_z]\hat{\imath} \\
+[(b_yc_z-b_zc_y)a_z-(b_xc_y-b_yc_x)a_x]\hat{\jmath} \\
+[(b_zc_x-b_xc_z)a_x-(b_yc_z-b_zc_y)a_y]\hat{k}
\end{gathered}$$

$$\boxed{1.5 \quad \vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) \quad \blacksquare}$$

Problem 1.6 Prove that

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = \mathbf{0}$$

Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?

$$[\vec{A} \times (\vec{B} \times \vec{C})] + [\vec{B} \times (\vec{C} \times \vec{A})] + [\vec{C} \times (\vec{A} \times \vec{B})] = 0 \quad ?$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\vec{B} \times (\vec{C} \times \vec{A}) = \vec{C}(\vec{B} \cdot \vec{A}) - \vec{A}(\vec{B} \cdot \vec{C})$$

$$\vec{C} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{C} \cdot \vec{B}) - \vec{B}(\vec{C} \cdot \vec{A})$$

$$\begin{gathered}
\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \\
\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) + \vec{C}(\vec{B} \cdot \vec{A}) - \vec{A}(\vec{B} \cdot \vec{C}) + \vec{A}(\vec{C} \cdot \vec{B}) - \vec{B}(\vec{C} \cdot \vec{A}) = \\
\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) + \vec{C}(\vec{A} \cdot \vec{B}) - \vec{A}(\vec{B} \cdot \vec{C}) + \vec{A}(\vec{B} \cdot \vec{C}) - \vec{B}(\vec{A} \cdot \vec{C})
\end{gathered}$$

$$\boxed{1.6 \quad [\vec{A} \times (\vec{B} \times \vec{C})] + [\vec{B} \times (\vec{C} \times \vec{A})] + [\vec{C} \times (\vec{A} \times \vec{B})] = 0 \quad \blacksquare}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \times \vec{C} \quad ?$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} \times \vec{B}) = \vec{B}(\vec{C} \cdot \vec{A}) - \vec{A}(\vec{C} \cdot \vec{B})$$

$$\boxed{\vec{A} \cdot \vec{B} \neq \vec{C} \cdot \vec{B} \neq 0}$$

$$\vec{C}(\vec{A} \cdot \vec{B}) = \vec{A}(\vec{C} \cdot \vec{B}) \quad \rightarrow \quad \vec{A} \parallel \vec{C} \quad \rightarrow \quad \vec{A} \cdot \vec{C} = |\vec{A}||\vec{C}|$$

1.6

$$(\vec{A} \times \vec{B}) \times \vec{C} = \vec{A} \times (\vec{B} \times \vec{C}) \quad : \quad \vec{A} \cdot \vec{B} = \vec{C} \cdot \vec{B} = 0 \quad \cup \quad \vec{A} \cdot \vec{C} = \pm|\vec{A}||\vec{C}|$$

La propiedad se cumple si $\vec{B}$ es perpendicular con $\vec{A}$ y $\vec{C}$, o que $\vec{A}$ y $\vec{C}$ sean paralelos.

Problem 1.7 Find the separation vector $\boldsymbol{\mathscr{r}}$ from the source point $(2, 8, 7)$ to the field point $(4, 6, 8)$. Determine its magnitude ($\mathscr{r}$), and construct the unit vector $\hat{\boldsymbol{\mathscr{r}}}$.

Vector de separación

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$\vec{r} = (4,6,8) \quad , \quad \vec{r}' = (2,8,7) \quad \rightarrow \quad \vec{\mathscr{r}} = (2,-2,1)$$

$$\mathscr{r} = |\ \vec{\mathscr{r}}\ | \quad \rightarrow \quad \mathscr{r} = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{1}{3}(2,-2,1)$$

$$\boxed{1.7 \quad \hat{\mathscr{r}} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)}$$

Problem 1.8

(a) Prove that the two-dimensional rotation matrix (Eq. 1.29) preserves dot products. (That is, show that $\overline{A}_y\overline{B}_y + \overline{A}_z\overline{B}_z = A_yB_y + A_zB_z$.)

(b) What constraints must the element (R_{ij}) of the three-dimensional rotation matrix (Eq. 1.30) satisfy, in order to preserve the length of $\mathbf{A}$ (for all vectors $\mathbf{A}$)?

Ecuación 1.29

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_z \end{pmatrix}$$

$$P_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$\vec{A}\cdot\vec{B} = \left(P_\phi\vec{A}\right)\cdot\left(P_\phi\vec{B}\right) \quad ?$$

$$\bar{A}_x = A_x\cos\phi + A_z\sin\phi \quad , \quad \bar{A}_z = -A_x\sin\phi + A_z\cos\phi$$
$$\bar{B}_x = B_x\cos\phi + B_z\sin\phi \quad , \quad \bar{B}_z = -B_x\sin\phi + B_z\cos\phi$$

$$\left(P_\phi\vec{A}\right)\cdot\left(P_\phi\vec{B}\right) = \bar{A}_x\bar{B}_x + \bar{A}_z\bar{B}_z =$$
$$(A_x\cos\phi + A_z\sin\phi)(B_x\cos\phi + B_z\sin\phi) + (-A_x\sin\phi + A_z\cos\phi)(-B_x\sin\phi + B_z\cos\phi)$$

$$\left(P_\phi\vec{A}\right)\cdot\left(P_\phi\vec{B}\right) =$$
$$A_xB_x\cos^2\phi + A_xB_z\cos\phi\sin\phi + A_zB_x\cos\phi\sin\phi + A_zB_z\sin^2\phi$$
$$A_xB_x\sin^2\phi - A_xB_z\cos\phi\sin\phi - A_zB_x\cos\phi\sin\phi + A_zB_z\cos^2\phi$$

$$\left(P_\phi\vec{A}\right)\cdot\left(P_\phi\vec{B}\right) = (A_xB_x + A_zB_z)(\cos^2\phi + \sin^2\phi) = A_xB_x + A_zB_z$$

$$\boxed{1.8\ (a) \quad \vec{A}\cdot\vec{B} = \left(P_\phi\vec{A}\right)\cdot\left(P_\phi\vec{B}\right) \quad \blacksquare}$$

Ecuación 1.30

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\bar{A}_i = \sum_j R_{ij} A_j$$

$$|\vec{A}|^2 = \vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2 = \bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2$$

$$\sum_i A_i^2 = \left(\sum_j R_{xj} A_j \right)^2 + \left(\sum_j R_{yj} A_j \right)^2 + \left(\sum_j R_{zj} A_j \right)^2$$

$$\sum_i A_i^2 = \sum_i \left(\sum_j R_{ij} A_j \right)^2 = \sum_i \left(\sum_j R_{ij} A_j \right) \left(\sum_k R_{ik} A_k \right)$$

$$\sum_i A_i^2 = \sum_i \sum_{j,k} R_{ij} R_{ik} A_j A_k = \sum_{j,k} A_j A_k \sum_i R_{ij} R_{ik}$$

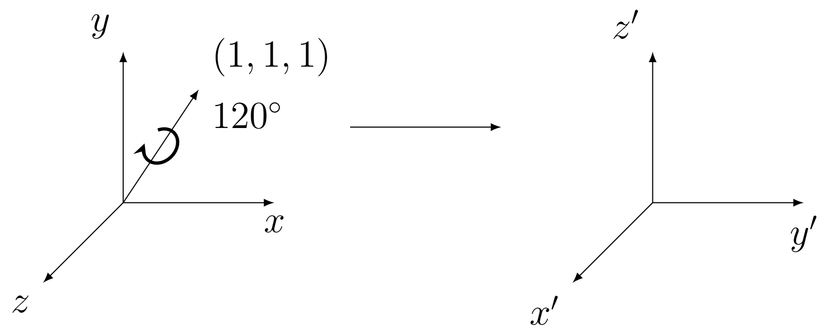
$$\boxed{\sum_i R_{ij} R_{ik} = \delta_{jk}}$$

$$\sum_i A_i^2 = \sum_{j,k} A_j A_k \delta_{jk} = \sum_i A_i A_i = \sum_i A_i^2$$

$$\boxed{\text{1.8 (b)} \quad \text{Restricción:} \quad \sum_i R_{ij} R_{ik} = \delta_{jk} = \left\{ \begin{array}{lcl} 0 & : & j \neq k \\ 1 & : & j = k \end{array} \right.}$$

Problem 1.9 Find the transformation matrix R that describes a rotation by $120°$ about an axis from the origin through the point $(1, 1, 1)$. The rotation is clockwise as you look down the axis toward the origin.

Transformación de la matriz R



$$\boxed{\therefore \quad \bar{A}_x = A_z \quad , \quad \bar{A}_z = A_y \quad , \quad \bar{A}_y = A_x}$$

$$\bar{A}_i = \sum_j R_{ij} A_j$$

$$\bar{A}_x = R_{xx}A_x + R_{xy}A_y + R_{xz}A_z = A_z \quad \rightarrow \quad R_{xx} = R_{xy} = 0 \quad , \quad R_{xz} = 1$$

$$\bar{A}_y = R_{yx}A_x + R_{yy}A_y + R_{yz}A_z = A_x \quad \rightarrow \quad R_{yy} = R_{yz} = 0 \quad , \quad R_{yx} = 1$$

$$\bar{A}_z = R_{zx}A_x + R_{zy}A_y + R_{zz}A_z = A_y \quad \rightarrow \quad R_{zx} = R_{zz} = 0 \quad , \quad R_{zy} = 1$$

$$\boxed{1.9 \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

Problem 1.10

(a) How do the components of a vector[5] transform under a **translation** of coordinates ($\overline{x} = x, \overline{y} = y - a, \overline{z} = z$, Fig. 1.16a)?

(b) How do the components of a vector transform under an **inversion** of coordinates ($\overline{x} = -x, \overline{y} = -y, \overline{z} = -z$, Fig. 1.16b)?

(c) How do the components of a cross product (Eq. 1.13) transform under inversion? [The cross-product of two vectors is properly called a **pseudovector** because of this "anomalous" behavoir.] Is the croos product of two pseudovectors a vector, or a pseudovector? Name two pseudovector quantities in classical mechanics.

(d) How does the scalar triple product of three vectors transform under inversions? (Such an object is called a **pseudoscalar.**)

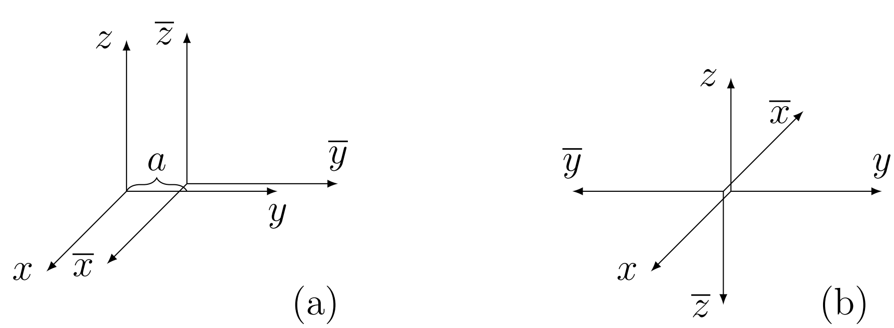


FIGURE 1.16

[5] *Beware:* The vector **r** (Eq. 1.19) goes from a specific point in space (the origin, $\mathcal{O}$) to the point $P = (x, y, z)$. Under translations the *new* origin ($\bar{\mathcal{O}}$) is at a different location, and the arrow from $\bar{\mathcal{O}}$ to P is a completely different vector. The original vector **r** still goes from $\mathcal{O}$ to P, regardless of the coordinates used to label these points.

1.10 (a)
Si ocurre una traslación del vector $\vec{A}$ a otro origen, las componentes en la nueva base no cambian:

$$\vec{A}' \to \vec{A}$$

$$\bar{A}_x = A_x \quad , \quad \bar{A}_y = A_y \quad , \quad \bar{A}_z = A_z$$

1.10 (b)
En el caso de una inversión, el sentido del vector se invierte al opuesto del original, lo que equivale a multiplicarlo por (-1):

$$\vec{A}' \to -\vec{A}$$

$$\bar{A}_x = -A_x \quad , \quad \bar{A}_y = -A_y \quad , \quad \bar{A}_z = -A_z$$

.

Ecuación 1.13

$$\mathbf{A} \times \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}$$

Inversión

$$\vec{A}' \to -\vec{A} \quad , \quad \vec{B}' \to -\vec{B}$$

$$\vec{C} = \vec{A} \times \vec{B}$$

$$\vec{C}' = \vec{A}' \times \vec{B}' = (-\vec{A}) \times (-\vec{B}) = \vec{A} \times \vec{B}$$

$$\therefore \quad \vec{C}' \to \vec{C}$$

1.10 (c)
En el caso de una transformación por inversión, tenemos que para el vector resultante del producto vectorial de dos vectores ($\vec{C} = \vec{A} \times \vec{B}$) no ocurre inversión, sus componentes permanecen iguales, por ello se considera un pseudovector.

$$\text{Tomamos a } \vec{A} \text{ y } \vec{B} \text{ como pseudovectores}$$

$$\vec{A}' \to \vec{A} \quad , \quad \vec{B}' \to \vec{B}$$

$$\vec{C} = \vec{A} \times \vec{B}$$

$$\vec{C}' = \vec{A}' \times \vec{B}' = (\vec{A}) \times (\vec{B}) = \vec{A} \times \vec{B}$$

$$\therefore \quad \vec{C}' \to \vec{C}$$

1.10 (c)
En el caso de una transformación por inversión, tenemos que para el vector resultante del producto vectorial de dos pseudovectores ($\vec{C} = \vec{A} \times \vec{B}$) no ocurre inversión, sus componentes permanecen iguales, por ello se considera un pseudovector.

.

$$\text{Tomamos a } \vec{A} \text{ como pseudovector}$$

$$\vec{A}' \to \vec{A} \quad , \quad \vec{B}' \to -\vec{B}$$

$$\vec{C} = \vec{A} \times \vec{B}$$

$$\vec{C}' = \vec{A}' \times \vec{B}' = (\vec{A}) \times (-\vec{B}) = -\vec{A} \times \vec{B}$$

$$\therefore \quad \vec{C}' \to -\vec{C}$$

1.10 (c)
En el caso de una transformación por inversión, tenemos que para el vector resultante del producto vectorial de un vector y un pseudovector ($\vec{C} = \vec{A} \times \vec{B}$) ocurre inversión, por ello se considera un vector “ordinario”.

1.10 (c)

Ejemplos de pseudovectores en la mecánica clásica:

$$\text{Momento angular:} \quad \vec{L} = \vec{r} \times \vec{p}$$
$$\text{Momento de torsión:} \quad \vec{M} = \vec{r} \times \vec{F}$$

.

Triple producto escalar

$$c = \vec{A} \cdot (\vec{B} \times \vec{C})$$

$$c' = \vec{A}' \cdot (\vec{B}' \times \vec{C}') = (-\vec{A}) \cdot [(-\vec{B}) \times (-\vec{C})] = -\vec{A} \cdot (\vec{B} \times \vec{C}) = c$$

$$\therefore \quad c' \rightarrow -c$$

1.10 (d)
En el caso de una transformación por inversión, el pseudoescalar resultante del producto triple producto escalar de tres vectores ($c = \vec{A} \cdot (\vec{B} \times \vec{C})$) invierte su signo.

Problem 1.11 Find the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Gradiente de una función

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

$$f(x,y,z) = x^2 + y^3 + z^4 \quad \rightarrow \quad \frac{\partial f}{\partial x} = 2x \quad , \quad \frac{\partial f}{\partial y} = 3y^2 \quad , \quad \frac{\partial f}{\partial z} = 4z^3$$

$$\boxed{1.11 \text{ (a)} \quad \nabla f = 2x\hat{\imath} + 3y^2\hat{\jmath} + 4z^3\hat{k}}$$

$$f(x,y,z) = x^2y^3z^4 \quad \rightarrow \quad \frac{\partial f}{\partial x} = 2xy^3z^4 \quad , \quad \frac{\partial f}{\partial y} = 3x^2y^2z^4 \quad , \quad \frac{\partial f}{\partial z} = 4x^2y^3z^3$$

$$\boxed{1.11 \text{ (b)} \quad \nabla f = 2xy^3z^4\hat{\imath} + 3x^2y^2z^4\hat{\jmath} + 4x^2y^3z^3\hat{k}}$$

$$f(x,y,z) = e^x \sin y \ln z$$

$$\frac{\partial f}{\partial x} = e^x \sin y \ln z \quad , \quad \frac{\partial f}{\partial y} = -e^x \cos y \ln z \quad , \quad \frac{\partial f}{\partial z} = \frac{1}{z}e^x \sin y$$

$$\boxed{1.11 \text{ (c)} \quad \nabla f = e^x \sin y \ln z\hat{\imath} - e^x \cos y \ln z\hat{\jmath} + \frac{1}{z}e^x \sin y\hat{k}}$$

Problem 1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

Punto critico

$$\nabla f = \vec{0} \quad \to |\nabla f| = 0$$

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

$$\frac{\partial h}{\partial x} = 10(2y - 6x - 18) \quad , \quad \frac{\partial h}{\partial y} = 10(2x - 8y + 28)$$

$$\therefore \quad \nabla h(x,y) = 10(2y - 6x - 18)\hat{\imath} + 10(2x - 8y + 28)\hat{\jmath}$$

$$\nabla h = \vec{0} = 0\hat{\imath} + 0\hat{\jmath}$$

$$10(2y - 6x - 18) = 0 \quad , \quad 10(2x - 8y + 28) = 0$$

$$y - 3x = 9 \quad , \quad x - 4y = -14$$

$$y = 3x + 9 \quad \to \quad x - 4(3x + 9) = -14 \quad \to \quad -11x = 22 \quad \to \quad x = -2$$

$$y = 3(-2) + 9 = 3$$

1.12 (a)

$$x = -2 \quad , \quad y = 3$$

La cima se encuentra 3 millas al norte y 2 al oeste de South Hadley.

$$h(-2,3) = 10[2(-2)(3) - 3(-2)^2 - 4(3)^2 - 18(-2) + 28(3) + 12] = 720$$

1.12 (b)

$$h(-2,3) = 720 \quad \rightarrow \quad h = 720\text{ft}$$

La altura en la cima es de 720 pies.

.

$$\nabla h(1,1) = 10[2(1){-}6(1){-}18]\hat{\imath}{+}10[2(1){-}8(1){+}28]\hat{\jmath} = 10[2{-}6{-}18]\hat{\imath}{+}10[2{-}8{+}28]\hat{\jmath}$$

$$\boxed{\nabla h(1,1) = -220\hat{\imath} + 220\hat{\jmath}}$$

$$|\nabla h(1,1)| = \sqrt{(-220)^2 + (220)^2} = \sqrt{2(220)^2}$$

$$\boxed{|\nabla h(1,1)| = 220\sqrt{2} \approx 311.127}$$

1.12 (c) La pendiente a una milla al norte y una al este de South Hadley es de aproximadamente $311.127\frac{\text{ft}}{\text{milla}}$ (pies por milla), en dirección noroeste.

Problem 1.13 Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$.

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the general formula for $\nabla(\mathscr{r}^n)$?

Vector de separación

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = (x - x')\hat{\imath} + (y - y')\hat{\jmath} + (z - z')\hat{k}$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\nabla(\mathscr{r}^2) = \nabla\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]$$

$$\nabla(\mathscr{r}^2) = \frac{\partial}{\partial x}(x - x')^2\hat{\imath} + \frac{\partial}{\partial y}(y - y')^2\hat{\jmath} + \frac{\partial}{\partial z}(z - z')^2\hat{k}$$

$$\nabla(\mathscr{r}^2) = 2(x - x')\hat{\imath} + 2(y - y')\hat{\jmath} + 2(z - z')\hat{k} = 2\left[(x - x')\hat{\imath} + (y - y')\hat{\jmath} + (z - z')\hat{k}\right]$$

$$\boxed{1.12\ (a) \quad \nabla(\mathscr{r}^2) = 2\,\vec{\mathscr{r}} \qquad \blacksquare}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = \nabla\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{1}{2}}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) =$$
$$\frac{\partial}{\partial x}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{1}{2}}\hat{\imath}$$
$$+\frac{\partial}{\partial y}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{1}{2}}\hat{\jmath}$$
$$+\frac{\partial}{\partial z}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{1}{2}}\hat{k}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) =$$
$$-\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{3}{2}}\left[\frac{\partial}{\partial x}(x-x')^2\hat{\imath}+\frac{\partial}{\partial y}(y-y')^2\hat{\jmath}+\frac{\partial}{\partial z}(z-z')^2\hat{k}\right]$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{3}{2}}\left[(x-x')\hat{\imath}+(y-y')\hat{\jmath}+(z-z')\hat{k}\right]$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\left(\mathscr{r}^{\,2}\right)^{-\frac{3}{2}}\vec{\mathscr{r}} = -\frac{1}{\mathscr{r}^{\,3}}\vec{\mathscr{r}} = -\frac{1}{\mathscr{r}^{\,2}}\frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

$$\boxed{1.13\ \text{(b)} \quad \nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}}}$$

$$\nabla(\mathscr{r}^{\,n}) = \nabla\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{n}{2}}$$

$$\begin{aligned}\nabla(\mathscr{r}^{\,n}) = {}& \\ &\frac{\partial}{\partial x}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{n}{2}}\hat{\imath} \\ &+\frac{\partial}{\partial y}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{n}{2}}\hat{\jmath} \\ &+\frac{\partial}{\partial z}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{n}{2}}\hat{k}\end{aligned}$$

$$\nabla(\mathscr{r}^{\,n}) =$$
$$\frac{n}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{n}{2}-1}\left[\frac{\partial}{\partial x}(x-x')^2\hat{\imath}+\frac{\partial}{\partial y}(y-y')^2\hat{\jmath}+\frac{\partial}{\partial z}(z-z')^2\hat{k}\right]$$

$$\nabla(\mathscr{r}^{\,n}) = n\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{n}{2}-1}\left[(x-x')\hat{\imath}+(y-y')\hat{\jmath}+(z-z')\hat{k}\right]$$

$$\nabla(\mathscr{r}^{\,n}) = n\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}(n-2)}\left[(x-x')\hat{\imath}+(y-y')\hat{\jmath}+(z-z')\hat{k}\right]$$

$$\nabla(\mathscr{r}^{\,n}) = n\,\mathscr{r}^{\,n-2}\,\vec{\mathscr{r}} = n\,\mathscr{r}^{\,n-1}\frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

$$\boxed{1.13\text{ (c)}\quad \nabla(\mathscr{r}^{\,n}) = n\,\mathscr{r}^{\,n-1}\,\hat{\mathscr{r}}}$$

Problem 1.14 Suppose that f is a function of two variables (y and z) only. Show that the gradient $\nabla f = (\partial f/\partial y)\hat{\mathbf{y}} + (\partial f/\partial z)\hat{\mathbf{z}}$ transforms as a vector under rotations. Eq. 1.29. [*Hint:* $(\partial f/\partial \bar{y}) = (\partial f/\partial y)(\partial y/\partial \bar{y}) + (\partial f/\partial z)(\partial z/\partial \bar{y})$, and the analogous formula for $\partial f/\partial \bar{z}$. We know that $\bar{y} = y\cos\phi + z\sin\phi$ and $\bar{z} = -y\sin\phi + z\cos\phi$; "solve" these equations for y and z (as functions of $\bar{y}$ and $\bar{z}$), and compute the needed derivatives $\partial y/\partial \bar{y}$, $\partial z/\partial \bar{y}$, etc.]

Ecuación 1.29

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_z \end{pmatrix}$$

$$\bar{y} = y\cos\phi + z\sin\phi \quad , \quad \bar{z} = -y\sin\phi + z\cos\phi$$

$$\bar{y}\cos\phi = y\cos^2\phi + z\cos\phi\sin\phi$$

$$\bar{z}\sin\phi = -y\sin^2\phi + z\cos\phi\sin\phi$$

$$\therefore \quad y = \bar{y}\cos\phi - \bar{z}\sin\phi$$

$$\frac{\partial y}{\partial \bar{y}} = \cos\phi \quad , \quad \frac{\partial y}{\partial \bar{z}} = -\sin\phi$$

$$\bar{y}\sin\phi = y\cos\phi\sin\phi + z\sin^2\phi$$

$$\bar{z}\cos\phi = -y\cos\phi\sin\phi + z\cos^2\phi$$

$$\therefore \quad z = \bar{y}\sin\phi + \bar{z}\cos\phi$$

$$\frac{\partial z}{\partial \bar{y}} = \sin\phi \quad , \quad \frac{\partial z}{\partial \bar{z}} = \cos\phi$$

$$\nabla f = \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \quad , \quad \nabla f = \frac{\partial f}{\partial \bar{y}}\bar{\hat{\jmath}} + \frac{\partial f}{\partial \bar{z}}\bar{\hat{k}}$$

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}} \quad , \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{z}}$$

$$\nabla f = \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}}\right)\bar{\hat{\jmath}} + \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{z}}\right)\bar{\hat{k}}$$

$$\nabla f = \left(\frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi\right)\bar{\hat{\jmath}} + \left(-\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi\right)\bar{\hat{k}}$$

$$\therefore$$

$$(\nabla f)_{\bar{y}} = (\nabla f)_y \cos\phi + (\nabla f)_z \sin\phi$$

$$(\nabla f)_{\bar{z}} = -(\nabla f)_y \sin\phi + (\nabla f)_z \cos\phi$$

$$\boxed{1.14 \quad \begin{pmatrix} (\nabla f)_{\bar{y}} \\ (\nabla f)_{\bar{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} (\nabla f)_y \\ (\nabla f)_z \end{pmatrix} \quad \blacksquare}$$

Problem 1.15 Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$.

Divergencia de una función

$$\nabla \cdot \vec{v}(x,y,z) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\vec{v}_a = x^2\hat{\imath} + 3xz^2\hat{\jmath} - 2xz\hat{k} \quad \rightarrow \quad \frac{\partial v_x}{\partial x} = 2x \quad , \quad \frac{\partial v_y}{\partial y} = 0 \quad , \quad \frac{\partial v_z}{\partial z} = -2x$$

$$\boxed{1.15\text{ (a)} \quad \nabla \cdot \vec{v}_a = 0}$$

$$\vec{v}_b = xy\hat{\imath} + 2yz\hat{\jmath} + 3zx\hat{k} \quad \rightarrow \quad \frac{\partial v_x}{\partial x} = y \quad , \quad \frac{\partial v_y}{\partial y} = z \quad , \quad \frac{\partial v_z}{\partial z} = x$$

$$\boxed{1.15\text{ (b)} \quad \nabla \cdot \vec{v}_b = x + 2y + 3z}$$

$$\vec{v}_c = y^2\hat{\imath} + (2xy + z^2)\hat{\jmath} + 2yz\hat{k} \quad \rightarrow \quad \frac{\partial v_x}{\partial x} = 0 \quad , \quad \frac{\partial v_y}{\partial y} = 2x \quad , \quad \frac{\partial v_z}{\partial z} = 2y$$

$$\boxed{1.15\text{ (c)} \quad \nabla \cdot \vec{v}_c = 2(x + y)}$$

Problem 1.16 Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you...can you explain it?

Vector de posición

$$\hat{r} = \frac{x\hat{\imath} + y\hat{\jmath} + x\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \quad , \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\vec{v} = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = \frac{x\hat{\imath} + y\hat{\jmath} + x\hat{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x}\left[x(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial y}\left[y(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial z}\left[z(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right]$$

$$\frac{\partial}{\partial x}\left[x(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right] = (x^2 + y^2 + z^2)^{-\frac{3}{2}} - 3x^2(x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

$$\therefore \quad \nabla \cdot \vec{v} = 3(x^2+y^2+z^2)^{-\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} - 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} - 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\nabla \cdot \vec{v} = 3(x^2 + y^2 + z^2)^{-\frac{3}{2}} - 3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

$$\nabla \cdot \vec{v} = 0$$

1.16

$$\nabla \cdot \vec{v} = 0 \quad : \quad \vec{v} = \frac{\hat{r}}{r^2}$$

La divergencia en todo el espacio es $\nabla \cdot \vec{v} = 0$, con excepción del origen que es una “fuente”, en el origen la función no esta definida, por lo que $\nabla \cdot \vec{v}$ tiende a infinito.