Introduction to Quantum Mechanics

Second Edition

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Solucionario

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Problem 1.1 For the distribution of ages in Section 1.3.1:

(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$.
(b) Determine Δj for each j, and use Equation 1.11 to compute the standard deviation.
(c) Use your results in (a) and (b) to check Equation 1.12.

Valor promedio de j

$$\langle j \rangle = \frac{\sum_{j=0}^{\infty} jN(j)}{N} = \sum_{j=0}^{\infty} jP(j)$$

$$N = \sum_{j=0}^{\infty} N(j)$$

$$N(14) = 1$$
$$N(15) = 1$$
$$N(16) = 3$$
$$N(22) = 2$$
$$N(24) = 2$$
$$N(25) = 5$$

$$N = 14$$

$$\langle j \rangle = \frac{14(1) + 15(1) + 16(3) + 22(2) + 24(2) + 25(5)}{14} = 21$$

$$\boxed{\textbf{1.1 (a)} \quad \langle j \rangle^2 = 441}$$

Valor promedio de j^2

$$\langle j \rangle = \frac{\sum_{j=0}^{\infty} j^2 N(j)}{N} = \sum_{j=0}^{\infty} j^2 P(j)$$

$$\langle j \rangle = \frac{14^2(1) + 15^2(1) + 16^2(3) + 22^2(2) + 24^2(2) + 25^2(5)}{14} = 21$$

$$\boxed{\textbf{1.1 (a)} \quad \langle j^2 \rangle \approx 459.571}$$

Desviación de la media

$$\Delta j = j - \langle j \rangle$$

$$j = 14 \quad \rightarrow \quad \Delta j = 14 - 21 = -7$$
$$j = 15 \quad \rightarrow \quad \Delta j = 15 - 21 = -6$$
$$j = 16 \quad \rightarrow \quad \Delta j = 16 - 21 = -5$$
$$j = 22 \quad \rightarrow \quad \Delta j = 22 - 21 = 1$$
$$j = 24 \quad \rightarrow \quad \Delta j = 24 - 21 = 3$$
$$j = 25 \quad \rightarrow \quad \Delta j = 25 - 21 = 4$$

Ecuación 1.11 (Definición de varianza)

$$\sigma^2 \equiv \langle (\Delta j)^2 \rangle$$

$$\sigma^2 = \frac{\sum_{j=0}^{\infty} (\Delta j)^2 N(j)}{N}$$

$$\sigma^2 = \frac{(-7)^2(1) + (-6)^2(1) + (-5)^2(3) + 1^2(2) + 3^2(2) + 4^2(5)}{14} = 21$$

$$\boxed{\sigma^2 \approx 18.571}$$

Desviación estándar

$$\sigma = \sqrt{\sigma^2}$$

$$\sigma \approx \sqrt{18.571}$$

$$\boxed{\textbf{1.1 (b)} \quad \sigma \approx 4.309}$$

Ecuación 1.12

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

$$\sigma \approx \sqrt{459.571 - 441} = \sqrt{18.571}$$

El resultado en (c) coincide con el resultado en (b)

$$\boxed{\textbf{1.1 (c)} \quad \sigma \approx 4.309}$$

Problem 1.2

(a) Find the standard deviation of the Example 1.1.
(b) What is the probability that a photograph, selected at random, would show a distance x more than one standard deviation away from average?

Función de densidad de probabilidad

$$\rho(x) = \frac{1}{2\sqrt{hx}} \quad : \quad x \in (0, h]$$

Valor de expectación de x

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx$$

$$\langle x \rangle = \int_0^h x\rho(x)dx = \int_0^h \frac{x}{2\sqrt{hx}}dx = \frac{1}{2\sqrt{h}}\int_0^h x^{\frac{1}{2}}dx = \frac{1}{2\sqrt{h}}\left|\frac{2}{3}x^{\frac{3}{2}}\right|_0^h = \frac{1}{3\sqrt{h}}\left|x^{\frac{3}{2}}\right|_0^h$$

$$\langle x \rangle = \frac{1}{3\sqrt{h}}\left(h^{\frac{3}{2}} - 0\right)$$

$$\boxed{\langle x \rangle = \frac{h}{3}}$$

Valor de expectación de x^2

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\rho(x)dx$$

$$\langle x^2 \rangle = \int_0^h x^2\rho(x)dx = \int_0^h \frac{x^2}{2\sqrt{hx}}dx = \frac{1}{2\sqrt{h}}\int_0^h x^{\frac{3}{2}}dx = \frac{1}{2\sqrt{h}}\left|\frac{2}{5}x^{\frac{5}{2}}\right|_0^h = \frac{1}{5\sqrt{h}}\left|x^{\frac{5}{2}}\right|_0^h$$

$$\langle x^2 \rangle = \frac{1}{5\sqrt{h}}\left(h^{\frac{5}{2}} - 0\right)$$

$$\boxed{\langle x^2 \rangle = \frac{h^2}{5}}$$

Desviación estándar

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\sigma = \sqrt{\frac{h^2}{5} - \left(\frac{h}{3}\right)^2} = \sqrt{\frac{h^2}{5} - \frac{h^2}{9}} = \sqrt{\frac{4h^2}{45}}$$

$$\boxed{\textbf{1.2 (a)} \quad \sigma = \frac{2h}{3\sqrt{5}}}$$

Probabilidad de encontrar x fuera de la desviación estándar

$$P = 1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x) dx$$

$$P = 1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \frac{1}{2\sqrt{hx}} dx = 1 - \frac{1}{2\sqrt{h}} \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} x^{-\frac{1}{2}} dx = 1 - \frac{1}{\sqrt{h}} \left| x^{\frac{1}{2}} \right|_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma}$$

$$P = 1 - \frac{1}{\sqrt{h}} \left| x^{\frac{1}{2}} \right|_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} = 1 - \frac{1}{\sqrt{h}} \left[(\langle x \rangle + \sigma)^{\frac{1}{2}} - (\langle x \rangle - \sigma)^{\frac{1}{2}} \right]$$

$$P = 1 - \frac{1}{\sqrt{h}} \left[\left(\frac{h}{3} + \frac{2h}{3\sqrt{5}} \right)^{\frac{1}{2}} - \left(\frac{h}{3} - \frac{2h}{3\sqrt{5}} \right)^{\frac{1}{2}} \right]$$

$$P \approx 1 - \frac{1}{\sqrt{h}} \left[(0.6315h)^{\frac{1}{2}} - (0.03519h)^{\frac{1}{2}} \right] = 1 - 0.607$$

$$\boxed{\textbf{1.2 (b)} \quad P \approx 0.393}$$

Problem 1.3 Consider the **gaussian** distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where A, a, and λ are real positive constants. (Look up any integrals you need.)

(a) Use Equation 1.16 to determinate A.
(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ.
(c) Sketch the graph of $\rho(x)$.

Ecuación 1.16

$$\int_{-\infty}^{\infty} \rho(x)dx = 1$$

$$\int_{-\infty}^{\infty} \rho(x)dx = A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}dx = 1$$

Función simétrica

$$\int_{-\infty}^{0} e^{-\lambda(x-a)^2}dx = \int_{0}^{\infty} e^{-\lambda(x-a)^2}dx$$

$$\therefore \quad \int_{-\infty}^{\infty} \rho(x)dx = 2A\int_{0}^{\infty} e^{-\lambda(x-a)^2}dx = 1$$

$$x = \sqrt{\frac{u}{\lambda}} + a \quad \rightarrow \quad u = \lambda(x-a)^2 \quad \rightarrow \quad du = 2\lambda(x-a)dx \quad \rightarrow \quad dx = \frac{1}{2\lambda(x-a)}du$$

$$dx = \frac{1}{2\sqrt{\lambda u}}du$$

$$\therefore \quad \int_{-\infty}^{\infty} \rho(x)dx = \frac{A}{\sqrt{\lambda}}\int_{0}^{\infty} u^{-\frac{1}{2}}e^{-u}du = 1$$

$$\frac{A}{\sqrt{\lambda}}\int_{0}^{\infty} u^{-\frac{1}{2}}e^{-u}du = \frac{A}{\sqrt{\lambda}}\Gamma\left(\frac{1}{2}\right) = \frac{A}{\sqrt{\lambda}}\sqrt{\pi} = 1$$

$$\boxed{\textbf{1.3 (a)} \quad A = \sqrt{\frac{\lambda}{\pi}}}$$

Valor de expectación de x

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)dx = A\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}dx$$

$$u = x - a \quad \rightarrow \quad du = dx$$

$$\langle x\rangle = A\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du = A\left[\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right]$$

Función impar

$$\int_{-\infty}^{0} ue^{-\lambda u^2}du = -\int_{0}^{\infty} ue^{-\lambda u^2}du \quad \rightarrow \quad \int_{-\infty}^{\infty} ue^{-\lambda u^2}du = 0$$

Función par

$$\int_{-\infty}^{0} e^{-\lambda u^2}du = \int_{0}^{\infty} e^{-\lambda u^2}du \quad \rightarrow \quad \int_{-\infty}^{\infty} e^{-\lambda u^2}du = 2\int_{0}^{\infty} e^{-\lambda u^2}du$$

$$\therefore \quad \langle x\rangle = 2aA\int_{0}^{\infty} e^{-\lambda u^2}du$$

$$u = \sqrt{\frac{t}{\lambda}} \quad \rightarrow \quad t = \lambda u^2 \quad \rightarrow \quad dt = 2\lambda u du \quad \rightarrow \quad du = \frac{1}{2\lambda u} dt$$

$$du = \frac{1}{2\sqrt{\lambda t}} dt$$

$$\langle x \rangle = \frac{aA}{\sqrt{\lambda}} \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \frac{aA}{\sqrt{\lambda}} \Gamma\left(\frac{1}{2}\right) = \frac{aA}{\sqrt{\lambda}} \sqrt{\pi}$$

$$A = \sqrt{\frac{\lambda}{\pi}} \quad \rightarrow \quad \langle x \rangle = a\sqrt{\frac{\lambda}{\pi}}\sqrt{\frac{\pi}{\lambda}}$$

$$\boxed{\textbf{1.3 (b)} \quad \langle x \rangle = a}$$

Valor de expectación de x^2

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \rho(x) dx = A \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx$$

$$u = x - a \quad \rightarrow \quad du = dx$$

$$\langle x^2 \rangle = A \int_{-\infty}^{\infty} (u+a)^2 e^{-\lambda u^2} du = A \left[\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du + a^2 \int_{-\infty}^{\infty} e^{-\lambda u^2} du + 2a \int_{-\infty}^{\infty} u e^{-\lambda u^2} du \right]$$

Función impar

$$\int_{-\infty}^{\infty} u e^{-\lambda u^2} du = 0$$

Funciones pares

$$\int_{-\infty}^{\infty} e^{-\lambda u^2} du = 2 \int_{0}^{\infty} e^{-\lambda u^2} du \quad , \quad \int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du = 2 \int_{0}^{\infty} u^2 e^{-\lambda u^2} du$$

$$\langle x^2 \rangle = A \left[2 \int_{0}^{\infty} u^2 e^{-\lambda u^2} du + 2a \int_{0}^{\infty} e^{-\lambda u^2} du \right]$$

$$u=\sqrt{\frac{t}{\lambda}} \quad \rightarrow \quad t=\lambda u^2 \quad \rightarrow \quad dt=2\lambda u du \quad \rightarrow \quad du=\frac{1}{2\lambda u}dt$$

$$du=\frac{1}{2\sqrt{\lambda t}}dt$$

$$\langle x^2\rangle=\frac{A}{\sqrt{\lambda}}\left[\int_0^\infty\left(\frac{t}{\lambda}\right)t^{-\frac{1}{2}}e^{-t}dt+a^2\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt\right]$$

$$\langle x^2\rangle=\frac{A}{\sqrt{\lambda}}\left[\frac{1}{\lambda}\int_0^\infty t^{\frac{1}{2}}e^{-t}dt+a^2\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt\right]=\frac{A}{\sqrt{\lambda}}\left[\frac{1}{\lambda}\Gamma\left(\frac{3}{2}\right)+a^2\Gamma\left(\frac{1}{2}\right)\right]$$

$$\langle x^2\rangle=\frac{A}{\sqrt{\lambda}}\left[\frac{\sqrt{\pi}}{2\lambda}+a^2\sqrt{\pi}\right]$$

$$A=\sqrt{\frac{\lambda}{\pi}} \quad \rightarrow \quad \langle x^2\rangle=\frac{\sqrt{\frac{\lambda}{\pi}}}{\sqrt{\lambda}}\left[\frac{\sqrt{\pi}}{2\lambda}+a^2\sqrt{\pi}\right]=\frac{1}{2\lambda}+a^2$$

$$\boxed{\textbf{1.3 (b)} \quad \langle x^2\rangle=a^2+\frac{1}{2\lambda}}$$

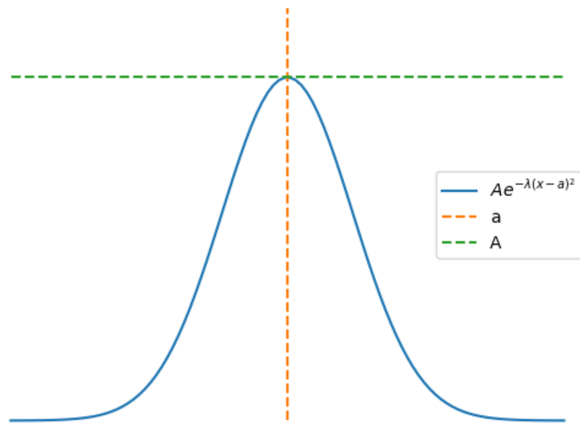
Desviación estándar

$$\sigma=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\left(a^2+\frac{1}{2\lambda}\right)-a^2}$$

$$\boxed{\textbf{1.3 (b)} \quad \sigma=\frac{1}{\sqrt{2\lambda}}}$$

Función de densidad de probabilidad

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$



1.3 (c)

Problem 1.4 At time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A\frac{x}{a}, & \text{if } 0 \leq x \leq a, \\ A\frac{(b-x)}{(b-a)}, & \text{if } a \leq x \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are constansts.

(a) Normalize Ψ (that is, find A, in termis of a and b).
(b) Sketch $\Psi(x,0)$. as a function of x.
(c) Where is the particle most likely to be found, at $t = 0$?
(d) What is the probability of finding the particle to the left of a? Check your results in the limiting cases $b = a$ and $b = 2a$.
(e) What is the expectation value of x?

Función de onda en $t = 0$

$$\Psi(x,0) = \begin{cases} A\frac{x}{a} & : \; 0 \leq x \leq a \\ A\frac{(b-x)}{(b-a)} & : \; a \leq x \leq b \\ 0 & : \; x < 0 \vee x > b \end{cases}$$

Función de densidad

$$|\Psi(x,0)|^2 = \begin{cases} A^2\frac{x^2}{a^2} & : \; 0 \leq x \leq a \\ A^2\frac{(b-x)^2}{(b-a)^2} & : \; a \leq x \leq b \\ 0 & : \; x < 0 \vee x > b \end{cases}$$

Normalización

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 1$$

$$\cancelto{0}{\int_{-\infty}^{0} |\Psi(x,0)|^2 dx} + \int_{0}^{a} |\Psi(x,0)|^2 dx + \int_{a}^{b} |\Psi(x,0)|^2 dx + \cancelto{0}{\int_{b}^{\infty} |\Psi(x,0)|^2 dx} = 1$$

$$\frac{A^2}{a^2}\int_{0}^{a} x^2 dx + \frac{A^2}{(b-a)^2}\int_{a}^{b} (b-x)^2 dx = 1$$

$$u(x) = b - x \quad \rightarrow \quad du = -dx$$

$$u(a) = b - a \quad , \quad u(b) = 0$$

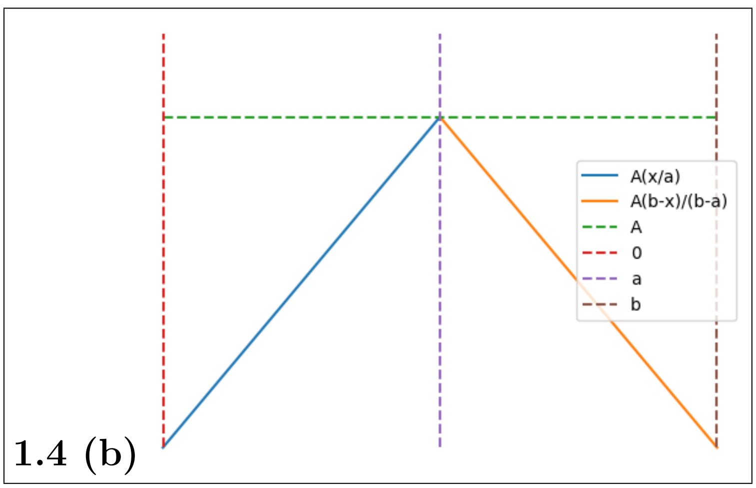
$$\frac{A^2}{a^2}\int_{0}^{a} x^2 dx - \frac{A^2}{(b-a)^2}\int_{b-a}^{0} u^2 du = 1$$

$$\frac{A^2}{a^2}\left|\frac{x^3}{3}\right|_0^a - \frac{A^2}{(b-a)^2}\left|\frac{u^3}{3}\right|_{b-a}^0 = \frac{A^2}{a^2}\left(\frac{a^3}{3}\right) + \frac{A^2}{(b-a)^2}\left[\frac{(b-a)^3}{3}\right] = 1$$

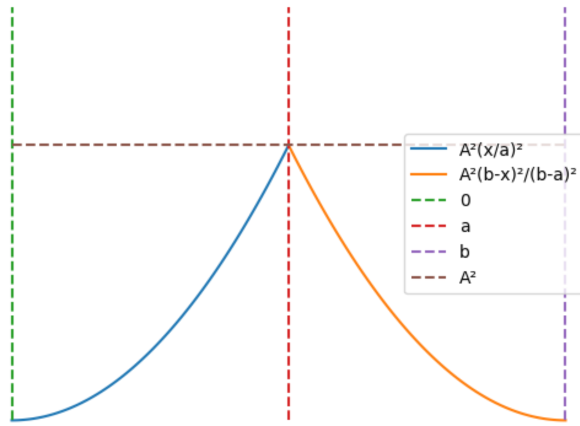
$$\frac{A^2}{3}[a+(b-a)] = 1 \quad \rightarrow \quad \frac{b}{3}A^2 = 1$$

$$\boxed{\textbf{1.4 (a)} \quad A = \sqrt{\frac{3}{b}}}$$

Bosquejo de $\Psi(x,0)$



Bosquejo de $|\Psi(x, 0)|^2$



Como se puede ver en el boceto de la función de densidad $|\Psi(x, 0)|^2$, el valor máximo y la posición donde es mas probable que se encuentre la partícula es en $x = a$, con una probabilidad de $P(x = a) = |\Psi(a, 0)|^2 = A^2 = \frac{3}{b}$.

$$\boxed{\textbf{1.4 (c)} \quad x = a}$$

Probabilidad de la posición de la partícula en un intervalo

$$P(a \leq x \leq b) = \int_a^b |\Psi(x,t)|^2 dx = \int_a^b |\Psi(x,0)|^2 dx$$

Probabilidad a la izquierda de a

$$P(x \leq a) = \int_{-\infty}^a |\Psi(x,t)|^2 dx = \int_{-\infty}^a |\Psi(x,0)|^2 dx$$

$$P(x \leq a) = \cancelto{0}{\int_{-\infty}^0 |\Psi(x,0)|^2 dx} + \int_0^a |\Psi(x,0)|^2 dx = \frac{A^2}{a^2}\int_0^a x^2 dx$$

$$P(x \leq a) = \frac{A^2}{a^2}\left|\frac{x^3}{3}\right|_0^a = \frac{A^2}{a^2}\left(\frac{a^3}{3}\right) = \frac{a}{3}A^2$$

$$A^2 = \frac{3}{b} \quad \rightarrow \quad P(x \leq a) = \frac{a}{3}\left(\frac{3}{b}\right)$$

$$\boxed{\therefore \quad P(x \leq a) = \frac{a}{b}}$$

Caso $b = a$

$$\boxed{\textbf{1.4 (d)} \quad P(x \leq a) = 1}$$

Caso $b = 2a$

$$\boxed{\textbf{1.4 (d)} \quad P(x \leq a) = \frac{1}{2}}$$

Valor de expectación de x

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,0)|^2 dx$$

$$\langle x\rangle = \int_0^a x|\Psi(x,0)|^2 dx + \int_a^b x|\Psi(x,0)|^2 dx$$

$$\langle x\rangle = \frac{A^2}{a^2}\int_0^a x^3 dx + \frac{A^2}{(b-a)^2}\int_a^b x(b-x)^2 dx$$

$$\langle x\rangle = \frac{A^2}{a^2}\int_0^a x^3 dx + \frac{A^2}{(b-a)^2}\int_a^b x(x^2-2bx+b^2) dx$$

$$\langle x\rangle = \frac{A^2}{a^2}\int_0^a x^3 dx + \frac{A^2}{(b-a)^2}\int_a^b (x^3-2bx^2+b^2x) dx$$

$$\langle x\rangle = \frac{A^2}{a^2}\int_0^a x^3 dx + \frac{A^2}{(b-a)^2}\int_a^b x^3 dx - \frac{2bA^2}{(b-a)^2}\int_a^b x^2 dx + \frac{b^2A^2}{(b-a)^2}\int_a^b x dx$$

$$\langle x\rangle = \frac{A^2}{a^2}\left|\frac{x^4}{4}\right|_0^a + \frac{A^2}{(b-a)^2}\left|\frac{x^4}{4}\right|_a^b - \frac{2bA^2}{(b-a)^2}\left|\frac{x^3}{3}\right|_a^b + \frac{b^2A^2}{(b-a)^2}\left|\frac{x^2}{2}\right|_a^b$$

$$\langle x\rangle = \frac{A^2}{a^2}\left(\frac{a^4}{4}\right) + \frac{A^2}{(b-a)^2}\left(\frac{b^4}{4}-\frac{a^4}{4}\right) - \frac{2bA^2}{(b-a)^2}\left(\frac{b^3}{3}-\frac{a^3}{3}\right) + \frac{b^2A^2}{(b-a)^2}\left(\frac{b^2}{2}-\frac{a^2}{2}\right)$$

$$\langle x\rangle = \left[\frac{1}{4}a^2 + \frac{(b^4-a^4)}{4(b-a)^2} - \frac{2b(b^3-a^3)}{3(b-a)^2} + \frac{b^2(b^2-a^2)}{2(b-a)^2}\right] A^2$$

$$A^2 = \frac{3}{b}$$

$$\langle x\rangle = \frac{3}{b}\left[\frac{1}{4}a^2 + \frac{(b^4-a^4)}{4(b-a)^2} - \frac{2b(b^3-a^3)}{3(b-a)^2} + \frac{b^2(b^2-a^2)}{2(b-a)^2}\right]$$

$$\langle x\rangle = \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + (b^4-a^4) - \frac{8}{3}b(b^3-a^3) + 2b^2(b^2-a^2)\right]$$

$$\langle x\rangle = \frac{3}{4b(b-a)^2}\left[a^2(a^2+b^2-2ab) + (b^4-a^4) - \frac{8}{3}(b^4-a^3b) + 2(b^4-a^2b^2)\right]$$

$$\langle x\rangle = \frac{3}{4b(b-a)^2}\left[(a^4+a^2b^2-2a^3b) + (b^4-a^4) - \frac{8}{3}(b^4-a^3b) + 2(b^4-a^2b^2)\right]$$

$$\langle x\rangle = \frac{3}{4b(b-a)^2}\left(\frac{2}{3}a^3b + \frac{1}{3}b^4 - a^2b^2\right) = \frac{2a^3+b^3-3a^2b}{4(b-a)^2} = \frac{(2a+b)(a^2+b^2-2ab)}{4(b-a)^2}$$

$$\langle x\rangle = \frac{(2a+b)(b-a)^2}{4(b-a)^2}$$

$$\boxed{\textbf{1.4 (e)} \quad \langle x\rangle = \frac{2a+b}{4}}$$

Problem 1.5 Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

where A, λ, and ω are positive real constans. (We'll see in Chapter 2 what potential (V) actually produces such a wave function.)

(a) Normalize Ψ.
(b) Determine the exprectation values of x and x^2.
(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x \rangle + \sigma)$ and $(\langle x \rangle - \sigma)$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

Función de onda

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t} = \begin{cases} Ae^{\lambda x}e^{-i\omega t} & : \; x \leq 0 \\ Ae^{-\lambda x}e^{-i\omega t} & : \; x > 0 \end{cases}$$

Función de densidad

$$|\Psi(x,t)|^2 = A^2e^{-2\lambda|x|} = \begin{cases} A^2e^{2\lambda x} & : \; x \leq 0 \\ A^2e^{-2\lambda x} & : \; x > 0 \end{cases}$$

Normalización

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{0} |\Psi(x,t)|^2 dx + \int_{0}^{\infty} |\Psi(x,t)|^2 dx = 1$$

$$A^2\int_{-\infty}^{0} e^{2\lambda x}dx + A^2\int_{0}^{\infty} e^{-2\lambda x}dx = 1 \quad \to \quad A^2\left|\frac{e^{2\lambda x}}{2\lambda}\right|_{-\infty}^{0} - A^2\left|\frac{e^{-2\lambda x}}{2\lambda}\right|_{0}^{\infty} = 1$$

$$\frac{A^2}{2\lambda}\left[\left(\cancelto{1}{e^0} - \cancelto{0}{\lim_{x\to-\infty} e^{2\lambda x}}\right) - \left(\cancelto{0}{\lim_{x\to\infty} e^{-2\lambda x}} - \cancelto{1}{e^0}\right)\right] = 1 \quad \to \quad \frac{A^2}{\lambda} = 1$$

$$\boxed{\therefore \quad A = \sqrt{\lambda}}$$

$$\boxed{\textbf{1.5 (a)} \quad \Psi(x,t) = \sqrt{\lambda}e^{-\lambda|x|}e^{-i\omega t}}$$

Valor de expectación de x

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx$$

$$\langle x \rangle = \lambda \int_{-\infty}^{0} xe^{2\lambda x} dx + \lambda \int_{0}^{\infty} xe^{-2\lambda x} dx$$

Función impar

$$\int_{-\infty}^{0} xe^{2\lambda x} dx = -\int_{0}^{\infty} xe^{-2\lambda x} dx$$

$$\therefore \quad \langle x \rangle = -\lambda \int_{0}^{\infty} xe^{-2\lambda x} dx + \lambda \int_{0}^{\infty} xe^{-2\lambda x} dx$$

$$\boxed{\textbf{1.5 (b)} \quad \langle x \rangle = 0}$$

Valor de expectación de x^2

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\Psi(x,t)|^2 dx$$

$$\langle x^2 \rangle = \lambda \int_{-\infty}^{0} x^2 e^{2\lambda x} dx + \lambda \int_{0}^{\infty} x^2 e^{-2\lambda x} dx$$

Función par

$$\int_{-\infty}^{0} x^2 e^{2\lambda x} dx = \int_{0}^{\infty} x^2 e^{-2\lambda x} dx$$

$$\therefore \quad \langle x^2 \rangle = 2\lambda \int_{0}^{\infty} x^2 e^{-2\lambda x} dx$$

$$x = \frac{1}{2\lambda} t \quad \to \quad t = 2\lambda x \quad \to \quad dt = 2\lambda dx \quad \to \quad dx = \frac{1}{2\lambda} dt$$

$$\langle x^2 \rangle = 2\lambda \int_{0}^{\infty} \left(\frac{1}{2\lambda} t \right)^2 e^{-t} \left(\frac{1}{2\lambda} dt \right) = \frac{1}{4\lambda^2} \int_{0}^{\infty} t^2 e^{-t} dt$$

Función Gamma

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

$$\langle x^2 \rangle = \frac{1}{4\lambda^2} \int_0^\infty t^{3-1} e^{-t} dt = \frac{1}{4\lambda^2} \Gamma(3) = \frac{(3-1)!}{4\lambda^2} = \frac{2!}{4\lambda^2}$$

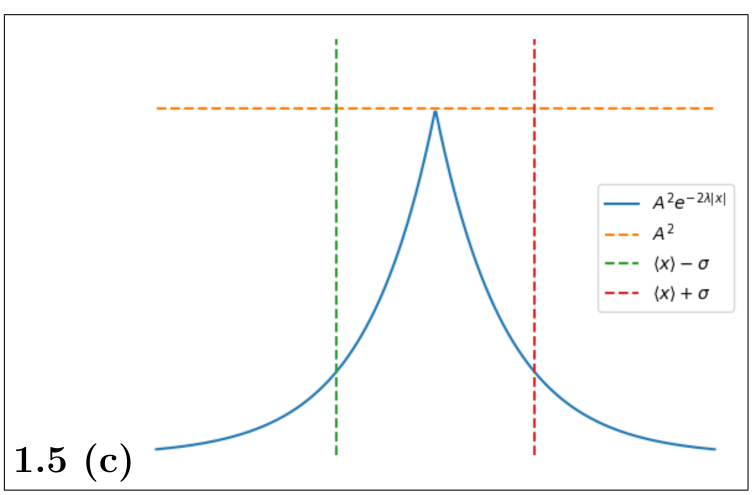
$$\boxed{\textbf{1.5 (b)} \quad \langle x^2 \rangle = \frac{1}{2\lambda^2}}$$

Desviación estándar

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{2\lambda^2} - 0}$$

$$\boxed{\textbf{1.5 (c)} \quad \sigma = \frac{1}{\sqrt{2}\lambda}}$$

Bosquejo de $|\Psi(x,t)|^2 = A^2 e^{-2\lambda|x|}$



Probabilidad de la posición de la partícula fuera de un intervalo

$$P(x < a \vee x > b) = \int_{-\infty}^{a} |\Psi(x,t)|^2 dx + \int_{b}^{\infty} |\Psi(x,t)|^2 dx$$

Probabilidad fuera de la desviación estándar

$$P = \int_{-\infty}^{a} |\Psi(x,t)|^2 dx + \int_{b}^{\infty} |\Psi(x,t)|^2 dx$$

$$P = \int_{-\infty}^{\langle x\rangle-\sigma} |\Psi(x,t)|^2 dx + \int_{\langle x\rangle+\sigma}^{\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{-\sigma} |\Psi(x,t)|^2 dx + \int_{\sigma}^{\infty} |\Psi(x,t)|^2 dx$$

$$P = A^2 \int_{-\infty}^{-\sigma} e^{2\lambda x} dx + A^2 \int_{\sigma}^{\infty} e^{-2\lambda x} dx$$

$$A^2 \int_{-\infty}^{-\sigma} e^{2\lambda x} dx = A^2 \int_{\sigma}^{\infty} e^{-2\lambda x} dx \quad \rightarrow \quad P = 2A^2 \int_{\sigma}^{\infty} e^{-2\lambda x} dx$$

$$A^2 = \lambda \quad \rightarrow \quad P = 2\lambda \int_{\sigma}^{\infty} e^{-2\lambda x} dx$$

$$P = -\left| e^{-2\lambda x} \right|_{\sigma}^{\infty} = e^{-2\lambda\sigma} - \cancelto{0}{\lim_{x\to\infty} e^{-2\lambda x}} = e^{-2\lambda\sigma}$$

$$\sigma = \frac{1}{\sqrt{2}\lambda} \quad \rightarrow \quad P = e^{-2\lambda\left(\frac{1}{\sqrt{2}\lambda}\right)} = e^{-\sqrt{2}}$$

$$\boxed{\textbf{1.5 (c)} \quad P \approx 0.2431}$$

Problem 1.6 Why can't you do integration-by-parts directly on the middle expression in Equation 1.29—pull the time derivative over onto x, note that $\partial x/\partial t = 0$, and conclude that $d\langle x\rangle/dt = 0$?

Ecuación 1.29

$$\frac{d\langle x\rangle}{dt} = \int x\frac{\partial}{\partial t}|\Psi|^2 dx = \frac{i\hbar}{2m}\int x\frac{\partial}{\partial x}\left(\bar{\Psi}\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\bar{\Psi}}{\partial x}\right)dx$$

No se puede recurrir a integración por partes, ya que para ello la variable de diferenciación e integración deben ser la misma, cosa que no ocurre en la expresión del medio en la ecuación 1.29.

$$\frac{\partial x}{\partial t} = 0 \quad \rightarrow \quad x\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}\left(x|\Psi|^2\right)$$

$$\therefore \quad \frac{d\langle x\rangle}{dt} = \int \frac{\partial}{\partial t}\left(x|\Psi|^2\right)dx$$

$$\rho(x) = |\Psi(x,t)|^2 \quad \rightarrow \quad \frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}|\Psi|^2 = 0$$

$$\therefore$$

$$\boxed{\textbf{1.6} \quad \frac{d\langle x\rangle}{dt} = 0}$$

Problem 1.7 Calculate $d\langle p\rangle/dt$. Answer:

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle .$$

Equations 1.32 (or the first part of 1.33) and 1.38 are instances of **Ehrenfest's theorem,** which tells us that *expectation values obey classic laws.*

Ecuación 1.33

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -i\hbar\int\left(\bar{\Psi}\frac{\partial\Psi}{\partial x}\right)dx$$

$$\frac{d\langle p\rangle}{dt} = -i\hbar\int\frac{\partial}{\partial t}\left(\bar{\Psi}\frac{\partial\Psi}{\partial x}\right)dx$$

$$\frac{\partial}{\partial t}\left(\bar{\Psi}\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\Psi}{\partial x} + \bar{\Psi}\frac{\partial^2\Psi}{\partial t\partial x}$$

Ecuación de Schrödinger

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi \quad \rightarrow \quad \frac{\partial\bar{\Psi}}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\bar{\Psi}}{\partial x^2} + \frac{i}{\hbar}V\bar{\Psi}$$

$$\frac{\partial}{\partial t}\left(\bar{\Psi}\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi}{\partial x}\left(-\frac{i\hbar}{2m}\frac{\partial^2\bar{\Psi}}{\partial x^2} + \frac{i}{\hbar}V\bar{\Psi}\right) + \bar{\Psi}\frac{\partial}{\partial x}\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)$$

$$\frac{\partial}{\partial t}\left(\bar{\Psi}\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi}{\partial x}\left(-\frac{i\hbar}{2m}\frac{\partial^2\bar{\Psi}}{\partial x^2} + \frac{i}{\hbar}V\bar{\Psi}\right) + \frac{i\hbar}{2m}\bar{\Psi}\frac{\partial^3\Psi}{\partial x^3} - \frac{i}{\hbar}\bar{\Psi}\frac{\partial}{\partial x}(V\Psi)$$

$$\frac{\partial}{\partial t}\left(\bar{\Psi}\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi}{\partial x}\left(-\frac{i\hbar}{2m}\frac{\partial^2\bar{\Psi}}{\partial x^2} + \frac{i}{\hbar}V\bar{\Psi}\right) + \frac{i\hbar}{2m}\bar{\Psi}\frac{\partial^3\Psi}{\partial x^3} - \frac{i}{\hbar}V\bar{\Psi}\frac{\partial\Psi}{\partial x} - \frac{i}{\hbar}|\Psi|^2\frac{\partial V}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\bar{\Psi}\frac{\partial \Psi}{\partial x}\right) = -\frac{i\hbar}{2m}\frac{\partial^2\bar{\Psi}}{\partial x^2}\frac{\partial \Psi}{\partial x} + \frac{i\hbar}{2m}\bar{\Psi}\frac{\partial^3\Psi}{\partial x^3} - \frac{i}{\hbar}|\Psi|^2\frac{\partial V}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\bar{\Psi}\frac{\partial \Psi}{\partial x}\right) = \frac{i\hbar}{2m}\left[\bar{\Psi}\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\bar{\Psi}}{\partial x^2}\frac{\partial \Psi}{\partial x}\right] - \frac{i}{\hbar}|\Psi|^2\frac{\partial V}{\partial x}$$

$$\therefore \quad \frac{d\langle p\rangle}{dt} = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left[\bar{\Psi}\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\bar{\Psi}}{\partial x^2}\frac{\partial \Psi}{\partial x}\right]dx - \int_{-\infty}^{\infty}|\Psi|^2\frac{\partial V}{\partial x}dx$$

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar^2}{2m}\left[\int_{-\infty}^{\infty}\bar{\Psi}\frac{\partial^3\Psi}{\partial x^3}dx - \int_{-\infty}^{\infty}\frac{\partial^2\bar{\Psi}}{\partial x^2}\frac{\partial \Psi}{\partial x}dx\right] - \int_{-\infty}^{\infty}|\Psi|^2\frac{\partial V}{\partial x}dx$$

$$u = \frac{\partial \Psi}{\partial x} \quad \rightarrow \quad du = \frac{\partial^2\Psi}{\partial x^2}dx$$
$$dv = \frac{\partial^2\bar{\Psi}}{\partial x^2}dx \quad \rightarrow \quad v = \frac{\partial \bar{\Psi}}{\partial x}$$

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar^2}{2m}\left[\int_{-\infty}^{\infty}\bar{\Psi}\frac{\partial^3\Psi}{\partial x^3}dx - \left|\frac{\partial\bar{\Psi}}{\partial x}\frac{\partial \Psi}{\partial x}\right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\partial\bar{\Psi}}{\partial x}\frac{\partial^2\Psi}{\partial x^2}dx\right] - \int_{-\infty}^{\infty}|\Psi|^2\frac{\partial V}{\partial x}dx$$

$$\lim_{|x|\to\infty}\frac{\partial \Psi}{\partial x} = \lim_{|x|\to\infty}\frac{\partial \bar{\Psi}}{\partial x} = 0 \quad \rightarrow \quad \left|\frac{\partial\bar{\Psi}}{\partial x}\frac{\partial \Psi}{\partial x}\right|_{-\infty}^{\infty} = 0$$

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar^2}{2m}\left[\int_{-\infty}^{\infty}\bar{\Psi}\frac{\partial^3\Psi}{\partial x^3}dx + \int_{-\infty}^{\infty}\frac{\partial\bar{\Psi}}{\partial x}\frac{\partial^2\Psi}{\partial x^2}dx\right] - \int_{-\infty}^{\infty}|\Psi|^2\frac{\partial V}{\partial x}dx$$

$$u = \frac{\partial^2 \Psi}{\partial x^2} \quad \rightarrow \quad du = \frac{\partial^3 \Psi}{\partial x^3} dx$$

$$dv = \frac{\partial \bar{\Psi}}{\partial x} dx \quad \rightarrow \quad v = \bar{\Psi}$$

$$\frac{d\langle p \rangle}{dt} = \frac{\hbar^2}{2m} \left[\int_{-\infty}^{\infty} \bar{\Psi} \frac{\partial^3 \Psi}{\partial x^3} dx + \left| \bar{\Psi} \frac{\partial^2 \Psi}{\partial x^2} \right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \bar{\Psi} \frac{\partial^3 \Psi}{\partial x^3} dx \right] - \int_{-\infty}^{\infty} |\Psi|^2 \frac{\partial V}{\partial x} dx$$

$$\lim_{|x| \to \infty} \Psi = \lim_{|x| \to \infty} \bar{\Psi} = 0 \quad \rightarrow \quad \left| \bar{\Psi} \frac{\partial^2 \Psi}{\partial x^2} \right|_{-\infty}^{\infty} = 0$$

$$\therefore \quad \frac{d\langle p \rangle}{dt} = - \int_{-\infty}^{\infty} |\Psi|^2 \frac{\partial V}{\partial x} dx = \int_{-\infty}^{\infty} \bar{\Psi} \Psi \left(- \frac{\partial V}{\partial x} \right) dx = \int_{-\infty}^{\infty} \bar{\Psi} \left(- \frac{\partial V}{\partial x} \Psi \right) dx$$

$$\boxed{\mathbf{1.7} \quad \frac{d\langle p \rangle}{dt} = \left\langle - \frac{\partial V}{\partial x} \right\rangle \quad \blacksquare}$$

Problem 1.8 Supose you add a constant V_0 to the potential energy (by "constant" I mean independent of x as well as t). In classical mechanics this doesn't change anything, but what about *quantum mechanics*? Show that the wave function picks up a time-dependent phase factor: $\exp(-iV_0/\hbar)$. What effect does this have on the expectation value of a dynamical variable?

Ecuación de Schrödinger

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial x^2} + (V + V_0)\Phi = i\hbar\frac{\partial\Phi}{\partial t}$$

$$\frac{\partial\Phi}{\partial t} + \frac{i}{\hbar}V_0\Phi = \frac{i\hbar}{2m}\frac{\partial^2\Phi}{\partial x^2} - \frac{i}{\hbar}V(x)\Phi$$

Método de factor integrante

$$\frac{df}{dt} + P(t)f(t) = g(t) \quad \rightarrow \quad \mu\frac{df}{dt} + \mu P(t)f(t) = \mu g(t)$$

$$\mu = e^{\int P(t)dt}$$

$$\mu\frac{\partial\Phi}{\partial t} + \mu\frac{i}{\hbar}V_0\Phi = \mu\frac{i\hbar}{2m}\frac{\partial^2\Phi}{\partial x^2} - \mu\frac{i}{\hbar}V(x)\Phi$$

$$\mu = e^{\int_0^t \frac{i}{\hbar}V_0 dt'} = e^{\frac{i}{\hbar}V_0\int_0^t dt'} = e^{\frac{i}{\hbar}V_0 t}$$

$$e^{\frac{i}{\hbar}V_0 t}\frac{\partial\Phi}{\partial t} + e^{\frac{i}{\hbar}V_0 t}\frac{i}{\hbar}V_0\Phi = e^{\frac{i}{\hbar}V_0 t}\frac{i\hbar}{2m}\frac{\partial^2\Phi}{\partial x^2} - e^{\frac{i}{\hbar}V_0 t}\frac{i}{\hbar}V(x)\Phi$$

$$e^{\frac{i}{\hbar}V_0 t}\frac{\partial \Phi}{\partial t} + e^{\frac{i}{\hbar}V_0 t}\frac{i}{\hbar}V_0\Phi = e^{\frac{i}{\hbar}V_0 t}\frac{i\hbar}{2m}\frac{\partial^2 \Phi}{\partial x^2} - e^{\frac{i}{\hbar}V_0 t}\frac{i}{\hbar}V(x)\Phi$$

$$\frac{\partial}{\partial t}\left(\Phi e^{\frac{i}{\hbar}V_0 t}\right) = \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\left(\Phi e^{\frac{i}{\hbar}V_0 t}\right) - \frac{i}{\hbar}V(x)\left(\Phi e^{\frac{i}{\hbar}V_0 t}\right)$$

$$\boxed{\Psi(x,t) = \Phi(x,t)e^{\frac{i}{\hbar}V_0 t} \quad \rightarrow \quad \Phi(x,t) = \Psi(x,t)e^{-\frac{i}{\hbar}V_0 t}}$$

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V(x)\Psi \quad \rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$

Medición de valores de expectación

$$\langle Q\rangle = \int_{-\infty}^{\infty}\bar{\Phi}\left[\hat{Q}\left(x, -i\hbar\frac{\partial}{\partial x}\right)\Phi\right]dx$$

$$\langle Q\rangle = \int_{-\infty}^{\infty}\left(\bar{\Psi}e^{\frac{i}{\hbar}V_0 t}\right)\left[\hat{Q}\left(x, -i\hbar\frac{\partial}{\partial x}\right)\left(\Psi e^{-\frac{i}{\hbar}V_0 t}\right)\right]dx$$

1.8

$$\langle Q\rangle = \int_{-\infty}^{\infty}\bar{\Psi}\left[\hat{Q}\left(x, -i\hbar\frac{\partial}{\partial x}\right)\Psi\right]dx$$

La medición del valor de expectación no presenta cambios al añadir una constante V_0 en el potencial del sistema.

Problem 1.9 A particle of mass m is in the state

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]}$$

where A and a are positive constants.

(a) Find A.
(b) For what potential energy function $V(x)$ does Ψ satisfy the Schrödinger equation?

(c) Calculate the expectation values of x, x^2, p, and p^2.
(d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

Normalización

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = A^2 \int_{-\infty}^{\infty} e^{-2a\frac{mx^2}{\hbar}} dx = 1$$

Función par

$$\int_{-\infty}^{0} e^{-2a\frac{mx^2}{\hbar}} dx = \int_{0}^{\infty} e^{-2a\frac{mx^2}{\hbar}} dx \quad \rightarrow \quad \int_{-\infty}^{\infty} e^{-2a\frac{mx^2}{\hbar}} dx = 2\int_{0}^{\infty} e^{-2a\frac{mx^2}{\hbar}} dx$$

$$\therefore \quad \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 2A^2 \int_{0}^{\infty} e^{-2a\frac{mx^2}{\hbar}} dx = 1$$

$$x = \sqrt{\frac{\hbar}{2am}u} \quad \rightarrow \quad u = \frac{2am}{\hbar}x^2 \quad \rightarrow \quad du = \frac{4am}{\hbar}xdx \quad \rightarrow \quad 2dx = \frac{\hbar}{2amx}du$$

$$2dx = \frac{\hbar}{2am}\sqrt{\frac{2am}{\hbar u}}du \quad \rightarrow \quad 2dx = \sqrt{\frac{\hbar}{2am}}u^{-\frac{1}{2}}du$$

$$2A^2 \int_{0}^{\infty} e^{-2a\frac{mx^2}{\hbar}} dx = A^2\sqrt{\frac{\hbar}{2am}} \int_{0}^{\infty} u^{-\frac{1}{2}} e^{-u} dx = A^2\sqrt{\frac{\hbar}{2am}}\Gamma\left(\frac{1}{2}\right) = A^2\sqrt{\frac{\hbar}{2am}}\sqrt{\pi} = 1$$

$$A^2\sqrt{\frac{\pi\hbar}{2am}} = 1 \quad \rightarrow \quad A^2 = \sqrt{\frac{2am}{\pi\hbar}}$$

$$\boxed{\textbf{1.9 (a)} \quad A = \left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{4}}}$$

Ecuación de Schrödinger

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t} \quad \rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi(x,t) = i\hbar\frac{\partial\Psi}{\partial t}$$

$$-A\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[e^{-a\left(\frac{mx^2}{\hbar}+it\right)}\right] + AV\left[e^{-a\left(\frac{mx^2}{\hbar}+it\right)}\right] = Ai\hbar\frac{\partial}{\partial t}\left[e^{-a\left(\frac{mx^2}{\hbar}+it\right)}\right]$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[e^{-a\left(\frac{mx^2}{\hbar}+it\right)}\right] + V\left[e^{-a\left(\frac{mx^2}{\hbar}+it\right)}\right] = i\hbar\frac{\partial}{\partial t}\left[e^{-a\left(\frac{mx^2}{\hbar}+it\right)}\right]$$

$$-\frac{\hbar^2}{2m}e^{-ait}\frac{\partial^2}{\partial x^2}\left(e^{-a\frac{mx^2}{\hbar}}\right) + Ve^{-a\left(\frac{mx^2}{\hbar}+it\right)} = i\hbar e^{-a\frac{mx^2}{\hbar}}\frac{\partial}{\partial t}\left(e^{-ait}\right)$$

$$-\frac{\hbar^2}{2m}e^{-ait}\frac{\partial}{\partial x}\left(-2a\frac{mx}{\hbar}e^{-a\frac{mx^2}{\hbar}}\right) + Ve^{-a\left(\frac{mx^2}{\hbar}+it\right)} = i\hbar e^{-a\frac{mx^2}{\hbar}}\left(-aie^{-ait}\right)$$

$$a\hbar e^{-ait}\frac{\partial}{\partial x}\left(xe^{-a\frac{mx^2}{\hbar}}\right) + Ve^{-a\left(\frac{mx^2}{\hbar}+it\right)} = a\hbar e^{-a\left(\frac{mx^2}{\hbar}+it\right)}$$

$$a\hbar e^{-ait}\left(e^{-a\frac{mx^2}{\hbar}} - \frac{2am}{\hbar}x^2e^{-a\frac{mx^2}{\hbar}}\right) + Ve^{-a\left(\frac{mx^2}{\hbar}+it\right)} = a\hbar e^{-a\left(\frac{mx^2}{\hbar}+it\right)}$$

$$a\hbar\left(1-\frac{2am}{\hbar}x^2\right)e^{-a\left(\frac{mx^2}{\hbar}+it\right)}+Ve^{-a\left(\frac{mx^2}{\hbar}+it\right)}=a\hbar e^{-a\left(\frac{mx^2}{\hbar}+it\right)}$$

$$a\hbar\left(1-\frac{2am}{\hbar}x^2\right)+V=a\hbar \quad \rightarrow \quad -2ma^2x^2+V=0$$

$$\boxed{\textbf{1.9 (b)} \quad V(x)=2ma^2x^2}$$

Valor de expectación de x

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = A^2 \int_{-\infty}^{\infty} x e^{-2a\frac{mx^2}{\hbar}} dx$$

Función impar

$$\int_{-\infty}^{0} x e^{-2a\frac{mx^2}{\hbar}} dx = -\int_{0}^{\infty} x e^{-2a\frac{mx^2}{\hbar}} dx \quad \rightarrow \quad \int_{-\infty}^{\infty} x e^{-2a\frac{mx^2}{\hbar}} dx = 0$$

$$\boxed{\textbf{1.9 (c)} \quad \langle x\rangle = 0}$$

Valor de expectación de x^2

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\Psi|^2 dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}} dx$$

Función impar

$$\int_{-\infty}^{0} x^2 e^{-2a\frac{mx^2}{\hbar}} dx = \int_{0}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}} dx \quad \rightarrow \quad \int_{-\infty}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}} dx = 2\int_{0}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}} dx$$

$$\langle x^2\rangle = 2A^2 \int_{0}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}} dx$$

$$x = \sqrt{\frac{\hbar}{2am}u} \quad \rightarrow \quad u = \frac{2am}{\hbar}x^2 \quad \rightarrow \quad du = \frac{4am}{\hbar}xdx \quad \rightarrow \quad 2x^2dx = \frac{\hbar}{2am}xdu$$

$$2x^2dx = \frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}}udu \quad \rightarrow \quad 2x^2dx = \left(\frac{\hbar}{2am}\right)^{\frac{3}{2}} u^{\frac{1}{2}}du$$

$$\therefore \quad \langle x^2 \rangle = A^2 \left(\frac{\hbar}{2am}\right)^{\frac{3}{2}} \int_0^\infty u^{\frac{1}{2}}e^{-u}du = A^2 \left(\frac{\hbar}{2am}\right)^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) = A^2 \left(\frac{\hbar}{2am}\right)^{\frac{3}{2}} \left(\frac{\sqrt{\pi}}{2}\right)$$

$$A^2 = \left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{2}} \quad \rightarrow \quad \langle x^2 \rangle = \left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{\hbar}{2am}\right)^{\frac{3}{2}} \left(\frac{\sqrt{\pi}}{2}\right) = \left(\frac{am}{2\hbar}\right)^{\frac{1}{2}} \left(\frac{\hbar}{2am}\right)^{\frac{3}{2}}$$

$$\boxed{\textbf{1.9 (c)} \quad \langle x^2 \rangle = \frac{\hbar}{4am}}$$

Valor de expectación de p

$$\langle p \rangle = \int_{-\infty}^{\infty} \bar{\Psi} \hat{p} \Psi dx = -i\hbar \int_{-\infty}^{\infty} \bar{\Psi} \frac{\partial \Psi}{\partial x} dx$$

$$\langle p \rangle = -i\hbar A^2 \int_{-\infty}^{\infty} e^{-a\frac{mx^2}{\hbar}} \frac{\partial}{\partial x} \left(e^{-a\frac{mx^2}{\hbar}} \right) dx = -i\hbar A^2 \int_{-\infty}^{\infty} e^{-2a\frac{mx^2}{\hbar}} \frac{\partial}{\partial x} \left(-a\frac{mx^2}{\hbar} \right) dx$$

$$\langle p \rangle = -i\hbar A^2 \int_{-\infty}^{\infty} e^{-2a\frac{mx^2}{\hbar}} \left(-\frac{2am}{\hbar} x \right) dx = 2imaA^2 \int_{-\infty}^{\infty} x e^{-2a\frac{mx^2}{\hbar}} dx$$

Función impar

$$\int_{-\infty}^{0} x e^{-2a\frac{mx^2}{\hbar}} dx = -\int_{0}^{\infty} x e^{-2a\frac{mx^2}{\hbar}} dx \quad \rightarrow \quad \int_{-\infty}^{\infty} x e^{-2a\frac{mx^2}{\hbar}} dx = 0$$

$$\boxed{\textbf{1.9 (c)} \quad \langle p \rangle = 0}$$

Valor de expectación de p^2

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \bar{\Psi}\hat{p}^2\Psi dx = -\hbar^2 \int_{-\infty}^{\infty} \bar{\Psi}\frac{\partial^2\Psi}{\partial x^2}dx$$

$$\langle p^2\rangle = -\hbar^2 A^2 \int_{-\infty}^{\infty} e^{-a\frac{mx^2}{\hbar}} \frac{\partial^2}{\partial x^2}\left(e^{-a\frac{mx^2}{\hbar}}\right) dx = -\hbar^2 A^2 \int_{-\infty}^{\infty} e^{-a\frac{mx^2}{\hbar}} \frac{\partial}{\partial x}\left(-\frac{2am}{\hbar} x e^{-a\frac{mx^2}{\hbar}}\right) dx$$

$$\langle p^2\rangle = 2\hbar ma A^2 \int_{-\infty}^{\infty} e^{-a\frac{mx^2}{\hbar}} \frac{\partial}{\partial x}\left(x e^{-a\frac{mx^2}{\hbar}}\right) dx$$

$$\langle p^2\rangle = 2\hbar ma A^2 \int_{-\infty}^{\infty} e^{-a\frac{mx^2}{\hbar}} \left(e^{-a\frac{mx^2}{\hbar}} - \frac{2am}{\hbar}x^2 e^{-a\frac{mx^2}{\hbar}}\right) dx$$

$$\langle p^2\rangle = 2\hbar ma A^2 \int_{-\infty}^{\infty} \left(1 - \frac{2am}{\hbar}x^2\right) e^{-2a\frac{mx^2}{\hbar}} dx$$

$$\langle p^2\rangle = 2\hbar ma A^2 \int_{-\infty}^{\infty} e^{-2a\frac{mx^2}{\hbar}} dx - 4m^2a^2A^2 \int_{-\infty}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}} dx$$

Funciones par

$$\int_{-\infty}^{0} e^{-2a\frac{mx^2}{\hbar}}dx = \int_{0}^{\infty} e^{-2a\frac{mx^2}{\hbar}}dx \quad \rightarrow \quad \int_{-\infty}^{\infty} e^{-2a\frac{mx^2}{\hbar}}dx = 2\int_{0}^{\infty} e^{-2a\frac{mx^2}{\hbar}}dx$$

$$\int_{-\infty}^{0} x^2 e^{-2a\frac{mx^2}{\hbar}}dx = \int_{0}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}}dx \quad \rightarrow \quad \int_{-\infty}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}}dx = 2\int_{0}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}}dx$$

$$\therefore \quad \langle p^2 \rangle = 4\hbar m a A^2 \int_{0}^{\infty} e^{-2a\frac{mx^2}{\hbar}}dx - 8m^2a^2A^2 \int_{0}^{\infty} x^2 e^{-2a\frac{mx^2}{\hbar}}dx$$

$$x = \sqrt{\frac{\hbar}{2am}u} \quad \rightarrow \quad u = \frac{2am}{\hbar}x^2 \quad \rightarrow \quad du = \frac{4am}{\hbar}xdx \quad \rightarrow \quad 4\hbar madx = \frac{\hbar^2}{x}du$$

$$4\hbar madx = \hbar^2\sqrt{\frac{2am}{\hbar u}}du \quad \rightarrow \quad \boxed{4\hbar madx = \hbar^2\sqrt{\frac{2am}{\hbar}}u^{-\frac{1}{2}}du}$$

$$du = \frac{4am}{\hbar}xdx \quad \rightarrow \quad 4m^2a^2x^2dx = \hbar maxdu \quad \rightarrow \quad 4m^2a^2x^2dx = \hbar ma\sqrt{\frac{\hbar}{2am}u}du$$

$$\boxed{8m^2a^2x^2dx = 2\hbar ma\sqrt{\frac{\hbar}{2am}}u^{\frac{1}{2}}du}$$

$$\therefore \quad \langle p^2 \rangle = \hbar^2\sqrt{\frac{2am}{\hbar}}A^2\int_{0}^{\infty} u^{-\frac{1}{2}}e^{-u}du - 2\hbar ma\sqrt{\frac{\hbar}{2am}}A^2\int_{0}^{\infty} u^{\frac{1}{2}}e^{-u}du$$

$$\langle p^2 \rangle = \hbar^2\sqrt{\frac{2am}{\hbar}}A^2\Gamma\left(\frac{1}{2}\right) - 2\hbar ma\sqrt{\frac{\hbar}{2am}}A^2\Gamma\left(\frac{3}{2}\right)$$

$$\langle p^2\rangle = \hbar^2\sqrt{\frac{2am}{\hbar}}A^2\sqrt{\pi} - 2\hbar am\sqrt{\frac{\hbar}{2am}}A^2\frac{\sqrt{\pi}}{2}$$

$$A^2 = \left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{2}} \quad \rightarrow \quad \langle p^2\rangle = \hbar^2\sqrt{\frac{2am}{\hbar}}\left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{2}}\sqrt{\pi} - 2\hbar am\sqrt{\frac{\hbar}{2am}}\left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2}$$

$$\langle p^2\rangle = 2am\hbar - am\hbar$$

$$\boxed{\textbf{1.9 (c)} \quad \langle p^2\rangle = am\hbar}$$

Desviación estándar de x

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{4am} - (0)^2}$$

$$\boxed{\textbf{1.9 (d)} \quad \sigma_x = \sqrt{\frac{\hbar}{4am}}}$$

Desviación estándar de p

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{am\hbar - (0)^2}$$

$$\boxed{\textbf{1.9 (d)} \quad \sigma_p = \sqrt{am\hbar}}$$

Principio de incertidumbre

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{4am}}\sqrt{am\hbar} = \sqrt{\frac{\hbar^2}{4}} \quad \rightarrow \quad \sigma_x \sigma_p = \frac{\hbar}{2}$$

1.9 (d)

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \quad \blacksquare$$

El valor del producto de las desviaciones es consistente con el principio de incertidumbre.