Tutorial 2

Combinatorics 1M020

Xing Shi Cai

05-02-2019

Department of Mathematics, Uppsala University, Sweden

Chocolate Championship

We will play the following game

1. You choose some of the problems in this document to work on for 25 minutes.
2. Volunteer/random students will be chosen to present solutions to problems of their choice.
3. Who speaks will get a number of chocolates according to the problem.
4. The champion of this game will also get a free coffee.

Types of problems

Binomial identities

Strings and Integers

Generating functions

Inclusion-Exclusion

Catalan Number

Binomial identities

Identity 1 🍫

For integers $m \geq 0$, $n \geq 0$ and $k \geq 0$, show that

$$\binom{m}{k}\binom{n}{m} = \binom{n-k}{m-k}\binom{n}{k}$$

Identity 2 🍫🍫

For an integer $n \geq 1$, find a simple form of

$$\sum_{k=0}^{n} \binom{2n}{2k} = 2^{2n-1}$$

and prove your answer.

Hint: How many ways can we color a $2 \times n$ grid with black and white so there are even number of black cells? Find a recursion.

Identity 3 🍫🍫🍫

For integers $m \geq 0$ and $n \geq 0$, show that

$$\sum_{0 \leq k \leq m} \binom{n-k}{m-k} = \binom{n+1}{m}.$$

Answer is here.

Strings and Integers

A simple digit 🍫

How many $2n$-digit positive integers can be formed if the digits in odd positions (counting the rightmost digit as position 1) must be odd and the digits in even positions must be even and positive?

Integer composition 🍫🍫

How many integer-valued solutions are there for

$$x_1 + x_2 + x_3 + x_4 + x_5 = 63,$$

such that $x_i \geq 0$ for all i and $x_2 \geq 10$.

Multiple of 8 🍫🍫🍫

Let (a, b, c) be a string with the alphabet $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $a + b + c$ is a multiple of 8. Examples

$$(1, 1, 6), \qquad (1, 2, 5), \qquad (7, 1, 8), ...$$

How many such strings are possible? Proof?

Generating functions

What is the GF and EGF 🍫

Let $a_n = 1$ for all odd n and $a_n = 0$ for all even n. What is the GF for $(a_n, n \geq 0)$? What is the EGF?

Integer composition 🍫🍫

How many integer-valued solutions are there for

$$x_1 + x_2 + x_3 + x_4 + x_5 = 63,$$

such that $x_i \geq 0$ for all i and $x_2 \geq 10$. Use GF to find the answer.

What is the GF? 🍫🍫🍫

Recall that the GF for integer partition is

$$P(x) = \prod_{m \geq 1} \frac{1}{1 - x^m}.$$

What is the GF for integer portion such that each part is at most 3? What is the GF for integer portion such that we have at most 3 parts?

Inclusion-Exclusion

Multiple of 5 and 2 🍫

How many positive integers less than or equal to 100 are divisible by 2? How many positive integers less than or equal to 100 are divisible by 5? Use this information to determine how many positive integers less than or equal to 100 are divisible by neither 2 nor 5.

Counting integers 🍫🍫

How many integers in $\{1, \ldots, 100\}$ are not divisible by $2, 3$ or 5?

Answer here.

Bad lock 🍫🍫🍫

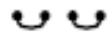
To enter your apartment building, you need to input 3 digits code into the lock. The lock is broken, so if you get any 2 digits right, the door will open. You forgot the code so you tried $(1, 2, 3)$ but it does not work. How many possible codes have been eliminated by this trial so you don't have to try them anymore.

Catalan Number

Non-intersecting arcs 🍫

Prove that $C(n)$ is the number of ways of connecting $2n$ points in a line by n non-intersecting arcs, and all the arcs are below the points. For example

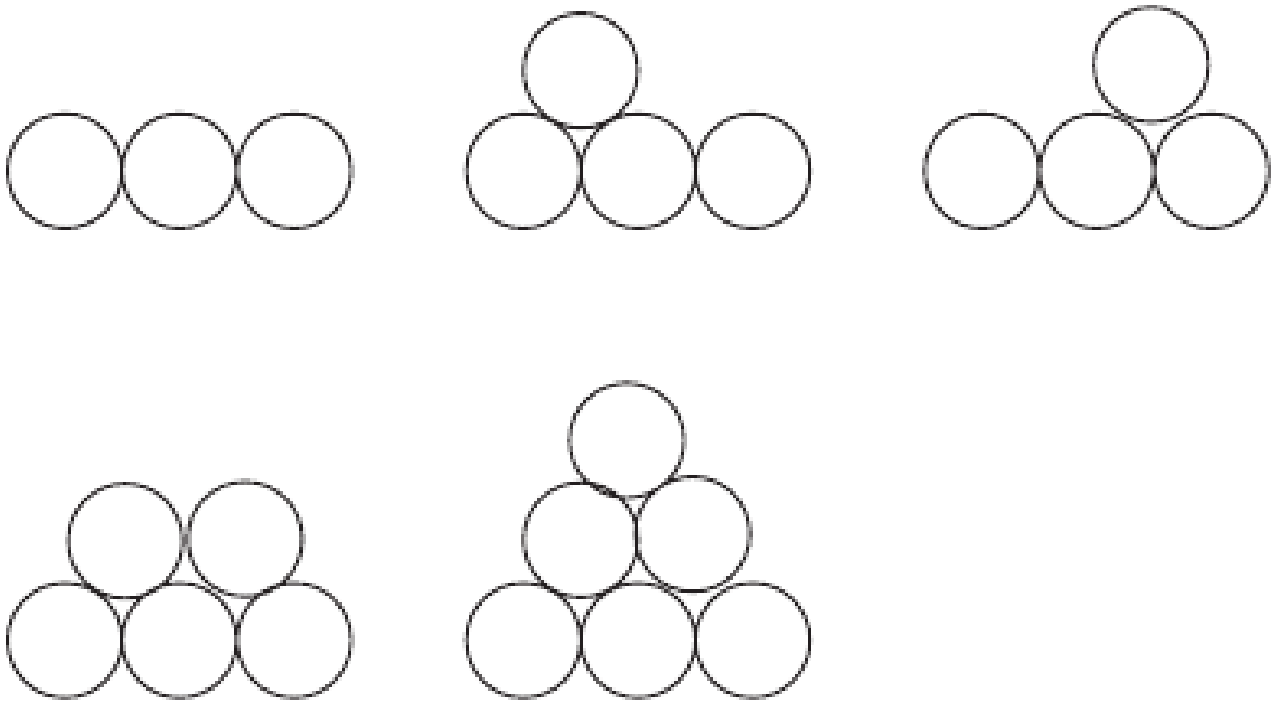
For $n = 1$, there is only 1 such planar diagram:

For $n = 2$, there are two such planar diagrams:  and 

For $n = 3$, there are five such planar diagrams:

Stack of coins 🍫🍫

Prove that $C(n)$ is the number of ways to stack coins in the plane, the bottom row consisting of n consecutive coins.



Stair-steps

Prove that $C(n)$ is the number of ways to tile a stair-step shape of height n with n rectangles. The following figure illustrates the case $n = 4$: