Tutorial 1

Combinatorics 1M020

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Format

In the first tutorial session, we will spend half of the time (45 min) solving the problems listed below.

For the second half of the session, you can either leave, or stay in class to do the assignment or solve the problems recommended at the end of this document. I will stay to answer your questions.

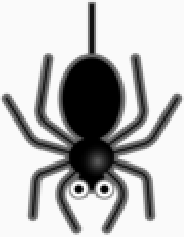
Problems of counting

A spider with shoes and socks

Problem

A spider needs a sock and a shoe for each of its eight legs. In how many ways can it put on the shoes and socks, if socks must be put on before the shoe?

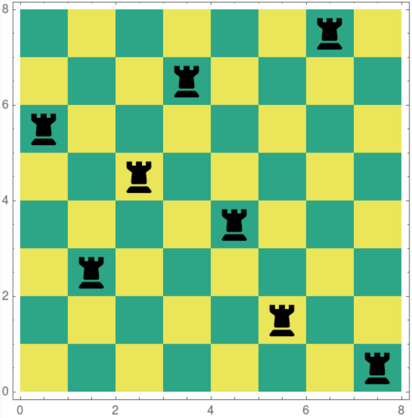
Hint: The answer is a multinomial coefficient.



Peaceful rooks

Problem

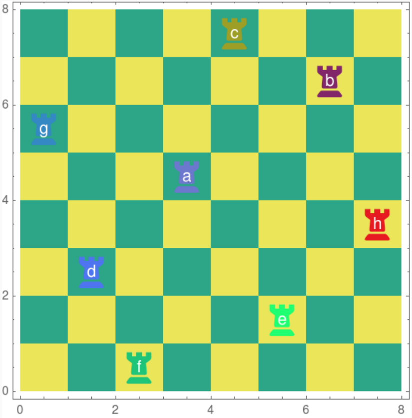
How many possibilities are there to put 8 identical rooks on a chessboard so they do not attack each other (i.e., no two rooks are in the same row or same column)?



Peaceful rooks

Problem

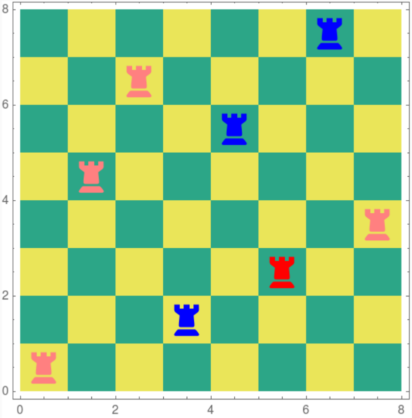
What if we have 8 distinct rook?



Peaceful rooks

Problem

What if we have one red, 3 blue, and 4 pink rooks?



Problems of binomial identities

Extend the defintion

For integers $n \geq m \geq 0$, we have defined that

$$\binom{n}{m} = \frac{P(n,m)}{m!} = \frac{n(n-1)(n-2)\dots(n-m+1)}{m!}$$

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But let's be crazier and let $r \in \mathbb{R}$ and $k \in \mathbb{Z}$ and define

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)(r-2)\dots(r-k+1)}{k!} & k \geq 0, \\ 0 & k < 0. \end{cases}$$

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For integers $k > r \geq 0$, $\binom{r}{k} = 0$ – there is no way to choose 5 🍌 out of 3 🍌.

For integers $r \geq 0$ and $k < 0$, $\binom{r}{k} = 0$ – there is no way to choose -1 🍌 out of 3 🍌.

If $r < 0$, no combinatorial interpretation.

Binomial identity

Problem

Show that $\binom{n}{k} = \binom{n}{n-k}$ for integers $n \geq 0$ and k. Find an example when $n < 0$ does not work.

Binomial identity

Problem

Show that
$$k\binom{r}{k} = r\binom{r-1}{k-1}$$
for all integers k.

Binomial identity – Vandermonde's convolution

Problem

For all integers m, n

$$\sum_{k\in\mathbb{Z}} \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$$

Note by the extended definition, we do not need to restrict k anymore.

Binomial identity

Problem

For all integers $l \geq 0, m, n$

$$\sum_{k \in \mathbb{Z}} \binom{l}{m+k} \binom{s}{n+k} = \binom{l+s}{l-m+n}$$

Hint – use for all integers m, n

$$\sum_{k \in \mathbb{Z}} \binom{r}{m+k} \binom{s}{n-k} = \binom{r+s}{m+n}$$

Binomial identity

Problem

Simplify the following for integers $n \geq 0$

$$\sum_{k \in \mathbb{Z}} \binom{n}{k} \binom{s}{k} k$$

Hint – use for all integers $l \geq 0, m, n$

$$\sum_{k \in \mathbb{Z}} \binom{l}{m+k} \binom{s}{n+k} = \binom{l+s}{l-m+n}$$

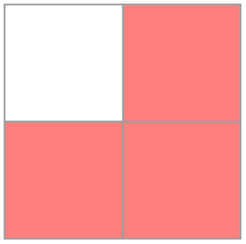
Problems of induction

L trominoes tiling

problem

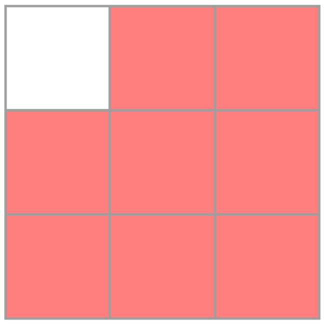
Proving that a $2^n \times 2^n$ chessboard with a single square missing can be covered using L-shaped tromino (made out of three squares).

Example, for n=1, one piece of L tromino is enough.



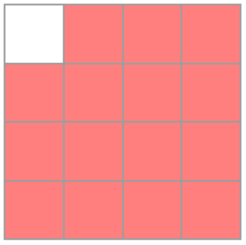
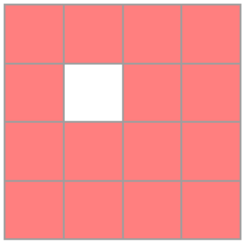
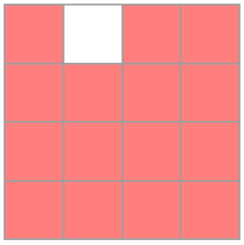
L trominoes tiling

We cannot do it for 3×3 chessboard. Why?



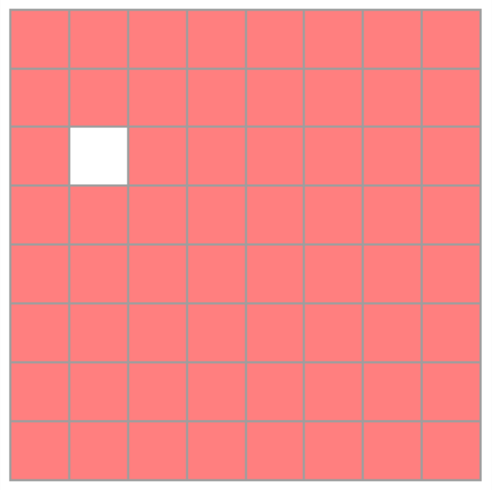
L trominoes tiling

Can you see how to do it for $n = 2$?

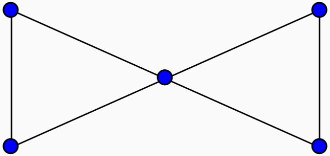




L trominoes tiling

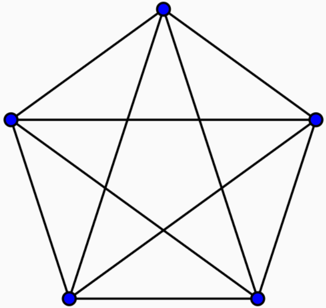
What about $n = 3$?



A planar graph is a graph that can be drawn on the plane in a way that its edges intersect only at their endpoints (no crossing).



A planar graph



Not a planar graph

Euler's formula

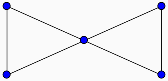
If a connected planar graph is drawn in the plane without any edge intersections, and

- v is the number of vertices
- e is the number of edges
- f is the number of faces

then

$$v - e + f = 2.$$

In this example $v = 5, e = 6, f = 3$ – the area outside the graph is also a face.



Problems of strings

Apples and bananas

Problem

Prove that the number of strings of m 🍎 and at most n 🍌 is

$$\binom{m+n+1}{m+1}.$$

Apples and bananas

Problem

Prove that the number of strings of at most m 🍎 and at most n 🍌 is

$$\binom{m+n+2}{m+1} - 1.$$

Hint: Either use the proof from the previous one, or use that for $n \geq 0, r \geq 0$

$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}.$$

Problems of integer composition

Bars and balls argument – review

Example

Amanda wants to give her 3 children \$10 so everyone has > 0

There are $\binom{9}{2}$ ways to insert 2 bars in 9 gaps.

For m objects, we can insert $n-1$ bars into $m-1$ gaps.

Bars and balls argument – empty cells allowed

Example

Amanda wants to give her 3 children \$10 so everyone has ≥ 0, i.e., the following is allowed

💵 💵 🚦 🚦 💵 💵 💵 💵 💵 💵 💵 💵

There are $\binom{12}{2}$ ways to do this, why?

More variations

We count the number of integer solutions to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 538$$

subject to various sets of restrictions on the values of $x_1, x_2, \ldots, x_6$. Some of these restrictions will require that the inequality actually be an equation.

The number of integer solutions is:

1. $C(537, 5)$, when all $x_i > 0$ and equality holds;
2. $C(543, 5)$, when all $x_i \geq 0$ and equality holds;
3. $C(291, 3)$, when $x_1, x_2, x_4, x_6 > 0$, $x_3 = 52$, $x_5 = 194$, and equality holds;
4. $C(537, 6)$, when all $x_i > 0$ and the inequality is strict (Imagine a new variable x_7 which is the balance. Note that x_7 must be positive.);
5. $C(543, 6)$, when all $x_i \geq 0$ and the inequality is strict (Add a new variable x_7 as above. Now it is the only one which is required to be positive.); and
6. $C(544, 6)$, when all $x_i \geq 0$.

More exercise

Try these exercises

Textbook

- Section 2.9, 2, 4, 11, 12, 16, 18, 27, 28
- Section 3.11, 6, 11, 12

Use the extended definition of binomial coefficients to show

- $2^{2n} = \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$ for all integers $n \geq 0$.
- $(r-k)\binom{r}{k} = r\binom{r-1}{k}$ for all integer k
- $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$ for all integer k
- Use induction on l to show that for integers $l \geq 0$, and integers m, n

$$\sum_{k\in\mathbb{Z}} \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}$$